Oberseminar

Modular Curves and the Eisenstein Ideal

Sommersemester 2010

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Ort: HS 4 **Zeit:** Donnerstag, 11 - 13 Uhr **Beginn:** 15.04.2010

Short Description

The goal of this seminar is to understand the proof of the following theorem on torsion points of elliptic curves defined over $\mathbb{Q}$ due to B. Mazur:

Theorem 1. *Let E be an elliptic curve over $\mathbb{Q}$ and suppose that the Mordell-Weil group $E(\mathbb{Q})$ contains a point of finite order N. Then either $1 \leq N \leq 10$ or $N = 12$. More precisely, the torsion subgroup of $E(\mathbb{Q})$ is isomorphic to one of the following 15 groups:*

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z}, &\quad \textit{for} \quad m \leq 10 \textit{ or } m = 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\nu\mathbb{Z}, &\quad \textit{for} \quad \nu \leq 4. \end{aligned}$$

Let $N \geq 1$ be an integer. The pairs (E, P), where E is an elliptic curve over $\mathbb{Q}$ and P is a point of order N of $E(\mathbb{Q})$ correspond to the rational points of the modular curve $Y_1(N)$. Thus the proof of Theorem 1 amounts to showing that for $N = 11$ or $N \geq 13$ the curve $Y_1(N)$ doesn't have any rational points, or equivalently, the only rational points of the compactified curve $X_1(N)$ are the rational cusps.

Our main subject is to study the action of the Hecke operators T_l on the Jacobian J of the modular curve $X_0(N)$. We consider the *Hecke algebra* $\mathbb{T}$, i.e. the subring of $\mathrm{End}(J)$ generated by the T_l and the involution ω and define the *Eisenstein ideal* $I \subset \mathbb{T}$ as the ideal generated by $1+\omega$ and $1+l-T_l$, l prime $\neq N$. The *Eisenstein quotient* of J is the quotient $\tilde{J} := J/\mathfrak{a}J$ where $\mathfrak{a}$ denotes the kernel of the natural map from T to the I-adic completion T_I. A crucial step in the proof will be to show that the following

Theorem 2. *The group $\tilde{J}(\mathbb{Q})$ is finite.*

This also implies that for primes $N \neq 2, 3, 5, 7, 13$ the modular curve $X_0(N)$ has only finitely many rational points. The condition on N is equivalent to the genus of $X_0(N)$ being greater than zero (otherwise $X_0(N) \cong \mathbb{P}^1$, in particular the curve would have infinitely many rational points).

After introducing modular curves and their fundamental properties, we are going to deal with parabolic modular forms and define the action of the Hecke algebra on the Jacobian J. By a subtle argument of Mazur, Theorem 2 follows from the fact that the model $J^0[\mathfrak{P}^m]_{/\,\mathrm{Spec}(\mathbb{Z})}$ is admissible if $\mathfrak{P}$ is a so-called *Eisenstein prime.* In order to show this, we will recall the most important facts from the theory of admissible p-group schemes.

The main reference of our seminar is Mazur's I.H.É.S. article [2]. In order to obtain a brief overview on the most important results we also recommend the preceding article by Mazur and Serre [3] (for the finiteness of $X_0(N)(\mathbb{Q})$; the exact statement of Theorem 1 is not treated here).

