Bachelorarbeit

The pentagrid as a device for the study of Penrose tilings

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Datum der Abgabe: 30. März 2017

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Declaration

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"We cannot find a lattice that goes into itself under other rotations, such as by $\frac{2\pi}{5}$ radians [...]"

C. Kittel, [4]

Abstract

When studying Penrose tilings, one can define the *pentagrid* which can be associated with a pattern arising from two fundemental building units: rhombuses with coloured and oriented edges.
It turns out that every pentagrid generates such a rhombus pattern. In general, there is a duality between pentagrids and these patterns.
Work with the pentagrid can be reduced to several parameters from which properties of Penrose tilings in the form of rhombus patterns can be derived. In this scenario the ring extension

$$\mathbb{Z}\left[e^{\frac{2\pi i}{5}}\right] := \left\{\sum_{j=0}^{4} a_j (e^{\frac{2\pi i}{5}})^j \mid a_j \in \mathbb{Z}\ \forall j\right\}$$

plays a central role as well as the ideal that is generated by $(1 - e^{\frac{2\pi i}{5}})$.
This work is concerned with the parameters of the pentagrid and with the information that can be deduced.

Zusammenfassung

Bei dem Studium von Penrose-Parkettierungen kann man das *Pentagrid* definieren, welches zu einem Muster assoziiert werden kann, das von von zwei fundamentalen Bausteinen, nämlich Rhombussen mit gefärbten und orientierten Kanten, ausgeht.
Es stellt sich heraus, dass ein jedes Pentagrid ein Muster aus solchen Rhombussen erzeugt und generell, dass eine Dualität zwischen diesen Gittern und Mustern herrscht.
Die Arbeit mit dem Pentagrid lässt sich auf wenige Parameter reduzieren, aus denen Eigenschaften von Penrose-Parkettierungen in Form von Rhombusmustern abgeleitet werden können. Eine Zentrale Rolle spielt hierbei die Ringerweiterung

$$\mathbb{Z}\left[e^{\frac{2\pi i}{5}}\right] := \left\{\sum_{j=0}^{4} a_j (e^{\frac{2\pi i}{5}})^j \mid a_j \in \mathbb{Z}\ \forall j\right\}$$

und das von $(1 - e^{\frac{2\pi i}{5}})$ erzeugte Ideal darin.
Diese Arbeit beschäftigt sich mit den Parametern des Pentagrids und den Informationen, die wir daraus gewinnen können.

Contents

1 Introduction

1.1 A brief history of Penrose tilings

Penrose tilings were introduced by Roger Penrose in 1974 after subdividing a pentagon into six smaller pentagons and five triangles which Penrose called "gaps" ([3], page 32). Iterating the process of subdivision would leave gaps of different shapes which had to be filled. The result was a set of six tiles which forced non-periodicity, after adding specific matching rules. According to ([3], page 33), this meant the absence of any period parallelogram.

Starting from these six tiles, Penrose later found a set of five tiles forcing non-periodicity. The number of tiles could even be reduced to two: "kites" and "darts", as John Conway suggested their names.
Tilings of this kind were used during medieval times in Islamic art, partially in order to avoid "images of life objects" ([2], page 998), which illustrates the somewhat hidden presence of mathematical issues in art.
When it comes to art and mathematics, some people might think of a certain number:

$$\frac{1}{2}(1+\sqrt{5}),$$

the *golden ratio*. Penrose tilings have several features involving this number as well, such as the inflation (or deflation) of a pattern or even proportions of edges and diameters of the prototiles themselves.

In 1981, Nicolaas Govert de Bruijn introduced a device to study algebraic properties of Penrose tilings: the pentagrid. This device serves as a global construction method of Penrose tesselations. In addition, it encodes attributes of these patterns, like symmetries and types of vertices.

Up to 1982, there was a paradigm in Physics and Chemistry that did not allow the existence of "a ten-fold scattering pattern" ([2], page 998) as it was found by Dan Shechtman from a substance which looked like a normal crystal.
Work concerning Penrose tilings helped to change the opinions of some scientists opposing Shechtman's observations. The discovery of quasicrytsals even was awarded the 2011 Nobel Prize in Chemistry ([2], page 998).

My personal motivation to work on the topic of this thesis grew soon, after perceiving the quasiperiodic aesthetics behind seemingly simple patterns, such as Penrose tesselations, and how patterns of this sort might appeal to many people. In a way, I consider Penrose tilings a mathematical curiosity and item that might even appeal to people who are usually less fond of this subject matter.
My perception was followed by the question concerning the possibilities of understanding these patterns in terms of mathematical formalism.
One way of answering this question was given by the results centered around the pentagrid, which were published by de Bruijn.

1.2 Conventions and first reflections

Throughout the following pages, this thesis focuses (not exclusively, but mostly) on working with the ring $\mathbb{Z}[\zeta]$; $\mathbb{Z}$ denotes the ring of integers and $\zeta = e^{\frac{2\pi i}{5}}$ is a fifth root of unity.
Generally speaking, an n-th root of unity is a solution of the equation $X^n - 1 = 0$ and n is a positive integer (see [5]).
In this, the set of fifth (n-th) roots of unity is a cyclic subgroup of the field of complex numbers, which will be denoted by $\mathbb{C}$; this subgroup is of order five (n).
All fifth roots of unity will be denoted by ζ^j; j is some integer. Since the set $\{\zeta^j \mid j \in \mathbb{Z}\}$ forms a cyclic group, it suffices to look at j as an element of $\{0, 1, 2, 3, 4\}$; in this, we can add elements modulo 5.
As it is a ring extension of $\mathbb{Z}$, we can define $\mathbb{Z}[\zeta]$ as follows:

$$\mathbb{Z}[\zeta] := \Big\{ \sum_{j=0}^{4} a_j \zeta^j \mid a_j \in \mathbb{Z} \ \forall j \Big\}.$$

Since most indices in this thesis revolve around $\{\zeta^j \mid j \in \mathbb{Z}\}$, we will abbreviate "for all j in $\{0, 1, 2, 3, 4\}$" by using "for all j" instead.
Conventionally, the field of real numbers will be denoted by $\mathbb{R}$, the field of rational numbers will be $\mathbb{Q}$ and the set of non-negative integers will be $\mathbb{N}_{\geq 0}$. As a consequence, the set of positive integers will be denoted by $\mathbb{N}_{>0}$.
For a complex number $z = a + bi$, with real numbers a and b, $\overline{a + bi} = a - bi$ is the complex conjugation and $\Re(z) = \frac{z + \overline{z}}{2}$ denotes the "real part" of z. Additionally, $\Im(z)$ is its "imaginary part".
For ζ^j,

$$\begin{aligned}
\overline{\zeta^j} &= \overline{e^{\frac{2j\pi i}{5}}} \\
&= \overline{\cos(\frac{2j\pi}{5}) + i \sin(\frac{2j\pi}{5})} \\
&= \cos(\frac{2j\pi}{5}) - i \sin(\frac{2j\pi}{5}) \\
&= \zeta^{-j}
\end{aligned}$$

holds.
In $\mathbb{F}^n$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, we use the notation $\langle v, w \rangle$ for the standard innner product of two vetors v and w of $\mathbb{F}^n$. We generally put $|v| = \langle v, v \rangle^{\frac{1}{2}}$ for a vector v.

2 Penrose tilings

Previously, Penrose tilings were introduced as “kites” and “darts” as prototiles (building blocks). An alternative access to Penrose tilings are so called “thick” and “thin” rhombuses.
These rhombuses can be turned into kites and darts, and vice versa, as introduced by Penrose, thus creating a duality between the two sets of prototiles. This duality enables the study of the tesselation by Penrose with thick and thin rhombuses.
This alternative can be used to derive a way to study algebraic properties of the tilings.

2.1 Arrowed rhombus patterns

Starting from two building blocks, thick and thin rhombuses, we can generate a pattern by using specific rules:
After orienting / colouring the edges of the rhombuses, we assemble them, following the principle that edges must have the same colour and the same orientation for the purpose of matching together.
We will introduce oriented double arrows and single arrows to indicate green edges (double arrows) and red edges (single arrows) in order to avoid problems with the colouring of the edges. Every edge of every building block is of length 1.
In total, we get a pattern which is made up of arrowed rhombuses: an “arrowed rhombus pattern”, or in short: “AR pattern”.

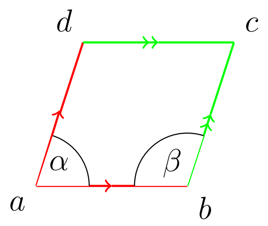

Figure 1: A thick rhombus, $\alpha = 72°$, $\beta = 108°$

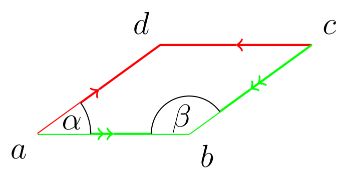

Figure 2: A thin rhombus, $\alpha = 36°$, $\beta = 144°$

3 The pentagrid

In the following section, we will work with a vector $\gamma = (\gamma_0, ..., \gamma_4)$ of real numbers with zero sum:

$$\sum_{j=0}^{4} \gamma_j = 0$$

Definition 3.1. *For $j \in \{0, ..., 4\}$, we define the* ***j-th grid***

$$\mathfrak{G}_{\gamma,j} := \left\{ z \in \mathbb{C} \mid \Re(z\zeta^{-j}) + \gamma_j \in \mathbb{Z} \right\},$$

where every γ_j is taken from γ, as mentioned before.
With γ, we associate the ***pentagrid***

$$\mathfrak{P}_\gamma := \bigcup_{j=0}^{4} \mathfrak{G}_{\gamma,j}.$$

A pentagrid $\mathfrak{P}_\gamma = \bigcup_j \mathfrak{G}_{\gamma,j}$ is called ***regular*** *if the number of j-grids containing z is at most two for all complex numbers z.*
Otherwise, $\mathfrak{P}_\gamma$ is called ***singular****.*
A ***mesh*** *is a connected component of the complement of the pentagrid in the plane.*

In the first part of this thesis, we will not be concerned with singular pentagrids but we will continue with building the theory of regular pentagrids.

When we talk about rhombus patterns arising from a pentagrid, it is important to build the theory towards a notion of the four vertices of a rhombus.
Starting from $\gamma = (\gamma_0, ..., \gamma_4)$ and an arbitrary complex numer z, we define:

$$K_j(z) := \lceil \Re(z\zeta^{-j}) + \gamma_j \rceil \in \mathbb{Z}, \tag{3.1}$$

where for a real number x, $\lceil x \rceil$ is the smallest following integer.

Naturally, we get a total of five values $K_j(z)$ for every z.
By

$$\Re(z\zeta^{-r}) + \gamma_r = k_r, \quad \Re(z\zeta^{-s}) + \gamma_s = k_s$$

with integers r, s, k_r, k_s so that $0 \leq r < s \leq 4$, a point z' in $\mathbb{C}$ solving both equations can be determined. In this case z' would be the point of intersection of a line of $\mathfrak{G}_{\gamma,r}$ and a line of $\mathfrak{G}_{\gamma,s}$.
Using (3.1), we get four different vectors in a small neighbourhood of z':

$$(K_j(z') + \varepsilon_1\delta_{j,r} + \varepsilon_2\delta_{j,s})_{j=0,\cdots,4} \tag{3.2}$$

with Kronecker's symbol $\delta_{a,b} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{else} \end{cases}$ and $(\varepsilon_1, \varepsilon_2) = (0,0), (0,1), (1,0), (1,1)$.
These four values will correspond to **the vertices of a rhombus**. Since these values are generated by points inside the four meshes surrounding an intersection point, we can already associate the meshes of the **regular** pentagrid with **vertices of rhombuses**.

3.1 The AR pattern associated to a regular pentagrid

Continuing where we left of at the beginning of this section, we are about to expand the idea of AR patterns arising from regular pentagrids by examining the tiling of the plane and the orientation as well as the colouring of the rhombuses. Motivated by previous reflections, we define:

Definition 3.2. *The **set of vertices of the rhombuses** is given by*

$$U_\gamma := \left\{ \sum_{j=0}^{4} K_j(z)\zeta^j \mid z \in \mathbb{C} \right\} \tag{3.3}$$

with $K_j(z)$ from (3.1).
The induced map $f : \mathbb{C} \longrightarrow \mathbb{C},\ z \longmapsto \sum_{j=0}^{4} K_j(z)\zeta^j$ is constant in every mesh of the regular pentagrid $\mathfrak{P}_\gamma$ according to (3.2).

