Topological Invariants of Stratified Maps

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OUTLINE:

- The Beilinson-Bernstein-Deligne decomposition in algebraic geometry.
- Cobordism of self-dual sheaves.
- The Cappell-Shaneson decomposition on spaces with only even-codimensional strata.
- L-classes of self-dual sheaves.
- The reductive Borel-Serre compactification of Hilbert modular surfaces.
- Generalized Poincaré duality on spaces with strata of odd-codimension (non-Witt spaces).

- The L-class of a non-Witt space.
- A decomposition theorem on non-Witt spaces.
- Implications for desingularization.
- Example: A 4-dimensional pseudomanifold not resolvable by a stratified map.

THE DECOMPOSITION THEOREM IN ALGEBRAIC GEOMETRY.

- conjectured spring 1980 by S. Gelfand and R. MacPherson.

- proved fall 1980 by Gabber-Deligne and indep. Beilinson-Bernstein.

- **Thm.** If $f : Y \longrightarrow X$ is a proper algebraic map of algebraic varieties in arbitrary characteristic, then

$$Rf_* \mathbf{IC}^{\bullet}_{\bar{m}}(Y) \cong \bigoplus_i j_* \mathbf{IC}^{\bullet}_{\bar{m}}(\overline{Z_i}; \mathcal{S}^i_f)[n_i],$$

  Z_i is a nonsingular, irreducible, locally closed subvariety of X, $\overline{Z_i}$ its closure, $\mathcal{S}^i_f$ a locally constant sheaf of vector spaces over Z_i, $n_i \in \mathbb{Z}$.

- Let $f : Y \longrightarrow X$ be a resolution of singularities: Y is nonsingular, f is a surjective map which restricts to an isomorphism from a dense open subset of Y to the nonsingular stratum of X. Then

  – $\mathbf{IC}^{\bullet}_{\bar{m}}(Y) = \mathbb{R}_Y$,

  – For some i_0: $\overline{Z_{i_0}} = X$, $n_{i_0} = 0$, $\mathcal{S}^{i_0}_f = \mathbb{R}_{Z_{i_0}}$,

  – $Rf_*\mathbb{R}_Y \cong \mathbf{IC}^{\bullet}_{\bar{m}}(X) \oplus \bigoplus_{i \neq i_0} j_* \mathbf{IC}^{\bullet}_{\bar{m}}(\overline{Z_i}; \mathcal{S}^i_f)[n_i]$,

  – Upon applying hypercohomology,

$$H_k(Y) = IH^{\bar{m}}_k(X) \oplus \bigoplus IH^{\bar{m}}_{k-n_i}(\overline{Z_i}; \mathcal{S}^i_f).$$

- **Cor.** If $f : Y \longrightarrow X$ is a resolution of singularities, then $IH^{\bar{m}}_k(X)$ is a direct summand of $H_k(Y)$.

- (Cor. conjectured by Kazhdan in 1979.)

TOPOLOGY:

- Geometric Category: X^n = closed, oriented pseudomanifold with DIFF-stratification $\{X_i\}$ (Whitney stratification).

- $X_i - X_{i-1}$ is a smooth i-manifold.

- P. Siegel: X is a *Witt space*, if

$$IH_k^{\bar{m}}(Link(x);\mathbb{Q}) = 0,$$

for all $x \in X_{n-2k-1} - X_{n-2k-2}$, all $k \geq 1$.

- $\bar{m} = (0,0,1,1,2,2,3,3,\ldots)$,
  $\bar{n} = (0,1,1,2,2,3,3,4,\ldots)$.

- X Witt $\Rightarrow \mathbf{IC}^\bullet_{\bar{m}}(X) \longrightarrow \mathbf{IC}^\bullet_{\bar{n}}(X)$ is an isomorphism in the derived category.

- For any X, $\mathcal{D}\mathbf{IC}^{\bullet}_{\bar{m}}(X)[n] \cong \mathbf{IC}^{\bullet}_{\bar{n}}(X)$ (GM).

- X Witt $\Leftrightarrow$ $\mathbf{IC}^{\bullet}_{\bar{m}}(X)$ is Verdier self-dual.

- Examples:

  1. Complex algebraic varieties (spaces with only even-codim strata).

  2. $\Sigma\mathbb{C}P^3$: $H^3(\mathbb{C}P^3) = 0$.

  3. $X^4 = S^1 \times \Sigma T^2$ is not Witt.

