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The Iwasawa Main Conjecture for elliptic modular forms (after Skinner and Urban).

Hauptseminar

Room: HS 4 **Time:** Th. 11.00-13.00 **c.t.** **First meeting:** 19.04.2012

Brief description: The aim of this seminar is the study of the work of Skinner and Urban on the Main Conjecture for elliptic modular forms. We start by giving a brief description of this conjecture and then present the main theorem of Skinner and Urban following the introduction of their work [5].

Let p be an odd prime. Let $f = \sum_{n>0} a(n,f)q^n \in S_k(N,\chi)$ be a weight $k \geq 2$ newform of conductor N and Nebentypus χ. Suppose that f is ordinary, that is $a(p,f)$ is a p-adic unit. Let L be a finite extension of $\mathbb{Q}_p$ containing $a(n,f)$ for all $n > 0$ and the roots of the p^{th} Euler factor $X^2 + a(p,f)X + \chi(p)p^{k-1} = 0$. Then it is known that there exists a p-adic L-function, that is a power series $\mathcal{L}_f \in \Lambda_{O_L} := O_L[[\mathbb{Z}_p]]$, O_L the ring of integers of L, such that for any primitive p^tth root of unity ζ and $0 \leq m \leq k-2$ an integer, we have

$$\mathcal{L}_f(\zeta(1+p)^m - 1) = e_p \frac{p^{t(m+1)} m! L(f, \psi^{-1}\omega^{-m}, m+1)}{(-2\pi i)^m G(\psi^{-1}\omega^{-m}) \Omega_f^{sgn((-1)^m)}},$$

where ψ is a Dirichlet character of conductor p^t such that $\psi(1+p) = \zeta$, ω is the cyclotomic character modulo p properly normalized, $G(\psi^{-1}\omega^{-m})$ is a Gauss sum, $\Omega_f^{\pm}$ the canonical periods associated to f, and e_p an interpolation factor that involves $\chi(p)$, k, m and the roots of $X^2 + a(p,f)X + \chi(p)p^{k-1} = 0$.

The p-adic L-function $\mathcal{L}_f$ is one of the two main ingredients of the Main Conjecture. The other is the characteristic ideal of a (p-)Selmer group associated to f, which we now briefly describe.

It is well known that there exists a p-adic Galois representation $\rho_f : G := Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(V_f)$, where V_f a two dimensional L vector space, such that $L(\rho_f, s) = L(f,s)$. Moreover our assumption on f being ordinary implies the existence of a unique G_p stable unramified line $V_f^+ \subset V_f$, where here G_p is the decomposition group of G at p. In V_f we can find a G-stable O_L lattice T_f and we define $T_f^+ := T_f \cap V_f$ as well as the twists $T := T_f(det(\rho_f^{-1})$ and $T^+ := T_f^+(det(\rho_f^{-1})$ of these lattices by $det(\rho_f^{-1})$. Then the (p-)Selmer group of f over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ is defined as the subgroup

$$Sel_{\mathbb{Q}_\infty}(f) \subset ker\left(H^1(\mathbb{Q}_\infty, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(\mathbb{Q}_{\infty,p}, T/T^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)\right),$$

of classes unramified outside p. It was shown by Kato that its Pontrjagin dual, $X_{\mathbb{Q}_\infty}(f)$, is a finitely generated torsion Λ_{O_L} module and hence, thanks to the structure theorem of finitely generated torsion Λ_{O_L} modules, we can write $Ch_{\mathbb{Q}_\infty}(f) \subseteq \Lambda_{O_L}$ for its characteristic element. Then the Main Conjecture for elliptic cusp forms is,

Main Conjecture for f: There is an equality of ideals in $\Lambda_{O_L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$Ch_{\mathbb{Q}_\infty}(f) = (\mathcal{L}_f),$$

and furthermore, if ρ_f is residually irreducible then this is an equality already in Λ_{O_L}.

Kato has proven that $(\mathcal{L}_f) \subseteq Ch_{\mathbb{Q}_\infty}(f)$. Under some technical assumptions on f (for example of trivial Nebentypus) and more seriously assuming the existence of four dimensional p-adic Galois representations associated to certain cuspidal automorphic representations of the unitary group $U(2,2)$, Skinner and Urban succeeded in proving also the other inclusion, that is $Ch_{\mathbb{Q}_\infty}(f) \subseteq (\mathcal{L}_f)$, and hence concluding the Main Conjecture for a large family of elliptic cusp forms (including cases of elliptic curves).

The plan of the Seminar: The main goal of the seminar is to understand the proof of Skinner and Urban. We will mainly follow their preprint [5]. Their proof can be divided into two parts. In the first part it is explained how one can construct non trivial elements of the p-adic Selmer group $Sel_{\mathbb{Q}_\infty}(f)$ from congruences between Eisenstein series and cusp forms of $GU(2,2)$ of an appropriate quadratic imaginary field . The Eisenstein series is properly selected so that its constant term involves L-values of f. This is a generalization of a strategy, initiated by Ribet and extended by Mazur-Wiles and later by Wiles in his proof of the cyclotomic Iwasawa's Main Conjecture. In the second part of Skinner and Urban's work this Eisenstein series is constructed. In this seminar we will focus on the first part. However we remark that the week before the Iwasawa 2012 conference, which will take place here in Heidelberg, there is going to be a preparatory course given by Xin Wan, where this second part of the proof will be discussed.

