ON SPECIAL VALUES OF SIEGEL MODULAR FORMS.

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In his admirable book "Arithmeticity in the Theory of Automorphic Forms" Shimura establishes various algebraicity results concerning special values of Siegel modular forms. These results are all stated over an algebraic closure of $\mathbb{Q}$. In this article we work out the field of definition of these special values. In this way we extend some previous results obtained by Sturm, Harris, Panchishkin, and Böcherer-Schmidt.

1. Introduction

Special values of L function play a central role in Iwasawa theory since they are indispensable for the formulation of the Main Conjectures. It is precisely this information which is encoded in the interpolation properties of the p-adic L-functions. The first step to construct these p-adic L-functions is to show that the L-functions under consideration evaluated at "critical" points have particular algebraic properties. These properties are usually described by Deligne's conjectures. In this article we address this kind of questions for L functions associated to Siegel modular forms.

This article grew out of the author's effort to read carefully the book of Shimura "Arithmeticity in the Theory of Automorphic Forms" ([11]) which means to do also the "exercises" left by Shimura to the reader. One of them is related to the algebraicity of various special values of Siegel modular forms (see page 239, Remark 28.13 in (loc. cit.)). As Shimura points out the results left as exercises should follow by using the various techniques and results obtained in his book and various papers of him. This is indeed the case since most ideas of this article can be found in the various works of Shimura, which of course in turn requires some familiarity with them. In any rate we believe that it is useful to have the results worked out in this paper documented in the literature, and for this reason we decided to write this article. In this paper we consider the special values of Siegel modular forms of integral weight. In [2], the continuation of this article, we consider also special value of half-integral weight.

Let us point out some results in this article that we believe deserve special mention. The first is the reciprocity law of the action of the Galois group on half-integral weight Eisenstein series. For integral weight Eisenstein series one can find the reciprocity laws in the book of Feit ([3]) (if not in the form that it is needed for our purposes). However to the best of our knowledge the reciprocity for half integral Eisenstein series has not been worked out for Siegel modular forms. Another interesting result is the definition of the period $\Omega_{\mathbf{f}}$ appearing in Theorem 7.3. These kind of periods have been first considered by Sturm and Harris [12, 5] (and later also by Panchishkin), based on an idea of Shimura. We follow the ideas of Sturm in defining them but using some new

The author acknowledges support by the ERC.

results of Shimura we are able to improve in some cases the bounds on the weight of the Siegel modular forms that the results are applicable. Also the fact that we use the more precise form of the Andrianov-Kalinin type identity proved by Shimura, we can obtain slightly finer results, since we need to remove less Euler factors of the L-function.

This paper is organized as follows. In section two we have a very brief introduction to Siegel modular forms. Then we move to section three where after presenting various results of Shimura with respect the theory of theta series and Eisenstein series for the symplectic group, we prove the various reciprocity laws of the action of the absolute Galois group on the Eisenstein series. Some of the result have already appeared in [12] and [3], and we use ideas of these works. For the case of half integral weight Eisenstein series we prove the reciprocity inspired by an idea of Shimura. In section 4 we introduce the L functions which are considered in this paper. All the material of this section is from Shimura's book. In section 5 we also present the work of Shimura on the generalization of the so-called Adrianov-Kalinin type identity. However for our purposes we use an integral expression that it is not in the book [11] but in a paper of Shimura [9]. The use of this integral expression will lead to study slightly different L-functions than in the ones studied in the book of Shimura (we explain more later on this). Also in this section all the material is taken from works of Shimura. In section 6 we define the periods that we will use to obtain the good reciprocity laws. The idea of defining this periods as values of an L function goes back to Shimura, and have been used by Sturm [12], Harris [5] and Panchishkin [7] in the case Siegel modular forms over the rationals and of even degree. We also note that we obtain a slightly different results than in these two works, partly because we use some newer results of Shimura that were not available when these works were written. Finally in the last section we present the various results on the field of rationality of the various special functions properly normalized and in some cases we provide some reciprocity results.

One last remark with respect to the notation used in this article. Since we are using as our main reference the book of Shimura [11] we decided to keep, the anyway excellent, notation used by him. In particular if some times we use some notions not defined in this paper the reader will find the exact same notation also in the reference. This allows to keep the length of this article reasonable since we do not need to introduce all the objects used here.

2. Siegel Modular forms

2.1. **Integral weight Siegel modular forms.** In this section we introduce the notion of a Siegel modular form (classicaly and adelically). We follow closely the book of Shimura [11].

For a positive inteher $n \in \mathbb{N}$ we define the matrix $\eta_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and for any commutative ring A with an identity the group $Sp_n(A) := \{\alpha \in GL_{2n}(A) | {}^t\alpha\eta_n\alpha = \eta_n\}$. The group $Sp_n(\mathbb{R})$ acts on the Siegel upper half space $\mathbb{H}_n := \{z \in \mathbb{C}_n^n | {}^tz = z,\ Im(z) > 0\}$ by linear fractional transformations, that is for $\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in Sp_n(\mathbb{R})$ and $z \in \mathbb{H}_n$ we have $\alpha \cdot z := (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1} \in \mathbb{H}_n$. Moreover if we define $\mu_\alpha(z) :=$

$\mu(\alpha, z) := c_\alpha z + d_\alpha$ then we have

$$\mu(\beta\alpha, z) = \mu(\beta, \alpha z)\mu(\alpha, z),\ \ \alpha, \beta \in Sp_n(\mathbb{R}), z \in \mathbb{H}_n$$

Let now F be a totally real field of degree $d := [F : \mathbb{Q}]$ and write $\mathfrak{g}$ for its ring of integers. We write $\mathbf{a}$ for the set of archimedean places of F and $G := Sp_n(F)$. We write $G_\mathbb{A}$ for the adelic group and we decompose $G_\mathbb{A} = G_\mathbf{h} G_\mathbf{a}$ where $G_\mathbf{a} := \prod_{v\in\mathbf{a}} G_v$. For two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{g}$, we define the subgroup of $G_\mathbb{A}$,

$$D[\mathfrak{a}, \mathfrak{b}] := \left\{ x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_\mathbb{A} | a_x \prec \mathfrak{g}_v, b_x \prec \mathfrak{a}_v, c_x \prec \mathfrak{b}_v, d_x \prec \mathfrak{g}_v, \forall v \in \mathbf{h} \right\},$$

where we use the notation "$\prec$" of Shimura, $x \prec \mathfrak{b}_v$ meaning that the v-component of x is a matrix with entries in the ideal $\mathfrak{b}_v$. We will mainly consider groups of the form $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ for a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$. Strong approximation for G implies that $G_\mathbb{A} = GqD[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ for any $\mathfrak{b}, \mathfrak{c}$ and $q \in G_\mathbf{h}$. We define $\Gamma^q(\mathfrak{b}, \mathfrak{c}) := G \cap qD[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]q^{-1}$. Given a Hecke character ψ of F with $\psi_v(a) = 1$ for all $a \in \mathfrak{g}_v^\times$, $v \in \mathbf{h}$ such that $a - 1 \in \mathfrak{c}_v$ we define a character on $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ by $\psi(x) = \prod_{v|\mathfrak{c}} \psi_v(det(d_x)_v)$ and a character which we still denote ψ on Γ^q by $\psi(\gamma) := \psi(q^{-1}\gamma q)$.

We now write $\mathbb{Z}^\mathbf{a} := \prod_{v\in\mathbf{a}} \mathbb{Z}$ and $\mathcal{H} := \prod_{v\in\mathbf{a}} \mathbb{H}_n$. For a function $f : \mathcal{H} \to \mathbb{C}$ and an element $k \in \mathbb{Z}^\mathbf{a}$ we define

$$(f|_k\alpha)(z) := j_\alpha(z)^{-k} f(\alpha z),\ \ \alpha \in G,\ \ z \in \mathcal{H}.$$

Here we write $z = (z_v)_{v\in\mathbf{a}}$ with $z_v \in \mathbb{H}_n$ and $\alpha_v \in Sp_n(\mathbb{R})$ and define $j_\alpha(z)^{-k} := \prod_v det(\mu_{\alpha_v}(z_v))^{-k_v}$. Let now Γ be group of the form Γ^q, $q \in G_\mathbf{h}$ as above and ψ a Hecke character. Then we define

Definition 2.1. A function $f : \mathcal{H} \to \mathbb{C}$ is called a Siegel modular form for the congruence subgroup Γ of weight $k \in \mathbb{Z}^\mathbf{a}$ and Nebentypus ψ if

(i) f is holomorphic,
(ii) $f|_k\gamma = \psi(\gamma)f$ for all $\gamma \in \Gamma$,
(iii) f is holomorphic at cusps.

The last condition is needed only if $F = \mathbb{Q}$ and $n = 1$. Then it is the classical condition of elliptic modular forms being holomorphic at cusps. The above defined space we will denote it by $\mathcal{M}_k(\Gamma, \psi)$. As it is explained in [11, page 33] for an element $f \in \mathcal{M}_k(\Gamma, \psi)$ and an element $\alpha \in G$ we have a Fourier expansion

$$(f|_k\alpha)(z) = \sum_{h\in S_+} c_\alpha(h) e^n_\mathbf{a}(hz),$$

where S_+ is the set of n by n symmetric matrices with entries in F which are positive semi-definite at every real place $v \in \mathbf{a}$ and $e^n_\mathbf{a}(x) = exp(2\pi i \sum_{v\in\mathbf{a}} tr(x_v))$. An element $f \in M_k(\Gamma, \psi)$ is called a cusp form if $c_\alpha(h) \neq 0$ for some $\alpha \in G$ implies h_v is positive definite for all $v \in \mathbf{a}$.

We now turn to the adelic Siegel modular forms. Let D be a group of the form $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ and ψ a Hecke character of F.

Definition 2.2. A function $\mathbf{f} : G_\mathbb{A} \to \mathbb{C}$ is called adelic Siegel modular form if

(i) $\mathbf{f}(\alpha x w) = \psi(w) j_w^k(\mathbf{i})\mathbf{f}(x)$ for $\alpha \in G$, $w \in D$ with $w(\mathbf{i}) = \mathbf{i}$,
(ii) For every $p \in G_{\mathbf{h}}$ there exists $f_p \in M_k(\Gamma^p, \psi_p)$, where $\Gamma^p := G \cap pCp^{-1}$ and $\psi_p(\gamma) = \psi(p\gamma p^{-1})$ such that $\mathbf{f}(py) = (f_p|_k y)(\mathbf{i})$ for every $y \in G_{\mathbf{a}}$.

We write $\mathcal{M}_k(D,\psi)$ for this space. Strong approximation theorem for Sp_n gives $\mathcal{M}_k(D,\psi) \cong M_k(\Gamma^q, \psi_q)$ for any $q \in G_{\mathbf{h}}$. We define the space of automorphic cusp form $\mathcal{S}_k(D,\psi)$ to be the subspace of $\mathcal{M}_k(D,\psi)$ that is in bijection with $S_k(\Gamma^q,\psi)$ for any $q \in G_{\mathbf{h}}$ in the above bijection. We may also sometimes write $\mathcal{M}_k(\mathfrak{b},\mathfrak{c},\psi)$ for $\mathcal{M}_k(D,\psi)$. Similarly we may write $M_k(\mathfrak{b},\mathfrak{c},\psi)$ for $M_k(\Gamma,\psi)$ where $\Gamma = G \cap D[\mathfrak{b}^{-1},\mathfrak{bc}]$, i.e $q = 1$.

2.2. **Half-integral weight Siegel modular forms.** Even though we will consider only algebraicity results for integral weight Siegel modular forms, in many case we will need to use half-integral weight modular forms. We denote by $M_{\mathbf{A}}$ the adelized metaplectic group siting in the exact sequence $0 \to \mathbb{T} \to M_{\mathbb{A}} \to G_{\mathbb{A}} \to 0$. The last projection we denote by pr. We write C^θ for the *theta* group defined for example in [9, page 536] and $\Gamma^\theta = G \cap C^\theta$. We also define the group $\mathfrak{M} = \{x \in M_{\mathbb{A}} | pr(x) \in P_{\mathbb{A}} C^\theta\}$, where P is the standard Siegel parabolic subgroup of G. Thanks to a canonical lift we may consider G as a subgroup of $M_{\mathbb{A}}$ and hence also Γ^θ a subgroup of $\mathfrak{M}$. For an element $\sigma \in \mathfrak{M}$ and $z \in \mathcal{H}$ we write $h_\sigma(z)$ for the holomorphic function defined by Shimura. By a half integral weight $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ we mean a tuple $(k_v)_{v\in\mathbf{a}}$ so that $k_v \in \mathbb{Z} + \frac{1}{2}$ for all $v \in \mathbf{a}$. For such a k we define the factor of automorphy

$$j_\sigma(z)^k := h_\sigma(z) j_{pr(\sigma)}(z)^{[k]}.$$

Then the definition of half integral weight modular forms, with congruence subgroup $\Gamma \leq \Gamma^\theta$ is the same as in integral case but using the new factor of automorphy. One may define also adelic automorphic forms, we refer to Shimura [11, page 166] for this.

3. Theta and Eisenstein Series

3.1. **Theta series.** Following Shimura we set $W = F_n^n$ and we let $\mathcal{S}(W_{\mathbf{h}})$ denote the space of Schwartz-Bruhat functions on $W_{\mathbf{h}}$. Let τ be an n by n symmetric matrix with entries in F such that $\tau_v > 0$ for all $v \in \mathbf{a}$. For an element $\lambda \in \mathcal{S}(W_{\mathbf{h}})$ and an element $\mu \in \mathbb{Z}^{\mathbf{a}}$ such that $0 \leq \mu_v \leq 1$ for all $v \in \mathbf{a}$ we define

$$\theta(z,\lambda) = \sum_{\xi \in W} \lambda(\xi_{\mathbf{h}}) det(\xi)^\mu \mathbf{e}_{\mathbf{a}}(tr({}^t\xi\tau\xi z)), \ \ z \in \mathcal{H}.$$

It is shown in the appendix of [11] that this is an element of $\mathcal{M}_l$ with $l := \mu + \frac{n}{2}\mathbf{a}$. Moreover it is also shown that if $\mu \neq 0$ then $\theta(z,\lambda)$ is actually a cusp form. We now introduce some extra notation following [11, Appendix A3.18]. We set

$$R = \prod_{v\mathbf{h}} (\mathfrak{g}_v)_n^n, \ E_v = GL_n(\mathfrak{g}_v), \ \ R^* = RW_{\mathbf{a}} \subset W_{\mathbf{A}}.$$

We let ω be now a Hecke character of F of conductor $\mathfrak{f}$ such that $\omega_{\mathbf{a}}(-1)^n = (-1)^{n\sum_v \mu_v}$. Let now r be an element of $GL_n(F)_{\mathbf{h}}$ and define

$$\theta(z) := \sum_{W \cap rR^*} \omega_{\mathbf{a}}(det(\xi))\omega^*(det(r^{-1}\xi)\mathfrak{g}) det(\xi)^\mu \mathbf{e}_{\mathbf{a}}^n({}^t\xi\tau\xi t),$$

where for a Hecke character ψ we denote by ψ^* the corresponding ideal character. Then Shimura proves the following proposition

Proposition 3.1 (Shimura)**.** *Let ρ_τ be the Hecke character of F corresponding to the extension $F(c^{1/2})/F$ with $c = (-1)^{[n/2]}det(2\tau)$; put $\omega' = \omega\rho_\tau$. Then there exist a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$, such that the conductor of ω' divides $\mathfrak{c}$, $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$ if n is odd, and*

$$\theta(\gamma z) = \omega'_{\mathfrak{c}}(det(a_\gamma))j^l_\gamma(z)\theta(z),\ \gamma \in G \cap D,$$

where $D = \{x \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. Moreover, if $\beta \in G \cap diag[q, \hat{q}]C$ with $q \in GL_n(F)_{\mathbf{h}}$, then

$$j^l_\beta(\beta^{-1}z)\theta(\beta^{-1}z) = \omega'(det(q))^{-1}\omega'_{\mathfrak{c}}(det(d_\beta q))|det(q)|^{n/2}_{\mathbf{A}} \times$$
$$\sum_{\xi \in W \cap rR^* q^{-1}} \omega_{\mathbf{a}}(det(\xi))\omega^*(det(\xi r^{-1}q)\mathfrak{g})det(\xi)^\mu \mathbf{e}^n_{\mathbf{a}}({}^t\xi\tau\xi z).$$

In particular, let $\mathfrak{x}$ and $\mathfrak{t}$ be fractional ideals of F such that ${}^tg2\tau g \in \mathfrak{x}$ for every $g \in r\mathfrak{g}^n_1$ and ${}^th(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ for every $h\hat{r}\mathfrak{g}^n_1$ and write $\mathfrak{h}$ for the conductor of ρ_τ. Then we can take

$$(\mathfrak{b}, \mathfrak{c}) = \begin{cases} (2^{-1}\mathfrak{d}\mathfrak{x}, \mathfrak{h} \cap \mathfrak{f} \cap \mathfrak{x}^{-1}\mathfrak{f}^2\mathfrak{t}), & \text{if } n \text{ is even;} \\ (2^{-1}\mathfrak{d}\mathfrak{a}^{-1}, \mathfrak{h} \cap \mathfrak{f} \cap 4\mathfrak{a} \cap \mathfrak{a}\mathfrak{f}^2\mathfrak{t}), & \text{if } n \text{ is odd.} \end{cases},$$

where $\mathfrak{a} = \mathfrak{x}^{-1} \cap \mathfrak{g}$.