- For a locally trivial fiber-bundle $F \to E \to B$ of oriented manifolds,

$$\sigma(E) = \sigma(B)\sigma(F),$$

provided $\pi_1(B)$ acts trivially on $H^{mid}(F)$ (Chern-Hirzebruch-Serre).

- Questions: How do invariants behave if

  1. the involved spaces are singular?

  2. the fiber of the map is allowed to change from point to point?

- Answer for target spaces with only even-codimensional strata and stratified maps: Cappell-Shaneson.

ALGEBRAIC COBORDISM OF SHEAVES:
As in algebraic L-theory (e.g. Ranicki: Algebraic L-Theory and Top. Manifolds).
Given

1. $\mathbf{Y}^\bullet$ self-dual: $\mathcal{D}\mathbf{Y}^\bullet[n] \cong \mathbf{Y}^\bullet$.

2. Morphisms $\mathbf{X}^\bullet \xrightarrow{u} \mathbf{Y}^\bullet \xrightarrow{v} \mathbf{Z}^\bullet$, $vu = 0$.

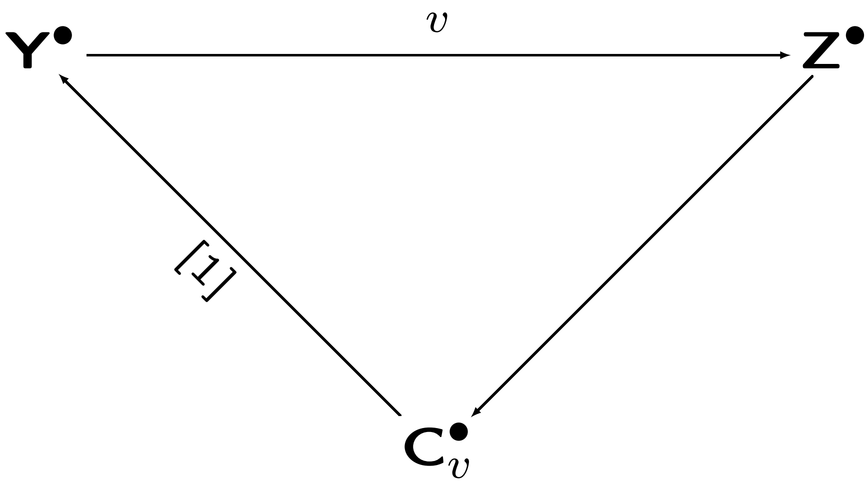
3. An isomorphism $\mathcal{D}\mathbf{X}^\bullet[n] \cong \mathbf{Z}^\bullet$ such that

$$\begin{array}{ccc} \mathbf{Y}^\bullet & \xrightarrow{\quad v \quad} & \mathbf{Z}^\bullet \\ \uparrow \cong & & \uparrow \cong \\ \mathcal{D}\mathbf{Y}^\bullet[n] & \xrightarrow{\mathcal{D}u[n]} & \mathcal{D}\mathbf{X}^\bullet[n] \end{array}$$

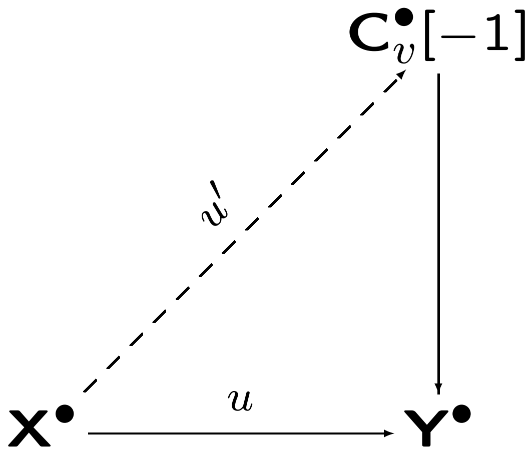
   commutes.

[Think of W, $\partial W = M \sqcup -N$ and $\mathbf{X}^\bullet = C_{*+1}(W, M)$, $\mathbf{Y}^\bullet = C_*(M)$, $\mathbf{Z}^\bullet = C_*(W, N)$.]

Let $\mathbf{C}_v^\bullet$ be a mapping cone on v:



As $vu = 0$, u can be lifted:



Let

$$\mathbf{C}_{u,v}^\bullet = \mathbf{C}_{u'}^\bullet.$$

$\mathbf{C}_{u,v}^\bullet$ is self-dual.

