Topological Invariants of Stratified Maps

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OUTLINE:

• The Beilinson-Bernstein-Deligne decomposition in algebraic geometry.

• Cobordism of self-dual sheaves.

• The Cappell-Shaneson decomposition on spaces with only even-codimensional strata.

• L-classes of self-dual sheaves.

• The reductive Borel-Serre compactification of Hilbert modular surfaces.

• Generalized Poincaré duality on spaces with strata of odd-codimension (non-Witt spaces).
• The L-class of a non-Witt space.

• A decomposition theorem on non-Witt spaces.

• Implications for desingularization.

• Example: A 4-dimensional pseudomanifold not resolvable by a stratified map.
THE DECOMPOSITION THEOREM IN ALGEBRAIC GEOMETRY.

• conjectured spring 1980 by S. Gelfand and R. MacPherson.

• proved fall 1980 by Gabber-Deligne and indep. Beilinson-Bernstein.

• Thm. If \( f : Y \rightarrow X \) is a proper algebraic map of algebraic varieties in arbitrary characteristic, then

\[
Rf_*IC^\bullet_m(Y) \cong \bigoplus_i j_*IC^\bullet_m(\overline{Z}_i; S^i_f)[n_i],
\]

\( Z_i \) is a nonsingular, irreducible, locally closed subvariety of \( X \), \( \overline{Z}_i \) its closure, \( S^i_f \) a locally constant sheaf of vector spaces over \( Z_i \), \( n_i \in \mathbb{Z} \).
• Let \( f : Y \rightarrow X \) be a resolution of singularities: \( Y \) is nonsingular, \( f \) is a surjective map which restricts to an isomorphism from a dense open subset of \( Y \) to the nonsingular stratum of \( X \). Then

\(- \quad \text{IC}_m^\bullet(Y) = \mathbb{R}Y, \)

\(- \quad \text{For some } i_0: \overline{Z}_{i_0} = X, n_{i_0} = 0, S_{i_0}^f = \mathbb{R}Z_{i_0}, \)

\(- \quad Rf_*\mathbb{R}Y \cong \text{IC}_m^\bullet(X) \oplus \bigoplus_{i \neq i_0} j_*\text{IC}_m^\bullet(\overline{Z}_i; S_f^i)[n_i], \)

\(- \quad \text{Upon applying hypercohomology, } H_k(Y) = IH^\bullet_k(X) \oplus \bigoplus IH^\bullet_{k-n_i}(\overline{Z}_i; S_f^i). \)

• **Cor.** If \( f : Y \rightarrow X \) is a resolution of singularities, then \( IH^\bullet_k(X) \) is a direct summand of \( H_k(Y) \).

• (Cor. conjectured by Kazhdan in 1979.)
TOPOLOGY:

- Geometric Category: $X^n = \text{closed, oriented pseudomanifold with DIFF-stratification } \{X_i\} \text{ (Whitney stratification).}$

- $X_i - X_{i-1}$ is a smooth $i$-manifold.

- P. Siegel: $X$ is a Witt space, if
  
  $$IH^m_k(\text{Link}(x); \mathbb{Q}) = 0,$$

  for all $x \in X_{n-2k-1} - X_{n-2k-2}$, all $k \geq 1$.

- $\bar{m} = (0, 0, 1, 1, 2, 2, 3, 3, \ldots)$,
  
  $\bar{n} = (0, 1, 1, 2, 2, 3, 3, 4, \ldots)$.

- $X \text{ Witt } \Rightarrow \mathbf{IC}^\bullet_{\bar{m}}(X) \longrightarrow \mathbf{IC}^\bullet_{\bar{n}}(X)$ is an isomorphism in the derived category.
• For any $X$, $\mathcal{D}\text{IC}_m^\bullet(X)[n] \cong \text{IC}_n^\bullet(X)$ (GM).

• $X$ Witt $\iff$ $\text{IC}_m^\bullet(X)$ is Verdier self-dual.