3.2. **Eisenstein series.** We follow Shimura [10, 11] and define various Eisenstein series of Siegel type. Let $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ be a weight, $\mathfrak{b}$ a fractional ideal of F, $\mathfrak{c}$ an integral ideal in F and a Hecke character χ of F with infinity type $\chi_{\mathbf{a}}(x) = x^\ell_{\mathbf{a}}|x_{\mathbf{a}}|^{-\ell}$, and

$$\chi_v(a) = 1,\ \text{if}\ v \in \mathbf{h},\ a \in \mathfrak{r}^\times_v,\ \text{and}\ a - 1 \in \mathfrak{r}_v\mathfrak{c}_v,\ \forall\ v \in \mathbf{h}.$$

When k is half integral we also assume that $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$, where $\mathfrak{d}$ the different ideal of F. Following the notation of Shimura we now define in the case of $k \in \mathbb{Z}^{\mathbf{a}}$

$$\widetilde{D} = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}],\ \widetilde{D}_0 = D_0[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}],$$

and otherwise

$$\widetilde{D} = \{x \in M_{\mathbf{A}} | pr(x) \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]\},\ \widetilde{D}_0 = \{x \in \widetilde{D} | pr(x) \in D_0[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]\}.$$

Write $P = \{x \in G | c_x = 0\}$ for the standard Siegel parabolic. We then define a function μ on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$ by

$$\mu(x) = 0,\ \text{if}\ x \notin P_{\mathbf{A}}\widetilde{D},$$
$$\mu(x) = \chi_{\mathbf{h}}(det(d_p))^{-1}\chi_{\mathfrak{c}}(det(d_w))^{-1}j^k_x(\mathbf{i})^{-1}|j_x(\mathbf{i})|^k,$$

if $x = pw$ with $p \in P_{\mathbf{A}}$ and $w \in \widetilde{D}$. Then for a pair $(x, s) \in G_{\mathbf{A}} \times \mathbb{C}$ if $k \in \mathbb{Z}^{\mathbf{a}}$ or in $M_{\mathbf{A}} \times \mathbb{C}$ otherwise, we define the Eisenstein series

$$E_{\mathbf{A}}(x, s) = E_{\mathbf{A}}(x, s; \chi, \widetilde{D}) = \sum_{\alpha \in P \backslash G} \mu(\alpha x)\epsilon(\alpha x)^{-s}.$$

We will need one more type of Eisenstein series. We define the element $\zeta \in Sp(n, F)_{\mathbf{A}}$ by

$$\zeta_{\mathbf{a}} = 1,\ \zeta_{\mathbf{h}} = \begin{pmatrix} 0 & -\delta^{-1}1_n \\ \delta 1_n & 0 \end{pmatrix},$$

where $\delta \in F_{\mathbf{h}}^{\times}$ such that $\delta\mathfrak{g} = \mathfrak{d}$. We further fix an element $\tilde{\zeta} \in M_{\mathbf{A}}$ such that $pr(\tilde{\zeta}) = \zeta$ and $h(\tilde{\zeta}, z) = 1$. Then we define the Eisenstein series

$$E^*_{\mathbf{A}}(x,s) = \chi(\delta)^{-n} \times \begin{cases} E_{\mathbf{A}}(x\zeta, s), & k \in \mathbb{Z}^{\mathbf{a}} \ ; \\ E_{\mathbf{A}}(x\tilde{\zeta}, s), & \text{otherwise.} \end{cases}$$

Finally we define the Eisenstein series

$$D_{\mathbf{A}}(x,s) = E^*_{\mathbf{A}}(x,s) \times \begin{cases} L_{\mathfrak{c}}(2s,\chi)\prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(4s-2i,\chi^2), & k \in \mathbb{Z}^{\mathbf{a}}; \\ \prod_{i=1}^{[(n+1)/2]} L_{\mathfrak{c}}(4s-2i-1,\chi^2), & k \notin \mathbb{Z}^{\mathbf{a}}; \end{cases},$$

Write $S = \{x \in F_n^n | {}^t x = x\}$. Then the q-expansion of $E^*_{\mathbf{A}}(x,s)$ is given by

$$E^*_{\mathbf{A}}\left(\left(\begin{array}{cc} q & \sigma\hat{q} \\ 0 & \hat{q} \end{array}\right), s\right) = \sum_{h \in S} c(h,q,s)\mathbf{e}^n_{\mathbf{A}}(h\sigma),$$

where $q \in GL_n(F)_{\mathbf{A}}$ and $\sigma \in S_{\mathbf{A}}$. We now define the Eisenstein series on $(z,s) \in \mathcal{H} \times \mathbb{C}$ by

$$E^*(x(\mathbf{i}), s) = j_x^k(\mathbf{i})E^*_{\mathbf{A}}(x,s),$$

and similarly we define $D(z,s$ and $D^*(z,s)$. When we want to indicate the dependence on the various input data we will write $E(z,s;k,\chi,\mathfrak{c})$ for $E(z,s)$ or in case we want also to indicate the dependency on $\mathfrak{b}$ we will write $E(z,s;k,\chi,\Gamma)$, where $\Gamma = G \cap D[\mathfrak{b}^{-1},\mathfrak{b}\mathfrak{c}]$.

We now note the q-expansion

$$E^*(z,s) = \sum_{h \in S} det(y)^{-k\mathbf{a}/2} c(h,q,s)\mathbf{e}^n_{\mathbf{a}}(hx),$$

where $q_{\mathbf{h}} = q_1$ and $q_{\mathbf{a}} = y^{1/2}$. For the coefficients $c(h,q,s)$ we have the following propositions of Shimura [11, Proposition 16.9, 16.10, 17.6],(for notation not introduced here we refer to (loc. cit)).

Proposition 3.2 (Shimura)**.** *Suppose that $\mathfrak{c} \neq \mathfrak{g}$ and $det(q_v) > 0$ for every $v \in \mathbf{a}$. Then $c(h,q,s) \neq 0$ only if $({}^t qhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v\tilde{S}_v$ for every $v \in \mathbf{h}$. In this case*

$$c(h,q,s) = C\chi_{\mathbf{h}}(det(-q))^{-1}|det(q)_{\mathbf{h}}|_{\mathbf{A}}^{n+1-2s}|D_F|^{-2ns+3n(n+1)/4}N(\mathfrak{b}\mathfrak{c})^{-n(n+1)/2}\times$$

$$det(y)^{s\mathfrak{a}}\Xi(y;h;s\mathbf{a}+k/2,s\mathbf{a}-k/2)\alpha^e_{\mathfrak{c}}(\epsilon_b^{-1}\cdot {}^t qhq, 2s,\chi),$$

where $C = 1$ and $e = 0$ if $k \in \mathbb{Z}^{\mathbf{a}}$, and $C = \mathbf{e}(n[F:\mathbb{Q}]/8)$ and $e = 1$ if $k \notin \mathbb{Z}^{\mathbf{a}}$; $\epsilon_b \in F_{\mathbf{h}}^{\times}$ such that $\epsilon_b\mathfrak{g} = \mathfrak{b}^{-1}\mathfrak{d}$ if $k \in \mathbb{Z}^{\mathbf{a}}$, and $\epsilon_b = 1$ otherwise; D_F is the discriminant of F.The function $\Xi(g;h;\alpha,\beta) = \prod_{v\in\mathbf{a}}\xi(y_v,h_v;\alpha_v,\beta_v)$ is given in [11, page 140].

Proposition 3.3 (Shimura)**.** *Consider q and h such that $c(h,q,s) \neq 0$. Set $r = rank(h)$ and let $g \in GL_n(F)$ such that $g^{-1}hg = diag[h',0]$ with $h' \in S^r$. Let ρ_h be the Hecke character corresponding to $F(c^{1/2})/F$ where $c = (-1)^{[r/2]}det(2h')$, if $r > 0$; let $\rho_h = 1$ if $r = 0$. Then*

$$\alpha^e_{\mathfrak{c}}(\epsilon_b^{-1}\cdot {}^t qhq, 2s,\chi) = \Lambda_{\mathfrak{c}}(s)^{-1}\Lambda_h(s)\prod_{v\in\mathbf{c}} f_{h,q,v}\left(\chi(\pi_v)|\pi_v|^{2s+e/2}\right),$$

where

$$\Lambda_{\mathfrak{c}}(s) = \begin{cases} L_{\mathfrak{c}}(2s,\chi)\prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(4s-2i,\chi^2), & \textit{if } k \in \mathbb{Z}^{\mathbf{a}}; \\ \prod_{i=1}^{[(n+1)/2]} L_{\mathfrak{c}}(4s-2i+1,\chi^2), & \textit{otherwise.} \end{cases}$$

$$\Lambda_h(s) = \begin{cases} L_{\mathfrak{c}}(2s-n+r/2,\chi\rho_h)\prod_{i=1}^{[(n-r)/2]} L_{\mathfrak{c}}(4s-2n+r+2i-1,\chi^2), & \text{if } k \in \mathbb{Z}^{\mathbf{a}}; \\ \prod_{i=1}^{[(n-r+1)/2]} L_{\mathfrak{c}}(4s-2n+r+2i-2,\chi^2), & \text{otherwise.} \end{cases}$$

Here $f_{h,q,v}$ are polynomials with coefficients in $\mathbb{Z}$, independent of χ. The set $\mathbf{c}$ is determined as follows: $\mathbf{c}=\emptyset$ if $r=0$. If $r>0$, then take $a \in \prod_{v\nmid\mathfrak{c}} GL_n(\mathfrak{g}_v)$ so that $(\epsilon_b^{-1}{}^t a^t qhqa)_v = diag[\tau_v, 0]$ with $\tau_v \in T_v^r$ for every $v \nmid \mathfrak{c}$. Then $\mathbf{c}$ consists of those v's not dividing $\mathfrak{c}$ such that τ_v is not regular.

For a number field W we follow Shimura and write $\mathcal{N}_k^r(W)$ for the space of W-rational nearly holomorphic forms of weight k (see [11, page 103 and page 110] for the definition). The theorem below is due to Shimura [11, Theorem 17.9].

Theorem 3.4 (Shimura). *Let Φ be the Galois closure of F over $\mathbb{Q}$ and let $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ with $k_v \geq (n+1)/2$ for all $v \in \mathbf{a}$ and $k_v - k_{v'} \in 2\mathbb{Z}$ for every $v, v' \in \mathbf{a}$. Let $\mu \in \frac{1}{2}\mathbf{Z}$ with $n+1-k_v \leq \mu \leq k_v$ and $|\mu - \frac{n+1}{2}| + \frac{n+1}{2} - k_v \in 2\mathbb{Z}$ for all $v \in \mathbf{a}$. Exclude the cases*

(i) $\mu = (n+2)/2$, $F=\mathbb{Q}$ *and* $\chi^2 = 1$,
(ii) $\mu = 0$, $\mathfrak{c} = \mathfrak{g}$ *and* $\chi = 1$,
(iii) $0 < \mu \leq n/2$, $\mathfrak{c} = \mathfrak{g}$ *and* $\chi^2 = 1$.

Then $D(z,\mu/2;k,\chi,\mathfrak{c})$ belongs to $\pi^{\beta}\mathcal{N}_k^r(\Phi\mathbb{Q}_{ab})$, where $r = (n/2)(k - |\mu-(n+1)/2|\mathbf{a} - \frac{n+1}{2}\mathbf{a})$ except in the case where $n=1$, $\mu=2$, $F=\mathbb{Q}$, $\chi=1$ and $n>1$, $\mu=(n+3)/2$, $F=\mathbb{Q}$, $\chi^2=1$. In these two case we have $r = n(k-\mu+2)/2$. Moreover we have that $\beta = (n/2)\sum_{v\in\mathbf{a}}(k_v+\mu) - [F:\mathbb{Q}]e$ where

$$e = \begin{cases} [(n+1)^2/4] - \mu, & \text{if } 2\mu+n \in 2\mathbb{Z} \text{ and } \mu \geq \lambda; \\ \left[n^2/4\right], & \text{otherwise.} \end{cases}$$

For an element $p \in \mathbb{Z}^{\mathbf{a}}$ and a weight $q \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ we write Δ_q^p for the differential operators defined by Shimura in [11, page 146]. In particular we have $\Delta_q^p \mathcal{N}_q^t(\Phi\mathbb{Q}_{ab}) \subset \pi^{n|p|}\mathcal{N}_{q+2p}^{t+np}(\Phi\mathbb{Q}_{ab})$. Moreover for any $f \in \mathcal{N}_q^t(\Phi\mathbb{Q}_{ab})$ and any $\sigma \in Gal(\Phi\mathbb{Q}_{ab}/\Phi)$ we have that

$$\left(\pi^{-n|p|}\Delta_q^p(f)\right)^{\sigma} = \pi^{-n|p|}\Delta_q^p(f^{\sigma}) \tag{3.1}$$

Let $\mu \in \frac{1}{2}\mathbb{Z}$ and $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ be as in the theorem above. If $\mu \geq (n+1)/2$ then Shimura shows that [11, page 146]

$$\Delta_{\mu\mathbf{a}}^p D(z,\mu/2;\mu\mathbf{a},\chi,\mathfrak{c}) = c_{\mu\mathbf{a}}^p(\mu/2)(i/2)^{n|p|}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}), \tag{3.2}$$

where $p = (k-\mu\mathbf{a})/2$. Here $c_{\mu\mathbf{a}}^p(\mu/2) \in \mathbb{Q}^{\times}$. If $\mu < (n+1)/2$ then we have

$$\Delta_{\nu\mathbf{a}}^p D(z,\mu/2;\nu\mathbf{a},\chi,\mathfrak{c}) = c_{\nu\mathbf{a}}^p(\mu/2)(i/2)^{n|p|}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}), \tag{3.3}$$

where $\nu = n+1-\mu$, $p = (k-\nu\mathbf{a})/2$ and again $c_{\nu\mathbf{a}}^p(\mu/2) \in \mathbb{Q}^{\times}$.

The following lemma is immediate from the above equations,

Lemma 3.5. *Assume there exists $A(\chi), B(\chi) \in \mathbb{Q}_{ab}$ and $\beta_1, \beta_2 \in \mathbb{N}$ such that for all $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$*

$$\left(\frac{D(z,\mu/2;\mu\mathbf{a},\chi,\mathfrak{c})}{\pi^{\beta_1}A(\chi)}\right)^\sigma = \frac{D(z,\mu/2;\mu\mathbf{a},\chi^\sigma,\mathfrak{c})}{\pi^{\beta_1}A(\chi^\sigma)}, \ \ \mu \geq (n+1)/2$$

and

$$\left(\frac{D(z,\mu/2;\nu\mathbf{a},\chi,\mathfrak{c})}{\pi^{\beta_2}B(\chi)}\right)^\sigma = \frac{D(z,\mu/2;\nu\mathbf{a},\chi^\sigma,\mathfrak{c})}{\pi^{\beta_2}B(\chi^\sigma)}, \ \ \mu \leq (n+1)/2.$$

Then we have for $\mu \geq (n+1)/2$ that

$$\left(\frac{D(z,\mu/2;k,\chi,\mathfrak{c})}{\pi^{\beta_1+n|p|}i^{n|p|}A(\chi)}\right)^\sigma = \frac{D(z,\mu/2;k,\chi^\sigma,\mathfrak{c})}{\pi^{\beta_1+n|p|}i^{n|p|}A(\chi^\sigma)}, \ \ p=(k-\mu\mathbf{a})/2 \in \mathbb{Z}^{\mathbf{a}},$$

and for $\mu \leq (n+1)/2$ that

$$\left(\frac{D(z,\mu/2;k,\chi,\mathfrak{c})}{\pi^{\beta_2+n|p|}i^{n|p|}B(\chi)}\right)^\sigma = \frac{D(z,\mu/2;k,\chi^\sigma,\mathfrak{c})}{\pi^{\beta_2+n|p|}i^{n|p|}B(\chi^\sigma)}, \ \ \nu = n+1-\mu \ \ p=(k-\nu\mathbf{a})/2 \in \mathbb{Z}^{\mathbf{a}},$$

We will be interested in algebraicity statements of the Eisenstein series of weight sufficient large it is enough to study the effect of the action of the Galois group of the full rank coefficients. More precisely we have the following lemma.

Lemma 3.6. *Let $f(z) = \sum_{h \in S} c(h)\mathbf{e}^n_{\mathbf{a}}(hz) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}^{ab})$ with $k \geq n/2$. Assume that for an element $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we have $c(h)^\sigma = ac(h)$ for all h with $det(h) \neq 0$ for some $a \in \mathbb{C}$. Then $c(h)^\sigma = ac(h)$ for all $h \in S$. In particular $f^\sigma = af$.*

Proof. We obviously have $f^\sigma \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. We consider $g := af - f^\sigma \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. We note that the form g has non-zero Fourier coefficients only for $h \in S$ with $det(h) = 0$. But then by [11, Proposition 6.16] we have that $g = 0$. □

We now want to consider the action of $Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ on the Eisenstein series. We first consider the holomorphic ones. That is, we consider the following two Eisenstein series

(i) $D(z,k/2;k\mathbf{a},\chi,\mathfrak{c}) \in \pi^\beta \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$ for $k \geq \frac{n+1}{2}$,
(ii) $D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \pi^\beta \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$ for $k := n+1-\mu$ and $\mu \leq \frac{n+1}{2}$,

where β is determined by Theorem 3.4. Note here that we take the field of definition to be $\mathbb{Q}_{ab}$, i.e. the extension Φ does not appear. For this we refer to [11, Theorem 17.7].

In the following lemma we collect some properties that we will need concerning the functions $\Xi(y,h;\alpha,\beta) = \prod_{v \in \mathbf{a}} \xi(y,h;\alpha,\beta)$.

Lemma 3.7. *Let $h \in S$ with $det(h) \neq 0$ and $y \in S^{\mathbf{a}}_+(\mathbb{R})$. Then we have for $k \in \frac{1}{2}\mathbb{Z}$ we have*

$$\Xi(y,h;k,0) = 2^{d(1-(n+1)/2)} i^{-dnk} (2\pi)^{dnk} \Gamma_n(k)^{-d} N(det(h))^{k-(n+1)/2} \mathbf{e}^n(iyh)$$

and for $\mu := n+1-k$ we have

$$\Xi(y,h;(n+1)/2,(\mu-k)/2) = i^{-nk} 2^{-(dn(\mu-k))/2} \pi^{dn(n+1)/2} \Gamma_n(\frac{n+1}{2})^{-d} \prod_{v \in \mathbf{a}} det(y_v)^{-(\frac{\mu-k}{2})} \mathbf{e}^n(iyh)$$

Proof. The first statement is in [11, Equation 17.12]. For the second we have $\Xi(y,h;(n+1)/2,\mu/2-k/2)=\prod_{v\in\mathbf{a}}\xi(y_v,h_v;(n+1)/2,\mu/2-k/2)$, where the function $\xi(\cdot)$ is given in [11, page 140]. By Shimura [8, Equation 4.35K] we have that $\omega(2\pi y_v,h_v;(n+1)/2,\mu/2-k/2)=2^{-n(n+1)/2}\mathbf{e}_v(iy_vh_v)$. We conclude that

$$\xi(y_v,h_v;(n{+}1)/2,\mu/2{-}k/2)=i^{-nk}2^{-(n(\mu-k))/2}\pi^{n(n+1)/2}\Gamma_n(\frac{n+1}{2})^{-1}det(y_v)^{-(\frac{\mu-k}{2})}\mathbf{e}_v(iy_vh_v),$$

where we have used the fact that $\delta_-(h_vy_v)=1$ (the product of the negative eigenvalues of h_vy_v). Indeed we have that $\delta_-(h_vy_v)=\delta_-(y_v^{1/2}h_v{}^ty_v^{1/2})$. But the last quantity has the same number of negative eigenvalues as the matrix h_v, but $h_v>0$.

□

We will need the following Theorem (for a proof see [10, Theorem A6.5].

Theorem 3.8. *Let F be a totally real field, and let ψ be a Hecke character of F with $\psi_{\mathbf{a}}(b)=\prod_{v\in\mathbf{a}}\left(\frac{b_v}{|b_v|}\right)^k$, with $0<k\in\mathbb{Z}$. For any integral ideal $\mathfrak{c}$ of F put*

$$P_{\mathfrak{c}}(k,\psi):=\mathbf{g}(\psi)^{-1}(2\pi i)^{-kd}|D_F|^{1/2}L_{\mathfrak{c}}(k,\psi),$$

where $d=[F:\mathbb{Q}]$ and $\mathbf{g}(\psi)$ is a Gauss sum (defined [10, page 240]*). Then $P_{\mathfrak{c}}(k,\psi)\in\mathbf{Q}(\psi)$ and for every $\sigma\in Gal(\mathbb{Q}(\psi)/\mathbb{Q})$ we have*

$$P_{\mathfrak{c}}(k,\psi)^\sigma=P_{\mathfrak{c}}(k,\psi^\sigma).$$

We also summarize in the following lemma some more properties of Gauss sums.

