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The Eigencurve

Oberseminar WS09

Ort: HS 4 **Zeit:** Donnerstag 11.00-13.00 Uhr **Beginn:** 15.10.2009

Short description: (See also the first few pages of [5] or [7].) Let p be a prime number and $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized eigenform for the congruence group $\Gamma_0(pN)$ with $(N,p) = 1$ and assume that its conductor is either N or Np. We define the slope of f to be the number $\sigma = ord_p(a_p)$. By the work of Hida it is known that if f is of weight $k \geq 2$ and of slope zero (usually called ordinary), then f is a member of a p-adic analytic family f_κ of p-adic modular eigenforms of slope 0 parameterized by their p-adic weights κ for κ ranging over a p-adic neighborhood of k in some properly defined p-adic weight space and such that $f_k = f$.

The theory of Hida has a geometric interpretation. Let us define $\Lambda_N := \mathbb{Z}_p[[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times]]$. Hida constructs a finite flat Λ_N-algebra, we call it $\mathbf{T}^\circ_{p,N}$, which is a universal deformation ring for two dimensional p-residual Galois representation associated to slope 0 eigenforms of tame level N. To this algebra one can associate a rigid analytic space, we denote it by $\mathbf{C}^\circ_{p,N}$ such that its $\mathbb{C}_p$ points parameterize normalized p-adic eigenforms of slope 0. If we write $\mathbf{W}_N$ for the rigid analytic space associated to Λ_N, the p-adic weight space, then the p-adic families f_κ mentioned above are obtained from the finite flat projection $\mathbf{C}^\circ_{p,N} \to \mathbf{W}_N$.

The main aim of this seminar is to understand how the above theory of Hida extends to eigenforms of finite slope. In particular we seek to construct a rigid analytic curve $\mathbf{C}_{p,N}$, the analogue of $\mathbf{C}^0_{p,N}$ in the ordinary case, which parameterizes all finite slope normalized (overconvergent) p-adic eigenforms of tame level N and to study its geometry, and in particular to understand its projection to the weight space $\mathbf{W}_N$ allowing us to construct families of overconvergent modular forms. Our main reference for this is the paper of Coleman and Mazur [5]. For technical reasons we will restrict ourselves to the cases of $p > 2$ and $N = 1$ (hopefully at the end of the seminar we will be able to understand which results can be extended to cover the general case, for this see [3]).

Talks:

1. The main aim of the first two talks is to introduce the needed theory from rigid analytic geometry that we need for the seminar. In this first talk the topics that should be covered are:
   (a) Tate algebras and their basic properties (ref. sections 1.2 and 1.3 of [1])
   (b) Affinoid algebras and affinoid spaces (ref. sections 1.4 and 1.5 of [1])
   (c) The canonical topology of affinoids and affinoids subdomains (ref. sections 1.6 of [1])

2. In this talk we continue with our introduction to rigid analytic geometry.
   (a) The presheaf of affinoid functions and Tate's acyclicity theorem (ref. sections 1.7 and 1.9 of [1])
   (b) The weak and strong Grothendieck topologies on affinoid spaces (ref. section 1.10 of [1]).
   (c) Rigid analytic varieties (ref. section 1.12 of [1])

3. The main aim of this talk is the construction of a natural functor $\mathcal{R} : A \mapsto X_A$ from the category of complete (semi-)local noetherian $\mathcal{O}_K$-algebras (here K a finite extension of $\mathbb{Q}_p$) to rigid analytic varieties such that $X_A(\mathbb{C}_p) = Hom_{ctn-\mathcal{O}_K}(A, \mathbb{C}_p)$ (ref. section 7 of [6] and also the second part of the notes of Bosch [1] is helpful).

4. The aim of this talk is the definition of overconvergent modular forms.
   (a) The modular curve and the definition of classical modular forms as sections of line bundles.
   (b) The affinoid subdomains $X_1(Np^n)(v)$ of the modular curve and the definition of overconvergent modular forms of integral weight (ref. pages 35-38 of [5] and part B of [4]).
   (c) The definition of overconvergent modular forms for p-adic weights with the help of the theory of Eisenstein series. (ref. pages 27-28 and 39-47 of [5] and also [3] section 6).

5. The main aim of this talk is to introduce the notion of pseudo-representations used in the theory of deformation of Galois representation (in this seminar we restrict ourselves to two dimensional representations). Further the notion of universal deformation rings will be introduced. We will apply this theory to p-modular representations. The rigid analytic space associated to

the corresponding universal deformation ring, through the above constructed functor $\mathcal{R}$, will be part of the ambient space of the eigencurve. We will define later in talk 8 the eigencurve as a Fredholm subvariety of this ambient space. (ref. Chapter 5 of [5]).

6. In this talk the theory of Fredholm determinats of operators on Banach spaces should be introduced. (ref. section A1 and A2 of [4] and pages 58-60 of [5]).

7. In this talk we extend the theory of Fredholm determinats defined above to systems of Banach spaces. Then we apply this theory to a system of Banach spaces obtained from overconvergent modular forms. The main result here is the analytic variation of the Fredholm determinats of Hecke operators belonging to the ideal of the Hecke algebra genereted by U_p (the $p-$Hecke operator). (ref. sections 4.3 and 4.4 of [5]).

8. In this talk the definition of the eigencurve will be given. It is a Fredholm subvariety of the "deformation" rigid variety constructed above. Its ideal of definition is defined using the theory of Fredholm determinants introduced in the previous talk. It will be shown that its $\mathbb{C}_p$-points are in one-one correspondence with the set of normalized overconvergent modular eigenforms (Ref. Chapter 6 of [5]).

9. In the next talks the aim is to construct, rather explicitly, another rigid analytic curve D and then prove that this is isomorphic with the reduced part of the eigencurve. This will allow us to derive many geometric properties of the eigencurve (as for example that the eigencurve is a curve!!). Also it is exactly this construction that has been in [3] generalized to include the case of $N \geq 1$ or(and) $p = 2$. In this talk for every element α in the Hecke algebra a rigid analytic curve D_α is constructed by gluing finite covers of the spectral curve Z_α. Then we set $D := D_1$ (ref. sections 7.1 and 7.2 of [5]).

10. In this talk the relationship between the various curves D_α is explored and various geometric properties of D are proved, in particular that D is reduced. (ref. section 7.3 and 7.4 in [5]).

11. Finally in this talk it is shown that the reduced part of the eigencurve is isomorphic to D. Various properties for the eigencurve are deduced from this isomorphism. (ref. section 7.5 and 7.6 of [5]).

References

[1] S.Bosch, *Lectures on formal and rigid geometry*, notes available at www.math.uni-muenster.de/sfb/about/publ/heft378.pdf.

[2] S.Bosch, U.Güntzer, R.Remmert, *Non-Archimedean Analysis*, A series of comperhensive studies in Mathematics, Springer, 1984.

[3] K. Buzzard, *Eigenvarieties*, In the proceedings of the LMS Durham conference on L functions and arithmetic, 2004.

[4] R.Coleman, *p-adic Banach spaces and families of modular forms*, Inv. Math. 127 (1997).

[5] R.Coleman and B.Mazur, *The Eigencurve*. In Galois representations in arithmetic algebraic geometry (Durham 1996). LMS vol 254 Cambridge University Press 1998.

[6] J. de Jong, *Crystalline Dieudonné module theory via formal and ridig geometry*, Inst. Hautes Etudes Sci. Publ. Math. No. 82 (1996).

[7] P. Kassaei, *The eigencurve: a brief survey*, available at www.uni-math.gwdg.de/tschinkel/SS05/school/kassaei.pdf.