Chapter 1
Knots, Singular Embeddings, and Monodromy

**Markus Banagl, Sylvain E. Cappell,
and Julius L. Shaneson**

Abstract The Goresky-MacPherson L-class of a PL pseudomanifold piecewise-linearly embedded in a PL manifold in a possibly nonlocally flat way, can be computed in terms of the Hirzebruch-Thom L-class of the manifold and twisted L-classes associated to the singularities of the embedding, as was shown by Cappell and Shaneson. These formulae are refined here by analyzing the twisted classes. We treat the case of Blanchfield local systems that extend into the singularities as well as cases where they do not extend. In the latter situation, we consider fibered embeddings of strata and 4-dimensional singular sets, using work of Banagl. Rho-invariants enter the picture.

1.1 Introduction

Let M^{n+2} be a closed, oriented, connected PL manifold of dimension $n+2$ and X^n a closed, oriented, connected PL pseudomanifold of dimension n. Let $i : X \hookrightarrow M$ be a

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The first author was partially supported by a grant of the Deutsche Forschungsgemeinschaft. The second and third authors were partially supported by grants of the Defense Advanced Research Projects Agency.

M. Banagl (✉)
Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany
e-mail: banagl@mathi.uni-heidelberg.de

S.E. Cappell
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA
e-mail: cappell@cims.nyu.edu

J.L. Shaneson
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
e-mail: shaneson@math.upenn.edu

M. Banagl, D. Vogel (eds.), *The Mathematics of Knots*,
Contributions in Mathematical and Computational Sciences 1,
DOI 10.1007/978-3-642-15637-3_1,

not necessarily locally flat PL embedding. Let $L^* = 1 + L^1(p_1) + L^2(p_1, p_2) + \cdots$ be the total Hirzebruch L-polynomial,

$$L^1(p_1) = \frac{1}{3}p_1, \qquad L^2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2), \ldots .$$

Let $P(M) \in H^*(M; \mathbb{Z})$ be the total Pontrjagin class of M and the Euler class $\chi \in H^2(M; \mathbb{Z})$ be the Poincaré dual of $i_*[X] \in H_n(M; \mathbb{Z})$, where $[X]$ is the fundamental class of X. Set

$$L_*(M, X) = [X] \cap i^* L^*(P(M) \cup (1 + \chi^2)^{-1}) \in H_*(X; \mathbb{Q}).$$

Recall that the sequence $L^1, L^2, \ldots$ of polynomials is the multiplicative sequence associated to the even power series defined by $x/\tanh(x)$. Thus

$$L^*(1 + \chi^2) = \frac{\chi}{\tanh(\chi)} = 1 + \frac{1}{3}\chi^2 - \frac{1}{45}\chi^4 \pm \cdots$$

and by the multiplicativity of $\{L^j\}$,

$$L^*((1 + \chi^2)^{-1}) = \frac{\tanh(\chi)}{\chi} = 1 - \frac{1}{3}\chi^2 + \frac{2}{15}\chi^4 \mp \cdots .$$

Hence the above defining expression for $L_*(M, X)$ may alternatively be written as

$$\begin{aligned} L_*(M, X) &= [X] \cap \left(\frac{\tanh i^*\chi}{i^*\chi} \cup i^* L^*(PM) \right) \\ &= [X] \cap \left(\left(1 - \frac{1}{3} i^*\chi^2 + \frac{2}{15} i^*\chi^4 \mp \cdots \right) \cup i^* L^*(PM) \right). \end{aligned}$$

When this formula is pushed on into M, one obtains

$$\begin{aligned} i_* L_*(M, X) &= i_* \left([X] \cap i^* \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \right) \\ &= i_*[X] \cap \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \\ &= ([M] \cap \chi) \cap \left(\frac{\tanh \chi}{\chi} \cup L^*(PM) \right) \\ &= [M] \cap (\tanh \chi \cup L^*(PM)). \end{aligned}$$

If the embedding is nonsingular, that is, X is a locally flat submanifold, then

$$L_*(X) = L_*(M, X),$$

where $L_*(X)$ is the Poincaré dual of the Hirzebruch L-class of X. In particular, the signature $\sigma(X) = L_0(X)$ is given by

$$\sigma(X) = L_0(M, X).$$

If the embedding is singular, the singularities of X and the singularities of the embedding induce a stratification of the pair (M, X). Under the assumption that there are no strata of odd codimension, it was shown in [CS91] that the Goresky-MacPherson L-class $L_*(X) \in H_*(X; \mathbb{Q})$ of X, defined using middle-perversity intersection homology, can be computed as

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*} L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}}), \tag{1.1}$$

where the sum ranges over all connected components V of pure strata of X that have codimension at least two, $i_V : \overline{V} \hookrightarrow X$ is the inclusion of the closure $\overline{V}$ of V into X and $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) \in H_*(\overline{V}; \mathbb{Q})$ is the Goresky-MacPherson L-class of $\overline{V}$ twisted by a local coefficient system $\mathcal{B}_V^{\mathbb{R}}$. This local system is endowed with a nonsingular symmetric or skew-symmetric form $\mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ and arises as Trotter's "scalar product" [Tro73] of a certain Blanchfield local system $\mathcal{B}_V \otimes \mathcal{B}_V^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$, $\Lambda = \mathbb{Q}[t, t^{-1}]$. The systems are defined on V and do not in general extend as local systems to the closure $\overline{V}$. They do, of course, extend as intersection chain sheaves by applying Deligne's pushforward/truncation-formula to $\mathcal{B}_V^{\mathbb{R}}$, and $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ is defined as the L-class of this self-dual sheaf complex on $\overline{V}$. (For an introduction to the L-class of self-dual sheaves see [Ban07].)

In the present paper, we refine formula (1.1) by computing the twisted classes $L_*(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ further. Two cases are to be distinguished: The systems $\mathcal{B}_V^{\mathbb{R}}$ either extend as local systems from V to $\overline{V}$ or they do not. In the former situation, the results of [BCS03] apply and yield the formula (Theorem 6)

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*}(\widetilde{\mathrm{ch}}[\mathcal{B}_V^{\mathbb{R}}]_K \cap L_*(\overline{V})), \tag{1.2}$$

where the modification $\widetilde{\mathrm{ch}}$ of the Chern character is given by precomposing with the second Adams operation, $\widetilde{\mathrm{ch}} = \mathrm{ch} \circ \psi^2$ and $[\mathcal{B}_V^{\mathbb{R}}]_K$ denotes the K-theory signature of $\mathcal{B}_V^{\mathbb{R}}$, an element of $\mathrm{KO}(X)$ if the form on $\mathcal{B}_V^{\mathbb{R}}$ is symmetric, and of $\mathrm{KU}(X)$ if it is skew-symmetric. In the situation of nonextendable systems, formulae of type (1.2), even when the right hand side is defined, cease to hold as counterexamples of [Ban08] show. The main results presented here, then, are concerned with understanding the twisted signatures $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$ when $\mathcal{B}_V^{\mathbb{R}}$ does not extend as a local system into the singularities of $\overline{V}$. Theorem 10 asserts that

$$\sigma(X) = L_0(M, X)$$

when all embeddings $\overline{V} - V \hookrightarrow \overline{V}$ are locally flat spherical fibered knots. In particular if $M = S^{n+2}$ is a sphere, we have $\sigma(X) = 0$, since $L_0(S^{n+2}, X) = 0$. The remaining results all assume that $i : X \hookrightarrow M$ has a 4-dimensional singular set such that the $\overline{V}$ are 4-manifolds and the bottom stratum consists of locally flat 2-spheres (see Examples 1 and 2). If the 2-spheres have zero self-intersection numbers and $\mathcal{B}_V^{\mathbb{R}}$ is positive ($\epsilon_V = 1$) or negative ($\epsilon_V = -1$) definite of rank r_V, then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}),$$

with V ranging over all connected components V of the pure 4-stratum (Theorem 7). Again we obtain a corollary for the case where M is a sphere:

$$\sigma(X) + \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}) = 0.$$

Similar corollaries for embeddings in spheres can be deduced for the following results as well.

More generally, if the structure group of the form on V is $O(p_V, q_V)$, then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} (p_V - q_V)\sigma(\overline{V}) - \sum_{V \subset X_4 - X_2} \langle 2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}] \rangle,$$

where $2(c_1^2 - 2c_2)$ is an $H^4(\overline{V}; \mathbb{Z})$-valued characteristic class (Theorem 8). As a corollary (Corollary 4) we deduce that $\sigma(X) - L_0(M, X)$ is divisible by 8 if every $\overline{V}$ is a 4-sphere. When the 2-spheres have nonzero self-intersection numbers, then rho-invariants enter. Theorem 9 for positive, say, definite forms asserts that

$$\begin{aligned}\sigma(X) &= L_0(M, X) \\ &\quad - \sum_{V \subset X_4 - X_2} \left(r_V \sigma(V) + \sum_{i=1}^{n_V} (\text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \, \text{sign}[S_i^2]^2 - \rho_{\alpha_i}(p_i, q_i)) \right),\end{aligned}$$

where $\sigma(V)$ denotes the (Novikov-) signature of the exterior of the 2-spheres

$$\bigsqcup_{i=1}^{n_V} S_i^2 \subset \overline{V},$$

$L_i = L(p_i, q_i)$, a lens space, is the boundary of a regular neighborhood of S_i^2 in $\overline{V}$, and α_i is obtained by restricting $\mathcal{B}_V^{\mathbb{C}}$ to L_i. The function $\rho_\alpha(p, q)$ is given by an explicit formula, see Sect. 1.9, p. 26, where the constancy-rank c-rk($\mathcal{S}$) of a local system $\mathcal{S}$ is defined as well.