Lemma 3.9. *Let χ and ψ be two finit order Hecke characters of F and $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we have*

(i) $\mathbf{g}(\chi)^\sigma=\chi^*(q\mathfrak{g})^{-1}\mathbf{g}(\chi^\sigma)$ *where $0<q\in\mathbb{Z}$ so that $e(1/N(\mathfrak{f}))^\sigma=e(q/N(\mathfrak{f}))$, where $\mathfrak{f}$ denotes the conductor of χ.*
(ii) $\left(\frac{\mathbf{g}(\chi\psi)}{\mathbf{g}(\chi)\mathbf{g}(\psi)}\right)^\sigma=\frac{\mathbf{g}(\chi^\sigma\psi^\sigma)}{\mathbf{g}(\chi^\sigma)\mathbf{g}(\psi^\sigma)}$.
(iii) *If χ is a quadratic character then $\mathbf{g}(\chi)=i^mN(\mathfrak{f})^{1/2}$ where m is the number of archimedean primes where $\chi_v\neq 1$.*

We remark here that if we pick an element $t\in\mathbb{Z}_{\mathbf{h}}^\times$ so that $\mathbf{e}_{\mathbf{h}}^{[t,\mathbb{Q}]}=\mathbf{e}_{\mathbf{h}}(t^{-1}x)$ for $x\in\mathbb{Q}/\mathbb{Z}$ then we have that we can pick the $q\in\mathbb{Z}$ above so that $rt_p-1\in N(\mathfrak{f})\mathbb{Z}_p$ for every prime p. Then we also obtain that $\chi^*(q\mathfrak{q})=\chi_{\mathfrak{f}}(t)$.

3.3. **Eisenstein series of integral weight:** We first consider the integral weight case. We have the following proposition.

Proposition 3.10. *For the Eisenstein series*

$$D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})=L_{\mathfrak{c}}(k,\chi)\prod_{i=1}^{[n/2]}L_{\mathfrak{c}}(2k-2i,\chi^2)E(z,k/2;k\mathbf{a},\chi,\mathfrak{c})$$

with $k\geq\frac{n+1}{2}$ we have that $\pi^{-\beta}D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})\in\mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$ and for all $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we have that

$$\left(\frac{D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{\pi^\beta P(\chi)}\right)^\sigma=\frac{D(z,k/2;k\mathbf{a},\chi^\sigma,\mathfrak{c})}{\pi^\beta P(\chi^\sigma)},\quad\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q}),$$

where $\beta = kd + \sum_{i=1}^{[n/2]}(2k-2i)$ *and* $P(\chi) := \frac{\mathbf{g}(\chi)(i)^{kd}}{|D_F|^{1/2}} \frac{\left(\prod_{i=1}^{[n/2]}(i)^{(2k-2i)d}\right)\mathbf{g}(\chi^{2[n/2]})}{|D_F|^{b(n)}}$, *with* $b(n) = 1/2$ *if* $[n/2]$ *odd and* 1 *otherwise.*

Proof. We observe that we have that $2k-2i > 0$ for all $i = 1 \dots [n/2]$. By definition we have that $\chi_{\mathbf{a}}(b) = \prod_{v \in \mathbf{a}} \left(\frac{b_v}{|b_v|}\right)^k$. By Theorem 3.8 above we have for

$$A(\chi) := \frac{|D_F|^{1/2} L_{\mathfrak{c}}(k,\chi)}{\mathbf{g}(\chi)(2\pi i)^{kd}} \prod_{i=1}^{[n/2]} \frac{|D_F|^{1/2} L_{\mathfrak{c}}(2k-2i,\chi^2)}{\mathbf{g}(\chi^2)(2\pi i)^{(2k-2i)d}} \in \mathbb{Q}_{ab}$$

and for all $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we have $A(\chi)^\sigma = A(\chi^\sigma)$. Using Lemma 3.9 we may define the quantity

$$B(\chi) = \frac{|D_F|^{1/2} L_{\mathfrak{c}}(k,\chi)}{\mathbf{g}(\chi)(2\pi i)^{kd}} \left(\prod_{i=1}^{[n/2]} \frac{L_{\mathfrak{c}}(2k-2i,\chi^2)}{(2\pi i)^{(2k-2i)d}}\right) \frac{|D_F|^{b(n)}}{\mathbf{g}(\chi^{2[n/2]})},$$

where $b(n) = 1/2$ if $[n/2]$ is odd and 1 otherwise. Then we have $B(\chi)^\sigma = B(\chi^\sigma)$. By [3, Theorem 15.1] we have $E(z,k/2;k\mathbf{a},\chi,\mathfrak{c})^\sigma = E(z,k/2;k\mathbf{a},\chi^\sigma,\mathfrak{c})$ for all $\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})$. In particular we conclude that

$$\left(\frac{D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{\pi^\beta P(\chi)}\right)^\sigma = \frac{D(z,k/2;k\mathbf{a},\chi^\sigma,\mathfrak{c})}{\pi^\beta P(\chi^\sigma)}, \quad \sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q}),$$

where $\beta = kd + \sum_{i=1}^{[n/2]}(2k-2i)$ and $P(\chi) := \frac{\mathbf{g}(\chi)(i)^{kd}}{|D_F|^{1/2}} \frac{\left(\prod_{i=1}^{[n/2]}(i)^{(2k-2i)d}\right)\mathbf{g}(\chi^{2[n/2]})}{|D_F|^{b(n)}}$.

□

Now we turn to the Eisenstein series

$$D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) = L_{\mathfrak{c}}(\mu,\chi) \prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(2\mu-2i,\chi^2) E(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}),$$

and

$$D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) = L_{\mathfrak{c}}(\mu,\chi) \prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(2\mu-2i,\chi^2) E^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}),$$

where we take $\mu \leq \frac{n+1}{2}$, and $k = n+1-\mu$.

We now prove

Lemma 3.11. *Let* $\beta \in \mathbb{N}$ *as in Theorem 3.4 so that* $\pi^{-\beta} D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. *Then we have that also* $\pi^{-\beta} D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. *Moreover for every* $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ *we have the reciprocity law*

$$\left(\frac{D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})}{\pi^\beta i^{-dnk} |D_F|^{-n\mu+3n(n+1)/4}}\right)^\sigma = \frac{D^*(z,\mu/2;k\mathbf{a},\chi^\sigma,\mathfrak{c})}{\pi^\beta i^{-dnk} |D_F|^{-n\mu+3n(n+1)/4}}.$$

Proof. The first statement i.e. that $\pi^{-\beta} D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$ follows from [11, Lemma 10.10]. Moreover by Lemma 3.6 it is enough to establish the action of $Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ on the full rank coefficients. By Proposition 3.2 and Lemma 3.7 we have that the h^{th} Fourier coefficient $c(h,\chi)$ of $\pi^{-\beta}D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})$ with $det(h) \neq 0$ is equal to

$$i^{-dnk}2^{-(dn(\mu-k))/2}\prod_{j=0}^{n}\Gamma(\frac{n+1}{2}-j/2)^{-d}|D_F|^{-n\mu+3n(n+1)/4}N(\mathfrak{bc})^{-n(n+1)/2}\times$$

$$\prod_{v\in\mathbf{c}} f_{h,v}\left(\chi(\pi_v)|\pi_v|^{\mu}\right)\times\begin{cases} L_{\mathfrak{c}}(\mu-n/2,\chi\rho_h), & n \text{ even };\\ 1, & n \text{ odd}.\end{cases}$$

If n is odd we have

$$\left(\frac{c(h,\chi)}{i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}}\right)^{\sigma}=\frac{c(h,\chi^{\sigma})}{i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}}.$$

Now we take $n=2m$ even. The character $\chi\rho_h$ has infinity type $(\chi\rho_h)_{\mathbf{a}}(b)=\prod_{v\in\mathbf{a}}\left(\frac{b_v}{|b_v|}\right)^{1-\mu+m}$ since the character ρ_h is the non-trivial character of the extension $F(c^{1/2})/F$ with $c:=(-1)^m det(2h)$ and $det(h)>>0$ as h is positive definite for all real embeddings of F. Since $1-\mu+m>0$ we have by [11, Theorem 18.12] that $L(1-(1-\mu+m),(\chi\rho_h)^{\sigma})=L(1-(1-\mu+m),(\chi\rho_h))^{\sigma}$ for all $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$. Hence we conclude also in the case of n even that

$$\left(\frac{c(h,\chi)}{i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}}\right)^{\sigma}=\frac{c(h,\chi^{\sigma})}{i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}}.$$

□

We now prove the following lemma

Lemma 3.12. *Assume that* $(\pi^{-\beta}D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}))^{\sigma}=a\pi^{-\beta}D^*(z,\mu/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c})$ *for* $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ $a\in\mathbb{Q}_{ab}^{\times}$. *Then*

$$(\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}))^{\sigma}=b\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c})$$

where $b=\chi(q\mathfrak{g})^{-n}a$, *where* $0<q\in\mathbb{Z}$ *such that* $e(1/N(\mathfrak{c}))^{\sigma}=e(q/N(\mathfrak{c}))$.

Proof. We use an argument due to Feit [3] and Sturm [12, Lemma 5] first introduced by Shimura in the case of $n=1$. We will need the reciprocity law of the action of the group $\mathcal{G}_+\times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ defined by Shimura in [11, Theorem 10.2]. We use the notation of Shimura in this theorem. Let t be an idele of F and as in Shimura we define $\imath(t):=\begin{pmatrix}1&0\\0&t^{-1}\end{pmatrix}$. For a $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we define the element $(\imath(t),\sigma)\in\mathcal{G}_+\times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ where $t\in\mathbb{Z}_{\mathbf{h}}^{\times}$ corresponds to σ by class field theory and we extend σ to an element of the absolute Galois group. Moreover we may consider also $\zeta_{\mathbf{h}}\in Sp_{\mathbb{A}}$ as an element of $\mathcal{G}_+\times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by taking $(\zeta_{\mathbf{h}},1)$. Then we have that

$$(\imath(t),\sigma)(\zeta_{\mathbf{h}},1)(\imath(t^{-1}),\sigma^{-1})(\zeta_{\mathbf{h}}^{-1},1)=\left(\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix},1\right)$$

In particular we have

$$\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})^{(\imath(t),\sigma)(\zeta_{\mathbf{h}},1)(\imath(t^{-1}),\sigma^{-1})(\zeta_{\mathbf{h}}^{-1},1)}=$$

$$\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})|_k\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix} = \chi_{\mathfrak{c}}(t)^n\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})$$

But then

$$(\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}))^{\sigma} = \pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})^{(\imath(t),\sigma)} =$$

$$\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})^{(\imath(t),\sigma)(\zeta_{\mathbf{h}},1)(\imath(t^{-1}),\sigma^{-1})(\zeta_{\mathbf{h}}^{-1},1)(\zeta_{\mathbf{h}},1)(\imath(t),\sigma)(\zeta_{\mathbf{h}}^{-1}),1)} =$$

(3.4)

$$\chi_{\mathfrak{c}}(t)^n\left(\left(\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})|_k(\zeta_{\mathbf{h}}\right)^{\sigma}\right)|_k\zeta_{\mathbf{h}}^{-1}) = \chi_{\mathfrak{c}}(t)^n a\pi^{-\beta}D(z,\mu/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c}).$$

□

We can now establish the following corollary

Corollary 3.13. *For the Eisenstein series $D(z,\mu/2;k\mathbf{a},\chi,D)$ we have*

$$\left(\frac{D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})}{\pi^{\beta}\mathbf{g}(\chi^n)i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}}\right)^{\sigma} = \frac{D(z,\mu/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c})}{\pi^{\beta}\mathbf{g}((\chi^n)^{\sigma})i^{-dnk}|D_F|^{-n\mu+3n(n+1)/4}},\ \ \sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q}).$$

Proof. This follows immediately by combining Lemma 3.9 ((i) and (ii)), and the last two lemmas. □

3.4. **Eisenstein series of half integral weight:** Now we consider the case of half-integral weight. We will need the theta series $\theta(z) := \sum_{a\in\mathfrak{g}^n}\mathbf{e}_{\mathbf{a}}({}^t aza/2)\in\mathcal{M}_{\frac{1}{2}\mathbf{a}}(\mathbb{Q},\phi)$, where the quadratic character ϕ of Γ^{θ} is defined by $h_{\gamma}(z)=\phi(\gamma)j_{\gamma}^{\mathbf{a}}(z)$ for $\gamma\in\Gamma^{\theta}$. Note that this is the series θ_F defined in [11, page 39, equation 6.16] by taking in the equation there, using Shimura's notation, $u=0$ and λ the characteristic function of $\mathfrak{g}^n\subset F^n$. Note in particular that since we are taking $u=0$ we have that $\phi_F=\theta_F$. In particular Theorem 6.8 in (loc. cit) gives the properties of the series θ. We now prove the following lemma.

Lemma 3.14. *For the theta series $\theta(z)$ and for $\sigma\in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have that*

$$\left(\theta|_{\frac{1}{2}\mathfrak{a}}(\zeta_{\mathbf{h}}\right)^{\sigma}|_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}^{-1}=\theta$$

Proof. This follows immediately after observing that $\zeta_{\mathbf{h}}\in C^{\theta}$ and from Theorem 6.8 (4) in [11]. Indeed since θ is invariant under $\Gamma^{\theta}=G\cap C^{\theta}$ we have that $\theta|_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}=\theta_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}^{-1}=\theta$. Since $\theta\in\mathcal{M}_{\frac{1}{2}\mathbf{a}}(\mathbb{Q})$, we conclude the proof.

□

Proposition 3.15. *Let λ be equal to k or μ. Let $\beta(\lambda)\in\mathbb{N}$ so that $\pi^{-\beta(\lambda)}D^*(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c})\in\mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. Let $\sigma\in Gal(\mathbb{Q}_{ab}/\mathbb{Q}$ and assume*

$$\left(\pi^{-\beta(\lambda)}D^*(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c})\right)^{\sigma}=\alpha(\lambda)\pi^{-\beta(\lambda)}D^*(z,\lambda/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c}),\ \ k=n+1-\mu,$$

for some $\alpha(\lambda)\in\overline{\mathbb{Q}}^{\times}$. Then we have $\pi^{-\beta(\lambda)}D(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c})\in\mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$ and

$$\left(\pi^{-\beta(\lambda)}D(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c})\right)^{\sigma}=\beta\pi^{-\beta(\lambda)}D(z,lambda/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c})$$

where $\beta=(\chi\phi)_{\mathfrak{c}}(t)^n\alpha(\lambda)$.

Proof. The fact that $\pi^{-\beta(\lambda)}D(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_k(\mathbb{Q}_{ab})$ follows from [11, Lemma 10.10]. The rest of the proof was inspired by the proof of Theorem 10.7 in [11]. We write $D(\chi,\lambda)$ for $\pi^{-\beta(\lambda)}D(z,\lambda/2;k\mathbf{a},\chi,\mathfrak{c})$. Let $k' = k+\frac{1}{2} \in \mathbb{Z}$. Then we note that $\theta D(\chi,\lambda) \in \mathcal{M}_{k'\mathbf{a}}(\mathbb{Q}_{ab})$ and for a $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ we have $\theta D(\chi,\lambda)^\sigma = (\theta D(\chi,\lambda))^\sigma$. Since $\theta D(\chi)$ is of integral weight we can apply the reciprocity-laws as before. Writing $t \in \mathbb{Z}_{\mathbf{h}}^\times$ corresponding to σ we have

$$(\theta D(\chi,\lambda))^\sigma = (\theta D(\chi,\lambda))^{\left((\imath(t),\sigma)(\zeta_{\mathbf{h}},1)(\imath(t^{-1}),\sigma^{-1})(\zeta_{\mathbf{h}}^{-1},1)\right)\left((\zeta_{\mathbf{h}},1)(\imath(t)\sigma)(\zeta_{\mathbf{h}}^{-1},1)\right)}$$

$$= \left((\theta D(\chi,\lambda))|_{k'}\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\right)^{(\zeta_{\mathbf{h}},1)(\imath(t),\sigma)(\zeta_{\mathbf{h}}^{-1},1)}$$

$$= (\phi\chi)_{\mathfrak{c}}(t)^n \left((\theta D(\chi)|_{k'\mathbf{a}}\zeta_{\mathbf{h}})^\sigma\right)_{k'\mathbf{a}}\zeta_{\mathbf{h}}^{-1} =$$

$$(\phi\chi)_{\mathfrak{c}}(t)^n\left(\phi(\zeta_0)\left(\theta|_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}\right)^\sigma (D(\chi,\lambda)|_{k\mathbf{a}}\zeta_{\mathbf{h}})^\sigma\right)|_{k'\mathbf{a}}\zeta_{\mathbf{h}}^{-1} =$$

$$\phi(\zeta_0)\phi(\zeta_0)^{-1}\left(\left(\theta|_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}\right)^\sigma|_{\frac{1}{2}\mathbf{a}}\zeta_{\mathbf{h}}^{-1}\right)\left((D(\chi,\lambda)|_{k\mathbf{a}}\zeta_{\mathbf{h}})^\sigma\right)|_{k\mathbf{a}}\zeta_{\mathbf{h}}^{-1} =$$

$$(\phi\chi)_{\mathfrak{c}}(t)^n\alpha(\lambda)\theta D(\chi^\sigma,\lambda).$$

The last equation follows from the last Lemma. However the previous equations deserve a comment. Note that for $f_1, f_2 \in \mathcal{M}_{\frac{1}{2}\mathbf{a}}$ and $\gamma \in \Gamma^\theta$ we have that $(f_1f_2)|_{\mathbf{a}}\gamma = \phi(\gamma)(f_1|_{\frac{1}{2}\mathbf{a}}\gamma)(f_2|_{\frac{1}{2}\mathbf{a}}\gamma)$ since $h_\gamma(z)^2 = \phi(\gamma)j_\gamma^{\mathbf{a}}(z)$.

So we obtain that $\theta D(\chi,\lambda)^\sigma = (\phi\chi)_{\mathfrak{c}}(t)^{-n}\alpha\theta D(\chi^\sigma)$. Since θ is not a zero divisor in the formal ring of the Fourier-expansion (see [11, page 74]) we conclude the proof.

□

We now establish also in the case of half-integral weight that

Proposition 3.16. *Let $\beta_1 \in \mathbb{N}$ so that $\pi^{-\beta_1}D^*(z,k/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. Let $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ Then for n even we have*

$$\left(\frac{\pi^{-\beta_1}D^*(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}}\right)^\sigma = \frac{\pi^{-\beta_1}D^*(z,k/2;k\mathbf{a},\chi^\sigma)}{i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}}$$

and for n odd

$$\left(\frac{\pi^{-\beta_1}D^*(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi)|D_F|^{1/2}(2i)^{-(k-n)d}b([n/2])}\right)^\sigma =$$

$$\frac{\pi^{-\beta_1}D^*(z,k/2;k\mathbf{a},\chi^\sigma)}{i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi^\sigma)|D_F|^{1/2}(2i)^{-(k-n)d}b([n/2])},$$

where $b(m) = i^d$ if m is m is odd and 1 otherwise.