- **Def.** $\mathbf{Y}^\bullet$ is *elementary cobordant* to $\mathbf{C}^\bullet_{u,v}$.

- Above analogy: $\mathbf{C}^\bullet_{u,v} = C_*(N)$.

- **Def.** Two self-dual sheaves $\mathbf{X}^\bullet$ and $\mathbf{Y}^\bullet$ are *cobordant* if there exist self-dual sheaves $\mathbf{X}^\bullet_0 = \mathbf{X}^\bullet, \mathbf{X}^\bullet_1, \ldots, \mathbf{X}^\bullet_k = \mathbf{Y}^\bullet$ such that there are elementary cobordisms from $\mathbf{X}^\bullet_i$ to $\mathbf{X}^\bullet_{i+1}$.

- $\Omega(X)$ = abelian group of algebraic cobordism classes of self-dual sheaves on X.

- Set-up:

  1. X^n, Y^m Whitney stratified.

  2. X^n has only even-codim strata.

  3. $m - n$ even.

  4. $f : Y \longrightarrow X$ a *stratified map*: f proper, $f^{-1}(\text{open stratum}) =$ $\cup$components of strata, $f|_{\text{comp}}$ is a smooth submersion.

  5. $\mathbf{S}^\bullet \in D^b_c(Y)$ self-dual (e.g. $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(Y)$ if Y is Witt).

- **Thm.**(Cappell-Shaneson, JAMS **4** 1991) In $\Omega(X)$,

$$[Rf_*\mathbf{S}^\bullet[\tfrac{1}{2}(m-n)]] =$$

$$[\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{S}_f^{X-\Sigma})] \oplus \bigoplus_{Z \in \mathcal{X}} [j_* \mathbf{IC}^\bullet_{\bar{m}}(\overline{Z}; \mathcal{S}_f^Z)[\tfrac{1}{2}\,\text{codim}\, Z]].$$

CHARACTERISTIC CLASSES – Thom-Pontrjagin construction.

- $X \subset M$, M a smooth manifold, $\mathbf{S}^\bullet \in D(X)$ self-dual.

- $f : X^n \longrightarrow S^k$ continuous.

- $f \simeq g$ such that

  1. g is the restriction of a smooth $G : M \longrightarrow S^k$.

  2. North pole $N \in S^k$ is a regular value of G.

  3. $G^{-1}(N)$ is transverse to each stratum of X.

- $g^{-1}(N)$ is Whitney stratified.

- $j^!\mathbf{S}^\bullet$ is self-dual on $g^{-1}(N)$, $j : g^{-1}(N) \hookrightarrow X$.

- Have signature $\sigma(g^{-1}(N); j^!\mathbf{S}^\bullet) \in \mathbb{Z}$.

- $\sigma : \pi^k(X) \longrightarrow \mathbb{Z}$.

- Serre: Hurewicz:$\pi^k(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^k(X; \mathbb{Q})$ for $2k - 1 > n$.

- Define

  $L_k(\mathbf{S}^\bullet) = \sigma \otimes \mathbb{Q} \in \mathrm{Hom}(H^k(X), \mathbb{Q}) \cong H_k(X; \mathbb{Q})$.

- In general, $L(\mathbf{S}^\bullet)$ is not in the image of $\cap[X] : H^*(X) \to H_*(X)$.

- Example: If X is Witt, then $L(X) = L(\mathbf{IC}^\bullet_{\bar{m}}(X))$ is the Goresky-MacPherson L-class of X.

- Algebraic cobordism of self-dual sheaves preserves L-classes.

- **Cor.** (CS) If Y is Witt, then

$$f_* L_i(Y) = L_i(X; \mathcal{S}_f^{X-\Sigma}) + \sum_{Z \in \mathcal{X}} L_i(\overline{Z}; \mathcal{S}_f^Z).$$

Non-Witt Spaces in Nature:

- $K =$ real quadratic number field.

- $\mathcal{O}_K =$ ring of algebraic integers in K.

- $\Gamma = PSL_2(\mathcal{O}_K)$, the Hilbert modular group.

- $Gal(K/\mathbb{Q}) = \{1, \sigma\}$.