Organization Section 1.2 reviews Blanchfield forms and their relation to Seifert manifolds. Fundamental results of Levine, Trotter, Kearton and Kervaire are recalled. Blanchfield and Poincaré local coefficient systems are defined. In Sect. 1.3 we review the Trotter trace $T : \mathbb{Q}(t)/\Lambda \to \mathbb{Q}$, which allows us to pass from Blanchfield local systems to Poincaré local systems. An important point here is that this passage reverses symmetry properties: if the Blanchfield form is Hermitian, then the real Poincaré form is skew-symmetric and if the Blanchfield form is skew-Hermitian, then the Poincaré form is symmetric. Section 1.4 serves mainly to set up notation concerning the complexification of real sheaf complexes, forms, etc. Various characterizations of extendability of a local system from the top stratum of a stratified pseudomanifold into the singular strata are discussed in Sect. 1.5. The K-theory signature of a Poincaré local system is recalled. Section 1.6 reviews the twisted L-class formula of [BCS03]. The Cappell-Shaneson L-class formula for singular embeddings, [CS91], is discussed in Sect. 1.7, where details on the stratification induced by a singular embedding, together with an example, are also to be found. Embeddings are always assumed to induce only strata of even codimension and to be of finite local type and of finite type. The final two sections contain the results of this paper; Sect. 1.8 for local systems that extend and Sect. 1.9 for systems that do not extend.

1.2 Blanchfield and Poincaré Local Systems

Let R be a Dedekind domain, for example $R = \Lambda = \mathbb{Q}[t, t^{-1}]$, the ring of Laurent polynomials. Let F be the quotient field of R and let A and B be finitely generated torsion R-modules. A pairing

$$A \otimes_R B \longrightarrow F/R$$

is called *perfect*, if the induced map

$$A \longrightarrow \mathrm{Hom}_R(B, F/R)$$

is an isomorphism. Suppose R is equipped with an involution $r \mapsto \bar{r}$. Then B^{op} will denote the R-module obtained by composing the module structure of B with the involution. A pairing

$$\beta : B \otimes_R B^{\mathrm{op}} \longrightarrow F/R$$

is called *Hermitian* if

$$\beta(a \otimes b) = \beta(b \otimes a)^{-},$$

and *skew-Hermitian* if

$$\beta(a \otimes b) = -\beta(b \otimes a)^{-}.$$

We will be primarily concerned with the ring $R = \Lambda$ of Laurent polynomials. For this ring, the quotient field F is $F = \mathbb{Q}(t)$, the rational functions. The involution on R is given by replacing t with t^{-1}.

Definition 1 An (abstract) *Blanchfield pairing* is a perfect Hermitian or skew-Hermitian pairing

$$B \otimes_\Lambda B^{\mathrm{op}} \longrightarrow \mathbb{Q}(t)/\Lambda,$$

where B is a finitely generated torsion Λ-module.

A locally flat knot $S^{2n-1} \subset S^{2n+1}$ possesses two related kinds of abelian invariants associated to the map of the knot group to its abelianization $\mathbb{Z}$: those arising from the infinite cyclic cover of the knot exterior and those arising from choices of Seifert manifolds. For later reference, we recall here some well-known facts about the relation between the Blanchfield pairing and Seifert matrices. A *Seifert manifold* for the knot is a codimension 1 framed compact submanifold of S^{2n+1} whose boundary is the knot. Every (locally flat) knot has a Seifert manifold. Let K be the exterior of the knot. The knot is *simple*, if $\pi_i(K) \cong \pi_i(S^1)$ for $1 \leq i < n$. The $(2n-1)$-knot is simple if and only if it bounds an $(n-1)$-connected Seifert manifold, [Lev65, Theorem 2]. A choice of Seifert manifold M^{2n} together with a choice of basis $\{b_i\}$ for the torsion-free part of $H_n(M)$ determines a Seifert matrix A by defining the (i, j)-entry to be the linking number of a cycle representing b_i with a translate in the positive normal direction to M of a cycle representing b_j. Any such A has the property that $A + (-1)^n A^T$ is unimodular. In fact, $A + (-1)^n A^T$ is the matrix of the intersection form of M. Two square integral matrices are *S-equivalent* if they can be obtained from each other by a finite sequence of elementary enlargements, reductions and unimodular congruences. An *elementary enlargement* of A is any matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \beta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where α is a row vector and β is a column vector. A matrix is an *elementary reduction* of any of its elementary enlargements.

Theorem 1 (Levine [Lev70]) *Seifert matrices of isotopic knots of any odd dimension are S-equivalent.*

Trotter [Tro73] abstractly calls a square integral matrix A with $A + A^T$ or $A - A^T$ unimodular a *Seifert matrix*. Such an A must be even dimensional. Any Seifert matrix A determines a $\mathbb{Z}[t, t^{-1}]$-module B_A presented by the matrix $tA + (-1)^n A^T$. The determinant of the latter matrix is the *Alexander polynomial* of the knot,

$$\Delta(t) = \det(tA + (-1)^n A^T),$$

defined up to multiplication with a unit of $\mathbb{Z}[t, t^{-1}]$. Moreover, A determines a non-singular $(-1)^{n+1}$-Hermitian pairing

$$\beta_A : B_A \otimes B_A^{\mathrm{op}} \longrightarrow F/\mathbb{Z}[t, t^{-1}],$$

given by the matrix $(1-t)(tA + (-1)^n A^T)^{-1}$, where F is the field of fractions of $\mathbb{Z}[t, t^{-1}]$.

Theorem 2 (Trotter [Tro73]) *If A_1 and A_2 are S-equivalent Seifert matrices, then there is an isometry $(B_{A_1}, \beta_{A_1}) \cong (B_{A_2}, \beta_{A_2})$.*

Thus (B_A, β_A) is an invariant of the knot; B_A is called the *knot module* of the knot. Let K_∞ be the infinite cyclic cover of the exterior K of the knot. The homology group $H_n(K_\infty)$ is a $\mathbb{Z}[t, t^{-1}]$-module via the action of the Deck-transformation group, generated by t. Assume that the knot is simple. The Blanchfield pairing

$$b : H_n(K_\infty) \otimes H_n(K_\infty)^{\mathrm{op}} \longrightarrow F/\mathbb{Z}[t, t^{-1}]$$

is nonsingular and $(-1)^{n+1}$-Hermitian.

Theorem 3 (Kearton [Kea73]) *If A is any Seifert matrix of a simple knot, then there is an isometry $(B_A, \beta_A) \cong (H_n(K_\infty), b)$.*

Particularly agreeable representatives of S-equivalence classes are provided by the following result.

Proposition 1 (Trotter [Tro73]) *Any Seifert matrix is S-equivalent to a nonsingular matrix.*

If A is a nonsingular Seifert matrix, then the $\mathbb{Q}$-vector space $B_A \otimes_{\mathbb{Z}} \mathbb{Q}$ has dimension

$$\dim_{\mathbb{Q}}(B_A \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rk} A. \tag{1.3}$$

We conclude this review with a geometric realization result due to Kervaire.

Theorem 4 (Kervaire [Ker65]) *Let $n > 2$ be an integer and A a square integral matrix such that $A + (-1)^n A^T$ is unimodular. Then there exists a simple locally flat $(2n-1)$-knot with Seifert matrix A.*

Let $(X^n, \partial X)$ be a pseudomanifold with (possibly empty) boundary and filtration

$$X^n = X_n \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0 \supset \varnothing,$$

where the strata are indexed by dimension, the $X_i \cap \partial X$ stratify ∂X, and the $X_i - \partial X$ stratify $X - \partial X$; $\Sigma = X_{n-2}$ is the singular set. For a ring R, let R_X denote the constant sheaf with stalk R on X.

Definition 2 A *Blanchfield local system on X* is a locally constant sheaf $\mathcal{B}$ on X together with a pairing

$$\beta : \mathcal{B} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow (\mathbb{Q}(t)/\Lambda)_X,$$

such that for every $x \in X$, the stalk $\mathcal{B}_x$ is a finitely generated torsion Λ-module and the restriction

$$\beta_x : \mathcal{B}_x \otimes \mathcal{B}_x^{\mathrm{op}} \longrightarrow \mathbb{Q}(t)/\Lambda$$

is a Blanchfield pairing.