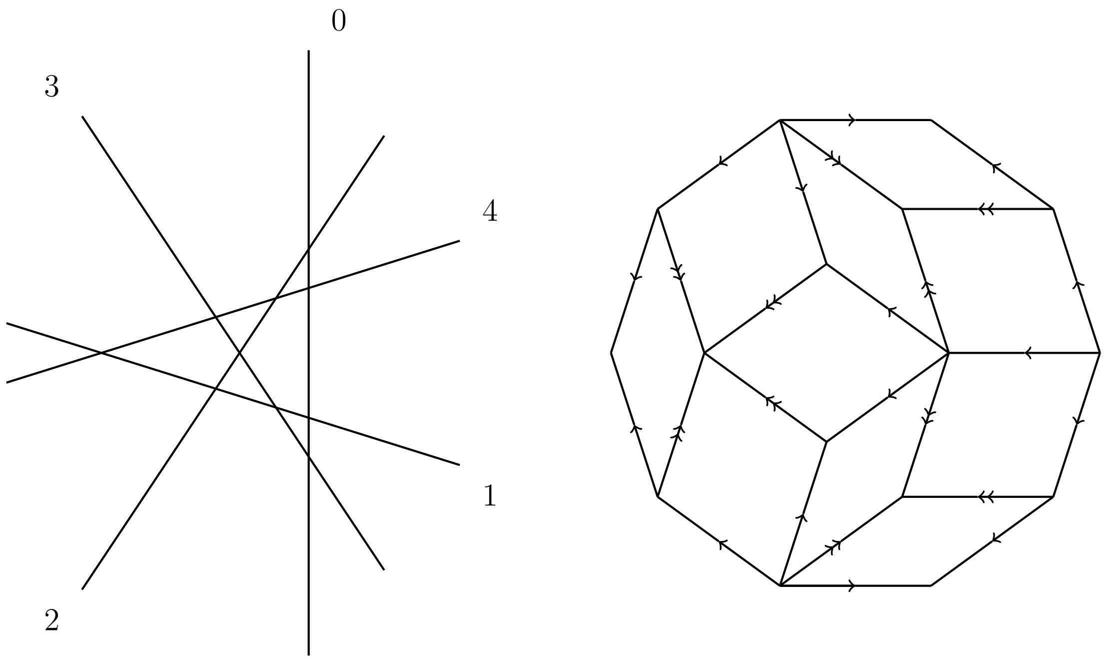

Figure 3: A section of a pentagrid and the corresponding AR pattern

Theorem 3.3. *The rhombuses constructed from the intersection points of a regular pentagrid $\mathfrak{P}_{\gamma=(\gamma_0,\ldots,\gamma_4)}$ with $\sum_{j=0}^{4} \gamma_j = 0$ form a tiling of the plane.*

Proof. Recall that the map $f(z)$ (see (3.3)) is constant in every mesh of $\mathfrak{P}_{\gamma=(\gamma_0,\ldots,\gamma_4)=:\gamma}$. Identifying these meshes with the vertices of rhombuses and intersection points in $\mathfrak{P}_\gamma$ with the faces of the rhombuses, we deduce that the rhombuses do not overlap locally.
We are then left with the task to cover every point in the complex plane by a rhombus. With

$$\lambda_j(z) := \underbrace{\lceil \Re(z\zeta^{-j}) + \gamma_j \rceil}_{=K_j(z);\ (3.1)} - \Re(z\zeta^{-j}) - \gamma_j, \tag{3.4}$$

we can rewrite f as follows:

$$\begin{aligned} f(z) &= \sum_{j=0}^{4} K_j(z)\zeta^j \\ &= \sum_{j=0}^{4} (\lambda_j(z) + \Re(z\zeta^{-j}) + \gamma_j)\zeta^j \\ &= \sum_{j=0}^{4} (\lambda_j(z) + \gamma_j)\zeta^j + \sum_{j=0}^{4} \Re(z\zeta^{-j})\zeta^j \\ &= \sum_{j=0}^{4} (\lambda_j(z) + \gamma_j)\zeta^j + \sum_{j=0}^{4} \frac{z\zeta^{-j} + \overline{z}\zeta^j}{2}\zeta^j \\ &= \sum_{j=0}^{4} (\lambda_j(z) + \gamma_j)\zeta^j + \underbrace{\sum_{j=0}^{4} \frac{z}{2}}_{=\frac{5z}{2}} + \frac{\overline{z}}{2} \underbrace{\sum_{j=0}^{4} \zeta^{2j}}_{=0} \end{aligned}$$

According to its definition (3.4),

$$0 \leq \lambda_j(z) < 1 \tag{3.5}$$

holds. As a result, $|\sum_{j=0}^{4}(\lambda_j(z) + \gamma_j)\zeta^j|$ has an upper bound, which we will call $m(\gamma)$ (since it only depends on γ), and $f(z) - \frac{5z}{2}$ is bounded for every z.

When we take an arbitrary complex number $\omega \in \mathbb{C}$, $\vartheta \in [0, 2\pi)$ and $r \in \mathbb{R}_{>0}$, $\vartheta \longmapsto \omega + re^{i\vartheta}$ describes a circle of radius r around ω which is run through counter-clockwise.
Substitution leads to

$$f(\omega + re^{i\vartheta}) = \sum_{j=0}^{4} (\lambda_j(\omega + re^{i\vartheta}) + \gamma_j)\zeta^j + \frac{5}{2}\omega + \frac{5}{2}re^{i\vartheta}.$$

To ensure that $f(\omega + re^{i\vartheta})$ runs around ω for $\vartheta \in [0, 2\pi)$, r must be chosen accordingly.
In this, ω must be covered by a rhombus, since f describes the vertices in the rhombus pattern.
Due to the upper bound $m(\gamma)$, we can make sure that every $\omega \in \mathbb{C}$ is covered by a rhombus. $\square$

Remark. *An example for a lower bound of the radius r could be the following:*

$$\begin{aligned} \frac{5}{2}r &> |\frac{5}{2}\omega + m(\gamma) - \omega| = |\frac{3}{2} + m(\gamma)| \\ \iff r &> |\frac{3}{5}\omega + \frac{2}{5}m(\gamma)|. \end{aligned}$$

When concerned with a regular pentagrid $\mathfrak{P}_{(\gamma_0,\ldots,\gamma_4)}$, $\Re(z\zeta^{-j}) + \gamma_j$ is an integer for at most two indices $j \in \{0, \ldots, 4\}$.

It follows (using (3.5)):

$$0 < \sum_{j=0}^{4} \lambda_j(z) < 5. \tag{3.6}$$

We now claim: $\sum_{j=0}^{4} \lambda_j(z) = \sum_{j=0}^{4} K_j(z)$ for all complex numbers z.
This holds indeed for we have:

$$\begin{aligned} \sum_{j=0}^{4} \Re(z\zeta^{-j}) &= \sum_{j=0}^{4} \frac{(z\zeta^{-j} + \overline{z}\zeta^{j})}{2} \\ &= \frac{z}{2} \underbrace{\sum_{j=0}^{4} \zeta^{-j}}_{=0} + \frac{\overline{z}}{2} \underbrace{\sum_{j=0}^{4} \zeta^{j}}_{=0} \\ &= 0 \end{aligned}$$

and $\sum_{j=0}^{4} \gamma_j = 0$.

Combined, we then get:

$$\begin{aligned} \sum_{j=0}^{4} \lambda_j(z) &= \sum_{j=0}^{4} K_j(z) - \underbrace{\sum_{j=0}^{4} \Re(z\zeta^{-j})}_{=0} - \underbrace{\sum_{j=0}^{4} \gamma_j}_{=0} \\ &= \sum_{j=0}^{4} K_j(z). \end{aligned}$$

Since all $K_j(z)$ are integers, their sum also is. Because of equality, $\sum_{j=0}^{4} \lambda_j(z)$ is an integer as well. Using (3.6), we conclude that

$$\sum_{j=0}^{4} K_j(z) \in \Big\{1, 2, 3, 4\Big\}.$$

Hence, every vertex in the rhombus pattern generated by $\mathfrak{P}_{(\gamma_0,\ldots,\gamma_4)}$ can be written as $\sum_{j=0}^{4} k_j\zeta^j$ with $\sum_{j=0}^{4} k_j \in \{1, 2, 3, 4\}$.
This serves as a motivation for the following

Definition 3.4. *The integer $\sum_{j=0}^{4} k_j \in \{1, 2, 3, 4\}$ is called the **index of a vertex**.*

Remark. *Starting from any given vertex, there are ten possible directional options for neighbouring vertices:*

$$\pm\zeta^j, \; j \in \Big\{0, \ldots, 4\Big\}.$$

The index changes according to the direction:
If the neighbouring vertex lies in any of the ζ^j-directions, the index changes by $+1$.
If the neighbouring vertex lies in any of the $-\zeta^j$-directions, the index changes by -1.

The argumentation in the proof of the following theorem will use the notion of the index of a vertex.
There are two options for a thick rhombus:

1. The vertices at the 72° angles have the indices 1 and 3. Every vertex at a 108° angle has the index 2.

2. The vertices at the 72° angles have the indices 2 and 4. Every vertex at a 108° angle has the index 3.

In a similar fashion, there are two cases for a thin rhombus:

1. The vertices at the 144° angles have the indices 1 and 3. Every vertex at a 36° angle has the index 2.

2. The vertices at the 144° angles have the indices 2 and 4. Every vertex at a 36° angle has the index 3.

When we appoint indices to vertices, the increase and decrease of neighbouring indices have to be taken into consideration.
Next, we divide the edges into **green** and **red** edges.

1. An edge connects an index 1 vertex and an index 2 vertex: **green**

2. An edge connects an index 2 vertex and an index 3 vertex: **red**

3. An edge connects an index 3 vertex and an index 4 vertex: **green**

We note that the above cases do not indicate the orientations of the edges in general.
It is possible to derive the orientations of **green** edges for there are two cases of those edges:
The **green** double arrow either points from 2 to 1 or from 3 to 4.
Since the enumeration above only states one option for **red** coloured edges, the task to orient these edges cannot be done as easily as the green coloured edges.

Theorem 3.5. *In the setting of Theorem 3.3, the rhombuses constructed from the intersection points can be provided with coloured and oriented edges in order to form an AR pattern.*

Proof. In order to orient the red edges, we choose two vertices X and Y from the AR pattern. The edge connecting these vertices is called $\overline{XY}$.
At X, the two rhombuses that have $\overline{XY}$ in common have the angles α and β.
Following the above schemes that put angles, vertices, indices, and colouring into context, we conclude:
If $\overline{XY}$ is red, both α and β are either bigger than 90° or smaller than 90°. We can thus orient the red edges.
An important remark is that this condition on the angles α and β holds if and only if $p+q$ is odd.

Without restriction, let l be a line of $\mathfrak{G}_{\gamma,0}$ and A and B be consecutive intersection points on l. Here, A is an intersection with a line of $\mathfrak{G}_{\gamma,p}$ and B is an

intersection with a line of $\mathfrak{G}_{\gamma,q}$, where p and q are taken from $\{1,...,4\}$. For later purposes, let l be the imaginary axis $i\mathbb{R}$.
The point A corresponds to the face of a rhombus which is adjacant to the face of the rhombus corresponding to B. The meshes on each side of the segment $\overline{AB}$ correspond to the vertices X and Y.
We generally call $\overline{AB}$ "red" if the index is 2 on one side of $\overline{AB}$ and 3 on the other side.
In addition to l being the imaginary axis $i\mathbb{R}$, we may also define γ_0 to be 0. For other cases, the pentagrid can be transformed (for details we refer to the fifth section). Using (3.1), we get for points iy on l:

$$\begin{aligned}\Re(iy\zeta^{-j}) &= y\Re(i\zeta^{-j}) = y\Re(i(\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5}))^{-j}) \\ &= y\Re(i(\cos(-j\frac{2\pi}{5}) + i\sin(-j\frac{2\pi}{5}))) = y\Re(i\cos(-j\frac{2\pi}{5}) - \sin(-j\frac{2\pi}{5})) \\ &= y\sin(j\frac{2\pi}{5})\end{aligned}$$

As a consequence,

$$K_j(iy) = \lceil y\sin(j\frac{2\pi}{5}) + \gamma_j\rceil.$$

In a regular pentagrid, $\gamma_1 + \gamma_4$ and $\gamma_2 + \gamma_3$ are not integers. It now holds that $p \neq q$ because the intersections of lines of $\mathfrak{G}_{\gamma,1}$ and $\mathfrak{G}_{\gamma,4}$ alternate. The same holds for $\mathfrak{G}_{\gamma,2}$ and $\mathfrak{G}_{\gamma,3}$.
Next, we assume that $p+q$ is even. It follows that

$$\{p,q\} \in \{\{1,3\},\{2,4\}\}. \tag{3.7}$$

After choosing $\gamma_0 = 0$, we get $\sum_{j=1}^{4}\gamma_j = 0$.

$$\begin{aligned} 0 \leq \lceil\gamma_1+\gamma_4\rceil - \gamma_1 - \gamma_4 &< 1 \\ 0 \leq \lceil\gamma_2+\gamma_3\rceil - \gamma_2 - \gamma_3 &< 1 \\ \Longrightarrow 0 \leq \underbrace{\lceil\gamma_1+\gamma_4\rceil + \lceil\gamma_2+\gamma_3\rceil}_{\in\{0,1\},\text{ because of integers}} &< 2\end{aligned}$$

The examination of the case $\lceil\gamma_1+\gamma_4\rceil + \lceil\gamma_2+\gamma_3\rceil = 0$ leads to

$$\begin{aligned}\lceil\gamma_1+\gamma_4\rceil &= \gamma_1+\gamma_4 \in \mathbb{Z} \\ \lceil\gamma_2+\gamma_3\rceil &= \gamma_2+\gamma_3 \in \mathbb{Z}.\end{aligned}$$

This cannot be the case for $\mathfrak{P}_{(\gamma_0,\ldots,\gamma_4)}$ is regular.
Hence,

$$\lceil\gamma_1+\gamma_4\rceil + \lceil\gamma_2+\gamma_3\rceil = 1. \tag{3.8}$$

We choose A to be ia on l and B to be ib for $b < a$.