- Γ acts on the product $H \times H$ of two upper half planes by

$$(z, w) \mapsto (\frac{az+b}{cz+d}, \frac{\sigma(a)w + \sigma(b)}{\sigma(c)w + \sigma(d)}).$$

- The *Hilbert modular surface* of K is the orbit space $X^4 = (H \times H)/\Gamma$.

- X is non-compact, has finitely many singular points, "cusps".

- Reductive Borel-Serre compactification $\overline{X}$: adjoin certain boundary circles to X,

$$\overline{X}_4 \supset \overline{X}_1 \supset \overline{X}_0 \supset \varnothing,$$

$\overline{X}_1 - \overline{X}_0 =$ disjoint union of the boundary circles
$\overline{X}_0 =$ cusps in X.

- $\overline{X}$ is not algebraic.

- Advantage: Hecke operators extend to $\overline{X}$.

- $Link(S^1) \cong T^2 \Rightarrow \overline{X}$ non-Witt.

INTERSECTION HOMOLOGY ON NON-WITT SPACES.

- $SD(X) \subset D(X^n)$ full subcategory, objects $\mathbf{S}^\bullet$ satisfy:

  (SD1) Top stratum:
  $\mathbf{S}^\bullet|_{X-\Sigma} \cong \mathbf{H}^{-n}(\mathbf{S}^\bullet)[n]|_{X-\Sigma}$

  (SD2) Lower bound:
  $\mathbf{H}^i(\mathbf{S}^\bullet) = 0$, for $i < -n$.

  (SD3) Stalk condition for $\bar{n}$:
  $\mathbf{H}^i(\mathbf{S}^\bullet|_{X-X_{n-k-1}}) = 0$,
  for $i > \bar{n}(k) - n, k \geq 2$.

  (SD4) Self-Duality: $d : \mathcal{D}\mathbf{S}^\bullet[n] \xrightarrow{\cong} \mathbf{S}^\bullet$.

- $SD(X)$ may or may not be empty.

There exist morphisms

$$\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathcal{S}) \xrightarrow{\alpha} \mathbf{S}^{\bullet} \xrightarrow{\beta} \mathbf{IC}^{\bullet}_{\bar{n}}(X;\mathcal{S}),$$

$\mathcal{S} = \mathbf{H}^{-n}(\mathbf{S}^{\bullet})|_{X-\Sigma}$:

$$\begin{array}{ccc}
\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathcal{S}) & \xrightarrow{\alpha} & \mathbf{S}^{\bullet} \\
\cong \uparrow & & \cong \uparrow d \\
\mathcal{D}\mathbf{IC}^{\bullet}_{\bar{n}}(X;\mathcal{S})[n] & \xrightarrow{\mathcal{D}\beta[n]} & \mathcal{D}\mathbf{S}^{\bullet}[n]
\end{array}$$

