The non-multiplicativity of the signature of a fibre bundle and its relation to asymmetric $L$-theory, $SK$ groups and $TQFT$

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Outline

1. Statement of the problem
2. Methods which have been used
3. Previous results
4. Algebraic approach
5. Relations to TQFT and signatures of Lefschetz fibrations
Statement of the problem and methods

The problem

Let $F^n \rightarrow E^{n+m} \rightarrow B^m$ be a fibre bundle with $n + m \equiv 0 \pmod{4}$, 
Problem: What is the relation between the signatures 

$$\sigma(E), \sigma(F), \sigma(B) \in \mathbb{Z}$$

Methods

- Spectral sequences
- Atiyah-Singer index theory
- Characteristic classes
- Group cohomology (cocycles)
- Algebraic $K$-theory
- Algebraic $L$-theory
The signature of a closed oriented $n$-dimensional manifold $M^n$ is denoted by $\sigma(M) \in \mathbb{Z}$.

- If $n = 4k$ then $\sigma(M)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the non-singular symmetric intersection form $(H^{2k}(M; \mathbb{R}), \phi)$, where

$$\phi : H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) \longrightarrow \mathbb{R}; (u, v) \mapsto \langle u \cup v, [M] \rangle.$$  

- If $n \neq 4k$ then $\sigma(M) = 0 \in \mathbb{Z}$. 

For an untwisted product of spaces $X$ and $Y$ the signature is multiplicative:

$$\sigma(X \times Y) = \sigma(X)\sigma(Y) \in \mathbb{Z}$$

In a fibre bundle $F \to E \to B$, the signature of the total space may not be the product of the signatures of the base space and the fibre, so that in general,

$$\sigma(E) \neq \sigma(B)\sigma(F) \in \mathbb{Z}$$
Sufficient conditions for multiplicativity

**Theorem (Chern, Hirzebruch and Serre, 1957)**

Let $F \to E \to B$ be a fibre bundle. *If the fundamental group $\pi_1(B)$ acts trivially on $H^*(F, \mathbb{Q})$ then*

$$\sigma(E) = \sigma(B)\sigma(F) \in \mathbb{Z}. \quad (\text{used spectral sequences})$$

**Example**

Atiyah 1969 and Kodaira 1967, constructed non-multiplicative examples of fibre bundles $F \to E \to B$ with $\pi_1(B)$ acting non-trivially on $H^*(F, \mathbb{Q})$: The total space $E$ is a 4-manifold which arises as a complex algebraic surface, and $B$ and $F$ are compact oriented surfaces of genus 129 and 6 respectively,

$$\sigma(E) = 2^8 \neq \sigma(B)\sigma(F) = 0 \in \mathbb{Z}. \quad (\text{used index theory})$$

More recent examples: Lefschetz fibrations $F^2 \to E^4 \to S^2$
Multiplicativity mod 4 for surface bundles. (Meyer, 1973)

**Theorem** If $F^2 \to E^4 \to B^2$ is a surface bundle then

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod{4}$$

(used group cohomology)

Multiplicativity mod 4. (Hambleton, Korzeniewski, Ranicki, 2007)

**Theorem** Let $F \to E \to B$ be a fibre bundle of closed, connected, compatibly oriented manifolds of **any dimension** then,

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod{4}$$

(used algebraic K-theory)
Motivation

Two closed $n$-dimensional manifolds $M$ and $M'$ are cobordant if there exists a manifold $W^{n+1}$ such that $\partial W = M \sqcup M'$. Cobordism groups of manifolds are denoted by $\Omega_n$.

Various algebraic $L$-theoretic groups can be defined in a similar way. Here we will review the definitions of:

- symmetric $L$-groups
- quadratic $L$-groups
- visible symmetric $L$-groups
- asymmetric $L\text{Asy}$-groups
An algebraic symmetric Poincaré complex \((C, \phi)\) over a ring with involution \(A\) is an \(A\)-module chain complex \(C\) with symmetric Poincaré duality \(\phi \cong \phi^* : C^{n-*} \cong C\).

The **symmetric \(L\)-groups** are cobordism groups of algebraic symmetric Poincaré complexes,

\[(C, \phi) \in L^n(A),\]

generalized Witt groups of symmetric forms. The symmetric signature \(\sigma^*(X) \in L^n(\mathbb{Z}[\pi_1(X)])\) of a geometric Poincaré complex \(X\) is the cobordism class of the symmetric Poincaré complex \((C(\tilde{X}), \phi)\), with \(\tilde{X} = \text{universal cover of } X\).