(i) Without restriction, we choose: $p = 1,\ q = 3$.
The case for $p = 2,\ q = 4$ works similarly.
Let c be an element of the open interval (b, a). Since $p = 1$, we get

$$\begin{aligned} K_1(ia) = \lceil a\sin(\frac{2\pi}{5}) + \gamma_1\rceil =^{\mathfrak{G}_{\gamma,p}} a\sin(\frac{2\pi}{5}) + \gamma_1 =: z \in \mathbb{Z} \\ \Updownarrow \\ a\sin(\frac{2\pi}{5}) = z - \gamma_1 \\ \implies K_4(ia) = \lceil -z + \gamma_1 + \gamma_4\rceil = -z + \lceil \gamma_1 + \gamma_4\rceil \\ \implies K_1(ia) + K_4(ia) - \lceil \gamma_1 + \gamma_4\rceil = 0 \end{aligned} \tag{3.9}$$

In addition, it holds that

$$K_2(ib) + K_3(ib) - \lceil \gamma_2 + \gamma_3\rceil = 0$$

for $\mathfrak{G}_{\gamma,q=3}$.
We know that the map f from (3.3) is constant in every mesh of a pentagrid. We now concentrate on the value of f on a grid line itself, i.e. on the boundary of a mesh. Orienting the lines of the 0-th grid along the imaginary axis (from $-\infty$ to $+\infty$) and the lines of the other grids of $\mathfrak{P}_\gamma$ via rotation in positive direction (counter-clockwise), we are able to tell that f has the same value on the left side of a boundary as on the line itself.
We therefore deduce:

$$\begin{aligned} K_1(ic) + K_4(ic) - \lceil \gamma_1 + \gamma_4\rceil = 0 \\ K_2(ic) + K_3(ic) - \lceil \gamma_2 + \gamma_3\rceil = 0 \end{aligned}$$

and using (3.8),

$$K_1(ic) + K_2(ic) + K_3(ic) + K_4(ic) = \lceil \gamma_1 + \gamma_4\rceil + \lceil \gamma_2 + \gamma_3\rceil = 1.$$

(ii) For $p = 3,\ q = 1$, and by applying the above techniques, we end up with:

$$K_1(ib) + K_4(ib) - \lceil \gamma_1 + \gamma_4\rceil = 0$$

$$K_2(ia) + K_3(ia) - \lceil \gamma_2 + \gamma_3\rceil = 0$$

and in conclusion, we get

$$\begin{aligned} K_1(ic) + K_4(ic) = \lceil \gamma_1 + \gamma_4\rceil + 1 \\ K_2(ic) + K_3(ic) = \rceil \gamma_2 + \gamma_3\rceil + 1 \\ \implies \sum_{j=1}^{4} K_j(ic) = 3 \end{aligned}$$

using (3.8) as well as the index change induced by the arrangement and orientation of the lines of $\mathfrak{P}_\gamma$.

In closing, these two cases show that

$$\sum_{j=1}^{4} K_j(iy) \in \{1, 3\}$$

between A and B. So, the indices on each side of $\overline{AB}$ are either 1 and 2 or 3 and 4.
Hence, $\overline{AB}$ is green.
This concludes our proof. $\square$

4 Global construction of AR patterns

This section illustrates an example of generating quasi-periodicity from "plain" periodicity:
Starting with a grid of unit cubes ($\mathbb{Z}^5$) in five dimensional space ($\mathbb{R}^5$), we will explain the process of "cutting through" this grid and projecting onto a plane in order to get a quasiperiodic pattern, the AR pattern.
For a better understanding of the expression "quasiperiodic" we refer to section 7.
For $\gamma = (\gamma_0, ..., \gamma_4)$ with zero sum, we assume $\mathfrak{P}_\gamma$ to be regular. Each unit cube in $\mathbb{R}^5$ can be determined by five integers as follows:

Definition 4.1. *For $k = (k_0, ..., k_4)$ in $\mathbb{Z}^5$ we define the interior of a unit cube to be the set*

$$C_k := \Big\{(x_0, ..., x_4) \in \mathbb{R}^5 \mid k_j - 1 < x_j < k_j \ \forall j\Big\}. \tag{4.1}$$

We call C_k the ***open unit cube of*** *k.*

Theorem 4.2. *In the setting of the paragraph above, the vertices of the AR pattern generated by $\mathfrak{P}_\gamma$ are given by $\sum_{j=0}^4 k_j \zeta^j$, $k = (k_0, ..., k_4)$, which run through all k in $\mathbb{Z}^5$ for which C_k has a non-empty intersection with the plane specified by*

(i) $\sum_{j=0}^4 x_j = 0$

(ii) $\sum_{j=0}^4 (x_j - \gamma_j)\Re(\zeta^{2j}) = 0$

(iii) $\sum_{j=0}^4 (x_j - \gamma_j)\Im(\zeta^{2j}) = 0$

Proof. (i), (ii) and (iii) state that the vector $(x_j - \gamma_j)_{j=0,\cdots,4}$ is orthogonal to $(1, ..., 1)^T =: (1)_{j=0,\cdots,4}$ ($\cdot^T$transposing a vector/ matrix), $(\zeta^{2j})_{j=0,\cdots,4}$ and $(\zeta^{-2j})_{j=0,\cdots,4}$ as it holds that

$$\langle (x_j - \gamma_j)_{j=0,\cdots,4}, (1)_{j=0,\cdots,4}\rangle = \sum_{j=0}^4 (x_j - \gamma_j) = \sum_{j=0}^4 x_j - \underbrace{\sum_{j=0}^4 \gamma_j}_{=0} =^{(i)} 0$$

(the cases for (ii) and (iii) work the same way, where every complex number is split up into its real and its imaginary part in order to obtain real vectors).
Consequently, $(x_j - \gamma_j)_{j=0,\cdots,4}$ can be expressed as a linear combination of vectors spanning the orthogonal complement.
In other words, complex numbers α and β exist so that

$$(x_j - \gamma_j)_{j=0,\cdots,4} = (\alpha\zeta^j + \beta\zeta^{-j})_{j=0,\cdots,4}.$$

By computation, let z be a complex number with $\frac{1}{2}\overline{z} = \alpha,\ \frac{1}{2}z = \beta$.
Hence, $\Re(z\zeta^{-j}) = \alpha\zeta^j + \beta\zeta^{-j} = x_j - \gamma_j$ for all j.
If $(x_j)_{j=0,\cdots,4}$ lies in $C_{k=(k_0,\cdots,k_4)}$, we can write every k_j as $\lceil \Re(z\zeta^{-j}) + \gamma_j \rceil$ after rearranging the identity for $x_j - \gamma_j$ above and taking (4.1) into consideration.
Furthermore, the same arguments show that $(x_j)_{j=0,\cdots,4}$ lies in $C_{(k'_j)_{j=0,\cdots,4}}$, where

$k_j^{'}$ is defined to be $\lceil \Re(z\zeta^{-j}) + \gamma_j \rceil$.
Since $\mathfrak{P}_\gamma$ is regular, two of its lines intersect in one point at the most. That way, z can be varied a little so it can be ensured that $(x_j)_{j=0,\cdots,4}$ lies in the interior of a unit cube.

$$\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \sqrt{\frac{2}{5}} \begin{pmatrix} 1 \\ \Re(\zeta^2) \\ \Re(\zeta^4) \\ \Re(\zeta^6) \\ \Re(\zeta^8) \end{pmatrix}, \sqrt{\frac{2}{5}} \begin{pmatrix} 0 \\ \Im(\zeta^2) \\ \Im(\zeta^4) \\ \Im(\zeta^6) \\ \Im(\zeta^8) \end{pmatrix}, \sqrt{\frac{2}{5}} \begin{pmatrix} 1 \\ \Re(\zeta) \\ \Re(\zeta^2) \\ \Re(\zeta^3) \\ \Re(\zeta^4) \end{pmatrix}, \sqrt{\frac{2}{5}} \begin{pmatrix} 0 \\ \Im(\zeta) \\ \Im(\zeta^2) \\ \Im(\zeta^3) \\ \Im(\zeta^4) \end{pmatrix} \right\}$$

is an orthonormal basis because $\sum_{j=0}^4 (\Re(\zeta^{2j}))^2 = \sum_{j=0}^4 (\Re(\zeta^j))^2 = \frac{5}{2}$, $\sum_{j=1}^4 (\Im(\zeta^{2j}))^2 = \sum_{j=1}^4 (\Im(\zeta^j))^2 = \frac{5}{2}$ and since for

$$A := \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \sqrt{2} & \sqrt{2}\Re(\zeta^2) & \sqrt{2}\Re(\zeta^4) & \sqrt{2}\Re(\zeta^6) & \sqrt{2}\Re(\zeta^8) \\ 0 & \sqrt{2}\Im(\zeta^2) & \sqrt{2}\Im(\zeta^4) & \sqrt{2}\Im(\zeta^6) & \sqrt{2}\Im(\zeta^8) \\ \sqrt{2} & \sqrt{2}\Re(\zeta) & \sqrt{2}\Re(\zeta^2) & \sqrt{2}\Re(\zeta^3) & \sqrt{2}\Re(\zeta^4) \\ 0 & \sqrt{2}\Im(\zeta) & \sqrt{2}\Im(\zeta^2) & \sqrt{2}\Im(\zeta^3) & \sqrt{2}\Im(\zeta^4) \end{pmatrix}$$

we have $AA^T = \mathbb{I}_5$, where $\mathbb{I}_5$ is the 5×5 unit matrix.
Projecting $k^T = (k_0, ..., k_4)^T$ onto the plane given by (i)-(iii), we get:

$$Ak^T = \frac{1}{\sqrt{5}} \begin{pmatrix} \sum_{j=0}^4 k_j \\ \sqrt{2} \sum_{j=0}^4 k_j \Re(\zeta^{2j}) \\ \sqrt{2} \sum_{j=1}^4 k_j \Im(\zeta^{2j}) \\ \sqrt{2} \sum_{j=0}^4 k_j \Re(\zeta^j) \\ \sqrt{2} \sum_{j=1}^4 k_j \Im(\zeta^j) \end{pmatrix}$$

The comparison with (3.3) provides $\sqrt{\frac{2}{5}} \sum_{j=0}^4 k_j \zeta^j$.
We therefore get U_γ after a change of scale in the projection of the vectors k, as mentioned in Theorem 4.2. $\square$

5 Properties of pentagrids and AR patterns

In the following section, we will explore some basic properties of AR patterns. Furthermore, the pentagrid will be studied in detail by introducing singular pentagrids more in depth and listing transformations that can be applied to a pentagrid.
An important result of this section will be the introduction of alternative parameters describing a pentagrid.

5.1 Understanding vertex types

Starting with the general idea of an AR pattern (meaning it does not have to be generated by a pentagrid), there is a finite number of different vertices in the pattern according to the matching rules.
These vertex types can be transformed into kite-and-dart patterns because of the duality. For comparison, see [1].
This work uses the notation given in [1], which was derived from the labelling in [9]: queen (Q), king (K), star (S), deuce (D), jack (J) and sun (S3, S4, S5).
The abbreviations we use are in parantheses.
The eight vertex types are shown in Figure 4 (the rotation of the vertices is not considered here).