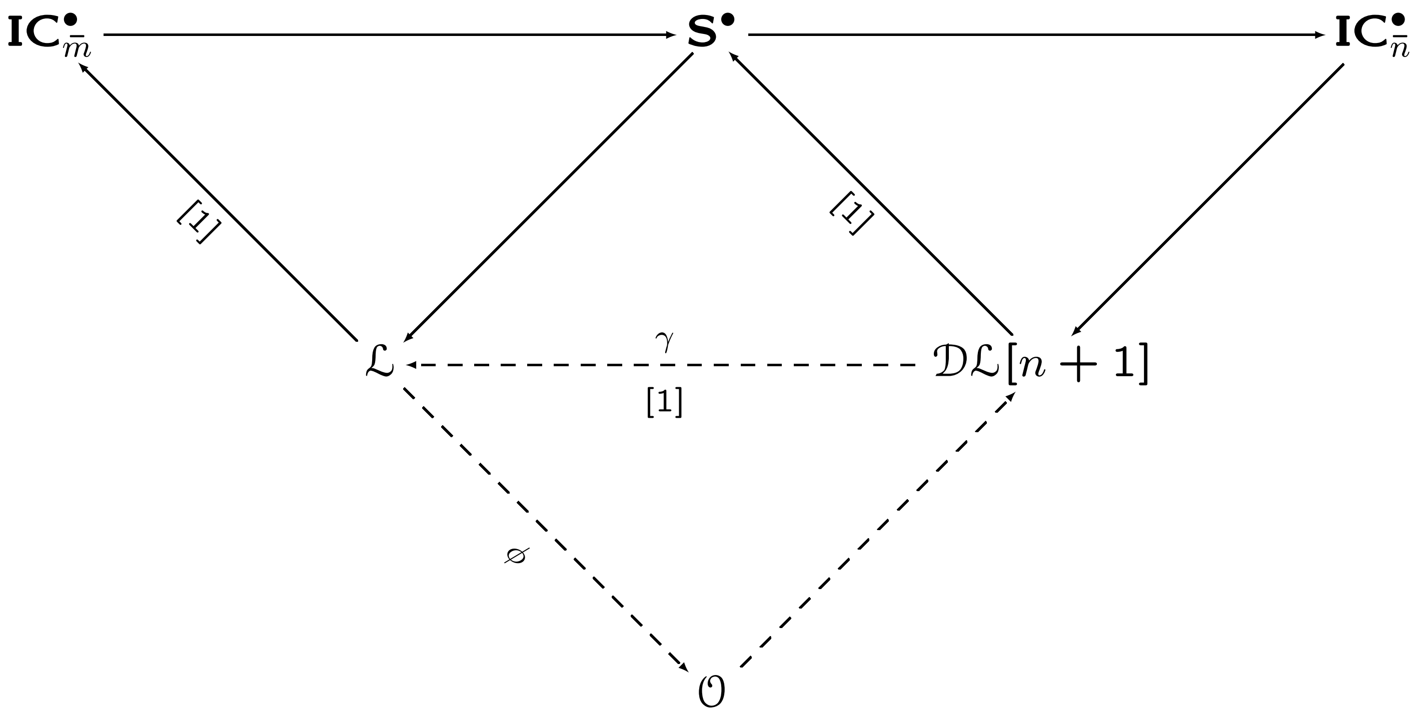
Example. 2-strata space $X_n \supset X_{n-k} \supset \varnothing$, k odd. Let $\mathbf{S}^{\bullet} \in SD(X)$. Define $\mathcal{O}$ by

$$\begin{array}{ccc}
\mathbf{IC}^{\bullet}_{\bar{m}}(X) & \longrightarrow & \mathbf{IC}^{\bullet}_{\bar{n}}(X) \\
{}_{[1]}\nwarrow & & \swarrow \\
& \mathcal{O} &
\end{array}$$

- $\mathcal{O}$ is concentrated in degree $\bar{n}(k) - n$.

- $supp(\mathcal{O}) \subset X_{n-k}$.

- Octahedral Axiom:



Properties:

1. ϕ is injective on stalks.

2. $\mathcal{D}\gamma[n+2] = \gamma$.

Stalks: Let $x \in X_{n-k}$, $s = \bar{n}(k) - n$. Then

$$\mathcal{L}_x^s \hookrightarrow \mathbf{H}^s(\mathcal{O})_x = H^{\text{mid}}(\text{Link}(x))$$

is a Lagrangian subspace wrt. the intersection form of Link(x).

Def. A *Lagrangian structure* is a morphism $\phi : \mathcal{L} \to \mathcal{O}$ satisfying 1, 2 above.

- Structure Thm: Postnikov-type decomposition into categories of Lagrangian structures.

Thm.(B, Memoirs AMS **160** 2002 no. 760) There is an equivalence of categories (say, for n even)

$$SD(X) \simeq \mathsf{Lag}(X_1 - X_0) \rtimes \mathsf{Lag}(X_3 - X_2) \rtimes \dots$$

$$\rtimes \mathsf{Lag}(X_{n-3} - X_{n-4}) \rtimes \mathsf{Coeff}(X - \Sigma).$$

(Similarly for n odd.)

Examples:

- $X^6 = S^1 \times \Sigma \mathbb{C}P^2$: $SD(X^6) = \varnothing$.

- $X^4 = S^1 \times \Sigma T^2$: $SD(X^4) \neq \varnothing$.

- Primary obstruction: Signature of the link, Secondary obstruction: Monodromy.

- **Thm.**(B-Kulkarni, to appear Geom. Ded.) Let $\overline{X}$ be the reductive Borel-Serre compactification of a Hilbert modular surface X. Then $SD(\overline{X}) \neq \varnothing$.

The L-Class of a Non-Witt Space:

- $\mathcal{S}$ = self-dual local coefficient system on $X - \Sigma$.

- $SD(X; \mathcal{S}) \subset SD(X)$: restriction to $X - \Sigma$ is $\mathcal{S}$.