Definition 3 A *Poincaré local system on* X is a locally constant sheaf $\mathcal{P}$ on X together with a pairing

$$\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X,$$

such that for every $x \in X$, the stalk $\mathcal{P}_x$ is a finite dimensional real vector space and the restriction

$$\phi_x : \mathcal{P}_x \otimes \mathcal{P}_x \to \mathbb{R}$$

is perfect and either symmetric or skew-symmetric.

1.3 Passage from Blanchfield Systems to Poincaré Systems

The method of partial fraction decomposition enables us to write any rational function $f \in \mathbb{Q}(t)$ uniquely in the form

$$f(t) = p(t) + \sum_{i=1}^{k} \frac{A_i}{t^i} + g(t),$$

where $p \in \mathbb{Q}[t]$, $A_i \in \mathbb{Q}$, and

$$g(t) = \sum_{j=1}^{l} \sum_{i=1}^{k_j} \frac{p_{i,j}(t)}{q_j(t)^i},$$

$p_{i,j}, q_j \in \mathbb{Q}[t]$, the q_j are distinct, irreducible, and prime to t, $\deg p_{i,j} < i \deg q_j$. Since t does not divide q_j, $q_j(0) \neq 0$ and thus $g(0) \in \mathbb{Q}$ is defined. The *Trotter trace* $T : \mathbb{Q}(t) \to \mathbb{Q}$ is the $\mathbb{Q}$-linear map

$$T(f) = g(0).$$

If $f \in \Lambda \subset \mathbb{Q}(t)$, then $g = 0$ and so $T(f) = 0$. Thus T passes from $\mathbb{Q}(t)$ to the quotient $\mathbb{Q}(t)/\Lambda$, $T : \mathbb{Q}(t)/\Lambda \to \mathbb{Q}$.

Let B be a Λ-module. By restricting the coefficients to the subring $\mathbb{Q} \subset \Lambda$, we may regard B as a $\mathbb{Q}$-vector space $B^{\mathbb{Q}}$. If B is finitely generated and torsion, then $B^{\mathbb{Q}}$ is finite dimensional. Using the standard embedding $\mathbb{Q} \subset \mathbb{R}$, we define the real vector space $B^{\mathbb{R}} = B^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\beta : B \otimes_{\Lambda} B^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ be a Blanchfield pairing. Define

$$\beta^{\mathbb{R}} : B^{\mathbb{R}} \otimes_{\mathbb{R}} B^{\mathbb{R}} \longrightarrow \mathbb{R}$$

by

$$\beta^{\mathbb{R}}((a \otimes_{\mathbb{Q}} \lambda) \otimes_{\mathbb{R}} (b \otimes_{\mathbb{Q}} \mu)) = \lambda\mu T\beta(a \otimes_{\Lambda} b),$$

$a, b \in B$, $\lambda, \mu \in \mathbb{R}$. Then $\beta^{\mathbb{R}}$ is a perfect pairing on $B^{\mathbb{R}}$. The passage from β to $\beta^{\mathbb{R}}$ reverses symmetry properties: if β is Hermitian, then $\beta^{\mathbb{R}}$ is skew-symmetric and if

β is skew-Hermitian, then $\beta^{\mathbb{R}}$ is symmetric. We denote the signature of this pairing by $\sigma(\beta^{\mathbb{R}})$; it is zero in the skew-symmetric case.

If $B = H_n(K_\infty)$ is the knot module of a locally flat knot $S^{2n-1} \subset S^{2n+1}$ and β the Blanchfield pairing for this knot, then the signature of $\beta^{\mathbb{R}}$ can be computed as

$$\sigma(\beta^{\mathbb{R}}) = \sigma(M^{2n}), \tag{1.4}$$

where M^{2n} is any Seifert manifold for the knot and $\sigma(M^{2n})$ denotes the (Novikov-) signature of its intersection form, see [CS91]. Note also that

$$\sigma(M^{2n}) = \sigma(A + (-1)^n A^T), \tag{1.5}$$

where A is the corresponding Seifert matrix, since $A + (-1)^n A^T$ is a matrix representation of the intersection form.

If $\mathcal{B}$ is a local system of Λ-modules on a space X, then $\mathcal{B}^{\mathbb{R}} = \mathcal{B}^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathcal{B}^{\mathbb{Q}}$ is the local system of $\mathbb{Q}$-vector spaces with stalks $(\mathcal{B}^{\mathbb{Q}})_x = (\mathcal{B}_x)^{\mathbb{Q}}$ obtained from restricting coefficients to $\mathbb{Q}$. Let $\beta : \mathcal{B} \otimes_\Lambda \mathcal{B}^{\mathrm{op}} \to (\mathbb{Q}(t)/\Lambda)_X$ be a Blanchfield local system on a pseudomanifold X. Then the pairings $(\beta_x)^{\mathbb{R}} : (\mathcal{B}_x)^{\mathbb{R}} \otimes_{\mathbb{R}} (\mathcal{B}_x)^{\mathbb{R}} \to \mathbb{R}$ define a Poincaré local system

$$\beta^{\mathbb{R}} : \mathcal{B}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{B}^{\mathbb{R}} \longrightarrow \mathbb{R}_X.$$

1.4 Passage from Poincaré Systems to Complex Hermitian Systems

Given a real vector space V of dimension n, let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexification, a complex vector space of dimension n. For example, $\mathbb{R}^n_{\mathbb{C}} = \mathbb{C}^n$. Taking complex conjugation as the involution to be composed with the scalar multiplication, we get the complex vector space $V_{\mathbb{C}}^{\mathrm{op}}$. In fact, $V_{\mathbb{C}}^{\mathrm{op}} = V \otimes_{\mathbb{R}} (\mathbb{C}^{\mathrm{op}})$. If $B = \{v_1, \dots, v_n\}$ is a basis for V, then $B_{\mathbb{C}} = \{v_1 \otimes 1, \dots, v_n \otimes 1\}$ is a basis for $V_{\mathbb{C}}$. As regards pairings, let us concentrate on the symmetric case, the skew-symmetric case is treated in a similar way. To a symmetric perfect pairing $\phi : V \otimes V \to \mathbb{R}$, we can associate a Hermitian perfect pairing $\phi_{\mathbb{C}} : V_{\mathbb{C}} \otimes V_{\mathbb{C}}^{\mathrm{op}} \to \mathbb{C}$ by setting

$$\phi_{\mathbb{C}}((v \otimes \lambda) \otimes (w \otimes \mu)) = \lambda\overline{\mu}\phi(v, w),$$

$v, w \in V$, $\lambda, \mu \in \mathbb{C}$. The canonical example is $\gamma : \mathbb{R}^{p+q} \otimes \mathbb{R}^{p+q} \to \mathbb{R}$ given by

$$\begin{aligned}&\gamma((x_1, \dots, x_{p+q}) \otimes (y_1, \dots, y_{p+q}))\\&\quad = x_1y_1 + \dots + x_py_p - x_{p+1}y_{p+1} - \dots - x_{p+q}y_{p+q}.\end{aligned}$$

For this pairing, $\gamma_{\mathbb{C}} : \mathbb{C}^{p+q} \otimes (\mathbb{C}^{p+q})^{\mathrm{op}} \to \mathbb{C}$ is

$$\begin{aligned}&\gamma_{\mathbb{C}}((z_1, \dots, z_{p+q}) \otimes (u_1, \dots, u_{p+q}))\\&\quad = z_1\overline{u}_1 + \dots + z_p\overline{u}_p - z_{p+1}\overline{u}_{p+1} - \dots - z_{p+q}\overline{u}_{p+q}.\end{aligned}$$

If $\mathrm{Mat}_B(\phi) = (\phi(v_i \otimes v_j))$ denotes the matrix representation of ϕ with respect to the basis B, then the matrix representation of $\phi_{\mathbb{C}}$ with respect to $B_{\mathbb{C}}$ is simply

$$\mathrm{Mat}_{B_{\mathbb{C}}}(\phi_{\mathbb{C}}) = \mathrm{Mat}_B(\phi),$$

viewed as a complex matrix. This is a Hermitian matrix because it is real and symmetric. In fact, we can choose B so that $\mathrm{Mat}_B(\phi)$ is diagonal. Then $\mathrm{Mat}_{B_{\mathbb{C}}}(\phi_{\mathbb{C}})$ is diagonal with the same entries, which shows that the signature does not change under complexification, $\sigma(\phi) = \sigma(\phi_{\mathbb{C}}) \in \mathbb{Z}$.