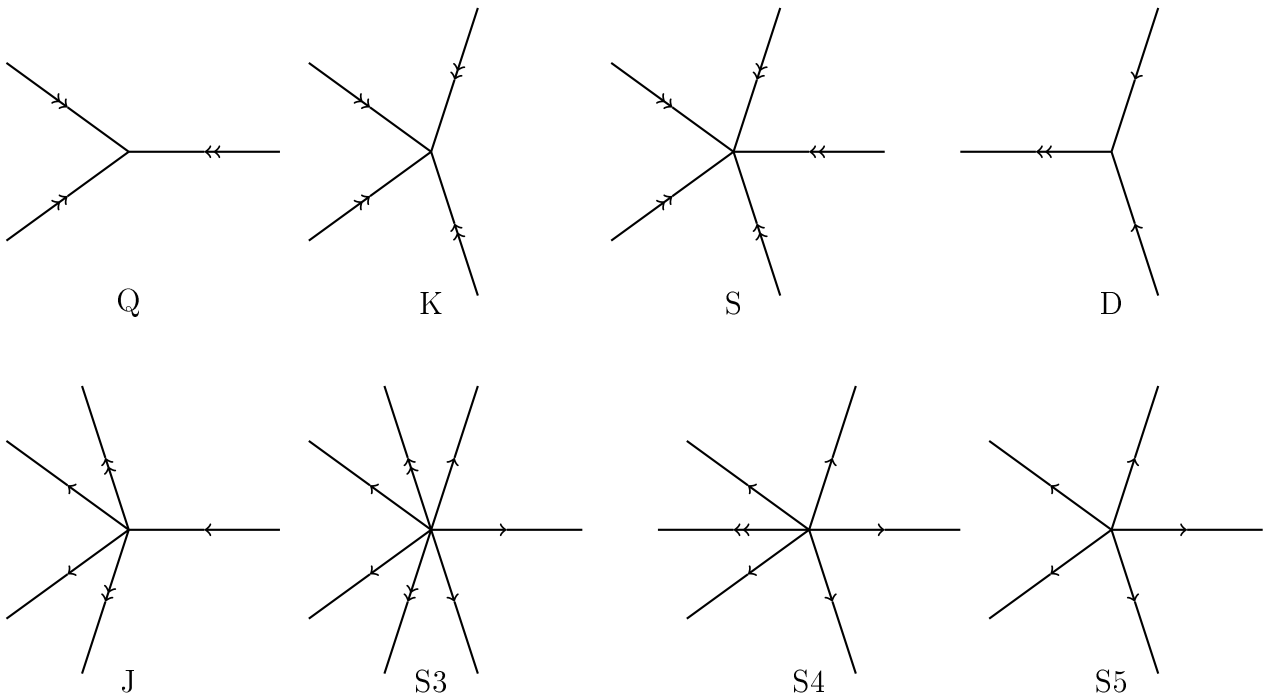

Figure 4: The eight different vertex types

Returning to AR patterns generated by a regular pentagrid $\mathfrak{P}_{\gamma=(\gamma_0,\cdots,\gamma_4)}$ (with $\sum_{j=0}^{4}\gamma_j = 0$), we continue by investigating whether for some $k = (k_0, ..., k_4) \in \mathbb{Z}^5$ there is a mesh in $\mathfrak{P}_\gamma$ with $k_j = K_j(z)$ for all j.
The reader is to be reminded of the fact that $f(z)$ from (3.3) is constant in every

mesh of the pentagrid.
Essentially, we want to know whether a complex number z exists such that

$$k_j - 1 < \Re(z\zeta^{-j} + \gamma_j) < k_j \tag{5.1}$$

holds for all j.
Expanding this idea, we can derive a method to check whether $\sum_{j=0}^{4} k_j\zeta^j$ is a vertex of the AR pattern since meshes in $\mathfrak{P}_\gamma$ correspond to the vertices of the rhombuses.

Theorem 5.1. *Let $\mathfrak{P}_\gamma$ be a regular pentagrid as above and let $(k_0, ..., k_4)$ be a vector of integers.*
Then the following statements are equivalent:

(i)

$$\exists z \in \mathbb{C}: \; k_j - 1 < \Re(z\zeta^{-j}) + \gamma_j < k_j \;\forall j \tag{5.2}$$

(ii)

$$(\sum_{j=0}^{4} k_j, \sum_{j=0}^{4}(k_j - \gamma_j)\zeta^{2j}) \in V := \{(\sum_{j=0}^{4} \lambda_j, \sum_{j=0}^{4} \lambda_j\zeta^{2j}) \mid 0 < \lambda_j < 1 \;\forall j\} \tag{5.3}$$

Proof. We start with examining "$(i) \implies (ii)$":
Let z be a complex number that fulfills (5.2).

$$\begin{aligned} k_j - 1 < \Re(z\zeta^{-j}) + \gamma_j & < k_j \\ \iff 0 < \underbrace{k_j - \Re(z\zeta^{-j}) - \gamma_j}_{=:\lambda_j \;\forall j} & < 1 \end{aligned}$$

It follows that $(\sum_{j=0}^{4} \lambda_j, \sum_{j=0}^{4} \lambda_j\zeta^{2j})$ is an element of V.
At this point, it suffices to show that $\sum_{j=0}^{4}(k_j - \gamma_j)\zeta^{2j} = \sum_{j=0}^{4} \lambda_j\zeta^{2j}$ and $\sum_{j=0}^{4} \lambda_j = \sum_{j=0}^{4} k_j$.

$$\begin{aligned} \sum_{j=0}^{4}(k_j - \gamma_j)\zeta^{2j} &= \sum_{j=0}^{4}(\lambda_j + \Re(z\zeta^{-j}))\zeta^{2j} \\ &= \sum_{j=0}^{4} \lambda_j\zeta^{2j} + \underbrace{\sum_{j=0}^{4} \Re(z\zeta^{-j})\zeta^{2j}}_{=0} \end{aligned} \tag{5.4}$$

$$\begin{aligned} \sum_{j=0}^{4} \lambda_j &= \sum_{j=0}^{4}(k_j - \Re(z\zeta^{-j}) - \gamma_j) \\ &= \sum_{j=0}^{4} k_j - \underbrace{\sum_{j=0}^{4} \Re(z\zeta^{-j})}_{=0} - \underbrace{\sum_{j=0}^{4} \gamma_j}_{=0} \end{aligned} \tag{5.5}$$

We go on with "$(ii) \Longrightarrow (i)$":
For all j there is a λ_j with $0 < \lambda_j < 1$ and $\sum_{j=0}^4 \lambda_j = \sum_{j=0}^4 k_j$

$$\sum_{j=0}^4 \lambda_j = \sum_{j=0}^4 k_j \Longleftrightarrow \sum_{j=0}^4 (\lambda_j - k_j) = 0.$$

With $\sum_{j=0}^4 \gamma_j = 0$, we then get:
$\sum_{j=0}^4 (k_j - \lambda_j - \gamma_j) = 0$ and $\sum_{j=0}^4 (k_j - \lambda_j - \gamma_j)\zeta^{2j} = 0$.
We therefore know that $((k_j - \lambda_j - \gamma_j))_{j=0,\cdots,4}$ is orthogonal to $(1)_{j=0,\cdots,4}$, $(\zeta^{2j})_{j=0,\cdots,4}$ and $(\zeta^{-2j})_{j=0,\cdots,4}$.
We can thus write $((k_j - \lambda_j - \gamma_j))_{j=0,\cdots,4}$ as a linear combination of $(\zeta^j)_{j=0,\cdots,4}$ and $(\zeta^{-j})_{j=0,\cdots,4}$.
Hence, there is an element z in $\mathbb{C}$ such that $k_j - \lambda_j - \gamma_j = \Re(z\zeta^{-j})$ for all j.
Since $0 < \lambda_j < 1$, we conclude:

$$k_j - 1 < \underbrace{\Re(z\zeta^{-j}) + \gamma_j}_{=k_j - \lambda_j} < k_j$$

for all j.
We note the similarities between these arguments and those used in the proof of Theorem 4.2. □

Remark. *A way to detect the vertex type of* $\sum_{j=0}^4 k_j\zeta^j$ *can be derived from (5.3): For* r *taken from* $\{1, 2, 3, 4\}$, V_r *is the set of all elements of* V *with* $\sum_{j=0}^4 \lambda_j = r$. *It holds, that* $V_4 = -V_1$ *and* $V_2 = -V_3$.
This is due to $-\sum_{j=0}^4 (1 - \lambda_j)\zeta^{2j} = \sum_{j=0}^4 \lambda_j\zeta^{2j}$.

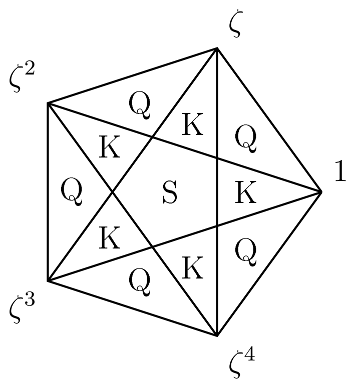

Figure 5: A depiction of V_1

The remaining vertex types can be detected through $V_2 = -V_3$. *In* V, *there are 64 subregions. Taking rotations into consideration, there are ten options for every vertex type apart from* S *and* $S5$. *For those two, there are only two options each.*

5.2 Alternative parameters

Thus far, we have been describing a pentagrid using five real numbers $\gamma_0, ..., \gamma_4$ with zero sum.

For practical purposes, this section aims at introducing parameters which will be used throughout the rest of this thesis:

$$\xi := \sum_{j=0}^{4} \gamma_j \zeta^{2j}, \ \eta := \sum_{j=0}^{4} \gamma_j \zeta^j \tag{5.6}$$

We claim that in the regular case the AR pattern only depends on ξ.
In combining (ii) and (iii) from Theorem 4.2, we deduce:

$$\begin{aligned} &\sum_{j=0}^{4}(x_j - \gamma_j)\zeta^{2j} = 0 \\ \Longleftrightarrow &\sum_{j=0}^{4} x_j \zeta^{2j} = \sum_{j=0}^{4} \gamma_j \zeta^{2j} = \xi. \end{aligned}$$

Our claim is correct since Theorem 4.2 serves to describe the vertices in the AR pattern.

Definition 5.2. *For $\gamma = (\gamma_0, ..., \gamma_4)$ and $\gamma^* = (\gamma_0^*, ..., \gamma_4^*)$ with $\sum_{j=0}^{4} \gamma_j = \sum_{j=0}^{4} \gamma_j^* = 0$, $\mathfrak{P}_\gamma$ and $\mathfrak{P}_{\gamma^*}$ are* ***shift equivalent*** *if there is a complex number a such that*

$$\Re(a\zeta^{-j}) + \gamma_j - \gamma_j^* \in \mathbb{Z}$$

for all j.

Definition 5.3. *Let $(R, +, \cdot)$ be a ring and I be a non empty subset of R. I is called an* ***ideal*** *in R if*

(i) $(I, +)$ is a subgroup of R

(ii) for every r in R and for every x in I, $r \cdot x$ is an element of I.

For a subset $A \subseteq R$, (A) denotes the ideal generated by A.

$$(A) = \bigcap_{A \subseteq I, I \text{ ideal in } R} I$$

In the case $A = \{a\}$ for some a in R, we write

$$(A) = (a)$$

Lemma 5.4. *For the ideal $(1-\zeta) \subset \mathbb{Z}[\zeta]$ and $P := \{\sum_{j=0}^{4} n_j \zeta^j \mid n_j \in \mathbb{Z}$ with $\sum_{j=0}^{4} n_j = 0\}$ we have:*

$$P = (1 - \zeta) \tag{5.7}$$

Proof. An arbitrary element of the ideal $(1 - \zeta)$ is of the form $(1 - \zeta)\sum_{j=0}^{4} z_j \zeta^j$ for integers z_j.

$$
\begin{aligned}
(1-\zeta)\sum_{j=0}^{4} z_j\zeta^j &= \sum_{j=0}^{4} z_j\zeta^j - \sum_{j=0}^{4} z_j\zeta^{j+1} \\
&= \sum_{j=0}^{4} \underbrace{(z_j - z_{j-1})}_{=:n_j}\zeta^j \quad \text{with } z_{-1} = z_4
\end{aligned}
$$

$$\sum_{j=0}^{4} n_j = \sum_{j=0}^{4} z_j - \sum_{j=0}^{4} z_{j-1} = 0$$

Conversely, for an element $\sum_{j=0}^{4} n_j\zeta^j$ of P with $\sum_{j=0}^{4} n_j = 0$, we have

$$n_j = -\sum_{i\neq j} n_i \quad \text{for all } j.$$

We can thus write $\sum_{j=0}^{4} n_j\zeta^j$ as

$$(1-\zeta)(n_0 - n_2\zeta - n_3\zeta(\zeta+1) - n_4\zeta(\zeta^2+\zeta+1)).$$

□

Theorem 5.5. *The pentagrids $\mathfrak{P}_{(\gamma_0,\ldots,\gamma_4)}$ and $\mathfrak{P}_{(\gamma_0^*,\ldots,\gamma_4^*)}$ are shift equivalent if and only if $\xi - \xi^*$ lies in P, where ξ^* is obtained from $\gamma_0^*, \ldots, \gamma_4^*$.*

Proof. First, we assume the pentagrids to be shift equivalent. According to Definition 5.2, there is an integer z_j with $z_j = \Re(a\zeta^{-j}) + \gamma_j - \gamma_j^*$ for some complex number a, for all j. Since $\sum_{j=0}^{4} \Re(a\zeta^{-j}) = 0$, we get the following two identities:

$$\sum_{j=0}^{4} z_j = \sum_{j=0}^{4}(\Re(a\zeta^{-j}) + \gamma_j - \gamma_j^*) = 0 \tag{5.8}$$

$$\xi - \xi^* = \sum_{j=0}^{4}(\gamma_j - \gamma_j^*)\zeta^{2j} = \sum_{j=0}^{4} z_j\zeta^{2j}. \tag{5.9}$$

With (5.8), (5.9) and Lemma 5.4 we conclude: $\xi - \xi^* \in P$.