- **Thm.**(B, to appear Annals of Math)
  If $SD(X; \mathcal{S}) \neq \varnothing$, then the L-classes

  $$L_i(X; \mathcal{S}) = L_i(\mathbf{IC}^\bullet_{\mathcal{L}}(X; \mathcal{S})) \in H_i(X; \mathbb{Q}),$$

  $\mathbf{IC}^\bullet_{\mathcal{L}}(X; \mathcal{S}) \in SD(X; \mathcal{S})$, are *independent* of the choice of Lagrangian structure $\mathcal{L}$.

- In particular: a non-Witt space has a well-defined L-class $L(X)$, provided $SD(X; \mathbb{R}) \neq \varnothing$.

Parity-Separated Spaces:

- Def. A *parity-separation* on X is a decomposition

$$X = {}^{o}X \cup {}^{e}X$$

into open subsets ${}^{o}X, {}^{e}X \subset X$ such that

  - ${}^{o}X$ has only strata of odd codimension.
  - ${}^{e}X$ has only strata of even codimension.

- Example: The reductive Borel-Serre compactification $\overline{X}$ possesses a parity-separation:
${}^{o}\overline{X} = \overline{X}_4 - \overline{X}_0$,
${}^{e}\overline{X}$ = union of small open neighborhoods of the cusps in X.

THE DECOMPOSITION THEOREM:

Thm.(B) Let X^n be a stratified pseudomanifold with parity-separation $X = {}^oX \cup {}^eX$ and $\mathbf{S}^\bullet \in D(X)$ a self-dual sheaf. Then

$$[\mathbf{S}^\bullet] = [\mathbf{P}^\bullet] \in \Omega(X)$$

with

$$\begin{aligned} \mathbf{P}^\bullet|_{{}^oX} &\cong \mathbf{IC}^\bullet_{\mathcal{L}}({}^oX; {}^o\mathcal{S}) \in SD({}^oX) \\ \mathbf{P}^\bullet|_{{}^eX} &\cong \mathbf{IC}^\bullet_{\bar{m}}({}^eX; {}^e\mathcal{S}) \oplus \\ &\quad \textstyle\bigoplus_{Z \in {}^e\mathcal{X}} j_* \mathbf{IC}^\bullet_{\bar{m}}(\overline{Z}; \mathcal{S}^Z)[\frac{1}{2}\,\mathrm{codim}\, Z]. \end{aligned}$$

$\Rightarrow$ NEW PHENOMENON:
The strata of odd codimension *never contribute any terms* in the decomposition.

Characteristic Class Formulae:

Cor.

- X^n, Y^m closed, oriented Whitney stratified pseudomanifolds,
- $m - n$ even,
- $SD(Y) \neq \varnothing$ (for instance Y a Witt space),
- $f : Y \to X$ a stratified map,
- X has only strata of odd codimension (except for the top stratum).

Then

$$f_* L(Y) = L(X; \mathcal{S}^{X-\Sigma}).$$

- Define a new category $SP(X)$ of perverse sheaves on X (possibly not Witt).

- Big enough so that every self-dual sheaf on X is cobordant to an object of $SP(X)$.

- Small enough so that on parity-separated spaces every self-dual object of $SP(X)$ is given by Lagrangian structures.

- Question: What is the set of perversities $\bar{r}$ such that every object of $SD(X)$ is $\bar{r}$-perverse?

Answer:

1. For $S \in \mathcal{X}$ with $k = \operatorname{codim} S$ even: three values,

   $\bar{r}(S) \in \{\bar{n}(k)-n,\ \bar{n}(k)-n+1,\ \bar{n}(k)-n+2\}.$

2. For $S \in \mathcal{X}$ with $k = \operatorname{codim} S$ odd: two values,

$$\bar{r}(S) \in \{\bar{n}(k) - n,\ \bar{n}(k) - n + 1\}.$$

If $S \in \mathcal{X}$ is not in the top stratum, set

$$\begin{aligned} \bar{p}(S) &= \begin{cases} \bar{n}(\operatorname{codim} S) - n + 1, & \operatorname{codim} S \text{ even} \\ \bar{n}(\operatorname{codim} S) - n, & \operatorname{codim} S \text{ odd} \end{cases}, \\ \bar{q}(S) &= \bar{n}(\operatorname{codim} S) - n + 1 \end{aligned}$$

and set $\bar{p}(S) = \bar{q}(S) = -n$ for S a component of the top stratum.