• Examples:

1. Complex algebraic varieties (spaces with only even-codim strata).

2. $\Sigma\mathbb{C}P^3$: $H^3(\mathbb{C}P^3) = 0$.

3. $X^4 = S^1 \times \Sigma T^2$ is not Witt.
For a locally trivial fiber-bundle $F \rightarrow E \rightarrow B$ of oriented manifolds,

$$\sigma(E) = \sigma(B)\sigma(F),$$

provided $\pi_1(B)$ acts trivially on $H^{mid}(F)$ (Chern-Hirzebruch-Serre).

Questions: How do invariants behave if

1. the involved spaces are singular?

2. the fiber of the map is allowed to change from point to point?

Answer for target spaces with only even-codimensional strata and stratified maps: Cappell-Shaneson.
ALGEBRAIC COBORDISM OF SHEAVES:
As in algebraic L-theory (e.g. Ranicki: Algebraic L-Theory and Top. Manifolds).
Given

1. \( Y^* \) self-dual: \( D Y^*[n] \cong Y^* \).

2. Morphisms \( X^* \xrightarrow{u} Y^* \xrightarrow{v} Z^* \), \( vu = 0 \).

3. An isomorphism \( D X^*[n] \cong Z^* \) such that

\[
\begin{array}{ccc}
Y^* & \xrightarrow{v} & Z^* \\
\downarrow \cong & & \downarrow \cong \\
DY^*[n] & \xrightarrow{Du[n]} & DX^*[n]
\end{array}
\]
commutes.

[Think of \( W, \partial W = M \sqcup -N \) and \( X^* = C_{*+1}(W, M), Y^* = C_* (M), Z^* = C_*(W, N) \).]
Let $C_v^\bullet$ be a mapping cone on $v$:

\[
\begin{array}{c}
Y^\bullet \\
\downarrow \quad [1] \\
C_v^\bullet \\
\downarrow u \\
X^\bullet \\
\end{array}
\quad \quad \begin{array}{c}
v \\
\downarrow u \\
Y^\bullet \\
\end{array}
\quad \begin{array}{c}
Z^\bullet \\
\end{array}
\]

As $vu = 0$, $u$ can be lifted:

Let $C_{u,v}^\bullet = C_{u'}^\bullet$.

$C_{u,v}^\bullet$ is self-dual.
• **Def.** $Y^\bullet$ is *elementary cobordant* to $C^\bullet_{u,v}$.

• Above analogy: $C^\bullet_{u,v} = C_\ast(N)$.

• **Def.** Two self-dual sheaves $X^\bullet$ and $Y^\bullet$ are *cobordant* if there exist self-dual sheaves $X^\bullet_0 = X^\bullet, X^\bullet_1, \ldots, X^\bullet_k = Y^\bullet$ such that there are elementary cobordisms from $X^\bullet_i$ to $X^\bullet_{i+1}$.

• $\Omega(X) =$ abelian group of algebraic cobordism classes of self-dual sheaves on $X$. 
• Set-up:

1. $X^n, Y^m$ Whitney stratified.

2. $X^n$ has only even-codim strata.

3. $m - n$ even.

4. $f : Y \to X$ a stratified map: $f$ proper, $f^{-1}(\text{open stratum}) = \bigcup \text{components of strata}$, $f|_{\text{comp}}$ is a smooth submersion.

5. $S^\bullet \in D^b_c(Y)$ self-dual (e.g. $S^\bullet = IC^\bullet_m(Y)$ if $Y$ is Witt).

• Thm. (Cappell-Shaneson, JAMS 4 1991)

In $\Omega(X)$,

$$[Rf_!S^\bullet[\frac{1}{2}(m - n)]] =$$

$$[IC^\bullet_m(X; s_f^{X - \Sigma})] \oplus \bigoplus_{Z \in X} [j_* IC^\bullet_m(Z; s_f^Z)[\frac{1}{2} \text{codim } Z]].$$
CHARACTERISTIC CLASSES – Thom-Pontrjagin construction.