Next, let us assume that $\xi - \xi^*$ lies in P.
Applying Lemma 5.4, we can tell that

$$\xi - \xi^* = \sum_{j=0}^{4}(\gamma_j - \gamma_j^*)\zeta^{2j} = \sum_{j=0}^{4} m_j\zeta^{2j}$$

with integers m_j so that $\sum_{j=0}^{4} m_j = 0$. Without loss of generalitiy, we can use the commutative property of $\mathbb{Z}[\zeta]$ in order to change the exponents in $\sum_{j=0}^{4} m_j\zeta^j$ to $2j$.
As a result, $\sum_{j=0}^{4}(\gamma_j - \gamma_j^* - m_j)\zeta^{2j} = 0$ and $(\gamma_j - \gamma_j^* - m_j)_{j=0,\cdots,4}$ therefore is orthogonal to $(1)_{j=0,\cdots,4}$, $(\zeta^{2j})_{j=0,\cdots,4}$ and $(\zeta^{-2j})_{j=0,\cdots,4}$.
Similar to the proof Theorem 4.2, we can deduce the existence of a complex number a so that $\Re(a\zeta^{-j}) + \gamma_j - \gamma_j^*$ is an integer for all j. □

Throughout the study of pentagrids, it can be useful to consider the following proposition:

Proposition 5.6. *For real numbers u and v with*

$$\xi = (1 - \zeta^2)u + (1 - \zeta^3)v,$$

where $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$ with real numbers γ_j such that $\sum_{j=0}^{4} \gamma_j = 0$, it holds that

$$(\gamma_0', ..., \gamma_4') := (u + v, -u, 0, 0, -v)$$

is another choice for a real vector that gives us the same ξ as $(\gamma_0, ..., \gamma_4)$.

Proof.

$$\begin{aligned} \xi &= (1 - \zeta^2)u + (1 - \zeta^3)v \\ &= (u + v) - u\zeta^2 - v\zeta^3 \\ &= (u + v)\zeta^0 - u\zeta^2 - v\zeta^8 \\ &= \sum_{j=0}^{4} \gamma_j' \zeta^{2j} \end{aligned}$$

The comparison of coefficients concludes this proof. □

Lemma 5.7 (Approximation Theorem, Kronecker)**.** *If for a positive integer n, $\vartheta_1, ..., \vartheta_n, 1$ are linearly independent over $\mathbb{Q}$, then for real numbers $\alpha_1, ..., \alpha_n$, $\varepsilon > 0$ and for an arbitrarily large integer t, we have*

$$\|t\vartheta_i - \alpha_i\| < \varepsilon \quad \textit{for all } i = 1, ..., n,$$

where for any real number x, $\|x\|$ denotes the distance between x and the nearest integer.

For the proof of the Approximation Theorem, see [7].

Lemma 5.8. *For real vectors $\gamma = (\gamma_0, ..., \gamma_4)$ and $\gamma^* = (\gamma_0^*, ..., \gamma_4^*)$ with $\sum_{j=0}^{4} \gamma_j = \sum_{j=0}^{4} \gamma_j^* = 0$, let U denote the set of vertices generated by $\mathfrak{P}_\gamma$ and let U^* be the one generated by $\mathfrak{P}_{\gamma^*}$.*

(i) $\mathfrak{P}_\gamma = \mathfrak{P}_{\gamma^*}$ if and only if $\gamma_j - \gamma_j^*$ is an integer for every j.

(ii) U^* is obtained from U by a shift with a fixed ϑ:

$$\alpha \longmapsto \alpha + \vartheta \quad \textit{for all complex numbers } \alpha$$

Then ϑ is in P.

Proof. (i) We start with assuming that $\mathfrak{P}_\gamma = \mathfrak{P}_{\gamma^*}$.
$\mathfrak{G}_{\gamma,j}$ equals $\mathfrak{G}_{\gamma^*,j}$ for all j. We can therefore choose z from $\mathfrak{G}_{\gamma,j}$ for some arbitrary j.
For this z, $\Re(z\zeta^{-j}) + \gamma_j$ and $\Re(z\zeta^{-j}) + \gamma_j^*$ are both integers.

$$\Re(z\zeta^{-j}) + \gamma_j - \Re(z\zeta^{-j}) - \gamma_j^* \in \mathbb{Z}$$

illustrates that $\gamma_j - \gamma_j^*$ is an integer.

Next, we assume that $\gamma_j - \gamma_j^*$ lies in $\mathbb{Z}$ for all j.
For $j \in \{0, ..., 4\}$, we look at $z \in \mathfrak{G}_{\gamma,j}$.
Since $\Re(z\zeta^{-j}) + \gamma_j$ and $-\gamma_j + \gamma_j^*$ are both integers, $\Re(z\zeta^{-j}) + \gamma_j^*$ is an integer as well: z therefore is an element of $\mathfrak{G}_{\gamma^*,j}$. That is why both grids are equal. Since we chose j arbitrarily, this holds for every grid and in total for their unions:
$\mathfrak{P}_\gamma = \mathfrak{P}_{\gamma^*}$.

(ii) Let $\sum_{j=0}^4 k_j^* \zeta^j$ be a vertex from U^* with index $\sum_{j=0}^4 k_j^*$ and let $\sum_{j=0}^4 k_j \zeta^j$ be a vertex from U with index $\sum_{j=0}^4 k_j$.
A parallel shift can now be expressed by subtracting these vertices:

$$\sum_{j=0}^4 k_j \zeta^j - \sum_{j=0}^4 k_j^* \zeta^j = \sum_{j=0}^4 \underbrace{(k_j - k_j^*)}_{=:n_j} \zeta^j$$

A shift does not affect indices. Hence,

$$\sum_{j=0}^4 k_j = \sum_{j=0}^4 n_j + \sum_{j=0}^4 k_j^* \equiv \sum_{j=0}^4 k_j^* \mod 5$$

And thus, $\sum_{j=0}^4 n_j \equiv 0 \mod 5$. In that case, integers m_j must exist, satisfying $\sum_{j=0}^4 m_j = 0$ as well as

$$\sum_{j=0}^4 n_j \zeta^j = \sum_{j=0}^4 m_j \zeta^j.$$

$\sum_{j=0}^4 m_j \zeta^j =: \vartheta$ is an element of P according to (5.7).

□

Theorem 5.9. *Let $\mathfrak{P}$ be the regular pentagrid determined by (ξ, η) and $\mathfrak{P}^*$ be the regular pentagrid determined by (ξ^*, η^*) (defined as above).*
$\mathfrak{P}$ and $\mathfrak{P}^*$ generate the same AR pattern if and only if $\xi = \xi^*$.
The two AR patterns are shift-equivalent if and only if $\xi - \xi^*$ lies in P.

Proof. The following proof is devided up into four major parts:

(i) Let $\xi - \xi^*$ be an element of P. It follows with the proof of Theorem 5.5 that $\gamma_j - \gamma_j^* - z_j = \Re(a\zeta^{-j})$ for integers z_j for all j and for some complex a. According to Lemma 5.8, the z_j's shift the AR pattern by some fixed $\vartheta \in P$. Since we assume $\xi - \xi^*$ to be an element of P, the two patterns are shift-equivalent.
In the case where $\xi - \xi^* = 0$, the patterns are the same.

(ii) With Lemma 5.7 and $n = 1$, for an irrational number ϑ, for every $\varepsilon > 0$, and for every real α, we find an integer t so that

$$\|t\vartheta - \alpha\| < \varepsilon,$$

using the notation from Lemma 5.7.
Hence, there exists an integer s with

$$|t\vartheta - \alpha - s| < \varepsilon.$$

For a fixed irrational number ϑ, we then find the set $\{n\vartheta + m \mid m, n \in \mathbb{Z}\}$ to be dense in $\mathbb{R}$
With $\vartheta = \cos(\frac{2\pi}{5})$ and by rotation, we deduce the density of $\{\sum_{j=0}^{4} k_j\zeta^{2j} \mid k_j \in \mathbb{Z}\ \forall j\}$ in $\mathbb{C}$.
More generally, the set $\{\sum_{j=0}^{4} k_j\zeta^{2j} \mid k_j \in \mathbb{Z}\ \forall j,\ \sum_{j=0}^{4} k_j = F\}$ is dense in $\mathbb{C}$.

(iii) We start with assumig $\mathfrak{P}$ and $\mathfrak{P}^*$ to generate the same pattern.
We then show $\xi = \xi^*$. Assume $\xi \neq \xi^*$:
Applying techniques implemented in Theorem 5.1, we get

$$(\sum_{j=0}^{4} k_j, \sum_{j=0}^{4} k_j\zeta^{2j} - \xi) \in V, \quad (\sum_{j=0}^{4} k_j, \sum_{j=0}^{4} k_j\zeta^{2j} - \xi^*) \notin V \tag{5.10}$$

because of the density of the points $\sum_{j=0}^{4} k_j\zeta^{2j}$ in $\mathbb{C}$ under the assumption that $\sum_{j=0}^{4} k_j$ equals some constant. (5.10) contradicts Theorem 5.1. So, $\xi = \xi^*$ holds.

(iv) We use arguments from the proof of Lemma 5.8 to show that we get $\xi - \xi^* \in P$ for shift-equivalent AR pattterns.
Let $\sum_{j=0}^{4} k_j\zeta^j$ be a vertex in U and $\sum_{j=0}^{4} k_j^*\zeta^j$ be a vertex in U^*. For the shift vector $\sum_{j=0}^{4} n_j\zeta^j$, we have $\sum_{j=0}^{4} n_j\zeta^j \equiv 0 \mod 5$. This gives evidence for the existence of integers m_j with $\sum_{j=0}^{4} m_j = 0$.
We then define $\gamma_j^{**} := \gamma_j^* - m_j$ for all j.
Because of $\sum_{j=0}^{4} m_j = 0$, this yields that $\mathfrak{P}^{**}$ is the same as $\mathfrak{P}$. So,

$$\xi = \xi^{**} \tag{5.11}$$

$\xi^{**} - \xi^* = \sum_{j=0}^{4} m_j\zeta^{2j} \in P$ because of (5.7).
(5.11) implies $\xi - \xi^* \in P$.

□

Having introduced all the parameters this work is concerned with, we can now list how several transformations of a real vector $\gamma = (\gamma_0, ..., \gamma_4)$ influence the pentagrid $\mathfrak{P}_\gamma$ and the vertices of the correspondig sets U_γ.
The transformed parameters will be marked with an asterisk (*).

Proposition 5.10. *(i) For $z \in \mathbb{C}$ and $\gamma_j^* := \gamma_j + \Re(z\zeta^{-j})$ we get*

$$\xi^* = \xi,\ u^* = u,\ v^* = v,\ \mathfrak{P}_{\gamma^*} = \mathfrak{P}_\gamma - z,\ U_{\gamma^*} = U_\gamma$$

for all j.

(ii) *For integers n_j with zero sum and for $\gamma_j^* := \gamma_j + n_j$ we get*

$$\xi^* = \xi + \sum_{j=0}^{4} n_j \zeta^{2j},\ u^* = u-(n_1+n_2)+n_4(\zeta+\zeta^4),\ v^* = v-(n_3+n_4)+n_4(\zeta+\zeta^4),$$

$$\mathfrak{P}_{\gamma^*} = \mathfrak{P}_\gamma,\ U_{\gamma^*} = U_\gamma + \sum_{j=0}^{4} n_j \zeta^j$$

for all j.

(iii) *For $\gamma_j^* = \gamma_{5-j}$ with $\gamma_5 = \gamma_0$, we get*

$$\xi^* = \overline{\xi},\ u^* = v,\ v^* = u,\ \mathfrak{P}_{\gamma^*} = \overline{\mathfrak{P}_\gamma},\ U_{\gamma^*} = \overline{U_\gamma}$$

for all j.

(iv) *For $\gamma_j^* = -\gamma_j$ we get*

$$\xi^* = -\xi,\ u^* = -u,\ v^* = -v,\ \mathfrak{P}_{\gamma^*} = -\mathfrak{P}_\gamma,\ U_{\gamma^*} = -U_\gamma$$

for all j.

(v) *For $\gamma_j^* = \gamma_{j+1}$ with $\gamma_5 = \gamma_0$, we get*

$$\xi^* = \zeta^{-2}\xi, u^* = v,\ v^* = -u + (\zeta^2+\zeta^3)v,\ \mathfrak{P}_{\gamma^*} = \zeta^{-1}\mathfrak{P}_\gamma,\ U_{\gamma^*} = \zeta^{-1}U_\gamma$$

for all j.

5.3 Singular pentagrids and their AR patterns

Up to this point, we have been concerned with the study of regular pentagrids and the corresponding rhombus patterns. But how can we easily determine whether more than two lines of a pentagrid intersect in one point? And if this should be the case, what can we tell about the relation between regular pentagrids and singular ones?
Once again, we are working with a real vector $\gamma = (\gamma_0, ..., \gamma_4)$ so that $\sum_{j=0}^{4} \gamma_j = 0$ throughout this subsection.