- $\bar{p}$ and $\bar{q}$ are dual.

- ${}^{\bar{p}}D^{\leq 0}(X) = \{\mathbf{X}^\bullet \in D(X) \mid \mathbf{H}^k(i_S^*\mathbf{X}^\bullet) = 0,$ $k > \bar{p}(S),\ \text{all } S\}.$

- ${}^{\bar{q}}D^{\geq 0}(X) = \{\mathbf{X}^\bullet \in D(X) \mid \mathbf{H}^k(i_S^!\mathbf{X}^\bullet) = 0,$ $k < \bar{q}(S),\ \text{all } S\}.$

- Def. $SP(X) = {}^{\bar{p}}D^{\leq 0}(X) \cap {}^{\bar{q}}D^{\geq 0}(X) \subset D(X).$

- If X is a space with only even-codimensional strata, for example a complex algebraic variety, then $\bar{p} = \bar{q}$ and $SP(X) = {}^{\bar{p}}P(X) = {}^{\bar{q}}P(X)$ is the usual category of perverse sheaves.

Implications for Desingularization:

Def. A closed, oriented, Whitney stratified pseudomanifold X^n is *resolvable by a stratified map* if there exists a closed, oriented, smooth manifold M^n which can be equipped with a Whitney stratification so that there exists a stratified map $f : M \to X$ whose restriction to the top stratum $f^{-1}(X - \Sigma)$ is an orientation preserving diffeomorphism.

Prop.(B) If X^n has a parity-separation, and there exists no Lagrangian structure (for constant real coefficients) at the strata of odd codimension, then X is not resolvable by a stratified map.

AN ILLUSTRATIVE EXAMPLE.

- $A \in SL_2(\mathbb{Z}) \quad \rightsquigarrow \quad \begin{array}{ccc} T^2 & \to & M(A)^3 \\ & & \downarrow^p \\ & & S^1 \end{array}$

- $cyl(p)$ is a non-Witt 4-dim pseudomanifold, $\partial cyl(p) = M(A)$, singular stratum $= S^1$.

- For $x \in S^1$, $\exists$ Lagrangian $L \subset H^1(Link(x)) = H^1(T^2)$.

- $\exists$ $\pi_1(S^1)$-invariant Lagrangian subspace $\Leftrightarrow$ A has a real eigenvalue.

- Take e.g. $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ (elliptic, with complex eigenvalues).

- Choose an oriented 4-manifold W with $\partial W = M(A)$.

- Set $X^4 = W \cup_{M(A)} cyl(p)$, a closed 4-pseudo-manifold, parity separated.

- Thm $\Rightarrow$ X^4 is not resolvable by a stratified map.

COMPARISON TO M. KATO'S THEORY.
(Topology **12** 1973)

- **Def. 1** A *PL-variety* (P, Σ) is an n-polyhedron P with singular set $\Sigma \subset P$ such that $P - \Sigma$ is a PL n-manifold without boundary.

- **Def. 2** A *PL blow-up* between oriented PL n-varieties (P', Σ') and (P, Σ) is a proper PL map of pairs $f : (P', \Sigma') \to (P, \Sigma)$ such that for some derived neighborhoods N' of $f^{-1}\Sigma$ in P' and N of Σ in P, the restriction $f| : P' - \mathrm{int}(N') \to P - \mathrm{int}(N)$ is an orientation preserving homeomorphism. If P' is an oriented PL n-manifold, such a blow-up is called a *PL resolution*.

- Given (P, Σ),
  $N :=$ regular nbhd of Σ in P.

- ∂N is an $(n-1)$-manifold.

- $p : \partial N \to \Sigma$ the restriction of the normal projection $N \to \Sigma$.

- $[p] \in \Omega_{n-1}(\Sigma)$.

- **Thm.** (Kato) (P, Σ) admits a PL resolution $\Leftrightarrow [p] = 0$.

- For our X^4, have

$$[M(A) \xrightarrow{p} S^1] \in \Omega_3(S^1).$$

- $H_*(S^1; \mathbb{Z})$ f.g., no odd torsion $\Rightarrow$ bordism spectral sequence collapses:

  $\Omega_3(S^1) \cong H_0(S^1; \Omega_3(pt)) \oplus H_1(S^1; \Omega_2(pt)) = 0.$

- Conclusion: X^4 admits a PL resolution, but not a resolution by a stratified map.