The symmetric perfect pairing ϕ may alternatively be described by a self-duality isomorphism $d : V \xrightarrow{\cong} V^*$, where $V^* = \mathrm{Hom}(V, \mathbb{R})$, given by $d(v) = \phi(v \otimes -)$. The symmetry property is equivalent to asserting that

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "\mathrm{ev}"', "\cong"] & V^* \\ V^{**} \arrow[ur, "d^*"'] & \end{tikzcd}$$

commutes, where ev is the canonical evaluation isomorphism. Similarly, if W is a complex vector space and $\psi : W \otimes W^{\mathrm{op}} \to \mathbb{C}$ a perfect Hermitian pairing, then ψ can alternatively be described by a self-duality isomorphism $D : W \xrightarrow{\cong} W^{\dagger}$, where $W^{\dagger} = \mathrm{Hom}(W^{\mathrm{op}}, \mathbb{C})$, by setting $D(w) = \psi(w \otimes -)$. The Hermitian symmetry is equivalent to asserting that

$$\begin{tikzcd} W \arrow[r, "D"] \arrow[d, "\mathrm{ev}"', "\cong"] & W^{\dagger} \\ W^{\dagger\dagger} \arrow[ur, "D^{\dagger}"'] & \end{tikzcd}$$

commutes. In particular, we get $D : V_{\mathbb{C}} \xrightarrow{\cong} V_{\mathbb{C}}^{\dagger}$ for $(W, \psi) = (V_{\mathbb{C}}, \phi_{\mathbb{C}})$.

Let X be a path-connected space and $(\mathcal{P}, \phi)$ a Poincaré local system on X, $\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X$. Applying complexification stalkwise, we obtain a Hermitian local system $\phi_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}} \otimes \mathcal{P}_{\mathbb{C}}^{\mathrm{op}} \to \mathbb{C}_X$. A monodromy-theoretic description of this passage runs as follows: Let p and q be such that $p + q = \mathrm{rk}\,\mathcal{P}$ and $p - q = \sigma(\phi_x)$, $x \in X$. Let $O(p, q)$ be the group of all matrices in $GL(p + q, \mathbb{R})$ that preserve the form γ, that is,

$$O(p, q) = \{A \in GL(p + q, \mathbb{R}) \; : \; A^T \cdot I_{p,q} \cdot A = I_{p,q}\},$$

where

$$I_{p,q} = \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix}.$$

Then $(\mathcal{P}, \phi)$ determines, and is determined by, a representation $\pi_1(X) \to O(p,q)$. Let $U(p,q)$ be the group of all matrices in $GL(p+q, \mathbb{C})$ that preserve the form $\gamma_{\mathbb{C}}$, that is,

$$U(p,q) = \{A \in GL(p+q, \mathbb{C}) \ : \ \overline{A}^T \cdot I_{p,q} \cdot A = I_{p,q}\}.$$

Note that $O(p,q) \subset U(p,q)$ is a subgroup. The Hermitian local system $(\mathcal{P}_{\mathbb{C}}, \phi_{\mathbb{C}})$ determines, and is determined by, a representation $\pi_1(X) \to U(p,q)$, and this representation is the composition

$$\pi_1(X) \longrightarrow O(p,q) \hookrightarrow U(p,q).$$

Let X^n be a PL stratified pseudomanifold. The real dualizing complex $\mathbb{D}^\bullet_X(\mathbb{R})$ on X may be defined as the complex of sheaves of real vector spaces which has the sheafification of the presheaf

$$U \mapsto C_j(X, X-U; \mathbb{R}), \qquad U \subset X \text{ open},$$

in degree $-j$, where C_j denotes singular chains of dimension j. Similarly, $\mathbb{D}^\bullet_X(\mathbb{C})$ is the sheafification of $U \mapsto C_j(X, X-U; \mathbb{C})$. Since $C_j(X, X-U; \mathbb{C}) = C_j(X, X-U; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $- \otimes_{\mathbb{R}} \mathbb{C}$ commutes with direct limits, it follows that

$$\mathbb{D}^\bullet_X(\mathbb{C}) = \mathbb{D}^\bullet_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}_X.$$

Let $\mathbf{S}^\bullet \in D^b_c(X; \mathbb{R})$ be an object of the constructible bounded derived category of sheaf complexes of real vector spaces on X. Since we are working over fields, $\overset{L}{\otimes} = \otimes$. Define the complexification of $\mathbf{S}^\bullet$ by $\mathbf{S}^\bullet_{\mathbb{C}} = \mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbb{C}_X \in D^b_c(X; \mathbb{C})$. Given $\mathbf{A}^\bullet \in D^b_c(X; \mathbb{C})$, we may apply composition with complex conjugation in a stalk-wise fashion to define $(\mathbf{A}^\bullet)^{\mathrm{op}} \in D^b_c(X; \mathbb{C})$. We have $(\mathbf{S}^\bullet_{\mathbb{C}})^{\mathrm{op}} = \mathbf{S}^\bullet \otimes_{\mathbb{R}} (\mathbb{C}^{\mathrm{op}}_X)$. Given $\mathbf{T}^\bullet \in D^b_c(X; \mathbb{R})$, there is a canonical isomorphism

$$\begin{aligned} \mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} &\cong (\mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet) \otimes_{\mathbb{R}} (\mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{op}}_X), \\ (v \otimes \lambda) \otimes (w \otimes \mu) &\mapsto (v \otimes w) \otimes (\lambda \otimes \mu). \end{aligned}$$

Let

$$\begin{aligned} m : \mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{op}}_X &\longrightarrow \mathbb{C}_X \\ \lambda \otimes \mu &\mapsto \lambda\overline{\mu} \end{aligned}$$

be the canonical multiplication. To a pairing

$$\phi : \mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet \longrightarrow \mathbb{D}^\bullet_X(\mathbb{R})$$

into the dualizing complex, we can associate a pairing

$$\phi_{\mathbb{C}} : \mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbb{D}^\bullet_X(\mathbb{C})$$

by taking $\phi_{\mathbb{C}}$ to be the composition

$$\mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}} = (\mathbf{S}^\bullet \otimes_{\mathbb{R}} \mathbf{T}^\bullet) \otimes_{\mathbb{R}} (\mathbb{C}_X \otimes_{\mathbb{C}} \mathbb{C}_X^{\mathrm{op}}) \xrightarrow{\phi \otimes m} \mathbb{D}^\bullet_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}_X = \mathbb{D}^\bullet_X(\mathbb{C}).$$

Under the canonical identifications

$$\mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet \otimes \mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R})) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet, \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R}))),$$
$$\mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C})) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{S}^\bullet_{\mathbb{C}}, \mathrm{R}\mathbf{Hom}^\bullet((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C}))),$$

the above procedure associates to a morphism

$$d : \mathbf{S}^\bullet \longrightarrow \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{T}^\bullet, \mathbb{D}^\bullet_X(\mathbb{R})) = \mathcal{D}_{X,\mathbb{R}}(\mathbf{T}^\bullet)$$

in $D^b_c(X; \mathbb{R})$ with codomain the real Verdier-dual of $\mathbf{T}^\bullet$ a morphism

$$D : \mathbf{S}^\bullet_{\mathbb{C}} \longrightarrow \mathrm{R}\mathbf{Hom}^\bullet((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}, \mathbb{D}^\bullet_X(\mathbb{C})) = \mathcal{D}_{X,\mathbb{C}}((\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}})$$

in $D^b_c(X; \mathbb{C})$ to the complex Verdier-dual of $(\mathbf{T}^\bullet_{\mathbb{C}})^{\mathrm{op}}$. If d is an isomorphism, then so is D. If d is symmetric, that is, $\mathcal{D}_{X,\mathbb{R}}(d) = d$, then D is Hermitian, that is, $\mathcal{D}_{X,\mathbb{C}}(D^{\mathrm{op}}) = D$. In particular, if $\mathbf{S}^\bullet$ is a symmetric self-dual real sheaf, then $\mathbf{S}^\bullet_{\mathbb{C}}$ is a Hermitian self-dual complex sheaf.

Suppose the dimension n of X is a multiple of 4 and that X is oriented, closed and has only even codimensional strata. Let $\phi : \mathcal{P} \otimes \mathcal{P} \to \mathbb{R}_X$ be a symmetric Poincaré local system on the top stratum of X. Then ϕ extends uniquely to a symmetric self-duality isomorphism

$$d : \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}) \cong \mathcal{D}_{X,\mathbb{R}} \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})[n].$$

Similarly, the associated Hermitian local system $\phi_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}} \otimes (\mathcal{P}_{\mathbb{C}})^{\mathrm{op}} \to \mathbb{C}_X$ extends uniquely to a Hermitian self-duality isomorphism

$$\delta : \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}}) \cong \mathcal{D}_{X,\mathbb{C}} \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}})^{\mathrm{op}}[n].$$

As, for an open inclusion i, the derived pushforward $\mathrm{R}\, i_*$ commutes with $- \otimes_{\mathbb{R}} \mathbb{C}$, and the truncation functor $\tau_{\leq k}$ commutes with $- \otimes_{\mathbb{R}} \mathbb{C}$ as well, we have

$$\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}}) = \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})_{\mathbb{C}}.$$

Moreover, $\delta = D$, where D is the complexification of d as described above.