The **quadratic \(L\)-groups** are the cobordism groups of algebraic quadratic Poincaré complexes,

\[(C, \psi) \in L_n(A)\]

These are the Wall surgery obstruction groups, generalized Witt groups of quadratic forms.
Algebraic cobordism groups III

The **visible symmetric L-groups** $VL^n(B)$ (Weiss, Ranicki, 1992) are the cobordism groups of visible symmetric Poincaré complexes over a space $B$. For the classifying space $K(\pi, 1)$ of a group $\pi$, $VL^{4k}(K(\pi, 1))$ is the Witt group of nonsingular symmetric forms over $\mathbb{Z}[\pi]$ with diagonal entries of the type

$$
\sum_{g \in \pi} a_g (g + g^{-1}) + b \in \mathbb{Z}[\pi], \ (a_g \in \mathbb{Z}, b = 0, 1)
$$

The forgetful map

$$
L_n(\mathbb{Z}[\pi]) \longrightarrow VL^n(K(\pi, 1)); (C, \psi) \mapsto (C, (1 + T)\psi)
$$

is an isomorphism modulo 8 torsion.
The **asymmetric LAsy-groups** are the cobordism groups of asymmetric Poincaré complexes,

\[(C, \lambda : C^{n-*} \simeq C) \in LAsy^n(A)\]

The asymmetric signature, \(\sigma Asy^*(X)\) is the cobordism class of the \(\mathbb{Z}[\pi_1(X)]\)-module chain complex \((C, \lambda)\).

It is important to note that the forgetful map

\[VL^n(K(\pi, 1)) \rightarrow LAsy^n(\mathbb{Z}[\pi])\]

is far from begin an isomorphism, not even rationally. Note that \(VL^0(\{\ast\})\) is finitely generated, whereas \(LAsy^0(\mathbb{Z})\) is infinitely generated.
Transfer maps in $L$-theory

For a fibre bundle $F^m \rightarrow E^{n+m} \xrightarrow{p} B^n$ there exist transfer maps:

\[
\begin{align*}
p^! : L_n(\mathbb{Z}[\pi_1(B)]) & \rightarrow L_{n+m}(\mathbb{Z}[\pi_1(E)]) \\
p^! : VL^n(B) & \rightarrow VL^{n+m}(E) \\
p^! : L\text{Asy}^n(\mathbb{Z}[\pi_1(B)]) & \rightarrow L\text{Asy}^{n+m}(\mathbb{Z}[\pi_1(E)])
\end{align*}
\]

The transfer maps use chain level parallel transport.

\[p^! = (C(\tilde{F}), \alpha, U)^!\]

with

- $C(\tilde{F})$ is a $\mathbb{Z}[\pi_1(E)]$-module chain complex and $\tilde{F}$ is the pullback from the universal cover $\tilde{E}$ of $E$,

- $\alpha : C(\tilde{F}) \rightarrow C(\tilde{F})^{n-*}$,

- $U : \mathbb{Z}[\pi_1(B)] \rightarrow H_0(\text{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}), C(\tilde{F})))$. $U$ is determined by the fibre transport and encodes the information about the action of $\pi_1(B)$ on the homotopy of the fibre $F$. 
A. Korzeniewski (geometric theorem, 2005)

**Theorem** Let $F^{4m} \rightarrow E^{4n+4m} \rightarrow B^{4n}$ be a fibre bundle such that the action of $\pi_1(B)$ on $(H_{2m}(F; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}_2$ is trivial then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod{8}$$

A. Korzeniewski (algebraic theorem, 2005)

**Theorem** Let $(C, \phi)$ be a $4n$ dimensional visible symmetric complex over $\mathbb{Z}[[\pi_1(B)]]$ and let $(A, \alpha, U)$ be a $\mathbb{Z}_2$-trivial $(\mathbb{Z}, 2m)$-symmetric representation. Then

$$\sigma((A, \alpha, U) \tilde{\otimes} (C, \phi)) \equiv \sigma(C, \phi)\sigma(A, \alpha) \pmod{8}$$

*Project: What happens if the $\mathbb{Z}_2$-triviality condition is not assumed?*
# Summary of multiplicativity