Let now $\beta_2 \in \mathbb{N}$ so that $\pi^{-\beta_2}D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. Then we have

$$\left(\frac{\pi^{-\beta_2}D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})}{i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}}\right)^\sigma = \frac{\pi^{-\beta_2}D^*(z,\mu/2;k\mathbf{a},\chi^\sigma,\mathfrak{c})}{i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}},\quad k=n+1-\mu$$

Proof. Arguing as before, it is enough to consider the action of σ on the full rank coefficients. We consider an h with $det(h) \neq 0$. Then we have that the h^{th} Fourier coefficient $c(h,\chi)$ of $\pi^{-\beta_1} D^*(z,k/2;k\mathbf{a},\chi,\mathfrak{c})$ is equal to

$$2^{d(nk+1-(n+1)/2)} i^{-dnk} \left(\prod_{j=0}^{n-1} \Gamma(k-j/2)\right)^{-d} N(det(h))^{k-(n+1)/2} C|D_F|^{nk/2+3n(n+1)/4} N(\mathfrak{bc})^{-n(n+1)/2} \times$$

$$\prod_{v \in \mathbf{c}} f_{h,v}\left(\chi(\pi_v)|\pi_v|^{k+1/2}\right) \times \begin{cases} \pi^{-d(k-n/2)} L_{\mathfrak{c}}(k-n/2, \chi\rho_h), & n \text{ odd };\\ 1, & n \text{ even.} \end{cases}$$

We now note that if n is even we have that $k-(n+1)/2 \in \mathbb{Z}$ and hence $N(det(h))^{k-(n+1)/2} \in \mathbb{Q}^\times$. Then we conclude that

$$\left(\frac{c(h,\chi)}{i^{-dnk} C|D_F|^{nk/2+3n(n+1)/4}}\right)^\sigma = \frac{c(h,\chi^\sigma)}{i^{-dnk} C|D_F|^{nk/2+3n(n+1)/4}}.$$

In the case where n is odd we have that

$$P_{\mathfrak{c}}(k-n/2,\chi\rho_h)^\sigma = P_{\mathfrak{c}}(k-n/2,\chi^\sigma\rho_h), \ \ \forall \sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q}),$$

with

$$P_{\mathfrak{c}}(k-n/2,\chi\rho_h) := \mathbf{g}(\chi\rho_h)^{-1}(2\pi i)^{-(k-n/2)d}|D_F|^{1/2} L_{\mathfrak{c}}(k-n/2,\chi\rho_h)$$

We have $\frac{\mathbf{g}(\chi\rho_h)^\sigma}{\mathbf{g}(\chi^\sigma\rho_h)} = \frac{\mathbf{g}(\chi)^\sigma \mathbf{g}(\rho_h)^\sigma}{\mathbf{g}(\chi^\sigma)\mathbf{g}(\rho_h)}$. Moreover we have that

$$\frac{\mathbf{g}(\rho_h)^\sigma}{\mathbf{g}(\rho_h)} = \begin{cases} \frac{\sqrt{N(2det(h))}^\sigma}{\sqrt{N(2det(h))}}, & \text{if } [n/2] \text{ even;}\\ \left(\frac{i^\sigma}{i}\right)^d \frac{\sqrt{N(2det(h))}^\sigma}{\sqrt{N(2det(h))}}, & \text{otherwise.} \end{cases}$$

In particular since $det(h) \in F_+$ we have

$$\frac{\left(\sqrt{N(2det(h))^{-1}}\mathbf{g}(\rho_h)\right)^\sigma}{\sqrt{N(2det(h))^{-1}}\mathbf{g}(\rho_h)} = \begin{cases} 1, & \text{if } [n/2] \text{ even;}\\ \left(\frac{i^\sigma}{i}\right)^d, & \text{otherwise.} \end{cases}$$

For n odd we have that $k-(n+1)/2$ is half integral. Hence we conclude that

$$\left(\frac{c(h,\chi)}{i^{-dnk} C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi)|D_F|^{1/2}(2i)^{-(k-n)d} b([n/2])}\right)^\sigma =$$
$$\frac{c(h,\chi^\sigma)}{i^{-dnk} C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi^\sigma)|D_F|^{1/2}(2i)^{-(k-n)d} b([n/2])},$$

where $b(i) = i^d$ if $[n/2]$ odd and 1 otherwise.

Now we turn to the Eisenstein series $D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})$. The Fourier coefficient $c(h,\chi)$ of $\pi^{-\beta_2} D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})$ for $det(h) \neq 0$ is equal to

$$i^{-dnk} 2^{-dn(\mu-k)/2} C|D_F|^{-n(n+1-k)+3n(n+1)/4} \prod_{j=0}^{n-1} \Gamma(\frac{n+1}{2} - j/2)^{-d} N(\mathfrak{bc})^{-n(n+1)/2} \times$$

$$\prod_{v\in\mathbf{c}} f_{h,v}\left(\chi(\pi_v)|\pi_v|^{n+1-k+1/2}\right)\times\begin{cases} L_{\mathfrak{c}}(n/2+1-k,\chi\rho_h), & n \text{ odd };\\ 1, & n \text{ even}.\end{cases}$$

Since we are taking $k \geq \frac{n+1}{2}$ we have that $L_{\mathfrak{c}}(n/2+1-k,\chi\rho_h) \in \mathbb{Q}$. Hence after observing that $n+1-k+1/2 \in \mathbb{Z}$ we conclude that

$$\left(\frac{c(h,\chi}{i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}}\right)^{\sigma} = \frac{c(h,\chi^{\sigma}}{i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}}$$

□

We can now conclude

Proposition 3.17. *Let $\beta_1 \in \mathbb{N}$ so that $\pi^{-\beta_1}D(z,k/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. Let $\sigma \in Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ Then for n even we have*

$$\left(\frac{\pi^{-\beta_1}D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{\mathbf{g}(\chi\phi)^n i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}}\right)^{\sigma} = \frac{\pi^{-\beta_1}D(z,k/2;k\mathbf{a},\chi^{\sigma})}{\mathbf{g}(\chi^{\sigma}\phi)^n i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}}$$

and for n odd

$$\left(\frac{\pi^{-\beta_1}D(z,k/2;k\mathbf{a},\chi,\mathfrak{c})}{\mathbf{g}((\chi\phi)^n)i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi)|D_F|^{1/2}(2i)^{-(k-n)d}b([n/2])}\right)^{\sigma} =$$

$$\frac{\pi^{-\beta_1}D(z,k/2;k\mathbf{a},\chi^{\sigma})}{\mathbf{g}((\chi^n)^{\sigma}\phi)i^{-dnk}C|D_F|^{nk/2+3n(n+1)/4}\mathbf{g}(\chi^{\sigma})|D_F|^{1/2}(2i)^{-(k-n)d}b([n/2])},$$

where $b(m) = i^d$ if m is m is odd and 1 otherwise.

Let now $\beta_2 \in \mathbb{N}$ so that $\pi^{-\beta_2}D^*(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}) \in \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}_{ab})$. Then we have

$$\left(\frac{\pi^{-\beta_2}D(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c})}{\mathbf{g}(\chi\phi^n)i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}}\right)^{\sigma} = \frac{\pi^{-\beta_2}D(z,\mu/2;k\mathbf{a},\chi^{\sigma},\mathfrak{c})}{\mathbf{g}(((\chi\phi)^n)^{\sigma})i^{-dnk}C|D_F|^{-n(n+1-k)+3n(n+1)/4}},$$

We now remark that the above proposition and Lemma [**?**] give a complete description of the reciprocity laws of the Eisenstein series which we are considering. We summarize all the above in the following Theorem.

Theorem 3.18. *Let $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ with $k_v \geq (n+1)/2$ for every $v \in \mathbf{a}$. Let $\mu \in \frac{1}{2}\mathbb{Z}$ such that $n+1-k_v \leq \mu \leq k_v$ and $|\mu-(n+1)/2|+(n+1)/2-k_v \in 2\mathbb{Z}$ for all $v \in \mathbf{a}$. Then with a $\beta \in \mathbb{N}$ as in Theorem 3.4 we have*

$$\pi^{-\beta}D(z,\mu/2;k,\chi,\mathfrak{c}) \in \mathcal{N}_k^r(\Phi\mathbb{Q}_{ab}),$$

and for every $\sigma \in Gal(\Phi\mathbb{Q}_{ab}/\Phi)$ we have

$$\left(\frac{\pi^{-\beta}D(z,\mu/2;k,\chi,\mathfrak{c})}{\omega(\chi)}\right)^{\sigma} = \frac{\pi^{-\beta}D(z,\mu/2;k,\chi^{\sigma},\mathfrak{c})}{\omega(\chi^{\sigma})},$$

where $\omega(\chi)$ is given as follows:

(i) *if $k \in \mathbb{Z}^{\mathbf{a}}$, $\mu \geq (n+1)/2$:*

$$\omega(\chi) = i^{n|p|}\mathbf{g}(\chi)i^{\mu d+2\mu[n/2]-[n/2]([n/2]+1)d}|D_F|^{-b(n)}\mathbf{g}(\chi^{2[n/2]}),$$

where $p := \frac{(k-\mu\mathbf{a})}{2}$ and $b(n) = 0$ if $[n/2]$ odd and $1/2$ otherwise.

(ii) *if* $k \in \mathbb{Z}^{\mathbf{a}}$, $\mu < (n+1)/2$*:*

$$\omega(\chi) = i^{n|p|}\mathbf{g}(\chi^n) i^{-dn\nu}|D_F|^{-n\mu+3n(n+1)/4},$$

where $\nu := n+1-\mu$ *and* $p := \frac{k-\nu\mathbf{a}}{2}$.

(iii) *if* $k \notin \mathbb{Z}^{\mathbf{a}}$ *and* $\mu \geq (n+1)/2$*:*

(a) *if* n *is even*

$$\omega(\chi) = i^{n|p|}\mathbf{g}(\chi^n) i^{-dnk} C|D_F|^{nk/2+3n(n+1)/4},$$

(b) *if* n *is odd*

$$\omega(\chi) = i^{n|p|}\mathbf{g}(\chi^n\phi) i^{-dnk} C|D_F|^{n\mu/2+3n(n+1)/4}\mathbf{g}(\chi)|D_F|^{1/2}(2i)^{-(\mu-n)d}b([n/2]),$$

where $p := \frac{(k-\mu\mathbf{a})}{2}$ *and* $b(m) = i^d$ *if* m *is* m *is odd and 1 otherwise and*

(iv) *if* $k \notin \mathbb{Z}^{\mathbf{a}}$ *and* $\mu < (n+1)/2$*:*

$$\omega(\chi) = i^{n|p|}\mathbf{g}(\chi\phi)^n i^{-dn\nu} C|D_F|^{-n(n+1-\nu)+3n(n+1)/4},$$

where $\nu := n+1-\mu$ *and* $p := \frac{k-\nu\mathbf{a}}{2}$.

In particular we have that

$$\frac{\pi^{-\beta}D(z,\mu/2;k,\chi,\mathfrak{c})}{\omega(\chi)} \in \mathcal{N}_k^r(\Phi(\chi)),$$

where $\Phi(\chi)$ *is the finite extension of* Φ *obtained by adjoining the values of the character* χ.

4. The L-function attached to a Siegel Modular Form

4.1. **The Hecke Algebra and the standard L-Function.** We start by discussing the Hecke algebras that we consider in this work. We follow Chapter V in [11]. As before we fix a fractional ideal $\mathfrak{b}$ of F and an integral ideal $\mathfrak{c}$. We write C for $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ Moreover we define

$$E = \prod_{v\in\mathbf{h}} GL_n(\mathfrak{g}_v),\ B = \{x \in GL_n(F)_{\mathbf{h}} | x \prec \mathfrak{g}\},\ \mathfrak{X} = CQC,\ Q = \{diag[\hat{r}, r] | r \in B\}$$

We write $\mathfrak{R}(C,\mathfrak{X})$ for the Hecke algebra corresponding to the pair $(C,\mathfrak{X})$ and for every place $v \in \mathbf{h}$ we we write $\mathfrak{R}(C_v,\mathfrak{X}_v)$ for the local Hecke algebra at v and hence $\mathfrak{R}(C,\mathfrak{X}) = \bigotimes_v \mathfrak{R}(C_v,\mathfrak{X}_v)$. We now consider the formal Dirichlet series with coefficients in the global Hecke algebra defined by $\mathfrak{T} = \sum_{C\backslash\mathfrak{X}/C} C\xi C[\nu_{\mathfrak{b}}(\xi)]$ and its local version at $v \in \mathbf{h}$ defined as $\mathfrak{T}_v = \sum_{C_v\backslash\mathfrak{X}_v/C_v} C_v\xi C_v[\nu_{\mathfrak{b}}(\xi)]$. Here $\nu_{\mathfrak{b}}(\xi)$ is defined by $det(q)\mathfrak{g}$ where $q \in B$ such that $\xi \in D[\mathfrak{b}^{-1},\mathfrak{b}]diag[q^{-1}, q^*]D[\mathfrak{b}^{-1},\mathfrak{b}]$. We have that $\mathfrak{T} = \prod_v \mathfrak{T}_v$. Moreover if we define for an integral $\mathfrak{g}$-ideal $\mathfrak{a}$ the elements $T(\mathfrak{a}) \in \mathfrak{R}(C,\mathfrak{X})$ and $T_v(\mathfrak{a}) \in \mathfrak{R}(C_v,\mathfrak{X}_v)$ for $v \in \mathbf{h}$ by

$$T(\mathfrak{a}) = \sum_{\xi\in\mathfrak{X},\nu_{\mathfrak{b}}(\xi)=\mathfrak{a}} C\xi C,\ \ T_v(\mathfrak{a}) = \sum_{\xi\in\mathfrak{X}_v,\nu_{\mathfrak{b}}(\xi)=\mathfrak{a}} C_v\xi C_v$$

then we have that $\mathfrak{T} = \sum_{\mathfrak{a}} T(\mathfrak{a})[\mathfrak{a}]$. The structure of the local Hecke algebra has been investigated by Shimura in [10, 11] where he proves

Theorem 4.1 (Shimura)**.** *Let $t_1,\dots,t_n$ be n indeterminates. Then for each $v \in \mathbf{h}$ there exists a $\mathbb{Q}$-linear ring injection*

$$\omega_v : \mathfrak{R}(C_v,\mathfrak{X}_v) \to \mathbb{Q}[t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1}]$$

such that $\omega(\mathfrak{T}_v) := \sum_{\xi\in C_v\backslash\mathfrak{X}/C_v}\omega(C_v\xi C_v)[\nu_{\mathfrak{b}}(\xi)]$ has the following expressions

$$\omega(\mathfrak{T}_v) = \begin{cases} \frac{1-[\mathfrak{p}]}{1-q^n[\mathfrak{p}]}\prod_{i=1}^n \frac{(1-(-q)^{2i}[\mathfrak{p}^2])}{(1-q^n t_i[\mathfrak{p}])(1-q^n t_i^{-1}[\mathfrak{p}])}, & \text{if } v\nmid\mathfrak{c};\\ \prod_{i=1}^n(1-q^n t_i[\mathfrak{p}])^{-1}, & \text{otherwise.}\end{cases}$$

For an element $\mathbf{f}\in\mathcal{M}_k(C,\psi)$ we have an action of the Hecke algebra $\mathfrak{R}(C,\psi)$ (see [10]). We denote this action by $\mathbf{f}|C\xi C$ for an element $C\xi C\in\mathfrak{R}(C,\psi)$. Assume now that for such an $\mathbf{f}\neq 0$ we have $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a})\in\mathbb{C}$ for all integral $\mathfrak{g}$-ideals. Then we have that there exists $\lambda_{v,i}\in\mathbb{C}$ such that

$$\mathfrak{L}\cdot\sum_{\mathfrak{a}}\lambda(\mathfrak{a})[\mathfrak{a}] = \prod_{v\in\mathbf{h}} Z_v,$$

where the factors Z_v are given by

$$Z_v = \begin{cases} (1-N(\mathfrak{p})^n[\mathfrak{p}])^{-1}\prod_{i=1}^n\Big((1-N(\mathfrak{p})^n\lambda_{v,i}[\mathfrak{p}])(1-N(\mathfrak{p})^n\lambda_{v,i}^{-1}[\mathfrak{p}])\Big)^{-1}, & \text{if } v\nmid\mathfrak{c};\\ \prod_{i=1}^n(1-N(\mathfrak{p})^n\lambda_{v,i}[\mathfrak{p}])^{-1}, & \text{otherwise.}\end{cases}$$

and $\mathfrak{L} := \prod_{\mathfrak{p}\nmid\mathfrak{c}}(1-[\mathfrak{p}])\prod_{i=1}^n\left(1-N(\mathfrak{p})^{2i}[\mathfrak{p}]^2\right)^{-1}$, where the product is over the prime $\mathfrak{g}$-ideals prime to $\mathfrak{c}$. For a Hecke character χ of F of conductor $\mathfrak{f}$ we put

$$Z(s,\mathbf{f},\chi) := \prod_{v\in\mathbf{h}} Z_v\left(\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s}\right), \tag{4.5}$$

where $Z_v\left(\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s}\right)$ is obtained from Z_v by substituting $(\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s}$ for $[\mathfrak{p}]$.

We will need another L-function which we will denote by $Z'(s,\mathbf{f},\chi)$ and we define by

$$Z'(s,\mathbf{f},\chi) := \prod_{v\in\mathbf{h}} Z_v\left(\chi^*(\mathfrak{q})(\psi/\psi_{\mathfrak{c}})(\pi_{\mathfrak{q}})N(\mathfrak{q})^{-s}\right), \tag{4.6}$$

where $\pi_{\mathfrak{q}}$ a uniformizer of $K_{\mathfrak{q}}$. We note here that we may obtain the first from the second up to a finite number of Euler factors by setting $\chi\psi^{-1}$ for χ.

5. The Rankin-Selberg Method

We now explain the integral representation of the zeta function introduced above due to Shimura. Everything in this section is taken from [11, paragraph 20 and 22] as well as [9].