Let $\mathbf{S}^\bullet \in D^b_c(X; \mathbb{R})$ be a symmetric self-dual sheaf on X, $d : \mathbf{S}^\bullet \cong \mathcal{D}_{X,\mathbb{R}} \mathbf{S}^\bullet[n]$, $\mathcal{D}_{X,\mathbb{R}} d[n] = d$. The isomorphism d induces a symmetric isomorphism on the middle-dimensional hypercohomology groups

$$\mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \xrightarrow{\cong} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet)^*,$$

i.e. a symmetric perfect pairing

$$\psi : \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \otimes_{\mathbb{R}} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet) \longrightarrow \mathbb{R}.$$

Let $\sigma(\mathbf{S}^\bullet) \in \mathbb{Z}$ denote the signature of this pairing. The complexification $\mathbf{S}^\bullet_{\mathbb{C}}$ is a Hermitian self-dual sheaf. Its self-duality isomorphism $D : \mathbf{S}^\bullet_{\mathbb{C}} \cong \mathcal{D}_{X,\mathbb{C}}(\mathbf{S}^\bullet_{\mathbb{C}})^{\mathrm{op}}[n]$ induces a perfect Hermitian pairing

$$\eta : \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbb{C}.$$

Let $\sigma(\mathbf{S}^\bullet_{\mathbb{C}}) \in \mathbb{Z}$ denote the signature of η. With $V = \mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet)$, we have $\mathcal{H}^{-n/2}(X; \mathbf{S}^\bullet_{\mathbb{C}}) = V_{\mathbb{C}}$ and $\eta = \psi_{\mathbb{C}}$, whence $\sigma(\mathbf{S}^\bullet_{\mathbb{C}}) = \sigma(\psi_{\mathbb{C}}) = \sigma(\psi) = \sigma(\mathbf{S}^\bullet)$. Given a Poincaré local system $(\mathcal{P}, \phi)$, the twisted signature $\sigma(X; \mathcal{P})$ is by definition the signature of the self-dual sheaf $(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), d)$. We conclude that

$$\begin{aligned} \sigma(X; \mathcal{P}) &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), d) \\ &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})_{\mathbb{C}}, D) \\ &= \sigma(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}_{\mathbb{C}}), \delta) \\ &= \sigma(X; \mathcal{P}_{\mathbb{C}}), \end{aligned}$$

where the last equality is a definition.

1.5 Strongly Transverse Poincaré Local Systems

Let $\epsilon \in \{\pm 1\}$, let $(\mathcal{P}, \phi)$ be an ϵ-symmetric Poincaré local system of stalk dimension m on the space X^n and let $\Pi_1(X)$ denote the fundamental groupoid of X. By $\mathfrak{Vect}_m$ denote the category whose objects are pairs (V, ψ), with V an m-dimensional real vector space and $\psi : V \times V \to \mathbb{R}$ a perfect ϵ-symmetric bilinear pairing, and whose morphisms are isometries of the pairings. The system $(\mathcal{P}, \phi)$ induces a covariant functor

$$\mu(\mathcal{P}) : \Pi_1(X) \longrightarrow \mathfrak{Vect}_m$$

as follows: For $x \in X$, let

$$\mu(\mathcal{P})(x) = (\mathcal{P}_x, \phi_x)$$

and for a path class $[\omega] \in \pi_1(X, x_1, x_2) = \mathrm{Hom}_{\Pi_1(X)}(x_2, x_1)$, $\omega : I \to X$, $\omega(0) = x_1$, $\omega(1) = x_2$, define the linear operator

$$\mu(\mathcal{P})[\omega] : \mu(\mathcal{P})(x_2) \longrightarrow \mu(\mathcal{P})(x_1)$$

to be the composition

$$\mu(\mathcal{P})(x_2) = \mathcal{P}_{\omega(1)} \cong (\omega^*\mathcal{P})_1 \underset{\mathrm{restr}}{\overset{\simeq}{\leftarrow}} \Gamma(I, \omega^*\mathcal{P}) \underset{\mathrm{restr}}{\overset{\simeq}{\rightarrow}} (\omega^*\mathcal{P})_0 \cong \mathcal{P}_{\omega(0)} = \mu(\mathcal{P})(x_1).$$

If we choose a base-point $x \in X$, then restricting $\mu(\mathcal{P})$ to the fundamental group $\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ gives an assignment of a linear automorphism on the stalk $\mathcal{P}_x$,

$$\mu(\mathcal{P})_x(g) : \mathcal{P}_x \longrightarrow \mathcal{P}_x,$$

preserving the pairing $\phi_x : \mathcal{P}_x \times \mathcal{P}_x \to \mathbb{R}$, to each $g \in \pi_1(X, x)$. Thus one obtains the monodromy representation

$$\mu(\mathcal{P})_x : \pi_1(X, x) \longrightarrow O(p, q)$$

when $\epsilon = 1$ ($p + q = m$ is the rank of $\mathcal{P}$, $p - q$ the signature of ϕ_x), and

$$\mu(\mathcal{P})_x : \pi_1(X, x) \longrightarrow Sp(2r; \mathbb{R})$$

when $\epsilon = -1$ ($m = 2r$ is the rank of $\mathcal{P}$). Conversely, a given functor $\mu : \Pi_1(X) \to \mathfrak{Vect}_m$ determines a Poincaré local system: Let X_0 be a path component of X, and $x_0 \in X_0$. Then $\pi(X_0, x_0)$ acts on $\mu(x_0) = (V, \phi)$ by the restriction μ_{x_0} and we have the associated local system

$$\mathcal{P}|_{X_0} = \widetilde{X_0} \times_{\pi_1(X_0, x_0)} V$$

over X_0 with an induced pairing ϕ, where $\widetilde{X_0}$ denotes the universal cover of X_0.

Definition 4 Let X be a stratified pseudomanifold with singular set Σ and let $\mathcal{X}$ denote the set of components of open strata of X of codimension at least 2. Each $Z \in \mathcal{X}$ has a link $\mathrm{Lk}(Z)$. Call a Poincaré local system $\mathcal{P}$ on $X - \Sigma$ *strongly transverse to* Σ if the composite functor

$$\Pi_1(\mathrm{Lk}(Z) - \Sigma) \xrightarrow{\mathrm{incl}_*} \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{P})} \mathfrak{Vect}_m$$

is isomorphic to the trivial functor for all $Z \in \mathcal{X}$.

On normal spaces, strong transversality of local systems characterizes those systems that extend as local systems over the whole space:

Proposition 2 *Let X^n be normal. A Poincaré local system $\mathcal{P}$ on $X - \Sigma$ is strongly transverse to Σ if and only if it extends as a Poincaré local system over all of X. Such an extension is unique.*

The normality assumption is not necessary for the "if"-direction. The assumption cannot be omitted in the "only if"-direction and in the uniqueness statement.

Corollary 1 *Let X^n be normal. A Poincaré local system $\mathcal{P}$ on $X - \Sigma$ is strongly transverse to Σ if and only if its monodromy functor $\mu(\mathcal{P}) : \Pi_1(X - \Sigma) \to \mathfrak{Vect}_m$ factors* (*up to isomorphism of functors*) *through* $\Pi_1(X)$:

$$\begin{tikzcd}
\Pi_1(X - \Sigma) \arrow[r, "\mathrm{incl}_*"] \arrow[dr, "\mu(\mathcal{P})"'] & \Pi_1(X) \arrow[d] \\
& \mathfrak{Vect}_m
\end{tikzcd}$$

Let X^n be normal. A Poincaré local system $(\mathcal{P}, \phi)$ on $X^n - \Sigma$ strongly transverse to Σ has a K-theory signature

$$[\mathcal{P}]_K \in \begin{cases} \mathrm{KO}(X), & \text{if } \epsilon = 1 \\ \mathrm{KU}(X), & \text{if } \epsilon = -1, \end{cases}$$

as we shall now explain. By Proposition 2, $(\mathcal{P}, \phi)$ has a unique extension to a Poincaré local system $(\bar{\mathcal{P}}, \bar{\phi})$ on X. We now proceed as in [Mey72]. Let $\mathcal{P}^c$ denote the flat vector bundle associated to the locally constant sheaf $\bar{\mathcal{P}}$, that is

$$\mathcal{P}^c|_{X_0} = \widetilde{X_0} \times_\pi \mathbb{R}^m$$

over a path component X_0 of X, where $\mathbb{R}^m$ is given the usual topology, $\pi = \pi_1(X_0)$, and π acts on $\mathbb{R}^m$ by means of the monodromy $\mu(\bar{\mathcal{P}})$ of $\bar{\mathcal{P}}$. A suitable choice of Euclidean metric on $\mathcal{P}^c$ induces (using $\bar{\phi}$) a vector bundle automorphism

$$A : \mathcal{P}^c \longrightarrow \mathcal{P}^c$$

such that $A^2 = 1$ (if $\bar{\phi}$ is symmetric, i.e. $\epsilon = 1$) or $A^2 = -1$ (if $\bar{\phi}$ is skew-symmetric, i.e. $\epsilon = -1$). Thus in the case $\epsilon = 1$, $\mathcal{P}^c$ decomposes as a direct sum of vector bundles

$$\mathcal{P}^c = \mathcal{P}_+ \oplus \mathcal{P}_-$$

corresponding to the ± 1-eigenspaces of A. Put

$$[\mathcal{P}]_K = [\mathcal{P}_+] - [\mathcal{P}_-] \in \mathrm{KO}(X).$$

In the case $\epsilon = -1$, A defines a complex structure on $\mathcal{P}^c$ and we obtain the complex vector bundle $\mathcal{P}_\mathbb{C}$ and its conjugate bundle $\mathcal{P}^*_\mathbb{C}$; we put

$$[\mathcal{P}]_K = [\mathcal{P}^*_\mathbb{C}] - [\mathcal{P}_\mathbb{C}] \in \mathrm{KU}(X).$$