<table>
<thead>
<tr>
<th></th>
<th>$\pi_1(B)$ acts trivially on $H^*(F, Q)$</th>
<th>$\pi_1(B)$ acts trivially on $H^{2m}(F, \mathbb{Z})/\text{torsion} \otimes \mathbb{Z}_2$</th>
<th>No assumption</th>
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<tbody>
<tr>
<td><strong>Double covers</strong></td>
<td>Multiplicative</td>
<td>Multiplicative</td>
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<tr>
<td>Manifolds</td>
<td>$\sigma(E) = 2\sigma(B)$</td>
<td>$\sigma(E) = 2\sigma(B)$</td>
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<tr>
<td>Geom. Poincaré cx</td>
<td>Multiplicative</td>
<td>Multiplicative mod 8(^{(1)})</td>
<td>Multiplicative mod 8</td>
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<tr>
<td></td>
<td>$2\sigma(W) - \sigma(\overline{W}) = 8[s(W)]$</td>
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<tr>
<td><strong>Fibrations in general</strong></td>
<td>Multiplicative</td>
<td>Multiplicative mod 8</td>
<td>Not mult. in general</td>
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<tr>
<td>Manifolds</td>
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<td>(Atiyah &amp; Kodaira)</td>
</tr>
<tr>
<td>Geom. Poincaré cx</td>
<td>Multiplicative</td>
<td>Multiplicative mod 8</td>
<td>Not mult. in general</td>
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<td>Mult. mod 4</td>
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\(^1\)The total surgery obstruction $s(W) \in S_{4k}(W)$ has image the codimension 1 splitting obstruction $[s(W)]$, which is given by the Browder-Livesay invariant.
Cut and paste operations on a manifold $M$ are realized as follows: Cut a closed $n$-dimensional smooth manifold $M$ along a codimension 1 manifold $F$ which has trivial normal bundle. After performing this cut we obtain a manifold with two boundary components, each of them a copy of $F$. Pasting back these boundary components by a diffeomorphism $h : F \to F$, results in a new manifold $M(F, h)$.

The set of equivalence classes of oriented manifolds in a space $X$ modulo the relation created by cutting and pasting gives rise to the definition of $SK$-groups.
The signature is a cut and paste invariant.

Jänich (1968)

Let $A = M_1 \cup_h M_2$ and $B = M_1 \cup_g M_2$ be two closed $n$-dimensional manifolds, and $h, g : \partial M_1 \to \partial M_2$ be orientation reversing diffeomorphisms. By the Novikov additivity of the signature:

$$\sigma(A) = \sigma(M_1 \cup_h M_2) = \sigma(M_1) + \sigma(M_2) = \sigma(M_1 \cup_g M_2) = \sigma(B).$$

Hence the signature is a cut and paste invariant.
Let $F_n(X) \subseteq \Omega_n(X)$ be the subgroup of the bordism classes of closed $n$-dimensional manifolds which fibre over $S^1$, then the cut and paste bordism groups are defined geometrically as $\overline{SK}_n(X) \cong \Omega_n(X)/F_n(X)$.

**Neumann (1975)**

**Theorem** (Neumann) If $F^m \to E^{4k} \to B^n$ is a fibration with $\sigma(B) = 0$ and $\sigma(E) \neq 0$, so that $\sigma(E) \neq \sigma(F)\sigma(B)$, then $[B, f : B \to BG]$ generates a free $\overline{SK}_*(BG)$-module.

The asymmetric signature of a mapping torus is zero:

$$\sigma Asy(T(h)) = 0 \in LAsy^*(\mathbb{Z}[\pi_1(T(h))])$$

**Ranicki (1998)**

$$\overline{SK}(X) \cong \text{Im}(\sigma Asy : \Omega_n(X) \to LAsy^n(\mathbb{Z}[\pi_1(X)])$$
Neumann’s theorem can be proved **algebraically** by using the transfer map in asymmetric $L$-theory:

$$
\text{LAsy}^n(\mathbb{Z}[\pi_1(B)]) \xrightarrow{p^!} \text{LAsy}^{4k}(\mathbb{Z}[\pi_1(E)]) \rightarrow \text{LAsy}^{4k}(\mathbb{Z}) \rightarrow L^{4k}(\mathbb{Z}) = \mathbb{Z}.
$$

Note that if $\sigma(E) \neq 0$ then $\sigma \text{Asy}(B)$ has infinite order in $\text{LAsy}^n(\mathbb{Z}[\pi_1(B)])$.

Consequently $0 \neq \sigma \text{Asy}(B) \in \text{Im}(\sigma \text{Asy} : \Omega_n(B) \rightarrow \text{LAsy}^n(\mathbb{Z}[\pi_1(B)]))$. 


From a topological point of view, a field theory gives a way to compute an invariant by cutting a manifold into simple pieces and making the function effectively computable. Which invariants of closed manifolds are partition functions of the field theory? No complete answer is known to this question, but SK invariants are such partition functions.

Very recently Markus Banagl has described how non-multiplicative fibre bundles give rise to certain TQFTs.

This relationship was described geometrically by Neumann and algebraically in terms of asymmetric L-theory by Ranicki.
Thank you!