• $X \subset M$, $M$ a smooth manifold, $S^\bullet \in D(X)$ self-dual.

• $f : X^n \to S^k$ continuous.

• $f \simeq g$ such that

  1. $g$ is the restriction of a smooth $G : M \to S^k$.

  2. North pole $N \in S^k$ is a regular value of $G$.

  3. $G^{-1}(N)$ is transverse to each stratum of $X$.

• $g^{-1}(N)$ is Whitney stratified.
• $j^!\mathcal{S}^\bullet$ is self-dual on $g^{-1}(N)$, $j : g^{-1}(N) \hookrightarrow X$.

• Have signature $\sigma(g^{-1}(N); j^!\mathcal{S}^\bullet) \in \mathbb{Z}$.

• $\sigma : \pi^k(X) \longrightarrow \mathbb{Z}$.

• Serre: Hurewicz: $\pi^k(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^k(X; \mathbb{Q})$ for $2k - 1 > n$.

• Define

$$L_k(S^\bullet) = \sigma \otimes \mathbb{Q} \in \text{Hom}(H^k(X), \mathbb{Q}) \cong H_k(X; \mathbb{Q}).$$
• In general, \( L(S^\bullet) \) is not in the image of 
\( \cap [X] : H^* (X) \to H_* (X) \).

• Example: If \( X \) is Witt, then \( L(X) = L(\text{IC}_{m}^\bullet (X)) \)
  is the Goresky-MacPherson L-class of \( X \).

• Algebraic cobordism of self-dual sheaves preserves L-classes.

• Cor. (CS) If \( Y \) is Witt, then
  \[
  f_* L_i (Y) = L_i (X; S^X_f - \Sigma) + \sum_{Z \in X} L_i (\overline{Z}; S^Z_f ).
  \]
Non-Witt Spaces in Nature:

- \( K \) = real quadratic number field.

- \( \mathcal{O}_K \) = ring of algebraic integers in \( K \).

- \( \Gamma = PSL_2(\mathcal{O}_K) \), the Hilbert modular group.

- \( Gal(K/\mathbb{Q}) = \{1, \sigma\} \).

- \( \Gamma \) acts on the product \( H \times H \) of two upper half planes by
  \[
  (z, w) \mapsto \left( \frac{az + b}{cz + d}, \frac{\sigma(a)w + \sigma(b)}{\sigma(c)w + \sigma(d)} \right).
  \]

- The \textit{Hilbert modular surface} of \( K \) is the orbit space \( X^4 = (H \times H)/\Gamma \).
• $X$ is non-compact, has finitely many singular points, “cusps”.

• Reductive Borel-Serre compactification $\overline{X}$: adjoin certain boundary circles to $X$,

$$\overline{X}_4 \supset \overline{X}_1 \supset \overline{X}_0 \supset \emptyset,$$

$$\overline{X}_1 - \overline{X}_0 = \text{disjoint union of the boundary circles}$$

$$\overline{X}_0 = \text{cusps in } X.$$

• $\overline{X}$ is not algebraic.

• Advantage: Hecke operators extend to $\overline{X}$.

• $\text{Link}(S^1) \cong T^2 \Rightarrow \overline{X}$ non-Witt.
INTERSECTION HOMOLOGY ON NON-WITT SPACES.