Similar remarks apply to perfect complex Hermitian local coefficient systems $\mathcal{S}$. They are determined over connected components $X_0 \subset X$ by monodromy representations $\mu(\mathcal{S}) : \pi_1(X_0) \to U(p, q)$ and their K-theory signature is defined as $[\mathcal{S}]_K = [\mathcal{S}_+] - [\mathcal{S}_-] \in \mathrm{KU}(X)$, where $\mathcal{S}^c = \mathcal{S}_+ \oplus \mathcal{S}_-$ is a nonflat splitting such that the Hermitian form is positive definite on $\mathcal{S}_+$ and negative definite on $\mathcal{S}_-$, corresponding to a reduction of the structure group from $U(p, q)$ to the maximal compact subgroup $U(p) \times U(q)$, see [Lus71].

1.6 Computing Twisted L-Classes for Strongly Transverse Coefficients

Let X be a closed Witt space with singular set Σ, and $(\mathcal{P}, \phi)$ a Poincaré local system on $X - \Sigma$ such that a self-dual extension $(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P}), \bar{\phi})$ exists. The *twisted L-classes*

$$L_k(X; \mathcal{P}) \in H_k(X; \mathbb{Q})$$

of X with coefficients in $\mathcal{P}$ are the L-classes of the self-dual sheaf $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{P})$. In [BCS03], we show:

Theorem 5 *Let X^n be a closed oriented Whitney stratified normal Witt space with singular set Σ, and let $(\mathcal{P}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Then*

$$L_*(X; \mathcal{P}) = \widetilde{\mathrm{ch}}[\mathcal{P}]_K \cap L_*(X). \tag{1.6}$$

Recall the

Definition 5 X^n is *supernormal*, if for any components Z, Z' of open strata with $\dim Z' > \dim Z \leq n-2$, the link $Lk(Z) \cap Z'$ is simply connected.

Theorem 5 implies

Corollary 2 *If X^n is supernormal, then for any Poincaré local system $(\mathcal{P}, \phi)$ on $X - \Sigma$*

$$L_*(X; \mathcal{P}) = \widetilde{\mathrm{ch}}[\mathcal{P}]_K \cap L_*(X).$$

To obtain the conclusion of the corollary, less than supernormality is actually needed. Indeed it is sufficient to require that X be normal and that the image of $\pi_1(Lk(Z) - \Sigma)$ in $\pi_1(X - \Sigma)$ vanishes for all $Z \in \mathcal{X}$.

In [Ban06], the first author has extended formula (1.6) to spaces that are not Witt, but still support self-dual perverse sheaves, given by Lagrangian structures, so that the L-class is still defined.

1.7 The Cappell-Shaneson L-Class Formula for Singular Embeddings

Let X^n be an oriented connected PL pseudomanifold of real dimension n, piecewise linearly embedded in an oriented, connected PL manifold M^m of dimension $m = n+2$. Since (M, X) is a PL pair, there exists a filtration

$$\begin{aligned} M = M_m \supset M_{m-1} = X \supset M_{m-2} = X \supset M_{m-3} = X \supset M_{m-4} \\ \supset M_{m-5} \supset \cdots \supset M_0 \supset M_{-1} = \varnothing, \end{aligned}$$

such that for each $y \in M_i - M_{i-1}$ there exists a distinguished neighborhood U of y in M, a compact Hausdorff pair (G, F), a filtration

$$G = G_{m-i-1} \supset \cdots \supset G_0 \supset G_{-1} = \varnothing,$$

and a PL homeomorphism

$$\phi : D^i \times c(G, F) \longrightarrow (U, U \cap X)$$

that maps $D^i \times c(G_{j-1}, G_{j-1} \cap F)$ onto $(M_{i+j}, M_{i+j} \cap X)$, where cY denotes the cone on a space Y. The link pair (G, F) depends up to PL homeomorphism only on the connected component V of $M_i - M_{i-1}$ that contains y. Since M is a manifold, $G = S^{m-i-1} = S^{n-i+1}$ is a sphere. As in [CS91], we will henceforth assume that embeddings are of finite local type and of finite type. (This guarantees finite dimensionality of intersection sheaf stalks and global intersection homology groups. Algebraic knots, for example, are always of finite type.) An induced PL stratification of X is given by

$$X = X_n \supset X_{n-1} = M_{m-3} = X \supset X_{n-2} = M_{m-4}$$
$$\supset X_{n-3} = M_{m-5} \supset \cdots \supset X_0 = M_0 \supset X_{-1} = \varnothing.$$

The link in X of a component V of a stratum $X_i - X_{i-1} = M_i - M_{i-1}$ at a point $y \in V$ is the above F. Let $\mathcal{X}$ be the collection of connected components of pure strata $X_i - X_{i-1}$, $i \leq n-2$. It is worthwhile to discuss the case of X a manifold. Since the embedding of X in M may not be locally flat, the pair (M, X) will in general still receive a nontrivial stratification, but the links of components in X will be spheres $F = S^{n-i-1}$. The link pairs in (M, X) will thus be knots (S^{n-i+1}, S^{n-i-1}). The closed strata $X_i \subset X$ induced by the embedding $X \subset M$ may or may not be submanifolds of X and the embeddings $X_i \subset X$ may or may not be locally flat.

Example 1 Let $S(Y)$ denote the unreduced suspension of a space Y. We shall discuss the stratification of $X^n = S^2 \times S(S^1 \times S^{n-4})$ induced by a certain nonlocally flat embedding $X^n \subset S^2 \times S^n = M^{n+2}$, where $n \geq 6$. We start out with a nontrivial locally flat PL knot $\kappa : S^{n-5} \hookrightarrow S^{n-3}$ and suspend it to obtain an embedding $S\kappa : S^{n-4} = S(S^{n-5}) \hookrightarrow S(S^{n-3}) = S^{n-2}$. Denote the two suspension points in S^{n-4} by p_+ and p_-. Think of S^{n-2} as the one-point compactification $S^{n-2} = \mathbb{R}^{n-2} \cup \{\infty\}$ of $\mathbb{R}^{n-2}$, with ∞ not in the image of $S\kappa$. Then by restricting $S\kappa$, we obtain an embedding $\sigma : S^{n-4} \hookrightarrow \mathbb{R}^{n-2}$, which is not flat at p_+ and p_- because the link pair at $p_\pm$ is the knot κ. On the complement $S^{n-4} - \{p_\pm\}$, σ is locally flat. Crossing with a circle S^1, we get an embedding $\mathrm{id}_{S^1} \times \sigma : S^1 \times S^{n-4} \hookrightarrow S^1 \times \mathbb{R}^{n-2}$ with link pair κ for the singular stratum $S^1 \times \{p_\pm\} \subset S^1 \times S^{n-4}$ where $\mathrm{id}_{S^1} \times \sigma$ is not locally flat. Embed S^1 in the plane $\mathbb{R}^2$ as the unit circle and $\mathbb{R}^2$ in $\mathbb{R}^{n-1}$ in the standard way, $(x, y) \mapsto (x, y, 0, 0, \ldots, 0)$. Then the normal bundle of $S^1 \hookrightarrow \mathbb{R}^{n-1}$ is $S^1 \times \mathbb{R}^{n-2}$ and defines an open embedding $S^1 \times \mathbb{R}^{n-2} \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow S^{n-1}$. The composition of $\mathrm{id}_{S^1} \times \sigma$ with this open embedding gives an embedding $f : S^1 \times S^{n-4} \hookrightarrow S^{n-1}$. Since the open embedding does not change the link types, f still has singular stratum $S^1 \times \{p_\pm\}$. Let q_+ and q_- be the two suspension points of $S(S^1 \times S^{n-4})$. Suspending f, we obtain an embedding $Sf : S(S^1 \times S^{n-4}) \hookrightarrow S(S^{n-1}) = S^n$. The points $q_\pm$ are singularities of the pseudomanifold $S(S^1 \times S^{n-4})$ and thus must appear as a stratum of the pair $(S^n, S(S^1 \times S^{n-4}))$. The stratification of $(S^n, S(S^1 \times S^{n-4}))$ is given by

$$S^n \supset S(S^1 \times S^{n-4}) \supset S(S^1 \times \{p_\pm\}) \supset \{q_\pm\}.$$

