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Prof. Dr. Gabriele Benedetti
&
Davide Legacci

Mathematisches Institut (Uni HD)

“Zero-sum evolutionary games and convex Hamiltonian systems“

November 20, 2020 1:30 PM

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ABSTRACT

Evolutionary games describe the time evolution of large sets of strategically interacting agents who adapt their choices in light of current payoff opportunities. Nowadays they are used to model plenty of phenomena from the emergence of conventions, norms, and institutions in economic, social, and technological environments to natural selection in biological environments.
In this talk we focus on the class of zero-sum replicator games, which model competitive interaction. If there is an equilibrium where all strategies coexist, these games admit a Hamiltonian formulation where the Hamiltonian function is given by the entropy relative to the equilibrium (Akin and Losert, 1984). This is an extremely significant result, which will allow us to use the principle of least action and the convexity of the Hamiltonian to find periodic time evolutions with prescribed relative entropy via numerical optimization.

By ZOOM video webinar system
Meeting ID: 935 6549 3662, Code: 928036
Contact: office@structures.uni-heidelberg.de

Zero-sum evolutionary games and convex Hamiltonian systems

Foundations

Gabriele Benedetti & Davide Legacci

Presentation of the EP 3.2

STRUCTURES Jour Fixe – November 20, 2020

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Idea

- The replicator dynamical system models the evolution of the aggregate behavior of individuals in a population.
- This system is Hamiltonian in the appropriate geometrical framework.

- Find periodic time evolutions with prescribed energy using the convexity of the Hamiltonian.

Dynamical system on categorical probability distributions[1]

- Discrete alphabet $\mathcal{S}(n+1) \ni i$
- $p \in P(\mathcal{S}(n+1)) = \Delta^n, \quad p \mapsto x \,:\, x^i = p(i)$

$$x \in \Delta^n \subset \mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} : \sum_i x^i = 1, x^i \geq 0\}$$

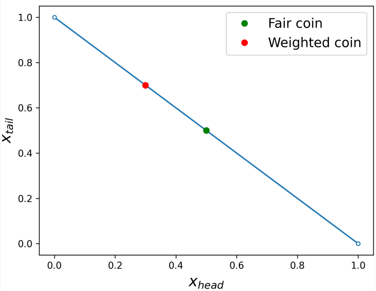

Figure 1: Space of PDs $x = (x_{head}, x_{tail})$

$$\dot{x}(t) = X_{\text{rep}}\left(x(t)\right)$$

Leaves interior $\mathring{\Delta}^n$ invariant

[1] Har09, p. 4.

Replicator dynamical system - Population Dynamics[2]

- Population composed of $n+1$ types or species
- The fitness or growth rate of each species $F : \mathbb{R}_+^{n+1} \to \mathbb{R}^{n+1}$ depends on the composition of the whole population

$$\dot{P}_i(t) = P_i(t)\, F_i\left(P(t)\right), \quad P \in \mathbb{R}_+^{n+1}$$

- Descend from $\mathbb{R}_+^{n+1}$ through a normalization map onto the simplex to the replicator equation, i.e. look at PD $x \in \Delta^n$ on the set of species, with $x^i = \frac{P_i}{\sum_j P_j}$

$$\dot{x}^i = x^i \left(f_i(x) - \sum_h x^h f_h(x) \right), \quad f_h(x) = F_h(P)$$

[2] HS88, p. 67; San10, p. 160.

Replicator dynamical system - Evolutionary Game Theory[3]

Population game $(\mathcal{S}(n+1), f)$: strategically interacting agents

- Agents choose a pure strategy from a finite set $\mathcal{S}(n+1)$
- The payoff of each pure strategy $f:\ \Delta^n \to \mathbb{R}^{n+1}$ depends on current population state $x \in P(\mathcal{S}(n+1)) = \Delta^n$

Mean dynamics via *revision protocol* $\rho:\ \Delta^n \to \mathbb{R}_+^{(n+1)\times(n+1)}$

$$\dot{x}^i = \left(\sum_j x^j \rho_{ji}(x)\right) - \left(x^i \sum_j \rho_{ij}(x)\right)$$

$$\rho_{ij}(x) = x^j \left(f_j(x) - f_i(x)\right)_+ \quad \text{[Imitation]}$$

[3] San10, p. 126.