• \( SD(X) \subset D(X^n) \) full subcategory, objects \( S^\bullet \) satisfy:

\((SD1)\) Top stratum:
\[
S^\bullet|_{X-\Sigma} \cong H^{-n}(S^\bullet)[n]|_{X-\Sigma}
\]

\((SD2)\) Lower bound:
\[
H^i(S^\bullet) = 0, \text{ for } i < -n.
\]

\((SD3)\) Stalk condition for \( \bar{n} \):
\[
H^i(S^\bullet|_{X-X_{n-k-1}}) = 0,
\]
for \( i > \bar{n}(k) - n, k \geq 2. \)

\((SD4)\) Self-Duality: \( d : DS^\bullet[n] \xrightarrow{\cong} S^\bullet. \)

• \( SD(X) \) may or may not be empty.
There exist morphisms

\[ \text{IC}_m^\bullet(X; S) \xrightarrow{\alpha} S^\bullet \xrightarrow{\beta} \text{IC}_n^\bullet(X; S), \]

\( S = H^{-n}(S^\bullet)|_{X - \Sigma}: \)

\[ \begin{array}{ccc}
\text{IC}_m^\bullet(X; S) & \xrightarrow{\alpha} & S^\bullet \\
\Downarrow \cong & & \Downarrow \cong d \\
\mathcal{D}\text{IC}_n^\bullet(X; S)[n] & \xrightarrow{\mathcal{D}\beta[n]} & \mathcal{D}S^\bullet[n]
\end{array} \]

Example. 2-strata space \( X_n \supset X_{n-k} \supset \emptyset, \) \( k \) odd. Let \( S^\bullet \in SD(X). \) Define \( \emptyset \) by

\[ \begin{array}{ccc}
\text{IC}_m^\bullet(X) & \xrightarrow{} & \emptyset \\
& \searrow \downarrow \mathcal{H} & \\
& \text{IC}_n^\bullet(X)
\end{array} \]
• $\emptyset$ is concentrated in degree $\bar{n}(k) - n$.

• $\text{supp}(\emptyset) \subset X_{n-k}$.

• Octahedral Axiom:
Properties:

1. $\phi$ is injective on stalks.

2. $D\gamma[n + 2] = \gamma$.

Stalks: Let $x \in X_{n-k}$, $s = \bar{n}(k) - n$. Then

$$L^s_x \hookrightarrow H^s(\emptyset)_x = H^{\text{mid}}(\text{Link}(x))$$

is a Lagrangian subspace wrt. the intersection form of $\text{Link}(x)$.

**Def.** A *Lagrangian structure* is a morphism $\phi : \mathcal{L} \rightarrow \emptyset$ satisfying 1, 2 above.
• Structure Thm: Postnikov-type decomposition into categories of Lagrangian structures.

**Thm.** (B, Memoirs AMS 160 2002 no. 760) There is an equivalence of categories (say, for \( n \) even)

\[
SD(X) \simeq \text{Lag}(X_1 - X_0) \times \text{Lag}(X_3 - X_2) \times \ldots \\
\times \text{Lag}(X_{n-3} - X_{n-4}) \times \text{Coeff}(X - \Sigma).
\]

(Similarly for \( n \) odd.)
Examples:

- $X^6 = S^1 \times \Sigma \mathbb{C}P^2$: $SD(X^6) = \emptyset$.

- $X^4 = S^1 \times \Sigma T^2$: $SD(X^4) \neq \emptyset$.

- Primary obstruction: Signature of the link, Secondary obstruction: Monodromy.

- **Thm.** (B-Kulkarni, to appear Geom. Ded.) Let $\overline{X}$ be the reductive Borel-Serre compactification of a Hilbert modular surface $X$. Then $SD(\overline{X}) \neq \emptyset$. 


The L-Class of a Non-Witt Space:

- $\mathcal{S} = \text{self-dual local coefficient system on } X - \Sigma.$

- $SD(X; \mathcal{S}) \subset SD(X): \text{restriction to } X - \Sigma \text{ is } \mathcal{S}.$

- **Thm.** (B, to appear Annals of Math)
  If $SD(X; \mathcal{S}) \neq \emptyset$, then the L-classes
  \[ L_i(X; \mathcal{S}) = L_i(\text{IC}_L^\bullet(X; \mathcal{S})) \in H_i(X; \mathbb{Q}), \]
  $\text{IC}_L^\bullet(X; \mathcal{S}) \in SD(X; \mathcal{S})$, are independent of
  the choice of Lagrangian structure $\mathcal{L}$.