Finally, $\mathrm{id}_{S^2} \times Sf : S^2 \times S(S^1 \times S^{n-4}) \hookrightarrow S^2 \times S^n$ defines an embedding $X^n \subset M^{n+2}$. The pair (M, X) is stratified by

$$\begin{aligned} M = S^2 \times S^n \supset X &= S^2 \times S(S^1 \times S^{n-4}) \\ &\supset X_4 = S^2 \times S(S^1 \times \{p_\pm\}) \supset X_2 = S^2 \times \{q_\pm\}. \end{aligned}$$

The collection $\mathcal{X}$ is given by

$$\mathcal{X} = \{S^2 \times \mathring{I} \times S^1 \times \{p_+\}, S^2 \times \mathring{I} \times S^1 \times \{p_-\}, S^2 \times \{q_+\},\ S^2 \times \{q_-\}\},$$

where $\mathring{I} = (0, 1)$ denotes the open unit interval. The closure $\overline{V}$ of the pure component $V = S^2 \times \mathring{I} \times S^1 \times \{p_+\}$ in X is

$$\overline{V} = S^2 \times S(S^1 \times \{p_+\}),$$

PL homeomorphic to the 4-manifold $S^2 \times S^2$, and the singular set $\overline{V} - V$ of $\overline{V}$ is $\overline{V} - V = S^2 \times \{q_\pm\}$, the disjoint union of two 2-spheres. The embedding $\overline{V} - V \subset \overline{V}$ is locally flat and has trivial normal bundle. The link pair of V is the knot κ that we started with.

We return to the general case of an oriented pseudomanifold $X \subset M$. Assume that all strata in $\mathcal{X}$ have even codimension in X. Let $V \in \mathcal{X}$ be a component of codimension $2c = m - i \geq 4$ in M and let $x \in V$ be a point with link pair $(G_x, F_x) = (S_x^{2c-1}, F_x)$. The fundamental class $[F_x]$ maps trivially to $H_{2c-3}(S_x^{2c-1}) = 0$. Thus $F_x \subset S_x^{2c-1}$ has a Seifert-pseudomanifold that can be used to define a linking number. For $\alpha \in \pi_1(S_x^{2c-1} - F_x)$, let $\mathrm{lk}(F_x, \alpha) \in \mathbb{Z}$ denote the linking number. The assignment $\alpha \mapsto t^{\mathrm{lk}(F_x,\alpha)}$ determines a local system $\mathcal{L}_x$ with stalks Λ on $S_x^{2c-1} - F_x$. The complex of sheaves $\mathbf{IC}^\bullet_{\bar{m}}(S_x^{2c-1}; \mathcal{L}_x)$ is defined by the Deligne extension process for the lower middle perversity $\bar{m}$ applied to $\mathcal{L}_x$. The pairing

$$\begin{aligned} \mathcal{L}_x \otimes \mathcal{L}_x^{\mathrm{op}} &\longrightarrow \Lambda, \\ f(t) \otimes g(t) &\mapsto f(t)g(t^{-1}) \end{aligned}$$

is perfect and Hermitian. Assuming that $F_x \subset S_x^{2c-1}$ is of finite local type, this pairing extends to a Verdier-superduality isomorphism

$$\mathbf{IC}^\bullet_{\bar{l}}(S_x^{2c-1}; \mathcal{L}_x)^{\mathrm{op}} \cong \mathcal{D}(\mathbf{IC}^\bullet_{\bar{m}}(S_x^{2c-1}; \mathcal{L}_x))[2c-1],$$

where $\bar{l}$ is the logarithmic perversity of [CS91], that is, $\bar{l}(s) = [(s+1)/2]$ so that $\bar{m}(s) + \bar{l}(s) = s - 1$ ($\bar{m}$ and $\bar{l}$ are "superdual"). If $F_x \subset S_x^{2c-1}$ is in addition of finite type, then this isomorphism induces upon taking hypercohomology an isomorphism

$$\begin{aligned} IH_i^{\bar{l}}(S_x^{2c-1}; \mathcal{L}_x)^{\mathrm{op}} &\cong \mathrm{Ext}(IH^{\bar{m}}_{2c-i-2}(S_x^{2c-1}; \mathcal{L}_x), \Lambda) \\ &= \mathrm{Hom}(IH^{\bar{m}}_{2c-i-2}(S_x^{2c-1}; \mathcal{L}_x), \mathbb{Q}(t)/\Lambda). \end{aligned}$$

With $(\mathcal{B}_V)_x = \mathrm{Image}(IH^{\bar{m}}_{c-1}(S^{2c-1}_x; \mathcal{L}_x) \to IH^{\bar{l}}_{c-1}(S^{2c-1}_x; \mathcal{L}_x))$, we thus get for $i = c-1$ a Blanchfield pairing

$$(\mathcal{B}_V)_x \otimes (\mathcal{B}_V)^{\mathrm{op}}_x \longrightarrow \mathbb{Q}(t)/\Lambda.$$

Remark 1 When $F_x = S^{2c-3}$ and $F_x \subset S^{2c-1}_x$ is locally flat, this is the classical Blanchfield pairing. If $F \subset S^{2c-1}$ is any locally flat submanifold, then according to [CS91, p. 339],

$$IH^{\bar{p}}_i(S^{2c-1}; \mathcal{L}) = \begin{cases} H_i(K; \mathcal{L}), & \bar{p}(2) = 0, \\ H_i(K, \partial K; \mathcal{L}), & \bar{p}(2) = 1, \end{cases}$$

where K is the exterior of F. In this situation, then, the above map $IH^{\bar{m}}_{c-1}(S^{2c-1}; \mathcal{L}) \to IH^{\bar{l}}_{c-1}(S^{2c-1}; \mathcal{L})$ becomes the map

$$H_{c-1}(K; \mathcal{L}) \longrightarrow H_{c-1}(K, \partial K; \mathcal{L})$$

induced by inclusion. For $F = S^{2c-3}$, $c \geq 3$, we have $\partial K = S^{2c-3} \times S^1$ and the map is an isomorphism.

Letting x vary over V, we obtain a Blanchfield local system

$$\mathcal{B}_V \otimes \mathcal{B}^{\mathrm{op}}_V \longrightarrow \mathbb{Q}(t)/\Lambda$$

over V. Again by the Deligne extension process, the associated Poincaré local system

$$\mathcal{B}^{\mathbb{R}}_V \otimes \mathcal{B}^{\mathbb{R}}_V \longrightarrow \mathbb{R}$$

extends to a self-duality isomorphism

$$\mathbf{IC}^{\bullet}_{\bar{m}}(\overline{V}; \mathcal{B}^{\mathbb{R}}_V) \cong \mathcal{D}\mathbf{IC}^{\bullet}_{\bar{m}}(\overline{V}; \mathcal{B}^{\mathbb{R}}_V)[m-2c].$$

This self-dual sheaf has L-classes

$$L_j(\overline{V}; \mathcal{B}^{\mathbb{R}}_V) \in H_j(\overline{V}; \mathbb{Q}),$$

assuming now that X is compact. Let $i_V : \overline{V} \hookrightarrow X$ be the inclusion. The Cappell-Shaneson L-class formula [CS91] for singular embeddings asserts that

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*} L_*(\overline{V}; \mathcal{B}^{\mathbb{R}}_V). \tag{1.7}$$

When $M = S^{n+2}$, $n > 0$, is a sphere, we have $i_*[X] = 0 \in H_n(S^{n+2})$ so that $\chi = 0$ and $L^*(P(M)) = 1$. Therefore,

$$L_*(S^{n+2}, X) = [X] \cap i^* L^*(P(M) \cup (1+\chi^2)^{-1}) = [X] \cap 1 = [X],$$

and in particular in degree 0,

$$L_0(S^{n+2}, X) = 0. \tag{1.8}$$

1.8 Embeddings and Strongly Transverse Coefficients

A synthesis of the characteristic class formula of Theorem 5 and the Cappell-Shaneson formula (1.7) yields the following result.

Theorem 6 *Let $i : X^n \hookrightarrow M^{n+2}$ be a PL embedding of an oriented compact pseudomanifold in an oriented compact manifold such that the pair (M, X) is stratifiable without odd-codimensional strata. Assume that all Poincaré local systems $\mathcal{B}_V^{\mathbb{R}}$ are strongly transverse to the singular set $\overline{V} - V$, $V \in \mathcal{X}$. Then*

$$L_*(X) = L_*(M, X) - \sum_{V \in \mathcal{X}} i_{V*}(\widetilde{\mathrm{ch}}[\mathcal{B}_V^{\mathbb{R}}]_K \cap L_*(\overline{V})). \tag{1.9}$$

Formula (1.9) holds automatically if X happens to be a manifold and the singular set of the embedded X has codimension at least 3. For in that case, the links in X are spheres of dimension 2 or higher which are simply connected. Thus we find ourselves in the supernormal situation of Corollary 2.