Theorem 5.11. *A pentagrid $\mathfrak{P}_\gamma$ is singular if and only if $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$ is an element of*

$$\left\{ iu\zeta^j + \vartheta \mid j = 0, ..., 4,\ \text{with } u \in \mathbb{R},\ \vartheta \in P \right\} \tag{5.12}$$

Proof. First, let $\mathfrak{P}_\gamma$ be singular. Without loss of generality, we look at three lines of the pentagrid intersecting in one point of the plane. Let the point of intersection be 0 and let the imaginary axis $i\mathbb{R}$ be the axis of symmetry.
In this scenario, we have either $\gamma_0,\ \gamma_1,\ \gamma_4$ in $\mathbb{Z}$ or $\gamma_0,\ \gamma_2,\ \gamma_3$ in $\mathbb{Z}$ (or both). This is due to 0 being an element of $\mathfrak{G}_{\gamma,j}$ for those j of the intersecting lines.
We have $\Re(0\zeta^{-j}) + \gamma_j \in \mathbb{Z}$ for $j = 0, 1, 4$ or $j = 0, 2, 3$ or both due to symmetry.
Next, we apply transformation (ii) from Proposition 5.10 to these two cases. Since both cases work similarly, we restrict this proof to the first case:

(i) $n_j = -\gamma_j$ for $j = 0, 2, 3$, $n_4 = -\gamma_1 - \gamma_4$ and $n_1 = 0$.
After transforming γ, we end up with $\gamma_j^* = 0$ for $j = 0, 2, 3$ and $\gamma_4^* = -\gamma_1^* = -\gamma_1$.

(ii) $\gamma_j^* = 0$ for $j = 0, 1, 4$ and $\gamma_2^* = -\gamma_3^* = -\gamma_3$.

Continuing with case (i), we compute:

$$\xi^* = \sum_{j=0}^{4} \gamma_j^* \zeta^{2j} = \gamma_4^*(\zeta^3 - \zeta^2) = i(\sin(\frac{6\pi}{5}) - \sin(\frac{4\pi}{5}))\gamma_4^* \ \in i\mathbb{R}$$

It follows for some $j \in \{0, ..., 4\}$ and $x \in \mathbb{R}$ with transformation (ii) and (5.7):

$$\xi^* \zeta^{-j} \equiv ix \mod P.$$

Furthermore, we consider $\xi = i\zeta^j + \vartheta$. After shifting and rotating the pentagrid (transformations (i) and (v) from Proposition 5.10), we may assume $\xi = i\omega$ for a real number ω.
By Proposition 5.6, there are real numbers u and v with

$$\begin{aligned} i\omega = \xi &= (1 - \zeta^2)u + (1 - \zeta^3)v \iff \\ i\omega &= (1 - \zeta^2)u + (1 - \zeta^{-2})v \\ \overline{i\omega} &= (1 - \zeta^{-2})u + (1 - \zeta^2)v \\ i\omega + \overline{i\omega} = 0 &= (2 - \zeta^2 - \zeta^{-2})(u + v). \end{aligned}$$

Hence, $u + v = 0$ and $\gamma = (0, -u, 0, 0, u)$.
Consequently, $\mathfrak{P}_\gamma$ cannot be regular. □

We now continue with the next question concerning the relation to regular pentagrids.

Definition 5.12. *Lines in a pentagrid with the property $\Re(z\zeta^{-j}) = c$ for a real number c are called j-**line**.*

For the following reflections, let $\gamma_0, ..., \gamma_4$ be real numbers with zero sum so that their pentagrid $\mathfrak{P}$ is singular.
$\mathfrak{P}$ will also be called the **unperturbed** grid, for we will vary our parameters a little so that we obtain real numbers $\gamma_0^{(p)}, ..., \gamma_4^{(p)}$ with $\sum_{j=0}^{4} \gamma_j^{(p)} = 0$. The resulting pentagrid $\mathfrak{P}_p$ will be the **perturbed** grid.
Our next assumption centers around the unperturbed grid. We want the imaginary axis $i\mathbb{R}$ to be a 0-line with a 1-line and a 4-line intersecting on it, thus turning $i\mathbb{R}$ into an axis of symmetry. Pairs of lines intersect on $i\mathbb{R}$: either a 1-line and a 4-line or a 2-line and a 3-line because of symmetry.
Additionally, we set $\gamma_0 = \gamma_1 + \gamma_4 = \gamma_2 + \gamma_3 = 0$.

Lemma 5.13. *For a pair of a 1-line and a 4-line intersecting the imaginary axis in the setting described above, the intersection of the perturbed lines lies on the left side of the perturbed 0-line if*

$$\gamma_0^{(p)} + (\gamma_1^{(p)} + \gamma_4^{(p)})(\zeta^2 + \zeta^3) < 0. \tag{5.13}$$

Remark. *Lemma 5.13 also holds for a 2-line and a 3-line with* $\gamma_0^{(p)} + (\gamma_2^{(p)} + \gamma_3^{(p)})(\zeta^1 + \zeta^4)$ *instead.*

Proof. In the unperturbed case where the imaginary axis $i\mathbb{R}$ is a 0-line and z' is the intersection point of a 1-line and a 4-line on it, we have $\Re(z') = 0$ and

$$\Re(z'\zeta^{-1}) + \gamma_1 = -\Re(z'\zeta^{-4}) - \gamma_4.$$

Perturbing the grid a little, we end up with z_0 as point of intersection of the perturbed 1-line and the perturbed 4-line and with an integer n such that

$$\begin{aligned} \Re(z_0\zeta^{-1}) + \gamma_1^{(p)} &= n \\ \Re(z_0\zeta^{-4}) + \gamma_4^{(p)} &= -n \\ \Longrightarrow \Re(z_0\zeta^{1}) + \gamma_4^{(p)} &= -n. \end{aligned}$$

Since $\zeta + \zeta^4$ is a real number, we deduce

$$\begin{aligned} &\Re(z_0)(\zeta + \zeta^4) + (\gamma_1^{(p)} + \gamma_4^{(p)}) = 0 \\ &\qquad \text{since } (\zeta + \zeta^4)(\zeta^2 + \zeta^3) = -1 : \\ &\qquad\qquad \Re(z_0) = (\gamma_1^{(p)} + \gamma_4^{(p)})(\zeta^2 + \zeta^3) < -\gamma_0^{(p)}. \end{aligned}$$

□

The intersection in the setting of the lemma above lies on the right side if

$$\gamma_0^{(p)} + (\gamma_1^{(p)} + \gamma_4^{(p)})(\zeta^2 + \zeta^3) > 0.$$

This can be shown similarly.
The sign of the expression (5.13) is the same as the sign of

$$\Re(\xi^{(p)}) = \underbrace{(1 - \frac{1}{2}(\zeta + \zeta^4))}_{>0}(\gamma_0^{(p)} + (\gamma_1^{(p)} + \gamma_4^{(p)})(\zeta^2 + \zeta^3))$$

In conclusion, the intersection of the 1-line and the 4-line lies left of the 0-line if the variation of the parameters moves $\xi^{(p)}$ leftwards.
Because of our choice for γ_j, $\Re(\xi)$ equals 0.
The same holds for the intersection of a 2-line and a 3-line.
Consequently, the intersections lie on the right side of the 0-line if $\xi^{(p)}$ is moved to the right.
As an important result, we note that **a singular pentagrid can be regarded as the limit of a sequence of regular pentagrids.**
This understanding of singular pentagrids can happen in two different ways, where the AR patterns have two different limits, depending on the impact of the variation of the parameters on $\xi^{(p)}$.

5.4 Symmetries of a pentagrid

This subsection will not be resricted to singular or regular pentagrids.
We will be examining different cases of symmetries, narrowing them down to their essence and describing them in terms of only one parameter:

$$\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}.$$

On examining symmetries of a pentagrid $\mathfrak{P}$, we investigate rotations r so that $r(\mathfrak{P})$ is shift-equivalent to $\mathfrak{P}$ or to $\overline{\mathfrak{P}}$.
As we have learned in Theorem 5.5 and Proposition 5.10, we need to study the following two cases:

(i)
$$\xi - (-1)^h \zeta^{2j} \xi \in P \tag{5.14}$$

(ii)
$$\xi - (-1)^h \zeta^{2j} \overline{\xi} \in P \tag{5.15}$$

with $h \in \{0, 1\}$ and $j = 0, ..., 4$.

First case:
First, let $j \neq 0$. In $\mathbb{Z}[\zeta]$, we have

$$(1 \mp \zeta^{2j})(1 \pm \zeta^{2j}) = 1 - \zeta^{4j}.$$

$1 - (-1)^h \zeta^{2j}$ thus divides $1 - \zeta^{4j}$.
Similarly, $1 - \zeta^{4j}$ divides $1 - \zeta$:

$$\begin{aligned}
&\underline{j = 1:} && -\zeta(1 - \zeta^4) &&= 1 - \zeta \\
&\underline{j = 2:} && (1 + \zeta^3)(1 - \zeta^3) &&= 1 - \zeta \\
&\underline{j = 3:} && (-\zeta - \zeta^3)(1 - \zeta^2) &&= 1 - \zeta \\
&\underline{j = 4:} && 1(1 - \zeta) &&= 1 - \zeta.
\end{aligned}$$

Ultimately, $1 - (-1)^h \zeta^{2j}$ divides $1 - \zeta$. Hence, there is a ϑ in $\mathbb{Z}[\zeta]$ with $\vartheta(1 - (-1)^h \zeta^{2j}) = 1 - \zeta$. In total, $(1 - \zeta)\xi$ lies in P because of (5.14).
As P is the ideal generated by $(1 - \zeta)$, there is an element $\theta \in \mathbb{Z}[\zeta]$ so that

$$(1 - \zeta)\xi = (1 - \zeta)\theta.$$

As a consequence, ξ is an element of $\mathbb{Z}[\zeta]$. Hence, ξ is congruent to $0, \pm 1, \pm 2$ modulo P.
Since -1 and 1 or -2 and 2 each lead to congruent pentagrids, it suffices to look at the cases $0, 1, 2$.
In Theorem 5.11, a way to find out whether a pentagird is singular or not has been introduced. Accordingly, $\mathfrak{P}$ is regular for $\xi \in \{1, 2\}$.

Remark. *For $\xi = 0$, $\mathfrak{P}$ is* ***exceptionally singular****: There is a point in the grid where five lines intersect.*
For $\xi \in P$, we generally get a pentagrid that is congruent to the one obtained by $\xi = 0$.

Next, we choose $h = 1,\ j = 0$. Following (5.14), we deduce $2\xi \in P$.
For integers n_j with $\sum_{j=0}^{4} n_j = 0$, we get:

$$\begin{aligned} 2\xi \in P \Longleftrightarrow 2\xi &= \sum_{j=0}^{4} n_j \zeta^j \\ \xi &= \frac{1}{2} \sum_{j=0}^{4} n_j \zeta^j. \end{aligned}$$

If all n_j's are even: $m_j := \frac{1}{2} n_j \in \mathbb{Z}$ with $\sum_{j=0}^{4} m_j = 0$ and $\xi = \sum_{j=0}^{4} m_j \zeta^j \in P$.
We then have a singular (exceptionally singular) pentagrid. The number of odd n_j's has to be even.
In the case where two of the n_j's are odd (for example $n_2,\ n_3$), we choose for integers m_j without restriction:

$$n_2 = 2m_2 - 1, \quad n_3 = 2m_3 + 1, \quad n_j = 2m_j \text{ for } j = 0, 1, 4$$

and we therefore get:

$$\xi = \frac{1}{2} \sum_{j=0}^{4} n_j \zeta^j = \sum_{j=0}^{4} m_j \zeta^j + \frac{1}{2}(\zeta^3 - \zeta^2).$$

We note that $0 = \sum_{j=0}^{4} n_j = 2 \sum_{j=0}^{4} m_j$ and thus,

$$\xi \equiv \frac{1}{2}(\zeta^3 - \zeta^2) \mod P.$$

Proceeding similarly with four odd n_j's yields $\xi = \frac{5}{2}$ and by rotation $\xi = \frac{5}{2}\zeta^j$.
Choosing n_1 and n_4 to be odd yields

$$\xi \equiv \frac{1}{2}(\zeta - \zeta^4) \mod P.$$

<u>Second case:</u>

Lemma 5.14. *For real numbers $\gamma_0, ..., \gamma_4$ with $\sum_{j=0}^{4} \gamma_j = 0$ and $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$, it holds:*

(i) $\xi \equiv \overline{\xi} \mod P$ *if and only if ξ is an element of $P + \mathbb{R} = \{\vartheta + x \mid \vartheta \in P,\ x \in \mathbb{R}\}$.*

(ii) $\xi \equiv -\overline{\xi} \mod P$ *if and only if ξ is an element of $P + i\mathbb{R} = \{\vartheta + ix \mid \vartheta \in P,\ x \in \mathbb{R}\}$.*

Proof. We only proof (i) because of similarity.
We start by assuming $\xi \equiv \overline{\xi} \mod P$.
There are integers n_j with $\sum_{j=0}^{4} n_j = 0$ and

$$\xi - \overline{\xi} = \sum_{j=0}^{4} \gamma_j (\zeta^{2j} - \overline{\zeta^{2j}}) = \sum_{j=0}^{4} n_j \zeta^j.$$

This holds for $n_0 = 0,\ n_1 = -n_4,\ n_2 = -n_3$.
For $\xi = x + iy$, we get $\xi - \overline{\xi} = 2iy$. We thus deduce with $iy = \frac{1}{2}(\xi - \overline{\xi})$:

$$\xi = x + \frac{1}{2} \underbrace{(\xi - \overline{\xi})}_{=\sum_{j=0}^{4} n_j \zeta^j} .$$