Replicator dynamical system - Information geometry[4]

Riemannian game = P.G. with Riemannian metric on $P(\mathcal{S}(n+1))$

- Gain $G(x,v) = \sum_i f_i(x)\, v^i, \quad v \in T_x\Delta, \quad f$ payoff
- Cost $C(x,v) = \frac{1}{2}\, \|v\|_x^2$

$$\dot{x} = \arg\max_{v \in T_x\Delta} \left(G(x,v) - C(x,v)\right)$$

- Replicator with *Fisher-Shahshahani metric* $g_{ij}(x) = \delta_{ij} \,/\, x^i$
- Replicator fields $\supset$ Fisher gradients
  - E.g. **linear symmetric payoff** replicator field
  - Wright and Fisher, classical population genetics

[4] MS18; Har09.

Zero-sum replicator systems

- Average payoff vanishes identically $\sum_i x^i f_i(x) \equiv 0$
- E.g. linear anti-symmetric payoff $f_i(x) = A_{ij} x^j$, $A + A^T = 0$

$$\dot{x}^i = x^i \left(f_i(x) - \sum_h x^h f_h(x) \right) = x^i f_i(x)$$

- Extensively studied in classical GT[5]
- Very restrictive assumption for real life applications
- *Discrete* zero-sum replicator: model for gene conversion[6]
- Interesting in its own right for Hamiltonian character

[5] Sig11, p. 4.

[6] Nag83b; Nag83a.

- General framework: *stratified space M*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \quad \text{[A.S., Leibnitz, Jacobi]}$$
$$\{f,g\} = \{x^i, x^j\}\, \partial_i f\, \partial_j g = \pi^{ij}\, \partial_i f\, \partial_j g$$

- π: $\binom{2}{0}$ tensor-field [A.S, Jacobi]
- Hamiltonian vector fields and dynamical systems

$$X_H = \pi(\mathrm{d}H, \cdot) \qquad X^i_H = \pi^{hi}\, \partial_h H$$
$$\dot{x} = X_H(x) \qquad \dot{x}^i = \{H, x^i\}$$

Poisson structure on a space M - degeneracy

- π, and equivalently $\{\cdot,\cdot\}$, can be degenerate
- No restriction on the dimension of M

$$M = \mathbb{R}^3, \{x^i, x^j\} = \begin{pmatrix} 0 & 1 & A \\ -1 & 0 & B \\ -A & -B & 0 \end{pmatrix}$$

- Casimir $f(x) = Bx^1 - Ax^2 + x^3$, namely $\{f,\cdot\} \equiv 0$
- Change coordinates to isolate degeneracy

$$y^1 = x^1, y^2 = x^2, y^3 = Bx^1 - Ax^2 + x^3$$
$$\{y^3,\cdot\} \equiv 0$$

Stratified Poisson structure for the standard simplex[7]

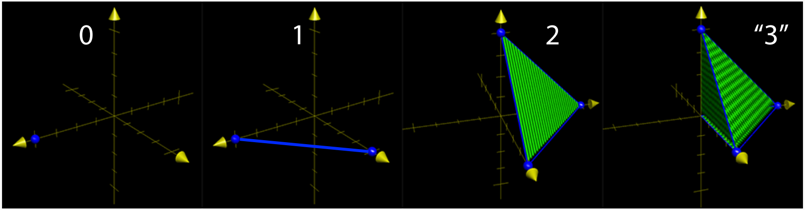

Figure 2: Simplices representable in three dimensions. Each face is a Poisson manifold.

Poisson structure on Δ^n with A anti-symmetric $(n+1)$ matrix

$$\{x^i, x^j\}_A = x^i x^j \left(\sum_h (A_{ih} + A_{hj}) x^h - A_{ij} \right)$$

[7] Regular and Singular Poisson Reduction Theorems [OR04, p. 364] [ORF09, p. 1273]

Interior Hamiltonian dynamics of zero-sum replicator

- Interior fixpoint $q \in \mathring{\Delta}^n$
- $H_q(x) = D_{KL}(q \| x) = \sum_i q^i \ln \frac{q^i}{x^i}$ Relative entropy
  - Provides the Fisher metric
  - Appears in EGT as Lyapunov function given ESS strategy

Theorem

Consider a replicator dynamical system with anti-symmetric payoff matrix A. If a fixpoint q exists in $\mathring{\Delta}^n$, then the system is Hamiltonian w.r.t $\{x^i, x^j\}_A$, with $H_q(x)$ as Hamiltonian function[8].

[8] AD14.

Interior Hamiltonian dynamics of zero-sum replicator

- Interior trajectories do not converge to the boundary nor to a fixpoint
- Bounded orbits, periodic or not?