Talks:

1. *Selmer groups for modular forms* [5, pages 16-30] (1 meeting): In this talk the basic definitions and properties of the Selmer groups that will play a role in this seminar should be given. In particular we will associate a (p-)Selmer group to a p-ordinary modular form. The aim is to cover sections §3.1, §3.2 and §3.3 inlcuding the statement of Kato's theorem 3.3.7.

2. *p-adic L-functions, the main theorems and applications to the arithmetic of elliptic curves* [5, pages 31-44] (1 meeting): This talk should start by recalling the definition of Hida families for elliptic modular forms and then explain the association of Selmer groups to these families. Then the interpolation properties of the various p-adic L-functions that show up in this seminar will be recalled. Then the Main Conjecture (Conjecture 3.5.3) should be explicitly stated as well as the precise formulation of the theorems of Skinner and Urban (Theorem 3.6.1, 3.6.5, 3.6.6) as well as their applications to the arithmetic of elliptic curves (Theorem 3.6.9, 3.6.11 and 3.6.14).

3. *Constructing cocycles* [5, pages 44-57] ($1\frac{1}{2}-2$ meetings): This is one of the most important parts of the seminar. Here, in a completely general way, Skinner and Urban describe a framework which gives rise to groups of extensions of representations and hence establish lower bounds for the size of Selmer groups. This talk should be given in full details and almost none of the proofs should be excluded. The proposition 4.5.6 will be a key proposition to prove the main theorem of Skinner and Urban.

4. *Shimura varieties for $GU(n,n)$ and the toroidal compactification* [5, pages 58-69]($1\frac{1}{2}$ meetings): In this talk a brief description of the theory of Shimura varieties associated to the unitary group $GU(n,n)$ should be given. The main goal of this talk is to discuss the theory of toroidal compactification. This will provide us the tools for stating the so-called q-expansion principle in an arithmetic setting and developing an arithmetic theory for the Siegel operator (to be introduced later). Of course since the theory of toroidal compactification could have been a seminar on its own, we will have to accept many parts of the theory as black boxes. In particular we will restrict ourselves to the special cases considered in the work of Skinner and Urban. (Additional references: For the theory of Shimura varieties over $\mathbb{C}$ see [4, section 4].)

5. *Automorphic forms and Hecke algebras for $GU(n,n)$* [5, pages 69-76]($1\frac{1}{2}$-2 meetings): In this talk the various ways of defining modular forms for the unitary group should be introduced. Classically, as functions on some hermitian symmetric space satisfying some modularity property and, especially for arithmetic applications, as sections of particular bundles. Then the theory of the Siegel operator should be discussed, especially in an arithmetic setting. Finally the Hecke algebra for the unitary group $U(n,n)$ as well as the associated L functions should be explained. (Additional references: For the theory of the Hecke algebras see also [3, section 11] and for the Euler factors [3, section 16]. For the complex theory of the Siegel operator see also [4, section 27])

6. *p-adic automorphic forms for $GU(n,n)$ and the control theorems* [5, pages 77-86](2 meetings): In this talk the theory of p-adic automorphic forms for the group $GU(n,n)$ should be discussed (section §6.1 of [5]). Then the ordinary operator of Hida should be introduced, §6.2 of [5]. Then the various control theorems should be proved (Corollary 6.2.6, Lemma 6.2.9 and Theorem 6.2.10) (Additional references: The book of Hida where almost the whole theory was inroduced [2] as well as the paper of Harris, Li and Skinner [1] for an introduction to the theory).

7. *Hida families for hermitian automorphic forms* [5, pages 85-90](1 meeting) In this talk Hida families for hermitian automorphic forms should be defined, and then their behavior with respect to the Siegel operator should be explained (Theorem 6.3.10). Finally, the ordinary Hecke algebra will be introduced and the notion of p-stabilization should be explained.

8. *The Eisenstein Ideal and Galois Representations, proof of the Main Theorem (modulo the existence of a proper Eisenstein series)* [5, pages 90-104] ($1\frac{1}{2}-2$ meetings): The notion of the Eisenstein ideal should be introduced and, what Skin-

ner and Urban call p-adic Eisenstein data. Then assuming the existence of the element $\mathbf{E_D}$ (its existence is the goal of the second part of their work mentioned above), in the notation of Skinner and Urban, the first fundamental inequality will be shown. That is, a product of two p-adic L-functions divides the Eisenstein ideal (Theorem 6.5.4). Then, using the theory of pseudo-representations, a conjectural existence of some Galois representations associated to cuspidal automorphic representations of $GU(2,2)$, and the general framework of constructing cocycles the second fundamental inequality will be proved. That is, the Eisenstein ideal divides (in some cases) the characteristic element of the p-Selmer group (Theorem 7.3.3.) Finally, in order to conclude the missing inclusion for the Main Conjecture (recall that there is a product of p-adic L-functions involved in the constant term of $\mathbf{E_D}$) the disjointness between the two p-adic L-functions should be explored.

Literatur

[1] M. Harris, J-S. Li and Ch. Skinner, *p-adic L-functions for unitary Shimura varieties I. Construction of the Eisenstein measure*, Doc. Math. (2006) no. Extra Vol., 393-464 (electronic),

[2] H. Hida, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, New Yrok 2004,

[3] G. Shimura, *Euler Products and Eisenstein Series*, CBMS Regional Conferences Series in Mathematics, No 93, Amer. Math. Soc. 1997

[4] G. Shimura, *Arithmeticity in the Theory of Automorphic forms*, Mathematical Surveys and Monographs, Volume 82, AMS, 2000,

[5] Ch. Skinner and E. Urban, *The Iwasawa Main Conjectures for GL_2.*, preprint, available on-line from www.math.columbia.edu/ urban/eurp/MC.pdf