- In particular: a non-Witt space has a well-defined L-class $L(X)$, provided $SD(X; \mathbb{R}) \neq \emptyset.$
Parity-Separated Spaces:

- **Def.** A *parity-separation* on $X$ is a decomposition

$$X = oX \cup eX$$

into open subsets $oX, eX \subset X$ such that

- $oX$ has only strata of odd codimension.
- $eX$ has only strata of even codimension.

- **Example:** The reductive Borel-Serre compactification $\overline{X}$ possesses a parity-separation:

  $o\overline{X} = \overline{X_4 - X_0}$,
  $e\overline{X} =$ union of small open neighborhoods of the cusps in $X$. 
THE DECOMPOSITION THEOREM:

Thm. (B) Let $X^n$ be a stratified pseudomanifold with parity-separation $X = oX \cup eX$ and $S^\bullet \in D(X)$ a self-dual sheaf. Then

$$[S^\bullet] = [P^\bullet] \in \Omega(X)$$

with

$$P^\bullet|_{oX} \equiv IC_{L}(oX; oS) \in SD(oX)$$

$$P^\bullet|_{eX} \equiv IC_{m}(eX; eS) \oplus \bigoplus_{Z \in eX} j^* IC_{m}(\overline{Z}; s^Z)[\frac{1}{2} \text{codim } Z].$$

⇒ NEW PHENOMENON:

The strata of odd codimension never contribute any terms in the decomposition.
Characteristic Class Formulae:

**Cor.**

- $X^n, Y^m$ closed, oriented Whitney stratified pseudomanifolds,
- $m - n$ even,
- $SD(Y) \neq \emptyset$ (for instance $Y$ a Witt space),
- $f : Y \to X$ a stratified map,
- $X$ has only strata of odd codimension (except for the top stratum).

Then

$$f_* L(Y) = L(X; S^{X - \Sigma}).$$
• Define a new category $SP(X)$ of perverse sheaves on $X$ (possibly not Witt).

• Big enough so that every self-dual sheaf on $X$ is cobordant to an object of $SP(X)$.

• Small enough so that on parity-separated spaces every self-dual object of $SP(X)$ is given by Lagrangian structures.

• Question: What is the set of perversities $\tilde{r}$ such that every object of $SD(X)$ is $\tilde{r}$-perverse?
Answer:

1. For $S \in \mathcal{X}$ with $k = \text{codim } S$ even: three values,
   \[ \bar{r}(S) \in \{\bar{n}(k) - n, \bar{n}(k) - n + 1, \bar{n}(k) - n + 2\}. \]

2. For $S \in \mathcal{X}$ with $k = \text{codim } S$ odd: two values,
   \[ \bar{r}(S) \in \{\bar{n}(k) - n, \bar{n}(k) - n + 1\}. \]

If $S \in \mathcal{X}$ is not in the top stratum, set
\[
\bar{p}(S) = \begin{cases} 
\bar{n}(\text{codim } S) - n + 1, & \text{codim } S \text{ even} \\
\bar{n}(\text{codim } S) - n, & \text{codim } S \text{ odd}
\end{cases},
\]
\[
\bar{q}(S) = \bar{n}(\text{codim } S) - n + 1
\]
and set $\bar{p}(S) = \bar{q}(S) = -n$ for $S$ a component of the top stratum.
• $\bar{p}$ and $\bar{q}$ are dual.

• $\bar{p} D^{\leq 0}(X) = \{ \mathbf{X}^\bullet \in D(X) \mid H^k(i^*_S \mathbf{X}^\bullet) = 0, \ k > \bar{p}(S), \ \text{all } S \}.$

• $\bar{q} D^{\geq 0}(X) = \{ \mathbf{X}^\bullet \in D(X) \mid H^k(i^!_S \mathbf{X}^\bullet) = 0, \ k < \bar{q}(S), \ \text{all } S \}.$

• Def. $SP(X) = \bar{p} D^{\leq 0}(X) \cap \bar{q} D^{\geq 0}(X) \subset D(X)$.