With the identities for the n_j's, we get

$$\xi = x + \frac{1}{2}(n_1(\zeta - \zeta^{-1}) + n_2(\zeta^2 - \zeta^{-2}))$$
$$\xi + \underbrace{n_1(1-\zeta) + n_2(1-\zeta^2)}_{\in P} = x + \frac{1}{2}(n_1 \underbrace{(-\zeta - \zeta^{-1} + 2)}_{\in \mathbb{R}} + n_2 \underbrace{(-\zeta^2 - \zeta^{-2} + 2)}_{\in \mathbb{R}})$$

As a result, ξ lies in $P + \mathbb{R}$.
Now, we assume that ξ is an element of $P + \mathbb{R}$.
Integers n_j with zero sum and a real number x therefore exist, with

$$\xi = \sum_{j=0}^{4} n_j \zeta^j + x$$
$$\Longrightarrow \overline{\xi} = \underbrace{\sum_{j=0}^{4} n_j \zeta^{-j}}_{\in P} + x.$$

This concludes our proof by subtraction. □

If (5.15) holds, there is an element ϑ in P so that $\xi = (-1)^h \zeta^{2j} \overline{\xi}$.
With $\xi_1 := \xi \zeta^{-j}$ and $\overline{\xi} = (\xi - \vartheta)(-1)^h \zeta^{-2j}$, we compute:

$$(-1)^h \overline{\xi_1} = (-1)^h \overline{\xi} \zeta^j = \xi_1 - \underbrace{\vartheta \zeta^{-j}}_{\in P} .$$

Hence, $\xi_1 \equiv (-1)^h \overline{\xi_1} \mod P$, and with Lemma 5.14, we get

$$\xi_1 \in i^h \mathbb{R} + P \implies \xi \in i^h \zeta^j \mathbb{R} + P.$$

We now only have to pay attention to the case $\xi \in \mathbb{R} + P$, for the other cases follow by rotation.
Let us assume $\mathfrak{P}$ to be singular, meaning $j = 1, 2, 3, 4$ in $\xi \in i^h \zeta^j \mathbb{R} + P$ in addition

to $\xi \in \mathbb{R} + P$.

$$\begin{aligned}
\xi &= a_1 + p_1 = i\zeta^j a_2 + p_2 \\
&\text{Conjugation:} \\
a_1 + \overline{p_1} &= -i\zeta^{-j} a_2 + \overline{p_2} \\
&\text{Subtraction:} \\
p_1 - \overline{p_1} &= \underbrace{i(\zeta^j + \zeta^{-j})a_2}_{\in P} + p_2 - \overline{p_2} \\
\implies i(\zeta^j + \zeta^{-j})a_2 &= p_1 - p_2 - \overline{(p_1 - p_2)} \\
&=^{\text{in } P} m(\zeta - \zeta^{-1}) + n(\zeta^2 - \zeta^{-2})
\end{aligned}$$

for integers m, n and for $a_1,\ a_2 \in \mathbb{R}$, $p_1,\ p_2 \in P$. We have $ia_2 = (m(\zeta - \zeta^{-1}) + n(\zeta^2 - \zeta^{-2}))(\zeta^j + \zeta^{-j})^{-1}$ by calculation as well as $\xi \equiv i\zeta^j a_2 \mod P$.
Altogether, we get

$$i\zeta^j a_2 = \zeta^j(\zeta^j + \zeta^{-j})^{-1}(m(\zeta - \zeta^{-1}) + n(\zeta^2 - \zeta^{-2})).$$

At this point, two cases can occur: If $j \neq 0$, we have $\zeta^j(\zeta^j + \zeta^{-j})^{-1} \in \mathbb{Z}[\zeta]$, which results in $\xi \equiv 0 \mod P$.
If $j = 0$, we get $\zeta^j(\zeta^j + \zeta^{-j})^{-1} = \frac{1}{2}$, which is not an element of $\mathbb{Z}[\zeta]$ and therefore, $2\xi \equiv 0 \mod P$.

Throughout this subsection, we have proved the following proposition:

Proposition 5.15. *For a pentagrid $\mathfrak{P}_\gamma$ determined by $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$, we have the following cases of symmetry (rotation is not considered):*

$$\xi = 0, \quad \xi = 1, \quad \xi = 2, \quad \xi = \frac{5}{2}, \quad \xi \in \mathbb{R},$$

$$\xi = \frac{1}{2}(\zeta^3 - \zeta^2), \quad \xi = \frac{1}{2}(\zeta - \zeta^4), \quad \xi \in i\mathbb{R}$$

6 The duality of AR patterns and pentagrids

The goal of this section is to indicate how every AR pattern is generated by a pentagrid and thus to deliver the final argument that shows the duality between AR patterns and pentagrids.
To do so, we introduce an operation on rhombus patterns, called “deflation”, and its inverse, “inflation”.
As an abbreviation, we introduce the notation ϕ for an AR pattern in the following section.

6.1 Deflation and inflation

Deflating an AR-pattern ϕ means to subdivide its building units, the rhombuses, in a specific way in order to obtain smaller tiles. As we will soon experience, this operation has an influence on the vertex types of ϕ.
Conversely, the tiling can grow in size - it is *inflated*. Every AR pattern can be regarded as the deflation of a pattern with bigger tiles, its inflation.

Definition 6.1. *For an original AR pattern ϕ, its **deflation** is called ψ whereas its **inflation** is called χ.*
If T_ϕ is the type of a vertex in ϕ (namely one of $\{Q, K, S, D, J, S3, S4, S5\}$), the deflated vertex type is called $T_\psi^{'}$ and the inflated type is T_χ^*.

Remark. *We note that no two elements of $\{T, T^{'}, T^*\}$ in the definition above are the same element in $\{Q, K, S, D, J, S3, S4, S5\}$.*

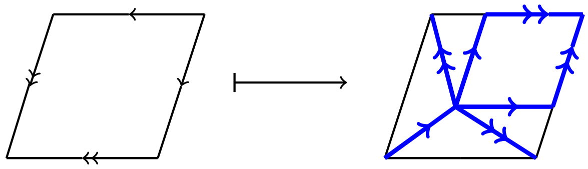

Figure 6: Deflating a thick rhombus

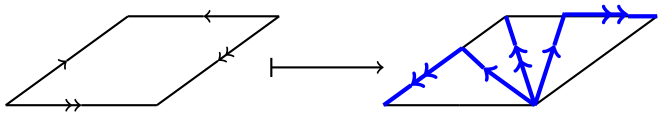

Figure 7: Deflating a thin rhombus

It can be shown that iterated deflation of an AR pattern acts on a vertex of some type from $\{Q, K, S, D, J, S3, S4, S5\}$, as it is depicted in figures 8 and 9.
A vertex from ϕ that produces a J_ψ does not exist. J_ψ is inflated to be a point in the interior of a thick rhombus. Generally speaking, deflating any thick rhombus produces a J_ψ and every red arrow (single arrow) in ϕ produces a D_ψ. This can be seen by comparing figures 4, 6, and 7.

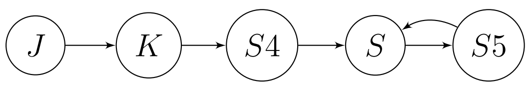

Figure 8: Iterated deflation starting with J

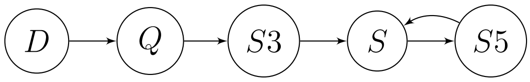

Figure 9: Iterated deflation starting with D

Continuing with the inflation χ of ϕ, we note that the pattern χ is obtained by erasing all vertices J_ϕ and D_ϕ and by connecting the remainig vertices if their distance is $\frac{1}{2}(1+\sqrt{5})$ times as long as their distance in ϕ.
An edge in χ will only be coloured red, if it passes through a vertex of the type D_ϕ in the original pattern ϕ.

Theorem 6.2. *Let ϕ_γ be the AR pattern generated by a regular pentagrid $\mathfrak{P}_{(\gamma_0,\cdots,\gamma_4)}$ and let χ be its inflation. For*

$$\delta_j := \gamma_{j+1} + \gamma_{j-1}, \quad \textit{with } \gamma_5 = \gamma_0,\ \gamma_{-1} = \gamma_4 \tag{6.1}$$

$$p := -(\zeta^2 + \zeta^{-2}) = \frac{1}{2}(1+\sqrt{5}) \tag{6.2}$$

let ϕ_δ be the AR pattern generated by $\mathfrak{P}_{(\delta_j)_{j=0,\cdots,4}}$.
It then follows:

(i) $\sum_{j=0}^4 \delta_j \zeta^{2j} = -p \sum_{j=0}^4 \gamma_j \zeta^{2j}$

(ii) $\chi = p\phi_\delta$

We refer back to the set V defined in Theorem 5.1: $V = \{(\sum_{j=0}^4 \lambda_j, \sum_{j=0}^4 \lambda_j \zeta^{2j}) \mid 0 < \lambda_j < 1\ \forall j\}$.
We claim that the map

$$H : V \longrightarrow V,\ (h,z) \longmapsto (3h \mod 5, -\frac{z}{p}) \tag{6.3}$$

for $h \in \{1,2,3,4\}$ is injective.

H is well-defined as for $(h = \sum_{j=0}^4 \lambda_j, z) \in V$ with $0 < \lambda_j < 1$ for all j and $z = \sum_{j=0}^4 \lambda_j \zeta^{2j}$, $3h$ is an element of $\{3,6,9,12\}$ and thus $3h \mod 5 = x \mod 5$ for $x \in \{1,2,3,4\}$.
We also have $-\frac{z}{p} = \frac{1}{2}(1-\sqrt{5}) \sum_{j=0}^4 \lambda_j \zeta^{2j}$.
H is injective because for two elements (h,z) and (h',z') from V with $3h \mod 5 = 3h' \mod 5$ and $-\frac{z}{p} = -\frac{z'}{p}$ we immediately deduce $z = z'$. $h = h'$ can be shown by assuming $h \neq h'$ and then computing $3h \mod 5$ and $3h' \mod 5$ for the different cases.

$V \setminus H(V)$ then is the set of J's and D's for vertices of index 1 or 4 according to Theorem 5.1 and Figure 5.

Lemma 6.3. *For the set*

$$W := \Big\{(k_0, ..., k_4) \in \mathbb{Z}^5 \mid 1 \leq \sum_{j=0}^4 k_j \leq 4\Big\},$$

the map $\Phi : W \longrightarrow W$ *defined by*

$$\Phi(k)_j = k_{j-1} + k_j + k_{j+1} - c,\ j = 0, ..., 4 \tag{6.4}$$

with $k_{-1} = k_4,\ k_5 = k_0$ *and* $c = \begin{cases} 0, & \text{if } \sum_{j=0}^{4} k_j = 1 \\ 1, & \text{if } \sum_{j=0}^{4} k_j = 2 \\ 1, & \text{if } \sum_{j=0}^{4} k_j = 3 \\ 2, & \text{if } \sum_{j=0}^{4} k_j = 4 \end{cases}$ *is a bijection.*

In addition, it holds that

$$-p^{-1} \sum_{j=0}^{4} k_j \zeta^{2j} = \sum_{j=0}^{4} \Phi(k)_j \zeta^{2j}. \tag{6.5}$$

Proof (Lemma 6.3). Φ is well-defined because for $k = (k_0, ..., k_4) \in W$ and $\Phi(k) = (m_0, ..., m_4)$ with integers m_j, we compute $\sum_{j=0}^{4} m_j = \sum_{j=0}^{4} k_{j-1} + \sum_{j=0}^{4} k_j + \sum_{j=0}^{4} k_{j+1} - 5c$.

$$\sum_{j=0}^{4} k_j = 1 \Longrightarrow c = 0 \Longrightarrow \sum_{j=0}^{4} m_j = 3$$
$$\sum_{j=0}^{4} k_j = 2 \Longrightarrow c = 1 \Longrightarrow \sum_{j=0}^{4} m_j = 1$$
$$\sum_{j=0}^{4} k_j = 3 \Longrightarrow c = 1 \Longrightarrow \sum_{j=0}^{4} m_j = 4$$
$$\sum_{j=0}^{4} k_j = 4 \Longrightarrow c = 2 \Longrightarrow \sum_{j=0}^{4} m_j = 2$$

Next, we show that Φ is injective:
We choose $k, k' \in W$ so that $\Phi(k) = k_{j-1} + k_j + k_{j-1} - c = k'_{j-1} + k'_j + k'_{j-1} - c' = \Phi(k')$. Consequently, $\Phi(k)_j = \Phi(k')_j$ for all j.
Assuming $\sum_{j=0}^{4} k_j \neq \sum_{j=0}^{4} k'_j$ yields $c \neq c'$ and thus $\sum_{j=0}^{4} \Phi(k)_j \neq \sum_{j=0}^{4} \Phi(k')_j$, a contradiction.
The case analysis for $\sum_{j=0}^{4} \Phi(k)_j = \sum_{j=0}^{4} \Phi(k')_j \in \{1, 2, 3, 4\}$ yields $\sum_{j=0}^{4} k_j = \sum_{j=0}^{4} k'_j$; hence, $c = c'$ holds.
With $k_{j-1} + k_j + k_{j-1} = k'_{j-1} + k'_j + k'_{j-1}$, $\sum_{j=0}^{4} k_j = \sum_{j=0}^{4} k'_j$ implies $k_j = k'_j$ for all j.
In order to show that Φ is surjective, we construct an element $k = (k_0, ..., 4) \in W$ so that for $m \in W$, we have $k_{j-1} + k_j + k_{j+1} = m_j$ for all j.
For this, we define $k_j := m_{j-1} + m_{m+1} - d$ and compute

$$-c + k_{j-1} + k_j + k_{j+1} = -3d + \sum_{j=0}^{4} m_j + m_j - c.$$

Now, we can derive the following condtions:

$$\sum_{j=0}^{4} m_j = 1 \Longrightarrow c = 1,\ d = 0$$
$$\sum_{j=0}^{4} m_j = 2 \Longrightarrow c = 2,\ d = 0$$
$$\sum_{j=0}^{4} m_j = 3 \Longrightarrow c = 0,\ d = 1$$
$$\sum_{j=0}^{4} m_j = 4 \Longrightarrow c = 1,\ d = 1.$$

A similar case analysis shows that $k = (k_j)_{j=0,\cdots,4}$ is an element of W, when $\sum_{j=0}^{4} k_j = 2\sum_{j=0}^{4} m_j - 5d$ is considered.