Talks

1. In this first talk of the seminar the arithmetic theory of the modular curve $X_0(N)$ should be introduced. That is, the concepts introduced in [2] Ch. II §1 should be discussed (i.e generalized elliptic curves, coarse moduli stack of generalized elliptic curves with extra structure and its minimal regular resolution, the Cohen-Macaulay property). Ref. [2] and [1]. Scheduled time: 1.5 meeting.
2. The goal of this talk is to introduce the arithmetic theory of parabolic (cuspidal) modular forms. Various definitions will be given (as regular differentials or as sections of particular sheaves over the moduli stack) and the relation between them will be studied (through the so-called Kodaira-Spencer morphism). Also the q-expansion principle should be explained. The references for this talk is [2] Ch II §4 and §5. Scheduled time: 2 meetings.
3. In this talk a particular Eisenstein series over $\mathbb{Z}$ (denoted by $\delta(q)$ in Mazur's paper) will be studied as well as its reduction modulo integers m. The main goal of this talk is to prove Proposition 5.12 in [2] Ch II §5 that provides a criterion when $\delta(q)$ is a parabolic modular form over $\mathbb{Z}/m\mathbb{Z}$. Ref. [2]. Scheduled time: 1.5 or 2 meetings.
4. The goal of this talk is to introduce the so-called Eisenstein ideal in the Hecke algebra and prove Proposition 9.7 in [2] Ch II §9. For this the theory of Hecke operators should be introduced (following Mazur's §6) and their operation on modular forms (with respect the various realizations of them). The references for this talk are §6 and §9 of Mazur. (In paragraph §6 without the operation of the Hecke algebra on the Jacobian of the modular curve). Scheduled time: 1 meeting.
5. In this talk we consider the operation of the Hecke algebra $\mathbb{T}$ on the Jacobian of the modular curve and understand the relation of the $Spec(\mathbb{T})$ of the Hecke algebra with the simple factors of the Jacobian. That is, in this talk the results of Ribet in [4] should be explained and the Eisenstein quotient of the Jacobian should be defined. References: [4] and Mazur's §10. Scheduled time: 1.5 meetings.
6. The goal of this talk is (i) to introduce (better say recall from Jakob's Vorlesung) the theory of finite commutative group schemes, (ii) to explain a theorem of Fontaine (see [2], Ch I Theorem 1.4) on associated Galois modules and (iii) introduce the theory of admissible p-groups. The goal is to go through §1 of Mazur's chapter I. The notes from Jakob's Vorlesung [5] could be helpful to prepare this talk. Scheduled time: 1.5 meetings.
7. In this talk we study the Galois action on the torsion points of the Jacobian of the modular curve. The main goal is to prove Prop. 14.1 in Mazur's Ch2 §14 on the admissibility of Eisenstein-torsion subgroups of the Jacobian. For these also the so-called Eichler-Shimura relations should be discussed (see for example Mazur's §6 in Chapter II. Scheduled time: 1.5 meetings
8. The main goal of this stalk is to show that the Mordell-Weil group $\tilde{J}(\mathbb{Q})$ of the Eisenstein part of the Jacobian is finite. This is done in Mazur's §3 of Chapter

III. The speaker should just follow the proof of theorem 3.1 there. Note that in our seminar we are not going to study the cuspidal group of the Jacobian that appears in the statement of this theorem. In addition as part of this talk theorem 4.1 on the finiteness of $X_0(N)(\mathbb{Q})$ in §4 should (could) be proved. Reference are Mazur's §3 in Chapter III as well as §8 of Chapter II. Scheduled time: 1.5 meetings.

9. In this talk we finally prove the main result of this seminar, that is we give the complete description of torsion in the Mordell-Weil group $E(\mathbb{Q})$ for elliptic curves defined over $\mathbb{Q}$. This is theorem 5.1 in Mazur's §5 ch. III. Remarks: This talk should be done!

References

[1] N. Katz and B. Mazur. *Arithmetic moduli of elliptic curves.* Princeton University Press, 1985.

[2] B. Mazur. Modular curves and the Eisenstein ideal. *Publ. Math. I.H.É.S.*, 47, 1978.

[3] B. Mazur and J.-P. Serre. Points rationnels des courbes modulaires $X_0(N)$. *Seminaire Bourbaki*, 469, 1974-1975.

[4] K. Ribet. Endomorphisms of semi-stable abelian varieties over number fields. *Annals of Math.*, 101, 1975.

[5] J. Stix. Notes from a course on finite flat group schemes and p-divisible groups.

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