We write $\mathcal{L}$ for the set of all $\mathfrak{g}$-lattices in F_1^n. We set $L_0 := \mathfrak{g}_1^n$ and we remark that for an element $L\in\mathcal{L}$ we can find an element $y\in GL_n(F)_{\mathbf{h}}$ such that $L = yL_0$. For an element $\tau\in S$ we define

$$L_\tau := \{L\in\mathcal{L}|\ell^*\tau\ell\in\mathfrak{b}\mathfrak{d}^{-1},\ \forall\ell\in L\}.$$

Let $\mathbf{f} \in \mathcal{M}_k(C,\psi)$, $\tau \in S_+$ and $q \in GL_n(F)_{\mathbf{h}}$. Following Shimura we define the following two formal Dirichlet series

$$D(\tau, q; \mathbf{f}) := \sum_{x \in B/E} \psi_{\mathbf{c}}(det(qx))|det(x)|_F^{-n-1} c(\tau, qx; \mathbf{f})[det(x)\mathfrak{g}], \tag{5.7}$$

and

$$D'(\tau, q; \mathbf{f}) := \sum_{x \in B/E} \psi(det(qx))|det(x)|_F^{-n-1} c(\tau, qx; \mathbf{f})[det(x)\mathfrak{g}]. \tag{5.8}$$

We note that the second is obtained from the first one by setting $(\psi/\psi_{\mathbf{c}})(t)[t\mathfrak{g}]$ for $[t\mathfrak{g}]$, $t \in F_{\mathbf{h}}^{\times}$ in $D(\tau, q; \mathbf{f})$ and multiplying by $(\psi/\psi_{\mathbf{c}})(det(q))$. For the formal Dirichlet series $D(\tau, q; \mathbf{f})$ Shimura [11, Theorem 20.4] proves the following theorem

Theorem 5.1 (Shimura)**.** *Given $\tau \in S_+$, $L \in \mathcal{L}_\tau$ and $\mathbf{f} \in \mathcal{M}_k(C,\psi)$, take $q \in GL_n(F)_{\mathbf{h}}$ so that $L = qL_0$ and define formal Dirichlet series $a(\tau; L)$ and $A(\tau, L)$ by*

$$A(\tau, L) := |det(q)|_F^{-\kappa}[det(qq^*)\mathfrak{g}] \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L) a(\tau, M),$$

and

$$a(\tau, L) := |det(q)|_F^{-\kappa}[(det(qq^*))^{-1}\mathfrak{g}]\alpha_{\mathfrak{c}^0}(\epsilon_b q^* \tau q).$$

Here ϵ_b is an element of $F_{\mathbf{h}}^{\times}$ such that $\epsilon_b \mathfrak{g} = \mathfrak{b}^{-1}\mathfrak{d}$. Then

$$[det(\widehat{q})\mathfrak{g}]\psi_{\mathfrak{c}}(det(q))c(\tau, q; \mathbf{f}|\mathfrak{T}) = \sum_{L < M \in \mathcal{L}_\tau} [det(\hat{y})\mathfrak{g}]A(\tau, M)D(\tau, y; \mathbf{f}),$$

where y is an element of $GL_n(F)_{\mathbf{h}}$ depending on M such that $M = yL_0$ and $y^{-1}q \in B$. In particular if $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})f$ for each integral $\mathfrak{r}$-ideal $\mathfrak{a}$, then

$$\psi_{\mathfrak{c}}(det(q))c(\tau, q; \mathbf{f}) \sum_{\mathfrak{a}} \lambda(\mathfrak{a})[\mathfrak{a}] = \sum_{L < M \in \mathcal{L}_\tau} [det(q^*\hat{y})\mathfrak{g}]A(\tau, M)D(\tau, y; \mathbf{f})$$

Here μ denote the Möbius function defined in [11, Lemma 19.10]. As Shimura remarks right after the Theorem 20.4 in [11] in the case that we consider the series $D(\tau, q; \mathbf{f})$ depend only on the lattice $L = qL_0$ so we can write $D(\tau, L; \mathbf{f})$ instead. The next important input from the theory of Shimura is the understanding of the series $A(\tau, L)$. Shimura proves the following lemma

Lemma 5.2 (Shimura)**.** *Let $\tau \in S_+ \cap GL_n(KF)$ and $L = qL_0 \in \mathcal{L}_\tau$; let $\mathbf{b}$ be the set of all primes $v \in \mathbf{h}$ prime to $\mathbf{c}$ such that $\epsilon_b q^* \tau q$ is not regular. Then*

$$A(\tau, L) = \prod_{v \in \mathbf{b}} g_v([\mathfrak{p}]) \prod_{v \nmid \mathfrak{c}} h_v([\mathfrak{p}])^{-1}(1 - [\mathfrak{p}]) \prod_{i=1}^{[n/2]} (1 - N(\mathfrak{p})^{2i}[\mathfrak{p}]^2),$$

Here $\mathfrak{p}$ is the prime ideal of F at v and g_v is a polynomial with coefficients in $\mathbb{Z}$ and constant term 1; $h_v = 1$ if n is odd and $h_v(t) = 1 - \rho_\tau^*(\mathfrak{p})N(\mathfrak{p})^{n/2}t$ with the Hecke character ρ_τ of F corresponding to $F(c^{1/2})$, $c = (-1)^{n/2}det(\tau)$, if n is even.

We define the series

$$\mathfrak{L}_0 = \prod_{v \nmid \mathfrak{c}} \left(\prod_{i=1}^{[(n+1)/2]} (1 - N(\mathfrak{p})^{2n+2-2i}[\mathfrak{p}]^2) \right)^{-1}.$$

Then we have

Theorem 5.3 (Shimura)**.** *Let $0 \neq \mathbf{f} \in \mathcal{M}_k(C,\psi))$ and such that $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for every $\mathfrak{a}$. Then for $\tau \in S_+ \cap GL_n(F)$ and $L = qL_0$ with $q \in GL_n(F)_{\mathbf{h}}$ we have*

$$D(\tau, q; \mathbf{f})\mathfrak{L}_0 \prod_{v \in \mathbf{b}} g_v[\mathfrak{p}] \prod_{v \nmid \mathfrak{c}} h_v([\mathfrak{p}])^{-1} =$$

$$\prod_{v \in \mathbf{h}} Z_v \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)\psi_{\mathfrak{c}}(det(y))[det(q^* \hat{y}\mathfrak{g}]c(\tau, y; \mathbf{f}).$$

Assume now that $k_v \geq n/2$ for some $v \in \mathbf{a}$. Then there exists $\tau \in S_+ \cap GL_n(F)$ and $r \in GL_n(F)_{\mathbf{h}}$ such that

$$0 \neq \psi_{\mathfrak{c}}(det(r))c_{\mathbf{f}}(\tau, r) \prod_{v \in \mathbf{h}} Z_v = D(\tau, r; \mathbf{f}) \cdot \mathfrak{L}_0 \prod_{v \nmid \mathfrak{c}} h_v([\mathfrak{p}\mathfrak{r}])^{-1} \cdot \prod_{v \in \mathbf{b}} g_v([\mathfrak{p}\mathfrak{r}])$$

Now given a Hecke character χ of F, $\tau \in S^+$ and $r \in GL_n(F)_{\mathbf{h}}$ we define a formal and an ordinary Dirichlet series as follows:

$$D'_{r,\tau}(\mathbf{f}, \chi) := \sum_{B/E} \psi(det(rx))\chi^*(det(x)\mathfrak{g})c_{\mathbf{f}}(\tau, rx)|det(x)|_F^{-(n+1)}[det(x)\mathfrak{g}], \tag{5.9}$$

and

$$D'_{r,\tau}(s, \mathbf{f}, \chi) := \sum_{B/E} \psi(det(rx))\chi^*(det(x)\mathfrak{g})c_{\mathbf{f}}(\tau, rx)|det(x)|_F^{s-n-1}. \tag{5.10}$$

This second series is obtained from the series in Equation 5.8 by putting $\chi^*(t\mathfrak{g})|t\mathfrak{g}|_F^s$ for $[t\mathfrak{g}]$. In particular we have the equation

$$D'_{r,\tau}(s, \mathbf{f}, \chi)\Lambda_{\mathfrak{c}}\left(\frac{2s-n}{4}\right) \prod_{v \in \mathbf{b}} g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^s) = \tag{5.11}$$

$$Z'(s, \mathbf{f}, \chi)(\psi/\psi_{\mathfrak{c}})^2(det(r)) \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathfrak{g})|det(r^*\hat{y}|_F^s c(\tau, y; \mathbf{f}).$$

where for an integral ideal $\mathfrak{a}$ we write

$$\Lambda_{\mathfrak{a}}(s) = \begin{cases} L_{\mathfrak{a}}(2s, \rho_\tau\psi\chi)\prod_{i=1}^{n/2} L_{\mathfrak{a}}(4s-2i, \psi^2\chi^2), & \text{if } n \text{ is even;} \\ \prod_{i=1}^{(n+1)/2} L_{\mathfrak{a}}(4s-2i+1, \psi^2\chi^2), & n \text{ is odd.} \end{cases},$$

Given χ as above we write $\mathfrak{f}$ for the conductor of χ. We define $t' \in \mathbf{Z}^{\mathbf{a}}$ by

$$(\psi\chi)_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'}.$$

and $\mu \in \mathbf{Z}^{\mathbf{a}}$ by the conditions $0 \leq \mu_v \leq 1$ for all $v \in \mathbf{a}$ and $\mu - [k] - t' \in 2\mathbf{Z}^{\mathbf{a}}$.

We now define a weight l and a Hecke character ψ' of F by $l = \mu + (n/2)\mathbf{a}$ and $\psi' = \chi^{-1}\rho_\tau$, where ρ_τ is the Hecke character of F corresponding to the extension $F(c^{\frac{1}{2}})/F$ with $c := (-1)^{[n/2]}det(2\tau)$. Let us write $\theta_\chi \in \mathcal{M}_l(C', \psi')$ for the theta series associated to the datum $(\overline{\chi}, \mu, \tau, r)$ in section 2. Write $C' = D[\mathfrak{b}'^{-1}, \mathfrak{b}'\mathfrak{c}']\}$ and define $\mathfrak{e} := \mathfrak{b} + \mathfrak{b}'$

Then we have (see [9, page 572])

Theorem 5.4 (Shimura)**.**

$$(4\pi)^{-n(su+(k+l)/2)}(\sqrt{D_F}N(\mathfrak{e})^{-1})^{n(n+1)/2}\prod_{v\in\mathbf{a}}\Gamma_n(s+(k_v+l_v)/2)D'_{r,\tau}(2s+3n/2+1;\mathbf{f},\chi) =$$

$$|det(r)|_F^{-2s-n/2}det(\tau)^{+(k+\mu+nu/2)/2+su}\int_\Phi f(z)\overline{\theta_\chi(z)E(z,\bar{s}+(n+1)/2,k-l,\epsilon\overline{\psi\chi\rho_\tau},\Gamma')}\delta(z)^k dz,$$

where $\Phi := \mathcal{H}/\Gamma'$ *and* $\Gamma' := G\cap D[\mathfrak{e}^{-1},\mathfrak{e}\mathfrak{h}]$*, where* $\mathfrak{h} = \mathfrak{e}^{-1}(\mathfrak{b}\mathfrak{c}\cap\mathfrak{b}'\mathfrak{c}')$*.*

In particular using the equation 5.11 we obtain

Theorem 5.5.

$$Z'(s,\mathbf{f},\chi)\prod_{v\in\mathbf{a}}\Gamma_n\left(\frac{s-n-1+k_v+\mu_v}{2}\right)\times$$

$$\left((\psi/\psi_{\mathfrak{c}})^2(det(r))\sum_{L<M\in\mathcal{L}_\tau}\mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathfrak{g})|det(r^*\hat{y})|_F^s c(\tau,y;\mathbf{f})\right) =$$

$$\left(D_F^{-1/2}N(\mathfrak{e})\right)^{n(n+1)/2}(4\pi)^{n||s'u+\lambda||}det(\tau)^{s'u+\lambda}|det(r)|_{\mathbf{A}}^{n+1-s}\times$$

$$\prod_{v\in\mathbf{b}}g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v\mathfrak{g})|\pi_v|^s)(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})((2s-n)/4)vol(\Phi) < f,\theta_\chi D((2s-n)/4) >_{\Gamma'},$$

where $s' = (2s-3n-2)/4$ *and for an integral ideal* $\mathfrak{a}$ *of* F*,*

$$\Lambda_{\mathfrak{a}}(s) = \begin{cases} L_{\mathfrak{a}}(2s,\rho_\tau\psi\chi)\prod_{i=1}^{n/2}L_{\mathfrak{a}}(4s-2i,\psi^2\chi^2), & \text{if } n \text{ is even;}\\ \prod_{i=1}^{(n+1)/2}L_{\mathfrak{a}}(4s-2i+1,\psi^2\chi^2), & n \text{ is odd.}\end{cases},$$

and

$$D(s) = \overline{\Lambda_{\mathfrak{h}}(s)}E(z,\bar{s};k-l,\epsilon,\rho_\tau\overline{\psi\chi},\Gamma').$$

We have normalized the Petersson inner product as follows

$$< f,\theta_\chi D((2s-n)/4) >_{\Gamma'} = \frac{1}{vol(\Phi)}\int_\Phi f(z)\overline{\theta_\chi(z)D(z,(2s-n)/4)}\delta(z)^k dz.$$

In particular there exists (τ, r) *with* $c(\tau,r;\mathbf{f})\neq 0$ *such that*

$$Z'(s,\mathbf{f},\chi)\prod_{v\in\mathbf{a}}\Gamma_n\left(\frac{s-n-1+k_v+\mu_v}{2}\right)\psi_{\mathfrak{c}}(det(r))c(\tau,r;\mathbf{f}) = \tag{5.12}$$

$$\left(D_F^{-1/2}N(\mathfrak{e})\right)^{n(n+1)/2}(4\pi)^{n||s'u+\lambda||}det(\tau)^{s'u+\lambda}|det(r)|_{\mathbf{A}}^{n+1-s}\times$$

$$\prod_{v\in\mathbf{b}}g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v\mathfrak{g})|\pi_v|^s)(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})((2s-n)/4)vol(\Phi) < f,\theta_\chi D((2s-n)/4) >_{\Gamma'}.$$

We note here that $vol(\Phi)\in\pi^{n(n+1)/2}\mathbb{Q}^\times$.

6. Petersson Inner Products and Periods

In this section we define some archimedean periods that we will use to normalize the special values of the function $Z'(s,\mathbf{f},\psi)$. The idea of defining these periods is due to Sturm [12] (building on previous work of Shimura), who considered the case of n even and $F=\mathbb{Q}$. However one should notice also the difference on the bounds of the weights that we impose. In what it follows we will call a Hecke operator $T(\mathfrak{a})$, relative to the group $C=D[\mathfrak{b}^{-1},\mathfrak{bc}]$, as "good" if $\mathfrak{a}$ is prime to $\mathfrak{c}$

Theorem 6.1. *Let $\mathbf{f}\in\mathcal{S}_k(\mathfrak{c},\psi)$ be an eigenform for all the "good" Hecke oparators of C. Let Φ be the Galois closure of F over $\mathbb{Q}$ and write Ψ for extension of Φ generated by the eigenvalues of $\mathbf{f}$ and their complex conjugation . Assume $m_0:=min_v(m_v)>[3n/2+1]+2$. Then there exists a period $\Omega_{\mathbf{f}}$ such that for any $\mathbf{g}\in\mathcal{S}_k(\overline{\mathbb{Q}})$ we have*

$$\left(\frac{<\mathbf{f},\mathbf{g}>}{\Omega_{\mathbf{f}}}\right)^{\sigma}=\frac{<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>}{\Omega_{\mathbf{f}^{\sigma}}},$$

for all $\sigma\in Gal(\overline{\mathbb{Q}}/\Phi)$, where $\sigma'=\rho\sigma\rho$. Moreover $\Omega_{\mathbf{f}}$ depends only on the eigenvalues of $\mathbf{f}$ and we have $\frac{<\mathbf{f},\mathbf{f}>}{\Omega_{\mathbf{f}}}\in\Psi^{\times}$.

Remark 6.2. As we remarked above, a theorem of this form has been firstly proved by Sturm [12], when $F=\mathbb{Q}$ and n is even. A similar theorem appears also in the work of Panchishkin [7]. It is also important to notice that in Panchishkin's theorem one can take also $\mathbf{g}$ not cuspidal. However for this he has to take the weight big enough in order to be in the range of absolute convergent for the Eisenstein series (see the Theorems after the proof). Our proof is modeled on that of Sturm [12, Theorem 3]and of Shimura [11, Theorem 28.5]. Maybe one should here remark that one of the differences with the proof here in comparison with the one of Sturm is that we use the identity **??** and not the Andrianov-Kalinin identity used by Sturm. Finally since we are using a stronger theorem of Shimura with respect to the absolute convergence of the function $Z(s,\mathbf{f},\chi)$ we also obtain better bounds for the weights. Finally we remark the slightly larger bound on m_0 than in Shimura [11, Theorem 28.5]. The reason for this is the above mentioned problem with the Eisenstein spectrum (i.e. separate it rationally from the cuspidal part).

Proof. We write $\{\lambda(\mathfrak{a})\}$ for the system of the eigenvalues of $\mathbf{f}$ (with respect to the "good" Hecke operators) and we define $\mathcal{V}:=\{\mathbf{h}\in\mathcal{S}_k(\mathfrak{c},\psi)|\mathbf{h}|T(\mathfrak{a})=\lambda(\mathfrak{a})\mathbf{h}\}$. Then as in Shimura we define $\mathcal{V}(\Psi)=\mathcal{V}\cap\mathcal{S}_k(\mathfrak{c},\psi;\Psi)$. By [4] we have that the space $\mathcal{V}(\Psi)$ is preserved by the operators $T(\mathfrak{a})$. Moreover the "good" Hecke operators generate a ring of semi-simple Ψ-linear transformations hence we have $\mathcal{V}=\mathcal{V}(\Psi)\otimes_{\Psi}\mathbb{C}$ and $\mathcal{S}_k(C,\Psi)=\mathcal{V}(\Psi)\oplus\mathcal{U}$, with $\mathcal{U}$ a vector space over Ψ which is stable under the action of the "good" Hecke operators. Since an eigenform in $\mathcal{U}\otimes_{\Psi}\mathbb{C}$ which is not contained in $\mathcal{V}$ must be orthogonal to it we have that the above decomposition is orthogonal with respect to the Petersson inner product.