• If $X$ is a space with only even-codimensional strata, for example a complex algebraic variety, then $\bar{p} = \bar{q}$ and $SP(X) = \bar{p} P(X) = \bar{q} P(X)$ is the usual category of perverse sheaves.
Implications for Desingularization:

**Def.** A closed, oriented, Whitney stratified pseudomanifold $X^n$ is *resolvable by a stratified map* if there exists a closed, oriented, smooth manifold $M^n$ which can be equipped with a Whitney stratification so that there exists a stratified map $f : M \to X$ whose restriction to the top stratum $f^{-1}(X - \Sigma)$ is an orientation preserving diffeomorphism.

**Prop.** (B) If $X^n$ has a parity-separation, and there exists no Lagrangian structure (for constant real coefficients) at the strata of odd codimension, then $X$ is not resolvable by a stratified map.
AN ILLUSTRATIVE EXAMPLE.

\[ T^2 \to M(A)^3 \]

- \( A \in SL_2(\mathbb{Z}) \)

\[ \sim \]

- \( \text{cyl}(p) \) is a non-Witt 4-dim pseudomanifold, \( \partial \text{cyl}(p) = M(A) \), singular stratum = \( S^1 \).

- For \( x \in S^1 \), \( \exists \) Lagrangian \( L \subset H^1(\text{Link}(x)) = H^1(T^2) \).

- \( \exists \pi_1(S^1) \)-invariant Lagrangian subspace \( \iff A \) has a real eigenvalue.

- Take e.g. \( A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \) (elliptic, with complex eigenvalues).
• Choose an oriented 4-manifold $W$ with $\partial W = M(A)$.

• Set $X^4 = W \cup_{M(A)\text{cyl}(p)} M(A)$, a closed 4-pseudo-manifold, parity separated.

• Thm $\Rightarrow X^4$ is not resolvable by a stratified map.
COMPARISON TO M. KATO’S THEORY.
(Topology 12 1973)

• **Def. 1** A *PL-variety* \((P, \Sigma)\) is an \(n\)-polyhedron \(P\) with singular set \(\Sigma \subset P\) such that \(P - \Sigma\) is a PL \(n\)-manifold without boundary.

• **Def. 2** A *PL blow-up* between oriented PL \(n\)-varieties \((P', \Sigma')\) and \((P, \Sigma)\) is a proper PL map of pairs \(f : (P', \Sigma') \to (P, \Sigma)\) such that for some derived neighborhoods \(N'\) of \(f^{-1}\Sigma\) in \(P'\) and \(N\) of \(\Sigma\) in \(P\), the restriction \(f| : P' - \text{int}(N') \to P - \text{int}(N)\) is an orientation preserving homeomorphism. If \(P'\) is an oriented PL \(n\)-manifold, such a blow-up is called a *PL resolution*. 
Given \((P, \Sigma)\),
\[ N := \text{regular nbhd of } \Sigma \text{ in } P. \]

- \(\partial N\) is an \((n - 1)\)-manifold.

- \(p : \partial N \to \Sigma\) the restriction of the normal projection \(N \to \Sigma\).

- \([p] \in \Omega_{n-1}(\Sigma)\).

\textbf{Thm.} (Kato) \((P, \Sigma)\) admits a PL resolution \(\Leftrightarrow [p] = 0\).
• For our $X^4$, have

$$[M(A) \xrightarrow{p} S^1] \in \Omega_3(S^1).$$

• $H_*(S^1; \mathbb{Z})$ f.g., no odd torsion $\Rightarrow$ bordism spectral sequence collapses:

$$\Omega_3(S^1) \cong H_0(S^1; \Omega_3(pt)) \oplus H_1(S^1; \Omega_2(pt)) = 0.$$

• Conclusion: $X^4$ admits a PL resolution, but not a resolution by a stratified map.