Lastly, we show $-p^{-1}\sum_{j=0}^{4} k_j\zeta^{2j} = \sum_{j=0}^{4}\Phi(k)_j\zeta^{2j}$:

$$\begin{aligned}
-p^{-1}\sum_{j=0}^{4} k_j\zeta^{2j} &= \sum_{j=0}^{4}\Phi(k)_j\zeta^{2j} \Longleftrightarrow \\
\sum_{j=0}^{4} k_j\zeta^{2j} &= -p\sum_{j=0}^{4}\Phi(k)_j\zeta^{2j} \\
&= \sum_{j=0}^{4}(k_{j-1}+k_j+k_{j+1}-c)\zeta^{2j+2} + \sum_{j=0}^{4}(k_{j-1}+k_j+k_{j+1})\zeta^{2j-2} \\
&= \sum_{j=0}^{4} k_j\zeta^{2j}
\end{aligned}$$

The last calculation was cut short. □

Proof (Theorem 6.2). (i) $\sum_{j=0}^{4}\delta_j = 0$ since $\sum_{j=0}^{4}\gamma_j = 0$.

$$\begin{aligned}
-p\sum_{j=0}^{4}\gamma_j\zeta^{2j} &= (\zeta^2+\zeta^{-2})\sum_{j=0}^{4}\gamma_j\zeta^{2j} \\
&= \sum_{j=0}^{4}\gamma_j\zeta^{2j+2} + \sum_{j=0}^{4}\gamma_j\zeta^{2j-2} \\
&= \sum_{j=0}^{4}\delta_j\zeta^{2j}.
\end{aligned}$$

(ii) We define a map

$$f(\gamma, k) = (\sum_{j=0}^{4} k_j, \sum_{j=0}^{4}(k_j-\gamma_j)\zeta^{2j}).$$

Theorem 5.1 tells us that $\sum_{j=0}^4 k_j\zeta^j$ is a vertex of ϕ_δ for $k \in W$ if and only if $f(\delta,k) \in V$.
It holds that $\sum_{j=0}^4 \Phi(k)_j = 3\sum_{j=0}^4 k_j - 5c$. Hence, $\sum_{j=0}^4 \Phi(k)_j = 3\sum_{j=0}^4 k_j \mod 5$.
$f(\delta,k) = (\sum_{j=0}^4 k_j, \sum_{j=0}^4 (k_j - \delta_j)\zeta^{2j})$ and with Lemma 6.3,

$$-p^{-1}\sum_{j=0}^{4}(k_j-\delta_j)\zeta^{2j} = \sum_{j=0}^{4}(\Phi(k)_j-\gamma_j)\zeta^{2j}.$$

We conclude:

$$\begin{aligned} f(\gamma,\Phi(k)) &= (\sum_{j=0}^{4}\Phi(k)_j, \sum_{j=0}^{4}(\Phi(k)_j-\gamma_j)\zeta^{2j}) \\ &= (3\sum_{j=0}^{4}k_j \mod 5, -p^{-1}\sum_{j=0}^{4}(k_j-\delta_j)\zeta^{2j}) \\ &= H(f(\delta,k)). \end{aligned}$$

Getting back to previous considerations, we erase J's and D's from ϕ_γ (in order to inflate the pattern), which leaves us with those vertices $\sum_{j=0}^4 k_j\zeta^j$ so that $k \in W$ and $f(\gamma,k) \in H(V)$.
By putting $\Phi(k)$ instead of k in the condition above, we get all the $\sum_{j=0}^4 \Phi(k)_j = p\sum_{j=0}^4 k_j\zeta^j$ with $k \in W$ and $f(\gamma,\Phi(k)) \in H(V)$ (H is an injection and $f(\gamma,\Phi(k)) = H(f(\delta,k))$) as a consequence. This is the case if and only if $f(\delta,k)$ lies in V, making $\sum_{j=0}^4 k_j\zeta^j$ a vertex of ϕ_δ and thus completing our proof.

□

6.2 How every AR pattern is generated by a pentagrid

Every AR pattern can be transformed in a way that the resulting pattern has vertices

$$\sum_{j=0}^{4}k_j\zeta^j$$

with $k = (k_0,...,k_4) \in W = \{(k_0,...,k_4) \in \mathbb{Z}^5 \mid 1 \leq \sum_{j=0}^4 k_j \leq 4\}$. We call a pattern of this kind "AR$_W$ pattern".
Arising thereby, the proof of the following theorem will use the notion of AR$_W$ patterns without restriction although it will not be stated in the theorem itself.
In preperation for the theorem, we build on the idea of deflation and inflation partially given by Theorem 6.2; if ϕ is an AR$_W$ pattern, its deflation ψ is given by $p^{-1}\phi^{(1)}$ for an AR pattern $\phi^{(1)}$. More so, $\phi^{(1)}$ is an AR$_W$ pattern as well. This is due to

$$\Big\{p\sum_{j=0}^{4}k_j\zeta^j \mid k \in W\Big\} = \Big\{\sum_{j=0}^{4}k_j\zeta^j \mid k \in W\Big\}.$$

Similarly, the inflation χ is given by $p\phi^{(-1)}$ for an AR_W pattern $\phi^{(-1)}$.
We put for positvie integers n by induction:

$$\phi^{(n+1)} = (\phi^{(n)})^{(1)}$$
$$\phi^{(-n-1)} = (\phi^{(-n)})^{(-1)}.$$

Theorem 6.4. *Every AR pattern is produced by a pentagrid $\mathfrak{P}$ which is either regular or singular.*

Proof. This proof mainly utilises three arguments.

(i) For $z_0 \in \mathbb{C}$, let ϕ and ψ be AR_W patterns which have z_0 in common as a vertex and generally have the set of neighbourig vertices in common as well. The union of the closed interior of those rhombuses sharing the vertex z_0 shall be K.
Since z_0 is the "centre" of one of eight possible vertex types, it remains the centre of another neighbourhood of rhombuses after deflating the pattern. $\phi^{(n)}$ and $\psi^{(n)}$ coincide at least inside $p^{-n}K$ by induction.

(ii) We claim that there is an AR_W pattern ψ which is generated by a regular pentagrid for all $R \in \mathbb{R}_{>0}$ and for every AR_W pattern ϕ, so that ϕ and ψ are the same pattern inside $\{z \in \mathbb{C} \mid |z| < R\}$.
As it can be visualized by using thin rhombuses (with edge length 1), the shortest distance between two edges in an AR pattern is $\sin(36°)$.
Let $n \in \mathbb{N}_{>0}$ with $p^n \sin(36°) > 2R$ and let z_0 be the vertex closest to 0 in $\phi^{(-n)}$. Once again, K shall denote the union of the closed rhombuses sharing z_0. Consequently, the distance between 0 and the boundary of K is at least $\frac{1}{2}\sin(36°)$.
From Theorem 5.1, we can deduce that every vertex type appears at least once in an AR pattern generated by a regular pentagrid. As a result, there is an AR_W pattern χ generated by a regular pentagrid that coincides with a neighbourhood of z_0 in $\phi^{(-n)}$. Hence, ϕ and χ conicide (at least) in

$$\{z \in \mathbb{C} \mid |z| < \frac{1}{2}\sin(36°)\}.$$

When we use the first argument and Theorem 6.2, ϕ and ψ coincide in

$$\{z \in \mathbb{C} \mid |z| < \frac{1}{2}p^n \sin(36°)\}$$

for an AR_W pattern ψ generated by a regular pentagrid.
Since $\frac{1}{2}p^n \sin(36°) > R$, the patterns coincide especially in

$$\{z \in \mathbb{C} \mid |z| < R\}.$$

For the n-th deflation of ψ, we get:

$$p^n \cdot \underbrace{p^{-n}\psi}_{n\text{-th deflation}} = \psi.$$

(iii) For an AR_W pattern ϕ and for every $n \in \mathbb{N}_{>0}$, there is, according to the second argument, an AR_W pattern ψ_n generated by a regular pentagrid $\mathfrak{P}_{(\gamma_{0n},\cdots,\gamma_{4n})}$ so that ψ_n and ϕ are the same in $\{z \in \mathbb{C} \mid |z| < n\}$.
For a fixed vertex $\sum_{j=0}^{4} k_j \zeta^j$ in ϕ (we note that ϕ is an AR_W pattern and as such, $(k_0, ..., k_4) \in W$) and for a large positive integer n, $\sum_{j=0}^{4} k_j \zeta^j$ is a vertex of ψ_n.
Hence, $z_n \in \mathbb{C}$ exists with

$$\lceil \Re(z_n \zeta^{-j}) + \gamma_{jn} \rceil = k_j.$$

Applying transoformation (i) from Proposition 5.10, we put $\Re(z_n \zeta^{-j}) + \gamma_{jn}$ instead of γ_{jn}; U_{γ_n} is not changed by this transformation.
Since $\lceil \gamma_{jn} \rceil = k_j$, $|\gamma_{jn}| \leq |k_j| + 1$ can be assumed, we thus have a bounded sequence.
By Bolzano-Weierstraß ([6]), a converging subsequence to $(\gamma_{0k}, ..., \gamma_{4k})$ exists with limit $\gamma := (\gamma_0, ..., \gamma_4)$ (without restriction, we put $\lim_{k\to\infty} \gamma_{ik} = \gamma_i$ for all i, with $\sum_{i=0}^{4} \gamma_{ik} = 0 \ \forall k \in \mathbb{N}$).

For our finite sums and for γ (as in the third argument), we get:

$$\sum_{j=0}^{4} \gamma_j = \sum_{j=0}^{4} \lim_{k\to\infty} \gamma_{jk} = \lim_{k\to\infty} \sum_{j=0}^{4} \gamma_{jk} = 0.$$

For $\sum_{j=0}^{4} \gamma_j = 0$, let $\mathfrak{P}_\gamma$ generate an AR pattern φ.
If $\mathfrak{P}_\gamma$ is regular, φ fully coincides with ϕ because ψ_n is generated by a regular pentagrid for all positive integers n: The radius of the region of coincidence grows for $n \to \infty$.
If $\mathfrak{P}_\gamma$ is singular, it is the limit of a sequence of regular pentagrids (as mentioned earlier) and φ thus fully coincides with ϕ accordingly. □

7 A short note on quasi-periodicity

AR patterns are an example of quasiperiodic tilings of the plane. With the knowledge aquired thus far, it is easy to show that every AR pattern is quasiperiodic, but first, the term "quasi-periodicity" has to be clarified.

Definition 7.1. *A tiling of the plane is* ***quasiperiodic*** *if there is no non-zero shift which leaves the pattern invariant.*

Remark. *The expression "quasiperiodic" is taken from [1], whereas this and other sources use the expression "nonperiodic" as well. In [8], proving non-periodicity works almost the same way as proving quasi-periodicity in [1]. This work therefore treats both expressions synonymously but uses "non-periodicity" exclusively in the introduction in order to relate to [3].*

Theorem 7.2. *Every AR pattern ϕ is quasiperiodic.*

Proof. Let $\gamma = (\gamma_0, ..., \gamma_4)$ be a real vector with $\sum_{j=0}^{4} \gamma_j = 0$ so that ϕ is generated by $\mathfrak{P}_\gamma$. A period would be expressed via a shift vector $\sum_{j=0}^{4} n_j \zeta^j$ for integers n_j with $\sum_{j=0}^{4} n_j = 0$ since the index of a vertex remains unchanged under a shift. Following Proposition 5.10 (ii), it holds $U_{\gamma^*} = U_\gamma$. Consequently, $\sum_{j=0}^{4} n_j \zeta^j = 0$ and for $\xi = \sum_{j=0}^{4} \gamma_j \zeta^{2j}$, we have $\xi^* = \xi + \underbrace{\sum_{j=0}^{4} n_j \zeta^{2j}}_{\neq 0}$ due to shift-equivalence.

Since the set $\{\zeta^j \mid j = 0, ..., 4\}$ is linearly independent over $\mathbb{Q}$, $\sum_{j=0}^{4} n_j \zeta^j = 0 \neq \sum_{j=0}^{4} n_j \zeta^{2j}$ cannot be the case. $\square$

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