We now pick an integer σ_0 so that $3n/2+1<\sigma_0<m_0$ and $m_0-\sigma_0\notin 2\mathbb{Z}$. Note that this is always possible thanks to our assumtption $m_0>[3n/2+1]+2$. Then we define $\mu\in\mathbb{Z}^{\mathbf{a}}$ by the conditions $0\leq\mu_v\leq 1$ and $\sigma_0-k_v+\mu_v\in 2\mathbb{Z}$ for all $v\in\mathbf{a}$. Our choice of σ_0 implies in particular that there exists an $v\in\mathbf{a}$ so that $\mu_v\neq 0$. We put $t':=\mu-k$. We now pick a quadratic character χ of F so that $(\psi\chi)_{\mathbf{a}}(x)=x_{\mathbf{a}}^{t'}|x_{\mathbf{a}}|^{-t'}$

and of conductor $\mathbf{f}$ such that $\mathfrak{c}|\mathfrak{f}$. Note that such a character can be obtained as the non trivial character of the quadratic extension $F(\sqrt{\Delta})$ by picking the sign of Δ properly at $v \in \mathbf{a}$ and Δ with non trivial valuation at all primes that divide $\mathfrak{c}$. The existence of such a Δ follows from the approxiamtion theorem for F. As in Shimura [11, page 236] we define $l := \mu + (n/2)\mathbf{a}$ and $\nu = \sigma_0 - (n/2)$. Then $\nu \geq (n+1)/2$ and $0 \leq k - l - \nu\mathbf{a} \in 2\mathbb{Z}^{\mathbf{a}}$. We consider the theta series θ_χ with respect to our choices of χ and μ. By Theorem 5.5, after evaluating at $s = \sigma_0$ we obtain

$$Z'(\sigma_0, \mathbf{f}, \chi) \prod_{v \in \mathbf{a}} \Gamma_n\left(\frac{\sigma_0 - n - 1 + k_v + \mu_v}{2}\right) \times$$

$$\left((\psi/\psi_{\mathfrak{c}})^2(det(r)) \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathfrak{g})|det(r^*\hat{y})|_F^{\sigma_0} c(\tau, y; \mathbf{f})\right) =$$

$$\left(D_F^{-1/2} N(\mathfrak{c})\right)^{n(n+1)/2} (4\pi)^{n||s_0'u+\lambda||} det(\tau)^{s_0'u+\lambda} |det(r)|_{\mathbf{A}}^{n+1-\sigma_0} \times$$

$$\prod_{v \in \mathbf{b}} g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v\mathfrak{g})|\pi_v|^{\sigma_0})(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})(\nu/2) vol(\Phi) < f, \theta_\chi D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}) >_{\Gamma'},$$

where $s_0' = (2\sigma_0 - 3n - 2)/4$. We now note (see [11, page 237]) that

$$\frac{\prod_{v \in \mathbf{a}} \Gamma_n\left(\frac{\sigma_0 - n - 1 + k_v + \mu_v}{2}\right)}{vol(\Phi)} \in \pi^{n||\frac{k-l-\nu\mathbf{a}}{2}|| - n||k|| + d\epsilon} \mathbb{Q}^\times$$

where $\epsilon = n^2/4$ if n even and $(n^2-1)/4$ otherwise. We now write δ for the rational part of $\frac{\prod_{v \in \mathbf{a}} \Gamma_n\left(\frac{\sigma_0 - n - 1 + k_v + \mu_v}{2}\right)}{vol(\Phi)}$. We now take $\beta \in \mathbb{N}$ so that $\pi^{-\beta} D(\nu/2) \in \mathcal{N}_{k-l}^p(\Phi\mathbb{Q}_{ab})$ with $p = \frac{k-l-\nu\mathbf{a}}{2}$ and we set $\gamma := n||\frac{k-l-\nu\mathbf{a}}{2}|| - n||k|| + d\epsilon - n||s_0'u + \lambda|| - \beta$. We further set

$$B(\chi, \psi, \tau, r, \mathbf{f}) := \delta\left((\psi/\psi_{\mathfrak{c}})^2(det(r)) \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathfrak{g})|det(r^*\hat{y})|_F^{\sigma_0} c(\tau, y; \mathbf{f})\right),$$

and

$$C(\chi, \psi, \tau, r) := (N(\mathfrak{c}))^{n(n+1)/2} |det(r)|_{\mathbf{A}}^{n+1-\sigma_0} \times$$
$$\prod_{v \in \mathbf{b}} g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v\mathfrak{g})|\pi_v|^{\sigma_0})(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})(\nu/2).$$

We then have for every $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ that

$$B(\chi, \psi, \tau, r, \mathbf{f})^\sigma = B(\chi^\sigma, \psi^\sigma, \tau, r, \mathbf{f}^\sigma) \text{ and } C(\chi, \psi, \tau, r)^\sigma = C(\chi^\sigma, \psi^\sigma, \tau, r).$$

We now note the equalities

$$< f, \theta_\chi D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}) >_{\Gamma'} = < f, \mathfrak{p}(\theta_\chi D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi})) >_{\Gamma'} = < f, Tr_{\Gamma'}^{\Gamma}(\mathfrak{p}(\theta_\chi D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}))) >_\Gamma,$$

where $\mathfrak{p} : \mathcal{R}_k^p \to \mathcal{S}_k$ is Shimura's holomorphic projection operators [11, Proposition 15.6](note that $\theta_\chi D(\nu/2) \in \mathcal{R}_k^p$ since θ_χ is a cusp form) and $Tr_{\Gamma'}^{\Gamma} : \mathcal{S}_k(\Gamma', \psi) \to S_k(\Gamma, \psi)$ is the usual trace operator attached to the groups $\Gamma' \leq \Gamma$. Moreover, since $\theta_\chi \pi^{-\beta} D(\nu/2) \in \mathcal{N}_k^p(\Phi\mathbb{Q}_{ab})$, we may consider the action of $\sigma \in Gal(\Phi\mathbb{Q}_{ab}/\Phi)$. Then

$$\mathfrak{p}(\theta_\chi \pi^{-\beta} D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}))^\sigma = \mathfrak{p}(\theta_\chi^\sigma (\pi^{-\beta} D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}))^\sigma), \text{ and } Tr_{\Gamma'}^{\Gamma}(\theta_\chi \pi^{-\beta} D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi}))^\sigma =$$
$$Tr_{\Gamma'}^{\Gamma}(\theta_\chi^\sigma D(\nu/2, \epsilon\rho_\tau\overline{\psi\chi})^\sigma),$$

where in the last equation the last trace is from the space $\mathcal{S}_k(\Gamma', \psi^\sigma)$ to $\mathcal{S}_k(\Gamma', \psi^\sigma)$. The equivariant property of the holomorphic projection operator is shown in Proposition 15.6 of (loc. cit.) and the one of the trace is exactly as in Sturm where he considers the case of $F = \mathbb{Q}$, but the arguments is valid also for general F since the strong approximation theorem also hold for the group $Sp_n(F)$, the essential argument in his proof. We make this more formal in the lemma following this proof.

Keeping now the character χ fixed we know that for any given $\mathbf{f} \in \mathcal{V}$ there exists (τ, r) such that

$$B(\chi, \psi, \tau, r, \mathbf{f}) = \delta\psi(det(r))c(\tau, r; \mathbf{f}) \neq 0.$$

We note here that the same pair (τ, r) can be used for the form $\mathbf{f}^\sigma$, as it follows from the proof of Theorem 20.9 in [11]. As in Shimura we write $\mathfrak{G}$ for the set of pairs (τ, r) for which such an $\mathbf{f}$ exists. From the observation above the set $\mathfrak{G}$ is the same also for the system of eigenvalues $\lambda(\mathfrak{a})^\sigma$, for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$. In particular for such an (τ, r)

$$0 \neq \pi^\gamma Z'(\sigma_0, \mathbf{f}, \chi)\delta\psi(det(r))c(\tau, r; \mathbf{f}) = \tag{6.13}$$

$$\left(D_F^{-1/2} N(\mathfrak{e})\right)^{n(n+1)/2} (4)^{n||s_0'u+\lambda||} det(\tau)^{s_0'u+\lambda} |det(r)|_{\mathbf{A}}^{n+1-\sigma_0} \times$$

$$\prod_{v \in \mathbf{b}} g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v \mathfrak{g})|\pi_v|^{\sigma_0})(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})(\nu/2) < f, \theta_\chi \pi^{-\beta} D(\nu/2, \epsilon\rho_\tau \overline{\psi\chi}) >_{\Gamma'} .$$

The fact that $Z'(\sigma_0, \mathbf{f}, \chi) \neq 0$ is in principle [11, Theorem 20.13]. Indeed in page 183 of (loc. cit) Shimura first proves the non-vanishing of $Z'(\sigma_0, \mathbf{f}, \chi)$ for any character χ with $\mu \neq 0$, as it is the case that we consider. Further we note that this in particular implies also that $C(\chi, \psi, \tau, r) \neq 0$ for all $(\tau, r) \in \mathfrak{G}$.

We now define an element $\mathbf{g}_{\tau,r,\psi} \in \mathcal{S}_k(\Gamma, \overline{\psi}; \Phi\mathbb{Q}_{ab})$ by

$$\mathbf{g}_{\tau,r,\psi} = \pi^{-\beta} Tr_{\Gamma'}^{\Gamma} \left(\mathfrak{p}(\theta_\chi \pi^{-\beta} D(\nu/2, \epsilon\rho_\tau \overline{\psi\chi}))\right),$$

and define the space $\mathcal{W}$ to be the space generated by $g_{\tau,r,\psi}$ for $(\tau, r) \in \mathfrak{G}$. We now consider the case n even or odd separately.

The case of n even: In this case we have that ϵ is the trivial character. We now claim that there exists an $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ such that any $\mathbf{f} \in \mathcal{V}$ and any $\mathbf{g}_{\tau,r,\psi}$

$$\left(\frac{< \mathbf{f}, \mathbf{g}_{\tau,r,\psi} >}{\Omega_{\mathbf{f}}}\right)^\sigma = \frac{< \mathbf{f}^\sigma, \mathbf{g}_{\tau,r,\psi}^{\sigma'} >}{\Omega_{\mathbf{f}^\sigma}},$$

where $\sigma' = \rho\sigma\rho$. First we observe that

$$\mathbf{g}_{\tau,r,\psi}^{\sigma'} = Tr_{\Gamma'}^{\Gamma}\left(\mathfrak{p}(\theta_\chi \pi^{-\beta} D(\nu/2, \epsilon\rho_\tau \overline{\psi\chi}))\right)^{\sigma'} = Tr_{\Gamma'}^{\Gamma}\left(\mathfrak{p}(\theta_\chi^{\sigma'}(\pi^{-\beta} D(\nu/2, \epsilon\rho_\tau \overline{\psi\chi}))^{\sigma'}\right) =$$

$$Tr_{\Gamma'}^{\Gamma}\left(\mathfrak{p}(\theta_\chi(\pi^{-\beta} D(\nu/2, \epsilon\rho_\tau \overline{\psi\chi}))^{\sigma'}\right),$$

where the last equality follows from the fact that χ is a quadratic character. We now recall that $D(\nu/2, \rho_\tau \overline{\psi\chi}) = D(z, \nu/2; k-l, \rho_\tau \overline{\psi\chi}, \Gamma)$ and we have seen that

$$\left(\frac{\pi^\gamma D(z, \nu/2; k-l, \rho_\tau \overline{\psi\chi}, \Gamma)}{P(\rho_\tau \overline{\psi\chi})}\right)^{\sigma'} = \frac{\pi^{-\beta} D(z, \nu/2; k-l, \rho_\tau \overline{\psi\chi}^{\sigma'}, \Gamma)}{P(\rho_\tau \overline{\psi\chi}^{\sigma'})},$$

where $P(\rho_\tau\overline{\psi}\chi) = \frac{i^{n|p|}\mathbf{g}(\rho_\tau\overline{\psi}\chi)(i)^{\nu d}}{|D_F|^{1/2}}\frac{\left(\prod_{i=1}^{[n/2]}(i)^{(2\nu-2i)d}\right)\mathbf{g}(\overline{\psi}^{2[n/2]})}{|D_F|^{b(n)}}$, with $p = \frac{k-l-\nu\mathbf{a}}{2}$. We conclude that

$$\mathbf{g}^{\sigma'}_{\tau,r,\psi} = \frac{P(\rho_\tau\overline{\psi\chi})^{\sigma'}}{P(\rho_\tau\overline{\psi}^{\sigma'}\chi)} Tr^{\Gamma}_{\Gamma'}\left(\mathfrak{p}(\theta_\chi\pi^{-\beta}D(\nu/2,\epsilon\rho_\tau\overline{\psi}^{\sigma'}\chi))\right) = \frac{P(\rho_\tau\overline{\psi\chi})^{\sigma'}}{P(\rho_\tau\overline{\psi}^{\sigma'}\chi)}\mathbf{g}_{\tau,r,\psi^\sigma}.$$

We set $R(\overline{\psi}) := \frac{i^{n|p|}(i)^{\nu d}}{|D_F|^{1/2}}\frac{\left(\prod_{i=1}^{[n/2]}(i)^{(2\nu-2i)d}\right)\mathbf{g}(\overline{\psi}^{2[n/2]})}{|D_F|^{b(n)}}$. We now consider the ratio

$$\frac{\mathbf{g}(\rho_\tau\overline{\psi}\chi)^{\sigma'}}{\mathbf{g}(\rho_\tau\overline{\psi}^{\sigma'}\chi)} = \frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})}\frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})}\frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi^{\sigma'})}.$$

We recall that ρ_τ is the non-trivial character of the quadratic extension $F(\sqrt{c})/F$ with $c = (-1)^{[n/2]}det(2\tau)$. Since we are considering $\tau > 0$ we have that

$$\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})} = \begin{cases} \frac{\sqrt{2det(\tau)}^{\sigma'}}{\sqrt{2det(\tau)}}, & \text{if } [n/2] \text{ even;} \\ \left(\frac{i^{\sigma'}}{i}\right)^d \frac{\sqrt{N(2det(\tau))}^{\sigma'}}{\sqrt{N(2det(\tau))}}, & \text{otherwise.} \end{cases}$$

Putting all these together we conclude that

$$\mathbf{g}^{\sigma'}_{\tau,r,\psi} = \frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})}\frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})}\frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi^{\sigma'})}\frac{R(\overline{\psi})^{\sigma'}}{R(\overline{\psi}^{\sigma'})}\mathbf{g}_{\tau,r,\psi^\sigma}$$

For any $\mathbf{g}_{\tau,r}$ we have

$$(4)^{-n||s_0'u+\lambda||}D_F^{n(n+1)/4}\pi^\gamma Z'(\sigma_0,\mathbf{f},\chi)B(\chi,\psi,\tau,r,\mathbf{f}) = det(\tau)^{s_0'u+\lambda}C(\chi,\psi,\tau,r) < \mathbf{f},\mathbf{g}_{\tau,r} >_\Gamma .$$

For any $(\tau,r) \in \mathfrak{G}$ we have seen that $C(\chi,\psi,\tau,r) \neq 0$. We obtain

$$\frac{< \mathbf{f},\mathbf{g}_{\tau,r} >_\Gamma}{(4\pi)^{-n||s_0'u+\lambda||}D_F^{n(n+1)/4}Z'(\sigma_0,\mathbf{f},\chi)} = det(\tau)^{-(s_0'u+\lambda)}\frac{B(\chi,\psi,\tau,r,\mathbf{f})}{C(\chi,\psi,\tau,r)}.$$

For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have then

$$\left(\frac{< \mathbf{f},\mathbf{g}_{\tau,r,\psi} >_\Gamma}{(4)^{-n||s_0'u+\lambda||}D_F^{n(n+1)/4}\pi^\gamma Z'(\sigma_0,\mathbf{f},\chi)}\right)^\sigma = \left(det(\tau)^{-(s_0'u+\lambda)}\frac{B(\chi,\psi,\tau,r,\mathbf{f})}{C(\chi,\psi,\tau,r)}\right)^\sigma =$$

$$(det(\tau)^{-(s_0'u+\lambda)})^\sigma\frac{B(\chi^\sigma,\psi^\sigma,\tau,r,\mathbf{f}^\sigma)}{C(\chi^\sigma,\psi^\sigma,\tau,r)} = \frac{(det(\tau)^{-(s_0'u+\lambda)})^\sigma}{det(\tau)^{-(s_0'u+\lambda)}}\frac{< \mathbf{f}^\sigma,\mathbf{g}_{\tau,r,\psi^\sigma} >_\Gamma}{(4\pi)^{-n||s_0'u+\lambda||}D_F^{n(n+1)/4}Z'(\sigma_0,\mathbf{f}^\sigma,\chi)}.$$

We remark that $s_0'u+\lambda = \frac{2\sigma_0-3n-2}{4}u + \frac{k+\mu+\frac{n}{2}u}{2} = \frac{\sigma_0u+k+\mu}{2} - \frac{n+1}{2}u$. By our choice of σ_0 we have that $\sigma_0u+k+\mu \in 2\mathbb{Z}^{\mathbf{a}}$. We obtain that $\frac{(det(\tau)^{-(s_0'u+\lambda)})^\sigma}{det(\tau)^{-(s_0'u+\lambda)}} = \frac{(det(\tau)^{\frac{1}{2}\mathbf{a}})^\sigma}{det(\tau)^{(\frac{1}{2}\mathbf{a})}}$ Now we note that since $det(\tau)$ is totally positive we have

$$\overline{\left(\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})}\right)}^{-1}\frac{(det(\tau)^{\frac{1}{2}\mathbf{a}})^\sigma}{det(\tau)^{(\frac{1}{2}\mathbf{a})}} = \begin{cases} 1, & \text{if } [n/2] \text{ even;} \\ \left(\frac{i^{\sigma'}}{i}\right)^d, & \text{otherwise.} \end{cases}$$

We have seen that

$$\mathbf{g}_{\tau,r,\psi^{\sigma}} = \left(\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})} \frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})} \frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi^{\sigma'})} \frac{R(\overline{\psi})^{\sigma'}}{R(\overline{\psi}^{\sigma'})} \right)^{-1} \mathbf{g}^{\sigma'}_{\tau,r,\psi}.$$

and hence

$$\left(\frac{< \mathbf{f}, \mathbf{g}_{\tau,r,\psi} >_{\Gamma}}{(4)^{-n||s_0' u+\lambda||} D_F^{n(n+1)/4} \pi^{\gamma} Z(\sigma_0, \mathbf{f}, \chi)} \right)^{\sigma} =$$

$$\overline{\left(\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau^{\sigma'})} \frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})} \frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi^{\sigma'})} \frac{R(\overline{\psi})^{\sigma'}}{R(\overline{\psi}^{\sigma'})} \right)^{-1}} \frac{(det(\tau)^{\frac{1}{2}\mathbf{a}})^{\sigma}}{det(\tau)^{(\frac{1}{2}\mathbf{a})}} \frac{< \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}_{\tau,r,\psi} >_{\Gamma}}{(4)^{-n||s_0' u+\lambda||} D_F^{n(n+1)/4} \pi^{\gamma} Z'(\sigma_0, \mathbf{f}^{\sigma}, \chi)},$$

or equivalently

$$\left(\frac{< \mathbf{f}, \mathbf{g}_{\tau,r,\psi} >_{\Gamma}}{\overline{\left(\mathbf{g}(\overline{\psi})\mathbf{g}(\chi)R(\overline{\psi})\right)^{-1}} B(n) 4^{-n||s_0' u+\lambda||} D_F^{n(n+1)/4} \pi^{\gamma} Z'(\sigma_0, \mathbf{f}, \chi)} \right)^{\sigma} =$$

$$\frac{< \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}_{\tau,r,\psi} >_{\Gamma}}{\overline{\left(\mathbf{g}(\overline{\psi}^{\sigma'})\mathbf{g}(\chi)R(\overline{\psi}^{\sigma'})B(n)\right)^{-1}} 4^{-n||s_0' u+\lambda||} D_F^{n(n+1)/4} \pi^{\gamma} Z'(\sigma_0, \mathbf{f}^{\sigma}, \chi)}.$$

where $B(n) = i^d$ if $[n/2]$ is odd and 1 otherwise. Hence we define

$$\Omega_{\mathbf{f}} := \overline{\left(\mathbf{g}(\overline{\psi})\mathbf{g}(\chi)R(\overline{\psi})\right)^{-1}} B(n) 4^{-n||s_0' u+\lambda||} D_F^{n(n+1)/4} \pi^{\gamma} Z'(\sigma_0, \mathbf{f}, \chi)$$