Interior Hamiltonian dynamics of zero-sum replicator

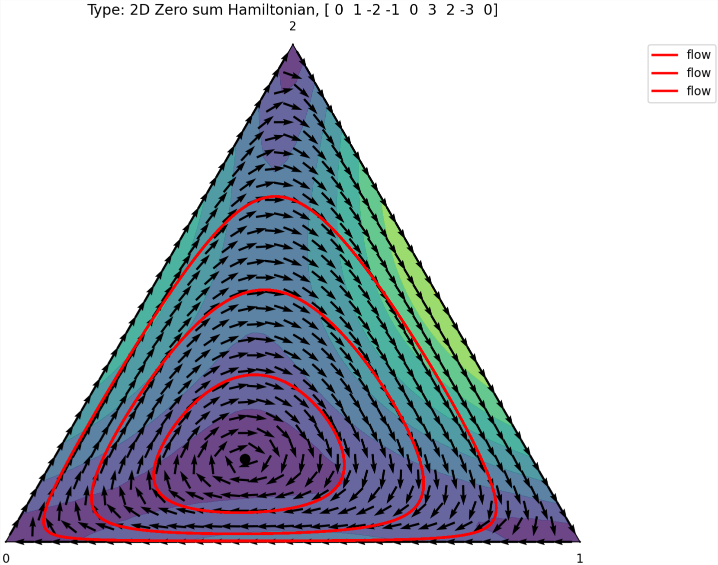

Figure 3: Three periodic orbits around the fixpoint.

Interior Hamiltonian dynamics of zero-sum replicator

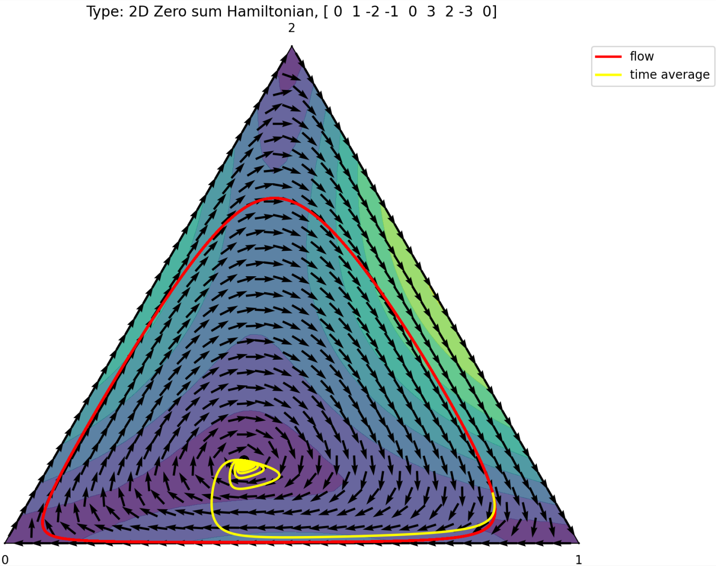

Figure 4: One periodic orbit. The time average converges to the fixpoint.

Interior Hamiltonian dynamics of zero-sum replicator

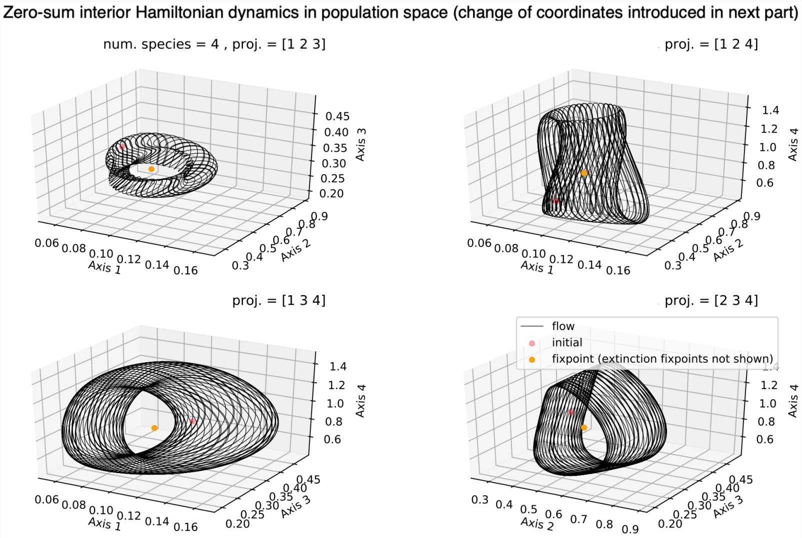

Figure 5: Non periodic bounded orbits

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Thanks