The case of n odd: We now repeat the considerations above but with the half-integral weight Eisenstein series.

$$\left(\frac{\pi^{-\beta} D(z, \nu/2; k-l, \epsilon\chi\overline{\psi}\rho_\tau, \mathfrak{c})}{\mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi\phi)^n i^{n|p|} i^{-dn\nu} C |D_F|^{n\nu/2+3n(n+1)/4} \mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi) |D_F|^{1/2} (2i)^{-(\nu-n)d} b([n/2])} \right)^{\sigma'} =$$

$$\frac{\pi^{-\beta} D(z, \nu/2; k-l, (\epsilon\chi\overline{\psi}\rho_\tau)^{\sigma})}{\mathbf{g}((\epsilon\chi\overline{\psi}\rho_\tau)^{\sigma'}\phi)^n i^{n|p|} i^{-dn\nu} C |D_F|^{n\nu/2+3n(n+1)/4} \mathbf{g}((\epsilon\chi\overline{\psi}\rho_\tau)^{\sigma'}) |D_F|^{1/2} (2i)^{-(\nu-n)d} b([n/2])},$$

where $b(m) = i^d$ if m is m is odd and 1 otherwise. We set

$$P(\epsilon\chi\overline{\psi}\rho_\tau) := \mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi\phi)^n i^{n|p|} i^{-dn\nu} C |D_F|^{n\nu/2+3n(n+1)/4} \mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi) |D_F|^{1/2} (2i)^{-(\nu-n)d} b([n/2])$$

and as before we have

$$\mathbf{g}^{\sigma'}_{\tau,r,\psi} = \frac{P(\epsilon\rho_\tau\overline{\psi\chi})^{\sigma'}}{P(\epsilon\rho_\tau\overline{\psi}^{\sigma'}\chi)} \mathbf{g}_{\tau,r,\psi^{\sigma}}.$$

We consider the ratio

$$\frac{\left(\mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi\phi)^n \mathbf{g}(\epsilon\rho_\tau\overline{\psi}\chi)\right)^{\sigma'}}{\mathbf{g}(\epsilon\rho_\tau\overline{\psi}^{\sigma'}\chi\phi)^n \mathbf{g}(\epsilon\rho_\tau\overline{\psi}^{\sigma'}\chi)} = \left(\frac{\mathbf{g}(\epsilon)^{\sigma'}}{\mathbf{g}(\epsilon)}\right)^{n+1} \left(\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau)}\right)^{n+1} \left(\frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})}\right)^{n+1} \left(\frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi)}\right)^{n+1} \left(\frac{\mathbf{g}(\phi)^{\sigma'}}{\mathbf{g}(\phi)}\right)^{n}.$$

Since $n+1$ is even and $\rho_\tau, \chi, \epsilon$ are quadratic characters we get that

$$\left(\frac{\mathbf{g}(\epsilon)^{\sigma'}}{\mathbf{g}(\epsilon)}\right)^{n+1} = \left(\frac{\mathbf{g}(\rho_\tau)^{\sigma'}}{\mathbf{g}(\rho_\tau)}\right)^{n+1} = \left(\frac{\mathbf{g}(\chi)^{\sigma'}}{\mathbf{g}(\chi)}\right)^{n+1} = 1.$$

We set $R := i^{n|p|}i^{-dn\nu}C|D_F|^{n\nu/2+3n(n+1)/4}|D_F|^{1/2}(2i)^{-(\nu-n)d}b([n/2])$, and then we have

$$\mathbf{g}^{\sigma'}_{\tau,r,\psi} = \left(\frac{\mathbf{g}(\overline{\psi})^{\sigma'}}{\mathbf{g}(\overline{\psi}^{\sigma'})}\right)^{n+1}\left(\frac{\mathbf{g}(\phi)^{\sigma'}}{\mathbf{g}(\phi)}\right)^{n}\frac{R^{\sigma'}}{R}\mathbf{g}_{\tau,r,\psi^{\sigma}}.$$

By the same calculations as in the case of n even, by no noticing that $s'_0u+\lambda \in \mathbb{Z}^{\mathbf{a}}$ we obtain For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have then

$$\left(\frac{<\mathbf{f},\mathbf{g}_{\tau,r,\psi}>_{\Gamma}}{4^{-n||s'_0u+\lambda||}D_F^{n(n+1)/4}\pi^{\gamma}Z'(\sigma_0,\mathbf{f},\chi)}\right)^{\sigma} = \frac{<\mathbf{f}^{\sigma},\mathbf{g}_{\tau,r,\psi^{\sigma}}>_{\Gamma}}{4^{-n||s'_0u+\lambda||}D_F^{n(n+1)/4}\pi^{\gamma}Z'(\sigma_0,\mathbf{f}^{\sigma},\chi)}.$$

Hence we conclude

$$\left(\frac{<\mathbf{f},\mathbf{g}_{\tau,r,\psi}>_{\Gamma}}{\left(\overline{\mathbf{g}(\overline{\psi})}^{n+1}\overline{\mathbf{g}(\overline{\phi})}^{n}\bar{R}\right)^{-1}4^{-n||s'_0u+\lambda||}D_F^{n(n+1)/4}\pi^{\gamma}Z'(\sigma_0,\mathbf{f},\chi)}\right)^{\sigma} =$$

$$\frac{<\mathbf{f}^{\sigma},\mathbf{g}_{\tau,r,\psi^{\sigma}}>_{\Gamma}}{\left(\overline{\mathbf{g}(\overline{\psi}^{\sigma'})}^{n+1}\overline{\mathbf{g}(\overline{\phi})}^{n}\bar{R}\right)^{-1}4^{-n||s'_0u+\lambda||}D_F^{n(n+1)/4}\pi^{\gamma}Z'(\sigma_0,\mathbf{f}^{\sigma},\chi)}.$$

So for n odd we define

$$\Omega_{\mathbf{f}} := \left(\overline{\mathbf{g}(\overline{\psi})}^{n+1}\overline{\mathbf{g}(\overline{\phi})}^{n}\bar{R}\right)^{-1}4^{-n||s'_0u+\lambda||}D_F^{n(n+1)/4}\pi^{\gamma}Z'(\sigma_0,\mathbf{f},\chi).$$

By $\mathcal{W}'$ we define the space generated by the projection of $\mathcal{W}$ on $\mathcal{V}$. By definition $\mathcal{W}' = \mathcal{V}$. Indeed for any element $\mathbf{g} \in \mathcal{V}$ there exists $\mathbf{h} \in \mathcal{W}'$ such that $<\mathbf{g},\mathbf{h}>_{\Gamma} \neq 0$, simply by taking the projection of the corresponding $\mathbf{g}_{\tau,r}$ to $\mathcal{W}'$. So the $\mathbb{C}$ span of $\mathbf{g}_{\tau,r}$ with $\tau, r \in \mathfrak{G}$ is equal to $\mathcal{V}$. Since $\mathbf{g}_{\tau,r}$ have algebraic coefficients we have that the $\overline{\mathbb{Q}}$-span is equal to $\mathcal{V}(\overline{\mathbb{Q}})$. We can now establish the theorem for any $\mathbf{g} \in \mathcal{V}(\overline{\mathbb{Q}})$ since after writing $\mathbf{g} = \sum_j c_j\mathbf{g}_{\tau_j,r_j,\mathcal{V}} \in \mathcal{V}(\overline{\mathbb{Q}})$, where $\mathbf{g}_{\tau_j,r_j,\mathcal{V}}$ is the projection of $\mathbf{g}_{\tau_j,r_j}$ to $\mathcal{V}$, we have

$$\left(\frac{<\mathbf{f},\mathbf{g}>}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \sum_j \overline{c_j}^{\sigma}\left(\frac{<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}_{\tau_j,r_j,\mathcal{V}}>}{\Omega_{\mathbf{f}^{\sigma}}}\right) = \frac{<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>}{\Omega_{\mathbf{f}^{\sigma}}}$$

We now take any $\mathbf{g} \in \mathcal{S}_k(\Gamma,\psi;\overline{\mathbb{Q}})$. We write $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ with $\mathbf{g}_1 \in \mathcal{V}$ and $\}_2 \in \mathcal{V}^{\perp}$. Then we have that

$$\left(\frac{<\mathbf{f},\mathbf{g}>}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \left(\frac{<\mathbf{f},\mathbf{g}_1>}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \frac{<\mathbf{f}^{\sigma},\mathbf{g}_1^{\sigma'}>}{\Omega_{\mathbf{f}^{\sigma}}} = \frac{<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>}{\Omega_{\mathbf{f}^{\sigma}}}$$

where the last equality follows from the fact that $<\mathbf{f},\mathbf{g}>=0$ implies that $<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>=0$. It is enough to show this for $\mathbf{g}$ an eigenform for all the good Hecke operators in an L-packet diffferent from that of $\mathbf{f}$'s. That is, there exists an ideal $\mathfrak{a}$ with $(\mathfrak{a},\mathfrak{c}) = 1$ so that $T(\mathfrak{a}\mathbf{f} = \lambda_{\mathbf{f}}\mathbf{f}$ and $T(\mathfrak{a}\mathbf{g} = \lambda_{\mathbf{g}}\mathbf{g}$ such that $\lambda_{\mathbf{f}} \neq \lambda_{\mathbf{g}}$. But then we have

$$\lambda_{\mathbf{f}}^{\sigma} <\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>=<T(\mathfrak{a})\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>=$$

$$<\mathbf{f}^{\sigma},T(\mathfrak{a})\mathbf{g}^{\sigma'}>=<\mathbf{f}^{\sigma},\lambda_{\mathbf{g}}^{\sigma'}\mathbf{g}^{\sigma'}>=<\mathbf{f}^{\sigma},\mathbf{g}^{\sigma'}>\lambda_{\mathbf{g}}^{\sigma}$$

and hence we conclude that $< \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'} >= 0$. Here we have used the facts that the good Hecke operators are self adjoint with respect to the Petersson inner produt, and that their Hecke eigenvalues are totally real (for both facts see [11, Lemma 23.15]).

Finally taking $\mathbf{g}$ equal to $\mathbf{f}$ we obtain that $\Omega_{\mathbf{f}}$ is equal to $< \mathbf{f}, \mathbf{f} >$ up to a non-zero element in the the Galois closure of the field generated by the Fourier coefficients of $\mathbf{f}$ (note that it also contains the eigenvalues).

□

We now give a proof of the equivariant property of the trace that we used in the proof of the theorem. The proof follows the proof given by Sturm [12, Lemma 11] extended to the totally real field situation.

Lemma 6.3. *With notation as in the proof of the above theorem we have for any* $f \in \mathcal{S}_k(\Gamma', \psi; \mathbb{Q}_{ab})$

$$Tr^{\Gamma}_{\Gamma'}(f)^{\sigma} = Tr^{\Gamma}_{\Gamma'}(f^{\sigma}),\ \ \sigma \in Gal(\Phi\mathbb{Q}_{ab}/\Phi).$$

Proof. Thanks to the strong approxiamation for $Sp_n(F)$ we may work adelically. We write D and D' for the corresponding to Γ and Γ' adelic groups (i.e. $\Gamma = G \cap D$). We fix elements $\{g_i\} \subset D_{\mathbf{h}}$ so that $D = \bigcup D' g_i$. For $t \in \mathbb{Z}_{\mathbf{h}}^{\times}$ corresponding to $\sigma|_{\mathbb{Q}_{ab}}$ we note that

$$\begin{pmatrix} 1_n & 0 \\ 0 & t^{-1}1_n \end{pmatrix} g_i \begin{pmatrix} 1_n & 0 \\ 0 & t1_n \end{pmatrix} \in Sp_n(\mathbb{A})_{\mathbf{h}}$$

and hence by strong approximation we can find elements $u_i \in D'$ with $f|u_i = f$ (i.e. $\psi(u_i) = 1$) and $w_i \in Sp_n(F)$ so that

$$\begin{pmatrix} 1_n & 0 \\ 0 & t^{-1}1_n \end{pmatrix} g_i \begin{pmatrix} 1_n & 0 \\ 0 & t1_n \end{pmatrix} = u_i w_i.$$

We moreover note that $w_{i\mathbf{a}} = u_{i\mathbf{a}}^{-1}$. Now we claim that since the g_i's form a set of representatives of the classes of D' in D, the same holds for $\begin{pmatrix} 1_n & 0 \\ 0 & t^{-1}1_n \end{pmatrix} g_i \begin{pmatrix} 1_n & 0 \\ 0 & t1_n \end{pmatrix}$, and hence also for w_i since $u_i \in D'$. Indeed since $t \in \mathbb{Z}_{\mathbf{h}}^{\times} \hookrightarrow F_{\mathbf{h}}^{\times}$ we have that

$$\begin{pmatrix} 1_n & 0 \\ 0 & t^{-1}1_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & t1_n \end{pmatrix} = \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} \in D[\mathfrak{a}, \mathfrak{b}]$$

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D[\mathfrak{a}, \mathfrak{b}]$, for some fractional ideals $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{g}$. In particular we have that $\imath(t) g_i \imath(t^{-1}) \in D$. We claim that the set $D = \coprod_i D' \imath(t) g_i \imath(t^{-1})$. Indeed let $d \in D$. Then $\imath(t^{-1}) d \imath(t) \in D$ and hence there exists $d' \in D'$ such that $\imath(t^{-1}) d \imath(t) = d' g_j$ for some j. Or equivalently $d = \imath(t) d' g_j \imath(t^{-1}) = \imath(t) d' \imath(t^{-1}) \imath(t) g_j \imath(t^{-1})$, which establishes our claim since $\imath(t) d' \imath(t^{-1}) \imath(t) \in D'$.

We now consider the elements $(\imath(t), \sigma), (w_i, id), (g_i, id) \in \mathcal{G}_+ \times Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we have

$$(Tr^{\Gamma}_{\Gamma'}(f^{\sigma}))^{\sigma^{-1}} = (\sum_i \psi(g_i)^{\sigma} f^{\sigma}|_k g_i)^{\sigma^{-1}} =$$

$$\sum_i (\psi(g_i) f)^{\left((\imath(t),\sigma)(g_i,1)(\imath(t^{-1}),\sigma^{-1})\right)} = \sum_i \psi(g_i) f^{(u_i w_i, 1)} = \sum_i \psi(g_i) f|_k w_i.$$

The proof of the lemma is now completed after observing that $\psi(g_i) = \psi(w_i)$. □

We also mention here the following theorem of Garrett [4].

Theorem 6.4 (Garrett)**.** *Let $k > 2n+1$ and $\mathbf{f}, \mathbf{g} \in \mathcal{S}_{k\mathbf{a}}$. Take $\mathbf{f}$ an eigenform for almost all Hecke operators. Then for all $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, we have*

$$\left(\frac{<\mathbf{f}^{\rho}, \mathbf{g}>}{<\mathbf{f}^{\rho}, \mathbf{f}>}\right)^{\sigma} = \frac{<\mathbf{f}^{\sigma\rho}, \mathbf{g}^{\sigma}>}{<\mathbf{f}^{\sigma\rho}, \mathbf{f}^{\sigma}>}$$

In particular if we take $\mathbf{f}, \mathbf{g} \in \mathcal{S}_{k\mathbf{a}}(\overline{\mathbb{Q}})$, and take $\mathbf{f}$ with totally real Fourier coefficients then we have that $\frac{<\mathbf{f}^{\rho},\mathbf{g}>}{<\mathbf{f}^{\rho},\mathbf{f}>} \in \overline{\mathbb{Q}}$ and

$$\left(\frac{<\mathbf{f}, \mathbf{g}>}{<\mathbf{f}, \mathbf{f}>}\right)^{\sigma} = \frac{<\mathbf{f}^{\sigma}, \mathbf{g}^{\sigma}>}{<\mathbf{f}^{\sigma}, \mathbf{f}^{\sigma}>}, \ \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}).$$

We note that if we combine the above result of Garrett with the following result of Harris on the Eisenstein spectrum

Theorem 6.5 (Harris)**.** *Let $k > 2n+1$ and write $\mathcal{E}_{k\mathbf{a}}$ for the orthogonal complement of $\mathcal{S}_{k\mathbf{a}}$ in $\mathcal{M}_{k\mathbf{a}}$ (the Eisenstein series). Define $\mathcal{E}_{k\mathbf{a}}(\mathbb{Q}) := \mathcal{M}_{k\mathbf{a}}(\mathbb{Q}) \cap \mathcal{E}_{k\mathbf{a}}$. Then we have*

$$\mathcal{M}_{k\mathbf{a}}(\mathbb{Q}) = \mathcal{E}_{k\mathbf{a}}(\mathbb{Q}) \oplus \mathcal{S}_{k\mathbf{a}}(\mathbb{Q}).$$

Proof. This follows from the work of Harris [6]. Indeed in general we have that (see [11, Theorems 27.14, and 27.16])

$$\mathcal{M}_{k\mathbf{a}}(\overline{\mathbb{Q}}) = \mathcal{E}_{k\mathbf{a}}(\overline{\mathbb{Q}}) \oplus \mathcal{S}_{k\mathbf{a}}(\overline{\mathbb{Q}})$$

and $\mathcal{E}_{k\mathbf{a}}(\overline{\mathbb{Q}}) = \oplus_{r=0}^{n} \mathcal{E}_{k\mathbf{a}}^{r}(\overline{\mathbb{Q}})$ where $\mathcal{E}_{k\mathbf{a}}^{r}$ the space of Klingen type Eisenstein series associated to a parabolic group P_r stabilizing an isotropic space of dimension r. Harris has shown that in the case of weight as above (i.e. the absolute convergence situation) we have that $\mathcal{E}_{k\mathbf{a}}^{r}(\overline{\mathbb{Q}}) = \mathcal{E}_{k\mathbf{a}}^{r}(\mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$.

□

Now this theorem allows us to take $\mathbf{g} \in \mathcal{M}_{k\mathbf{a}}$ in Theorem 6.4.

7. Algebraicity results for Siegel Modular Forms over totally real fields

In this section we present various results regarding special values of the function $Z'(s, \mathbf{f}, \chi)$, with $\mathbf{f} \in \mathcal{S}_k(\mathfrak{b}, \mathfrak{c}, \psi)$, an eigenform for all Hecke operators. We remind that we have also considered the function $Z(s, \mathbf{f}, \chi)$. The two coincide when the Nebentypus of $\mathbf{f}$ is trivial. Indeed if we write $Z_v(\chi^*(\pi_v \mathfrak{g})|\pi_v|^s)$ for the Euler factor of $Z(s, \mathbf{f}, \chi)$ at some prime $v \in \mathbf{h}$ then the corresponding Euler factor of $Z'(s, \mathbf{f}, \chi)$ is equal to $Z_v((\psi/\psi_{\mathfrak{c}})\chi^*(\pi_v \mathfrak{g})|\pi_v|^s)$. We note the equation

$$Z'(s, \mathbf{f}, \chi\psi^{-1}) = Z_{\mathfrak{c}}(s, \mathbf{f}, \chi),$$

where the subindex on the right hand side indicates that we have removed the Euler factors all primes in the support of $\mathfrak{c}$. In particular if we take the character χ trivial (may not primitive) at the primes dividing $\mathfrak{c}$ then we have that the two functions are the same.

We start by stating a result of Shimura [11, Theorem 28.8]. We take an $\mathbf{f} \in \mathcal{S}_k(C;\overline{\mathbb{Q}})$, where

$$C = \{x \in D[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}\mathfrak{c}] | a_x - 1 \prec \mathfrak{c}\}$$

We moreover take $\mathbf{f}$ of trivial Nebentypus and assume that it is an eigenform for all Hecke operators away from the primes in the support of $\mathfrak{c}$. In the notation of Shimura in Chapter V of his book, we take $\mathfrak{e} = \mathfrak{c}$, and not $\mathfrak{e} = \mathfrak{g}$. In particular here we take the Euler factors Z_v trivial for v in the support of $\mathfrak{c}$. The theorem below is stated only for $k \in \mathbb{Z}^{\mathbf{a}}$.

Theorem 7.1 (Shimura)**.** *With notation as above define* $m_0 := min\{k_v | v \in \mathbf{a}\}$ *and assume* $m_0 > (3n/2) + 1$*. Let* χ *be a character of* F *such that* $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ *with* $t \in \mathbb{Z}^{\mathbf{a}}$*. Set* $\mu_v := 0$ *if* $k_v - t_v \in 2\mathbb{Z}$ *and* $\mu_v = 1$ *if* $k_v - t_v \notin 2\mathbb{Z}$*. Let* $\sigma_0 \in \mathbb{Z}^{\mathbf{a}}$ *such that*

(i) $2n + 1 - k_v + \mu_v \leq \sigma_0 \leq k_v - \mu_v$,
(ii) $\sigma_0 - k_v + \mu_v \in 2\mathbb{Z}$ *for every* $v \in \mathbf{a}$ *if* $\sigma_0 > n$,
(iii) $\sigma_0 - 1 + k_v - \mu_v \in 2\mathbb{Z}$ *for every* $v \in \mathbf{a}$ *if* $\sigma_0 \leq n$.

We exclude the cases

(i) $\sigma_0 = n + 1$, $F = \mathbb{Q}$ *and* $\chi^2 = 1$,
(ii) $\sigma_0 = 0$, $\mathfrak{c} = \mathfrak{g}$ *and* $\chi = 1$,
(iii) $0 < \sigma_0 \leq n$, $\mathfrak{c} = \mathfrak{g}$, $\chi^2 = 1$ *and the conductor of* χ *is* $\mathfrak{g}$.

Then we have

$$\frac{Z(\sigma_0, \mathbf{f}, \chi)}{< \mathbf{f}, \mathbf{f} >} \in \pi^{n(\sum_v k_v) + de}\overline{\mathbb{Q}}$$

where $d = [F : \mathbb{Q}]$ *and*

$$e := \begin{cases} (n+1)\sigma_0 - n^2 - n, & \sigma_0 > n; \\ n\sigma_0 - n^2, & \text{otherwise.} \end{cases}$$

We now take $\mathbf{f} \in \mathcal{S}_k(C, \psi; \overline{\mathbb{Q}})$ with C of the form $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ (i.e. the standard setting in this paper). We are interested in special values of $Z'(s, \mathbf{f}, \chi)$ for a Hecke character χ of F of conductor $\mathfrak{f}$.

Theorem 7.2. *Let* $\mathbf{f} \in \mathcal{S}_k(\mathfrak{b}, \mathfrak{c}, \psi; \overline{\mathbb{Q}})$ *be an eigenform for all Hecke operators. Assume that either*

(i) *there exists* $v, v' \in \mathbf{a}$ *such that* $k_v \neq k_{v'}$*, and* $m_0 = min\{k_v | v \in \mathbf{a}\} > [3n/2 + 1] + 2$ *or*
(ii) k *is a parallel weight with* $k > 2n + 1$.

Let χ *be a character of* F *such that* $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ *with* $t \in \mathbb{Z}^{\mathbf{a}}$*. Define* $t' \in \mathbb{Z}^{\mathbf{a}}$ *by* $(\psi\chi)_{\mathbf{a}}(x) = x_{\mathbf{a}}^{t'} |x_{\mathbf{a}}|^{t'}$*. Set* $\mu_v := 0$ *if* $k_v - t'_v \in 2\mathbb{Z}$ *and* $\mu_v = 1$ *if* $k_v - t'_v \notin 2\mathbb{Z}$*. Let* $\sigma_0 \in \mathbb{Z}^{\mathbf{a}}$ *such that*

(i) $2n + 1 - k_v + \mu_v \leq \sigma_0 \leq k_v - \mu_v$ *for all* $v \in \mathbf{a}$,
(ii) $|\sigma_0 - n - \frac{1}{2}| + n + \frac{1}{2} - k + \mu \in 2\mathbb{Z}^{\mathbf{a}}$.
(iii) *if* n *is even, and* $\sigma_0 = n/2 + i$ *for* $i = 0, \ldots n/2$, $i \in \mathbb{N}$ *or if* n *is odd and* $\sigma_0 = n/2 - 1 + i$, $i = 1, \ldots, (n+1)/2$*, then we assume the* ***Assumption*** *below.*

We exclude the cases

(i) *$\sigma_0 = n+1$, $F = \mathbb{Q}$ and $(\chi\psi)^2 = 1$,*
(ii) *$\sigma_0 = \frac{n}{2}$, $\mathfrak{c} = \mathfrak{g}$, n is even and there is no (τ, r) that satisfy our assumption such that $\rho_\tau \neq 1$ and $\chi\psi = 1$,*
(iii) *$n/2 < \sigma_0 \leq n$, $\mathfrak{c} = \mathfrak{g}$ and $(\psi\chi)^2 = 1$.*

Then with notation as in the previous theorem we have

$$\frac{Z'(\sigma_0, \mathbf{f}, \chi)}{< \mathbf{f}, \mathbf{f} >} \in \pi^{n(\sum_v k_v)+de}\overline{\mathbb{Q}}$$

Moreover, if we take a number field W so that $\mathbf{f}, \mathbf{f}^\rho \in \mathcal{S}_k(W)$ and $\Phi \subset W$, where Φ is the Galois closure of F in $\overline{\mathbb{Q}}$, then

$$\frac{Z'(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta(\sqrt{D_F}^{n(n+1)/4})i^m\omega(\epsilon\overline{\chi\psi})^\rho < \mathbf{f}, \mathbf{f} >} \in \mathcal{W} := W(\chi\psi),$$

where $\omega(\cdot)$ is defined by using the Theorem 3.18 as follows

(i) *for $\sigma_0 > n$ and n even then $\omega(\cdot)$ is as in Theorem 3.18 (i),*
(ii) *for $\sigma_0 > n$ and n odd then $\omega(\cdot)$ is as in Theorem 3.18 (iii) (b),*
(iii) *for $\sigma_0 \leq n$ and n even then $\omega(\cdot)$ is as in Theorem 3.18 (ii),*
(iv) *for $\sigma_0 \leq n$ and n odd thenthen $\omega(\cdot)$ is as in Theorem 3.18 (iv).*

and $m = d$ if $[n/2]$ is odd and 0 otherwise.

Assumption: Let $\theta \in F_{\mathbf{h}}^\times$ so that $\theta\mathfrak{g} = \mathfrak{b}^{-1}\mathfrak{d}$. Write $\mathfrak{f}'$ for the conductor of χ^2. We assume that we can find $\tau \in S_+ \cap GL_n(F)$ and $r \in GL_n(F)_{\mathbf{h}}$ so that $c(\tau, r; \mathbf{f}) \neq 0$, equation 5.12 in Theorem 5.5 holds and

(i) if n is even and $v \nmid \mathfrak{c}\mathfrak{f}'$ then $(\theta^t r\tau r)_v$ is regular and $v \nmid \mathfrak{f}$,
(ii) if n is odd and $v \nmid \mathfrak{c}\mathfrak{f}'$ then $(\theta^t r\tau r)_v$ is regular and $v \nmid 2\mathfrak{f} \cap \mathfrak{b}^{-1}\mathfrak{d}$.

We note that this assumption implies that in Theorem 5.5 we have that $\Lambda_{\mathfrak{c}}(s)/\Lambda_{\mathfrak{h}}(s) = 1$ (see [9, Proposition 8.3]).

Proof. (of Theorem 7.2) We first consider the Gamma factors that appear in Theorem 5.5. We first recall that

$$\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{j=0}^{n-1}\Gamma(s - \frac{j}{2}).$$

Hence for $\prod_{v\in\mathbf{a}}\Gamma_n(\frac{\sigma_0-n-1+k_v+\mu_v}{2})$ we need the condition that $\sigma_0 > 2n - k_v + \mu_v$ for all $v \in \mathbf{a}$, which is the lower bound appearing in the theorem. Moreover the Eisenstein series $D(\frac{\nu}{2})$ of weight $k - \mu - \frac{n}{2}$ for $\nu = \sigma_0 - \frac{n}{2}$ is nearly holomorphic if and only if $n+1-(k_v - \mu_v - \frac{n}{2}) \leq \sigma_0 - \frac{n}{2} \leq k_v - \mu_v - \frac{n}{2}$ and $|\nu - \frac{n+1}{2}| + \frac{n+1}{2} - k_v + \mu_v + \frac{n}{2} \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$. These inequalities give the upper bound in the (i) condition for σ_0 and (ii). The third condition for σ_0 is imposed so that in the range where the fraction $\Lambda_{\mathfrak{c}}(s)/\Lambda_{\mathfrak{h}}(s)$ (a finite product of Euler factors associated to finite order characters) could have a pole it is equal to 1. Finally the various exclusion follows from various cases where the Eisenstein series $D(\frac{\nu}{2})$ is not nearly holomorphic.

We take $\beta \in \mathbb{N}$ so that $\pi^{-\beta}D(\frac{v}{2}) \in \mathcal{N}_{k-l}(\Phi\mathbb{Q}_{ab})$. Now using Theorem 5.5 after a proper choice of (τ, r) we have

$$\pi^{\gamma} Z'(\sigma_0, \mathbf{f}, \chi)\psi_{\mathfrak{c}}(det(r))c(\tau, r; \mathbf{f}) =$$
$$\alpha \left(D_F^{-1/2}\right)^{n(n+1)/2} det(\tau)^{s_0' u+\lambda}|det(r)|_{\mathbf{A}}^{n+1-\sigma_0} \times$$
$$\prod_{v \in \mathbf{b}} g_v((\psi/\psi_{\mathbf{c}})(\pi_v)\chi^*(\pi_v\mathfrak{g})|\pi_v|^{\sigma_0})(\Lambda_{\mathfrak{c}}/\Lambda_{\mathfrak{h}})((2\sigma_0 - n)/4) < f, \theta_\chi(\pi^{-\beta}D(\nu/2)) >,$$

where $\alpha \in \mathbb{Q}^\times$, and $\gamma := n||\frac{k-l-\nu\mathbf{a}}{2}|| - n||k|| + d\epsilon - n||s_0'u + \lambda|| - \beta$ where we recall $\epsilon = n^2/4$ if n even and $(n^2-1)/4$ otherwise.

We now note that $\theta_\chi \in \mathcal{M}_l(\mathcal{W})$ and $\pi^{-\beta}D(\nu/2) \in \mathcal{N}^r_{k-l}(W\mathbb{Q}_{ab})$ where $r = (k-l-\nu\mathbf{a})/2$ if $\nu > (n+1)/2$ and $r = (k-l-(n+1-\nu)\mathbf{a})/2$ otherwise.

Moreover we have $s_0'u + \lambda = \frac{\sigma_0 u + k + \mu}{2} - \frac{n+1}{2}u$. In particular

(i) for $\sigma_0 > n$ and n even we have that $s_0'u + \lambda \notin 2\mathbb{Z}$,
(ii) for $\sigma_0 > n$ and n odd we have that $s_0'u + \lambda \in 2\mathbb{Z}$,
(iii) for $\sigma_0 \leq n$ and n even we have that $s_0'u + \lambda \in 2\mathbb{Z}$,
(iv) for $\sigma_0 \leq n$ and n odd we have that $s_0'u + \lambda \notin 2\mathbb{Z}$.

We now note that $\mathbf{g}(\rho_\tau) = i^m\sqrt{N_{F/\mathbb{Q}}det(\tau)}$, with $m = d$ if $[n/2]$ is odd and 0 otherwise.

Now we set $P := \sqrt{D_F}^{n(n+1)/4} i^m \omega(\epsilon\overline{\chi\psi})$ where $\omega(\cdot)$ is defined as in the statement of the theorem. Then by Theorem 3.18 we conclude that

$$\left(D_F^{-1/2}\right)^{n(n+1)/2} det(\tau)^{s_0'u+\lambda}\pi^{-\beta}P^{-1}D(\nu/2) \in \mathcal{N}^r_{k-l}(\mathcal{W}).$$

We set $a := \left(D_F^{-1/2}\right)^{n(n+1)/2} det(\tau)^{s_0'u+\lambda}\pi^{-\beta}P^{-1}$. By Lemma 15.8 in [11] we have that there exists a $q \in \mathcal{M}_k(\mathcal{W})$ so that $< f, \theta_\chi a D(\nu/2) > = < f, q >$. If k is not a parallel weight, then we have that actually $q \in \mathcal{S}_k(\mathcal{W})$ since in this case $\mathcal{M}_k = \mathcal{S}_k$. Then by Theorem 6.1 we have that $\frac{<f,q>}{<f,f>} \in \mathcal{W}$. In the other case, that is of k being a parallel weight we can use Theorem 6.4 combined with the Theorem 6.5 to conclude again $\frac{<f,q>}{<f,f>} \in \mathcal{W}$ and hence conclude the proof. ☐

We now obtain also some results with reciprocity laws.

Theorem 7.3. *Let* $\mathbf{f} \in \mathcal{S}_k(\mathfrak{b}, \mathfrak{c}, \psi; \overline{\mathbb{Q}})$ *be an eigenform for all Hecke operators. With notation as before we take* $m_0 > [3n/2+1]+2$. *Let* χ *be a character of* F *such that* $\chi_{\mathbf{a}}(x) = x^t_{\mathbf{a}}|x_{\mathbf{a}}|^{-t}$ *with* $t \in \mathbb{Z}^{\mathbf{a}}$. *Define* $t' \in \mathbb{Z}^{\mathbf{a}}$ *by* $(\psi\chi)_{\mathbf{a}}(x) = x^{t'}_{\mathbf{a}}|x_{\mathbf{a}}|^{t'}$. *Set* $\mu_v := 0$ *if* $k_v - t'_v \in 2\mathbb{Z}$ *and* $\mu_v = 1$ *if* $k_v - t'_v \notin 2\mathbb{Z}$. *Assume that* $\mu \neq 0$.

Let $\sigma_0 \in \mathbb{Z}^{\mathbf{a}}$ *be as in the previous Theorem. Then with* $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ *as defined in the previous section in Theorem 6.1 we have for all* $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ *that*

$$\left(\frac{Z'(\sigma_0, \mathbf{f}, \chi)}{\pi^{n(\sum_v k_v)+de}\sqrt{D_F^{n(n+1)/4}} i^m \omega(\epsilon\overline{\chi\psi})^\rho \Omega_{\mathbf{f}}}\right)^\sigma = \frac{Z'(\sigma_0, \mathbf{f}^\sigma, \chi^\sigma)}{\pi^{n(\sum_v k_v)+de} i^m \sqrt{D_F}^{n(n+1)/4}\omega(\epsilon\overline{\chi^\sigma\psi^\sigma})^\rho \Omega_{\mathbf{f}^\sigma}}.$$

Proof. We first observe that thanks to the assumption that $\mu \neq 0$ we have that $\theta_\chi \in \mathcal{S}_l$. Moreover for $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ we have $\theta_\chi^{\sigma'} = \theta_{\chi^\sigma}$, as it follows from the explicit Fourier expansion of θ_χ. Moreover arguing as in the theorem above and using the reciprocity laws for Eisenstein series in Theorem 3.18 we have that

$$\left(\frac{\pi^{-\beta} \sqrt{D_F}^{n(n+1)/2} det(\tau)^{s'_0 u + \lambda} D(\nu/2, \epsilon \overline{\psi\chi} \rho_\tau)}{\omega(\epsilon \overline{\psi\chi})} \right)^{\sigma'} =$$

$$\frac{\pi^{-\beta} \sqrt{D_F}^{n(n+1)/2} det(\tau)^{s'_0 u + \lambda} D(\nu/2, \epsilon \overline{\psi^\sigma \chi^\sigma})}{\omega(\epsilon \overline{\psi^\sigma \chi^\sigma})}, \quad \sigma \in Gal(\overline{\mathbb{Q}}/\Phi).$$

Moreover we have that $\theta_\chi D(\nu/2, \epsilon \overline{\psi^\sigma \chi^\sigma} \rho_\tau) \in \mathcal{R}_k$. By Proposition 15.6 in [11] we have that there exists $q = \mathfrak{p}\left(\theta_\chi D(\nu/2, \epsilon \overline{\psi^\sigma \chi^\sigma} \rho_\tau)\right) \in \mathcal{S}_k$ so that $< f, \theta_\chi D(\nu/2, \epsilon \overline{\psi\chi} \rho_\tau) >=< f, q >$ and $q^\sigma = \mathfrak{p}\left(\theta_\chi^\sigma D(\nu/2, \epsilon \overline{\psi\chi} \rho_\tau)^\sigma\right)$ for all $\sigma \in Aut(\mathbb{C}/\Phi)$. In particular we have that

$$\left(\frac{\sqrt{D_F}^{n(n+1)/2} det(\tau)^{s'_0 u + \lambda} < f, \theta_\chi \pi^{-\beta} D(\nu/2, \epsilon \overline{\psi\chi} \rho_\tau) >}{\omega(\epsilon \overline{\psi\chi})^\rho \Omega_{\mathbf{f}}} \right)^{\sigma} =$$

$$\frac{\sqrt{D_F}^{n(n+1)/2} det(\tau)^{s'_0 u + \lambda} < f^\sigma, \theta_{\chi^\sigma} \pi^{-\beta} D(\nu/2, \epsilon \overline{\psi^\sigma \chi^\sigma} \rho_\tau) >}{\omega(\epsilon \overline{\psi^\sigma \chi^\sigma})^\rho \Omega_{\mathbf{f}^\sigma}},$$

from which we conclude the proof of the theorem

□

As we have remarked in the introduction results similar to the ones proved in this have been obtained by Sturm [12], Harris [6] and Panchishkin [7] in the case of $F = \mathbb{Q}$ and n even. Our proofs are just generalizations of theirs building in some new results of Shimura. We close this article by mentioning that the perhaps strongest result concerning the special values of Siegel modular forms, at least when $F = \mathbb{Q}$ and under some other technical assumptions, is due to Böcherer and Schmidt [1]. Using the doubling method and some holomorphic differential operators of Böcherer they obtained results as above but assuming only that the weight of the Siegel modular form is larger than n and not $3n/2 + 1$ as above. It is of course very interesting to extend their results to the totally real field case, however the generalization of their result seems to be a quite challenging task.

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