1.9 Nontransverse Coefficient Systems

We shall first consider singular embeddings $X^n \subset M^{n+2}$ which have at most 4-dimensional singularities whose pure components have definite real Blanchfield form. The 4-stratum may contain a 2-stratum which is a disjoint union of 2-spheres, embedded in the 4-stratum in a locally flat way and with zero self-intersection number. The following is an example of such a situation.

Example 2 Let A be a square integral matrix such that $A + A^T$ is unimodular. According to the realization theorem of Kervaire (Theorem 4), there exists a simple locally flat 7-knot $\kappa : S^7 \hookrightarrow S^9$ with Seifert matrix A. Applying the construction of Example 1, we obtain an embedding

$$i : X^{12} = S^2 \times S(S^1 \times S^8) \subset S^2 \times S^{12} = M^{14}.$$

The induced stratification has the form

$$M \supset X \supset X_4 = S^2 \times S(S^1 \times \{p_\pm\}) \supset X_2 = S^2 \times \{q_\pm\},$$

where $X_4 - X_2$ has 2 connected components $V_\pm = S^2 \times \mathring{I} \times S^1 \times \{p_\pm\}$ with closures $\overline{V}_\pm = S^2 \times S^2$, a 4-manifold, and $\overline{V}_+ - V_+ = S^2 \times \{q_\pm\} = \overline{V}_- - V_-$. The embeddings $S^2 \times \{q_\pm\} \subset \overline{V}_\pm = S^2 \times S^2$ are locally flat and have zero self-intersection number. The link pair of both V_- and V_+ is the 7-knot κ. Taking for instance the nonsingular matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we have $A + A^T = E_8$, which is unimodular with $\sigma(E_8) = 8$. By Kervaire's theorem, there exists a simple locally flat knot $\kappa : S^7 \subset S^9$ with Seifert matrix A. By formulae (1.4) and (1.5), the signature of the skew-Hermitian Blanchfield pairing β of κ is

$$\sigma(\beta^{\mathbb{R}}) = \sigma(A + A^T) = 8.$$

Since A is nonsingular, formula (1.3) shows that the knot $\mathbb{Z}[t, t^{-1}]$-module B_A determined algebraically by A as described in Sect. 1.2 has rational dimension

$$\dim_{\mathbb{Q}}(B_A \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rk} A = 8.$$

By Kearton's theorem (Theorem 3), $B_A \cong H_4(K_\infty)$, where K_∞ is the infinite cyclic cover of the exterior K of κ. Thus $H_4(K_\infty; \mathbb{Q})$ has dimension 8 over $\mathbb{Q}$. We conclude that the symmetric real Blanchfield form of κ is positive definite.

Theorem 7 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X = X_n \supset X_4 \supset X_2 \supset X_{-1} = \varnothing,$$

such that

(i) *for every connected component V of $X_4 - X_2$, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the link pair of every such V is a (necessarily nontrivial but locally flat) spherical knot (S^{n-3}, S^{n-5}) with definite real Blanchfield form of rank r_V,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2 \subset \overline{V}$, the latter embedding is locally flat with zero self-intersection number.*

Then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}),$$

where the sum ranges over all connected components V of $X_4 - X_2$ and $\epsilon_V = 1$ if the real Blanchfield form on V is positive definite and $\epsilon_V = -1$ if it is negative definite.

Proof Write $n = 4k$. Let V be a connected component of $X_4 - X_2$ and $\beta_V : \mathcal{B}_V \otimes \mathcal{B}_V^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ the associated Blanchfield local system with stalk

$$(\mathcal{B}_V)_x = IH^{\bar{m}}_{2k-2}(S_x^{4k-3}; \mathcal{L}_x) = H_{2k-2}(K_x; \mathcal{L}_x) \cong H_{2k-2}(K_x, \partial K_x; \mathcal{L}_x)$$

by Remark 1. At $x \in V$, $(\beta_V)_x$ is the classical Blanchfield pairing of the locally flat link pair (S_x^{4k-3}, S_x^{4k-5}). This pairing is skew-Hermitian. Its Poincaré local system $\beta_V^{\mathbb{R}} : \mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ is obtained using the Trotter trace as described in Sect. 1.3. This system is symmetric and by assumption definite of rank r_V. In principle, we shall use Theorem 4.1 of [Ban08] to compute the twisted signature $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$. That theorem was proven in a slightly different context, namely for complex Hermitian local systems and for smooth embeddings. The first issue is easily resolved by passing to the complexification $\mathcal{B}_{\mathbb{C}}$ of $\mathcal{B}_V^{\mathbb{R}}$ as described in Sect. 1.4. As we have seen, the signature does not change under complexification,

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) = \sigma(\overline{V}; \mathcal{B}_{\mathbb{C}}).$$

The second issue presents no problem either, since the proof of Theorem 4.1 [Ban08] essentially carries over to the PL category, with one minor addition concerning smoothability. Let us recall the argument. The stratum X_2 is comprised of pairwise disjoint two-spheres. Those two-spheres that lie in $\overline{V}$ are embedded there in a locally flat manner, whence they have a normal (block) bundle. That bundle is trivial by the assumption on the self-intersection number. We can thus do surgery on these two-spheres in $\overline{V}$ and obtain a closed PL manifold M^4. The surgery replaces each two-sphere by a circle, and M minus these circles is homeomorphic to V. Thus the local system $\mathcal{B}_{\mathbb{C}}$ on V is naturally defined on M minus the circles. The key observation is that the circles have high enough codimension (namely 3) in M in order for $\mathcal{B}_{\mathbb{C}}$ to extend (uniquely) onto all of M. (The link of a circle in M is a 2-sphere, which is simply connected. Thus $\mathcal{B}_{\mathbb{C}}$ is constant on the links of the circle and extends (uniquely) to the cone on the link.) Let us call this unique extension $\overline{\mathcal{B}}_{\mathbb{C}}$. In fact it is not hard to see that $\mathcal{B}_{\mathbb{C}}$ and $\overline{\mathcal{B}}_{\mathbb{C}}$ extend further as local systems over the trace W of the surgery, so that W together with this extension is a bordism between $(\overline{V}; \mathcal{B}_{\mathbb{C}})$ and $(M; \overline{\mathcal{B}}_{\mathbb{C}})$. By bordism invariance of the twisted signature,

$$\sigma(\overline{V}; \mathcal{B}_{\mathbb{C}}) = \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}).$$

At this point, the proof of Theorem 4.1 [Ban08] is able to invoke W. Meyer's twisted signature formula [Mey72] because in that context the manifold M is smooth. Our present M however is piecewise linear. The Hirsch-Mazur obstructions to smoothing M lie in $H^i(M; \pi_{i-1}(PL/O))$. They all vanish because PL/O is 6-connected and M is 4-dimensional. Thus M is smoothable and we may indeed call on Meyer's formula

$$\sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) = \langle \widetilde{\mathrm{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K \cup L^*(M), [M] \rangle,$$

where $L^*(M) = L^*(P(M))$. Since $\beta_V^{\mathbb{R}}$, and thus also the complexification of $\beta_V^{\mathbb{R}}$, is definite of rank r_V, the K-theory signature $[\overline{\mathcal{B}}_{\mathbb{C}}]_K$ of $\overline{\mathcal{B}}_{\mathbb{C}}$ is given by

$$[\overline{\mathcal{B}}_{\mathbb{C}}]_K = \epsilon_V [\overline{\mathcal{B}}_{\mathbb{C}}] \in \mathrm{KU}^0(M),$$

where we regard $\overline{\mathcal{B}}_{\mathbb{C}}$ as a flat complex vector bundle of rank r_V. The positive dimensional rational Chern classes of $\overline{\mathcal{B}}_{\mathbb{C}}$ vanish by flatness, so that $\widetilde{\mathrm{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K = \epsilon_V r_V$. Therefore,

$$\begin{aligned}\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) &= \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) \\ &= \epsilon_V r_V \langle L^*(M), [M] \rangle \\ &= \epsilon_V r_V \sigma(M) \\ &= \epsilon_V r_V \sigma(\overline{V}),\end{aligned}$$

using the Hirzebruch signature theorem and bordism invariance.

Let S be a connected component of X_2 and $\beta_S : \mathcal{B}_S \otimes \mathcal{B}_S^{\mathrm{op}} \to \mathbb{Q}(t)/\Lambda$ the associated Blanchfield local system with stalk

$$(\mathcal{B}_S)_x = \mathrm{Image}(IH_{2k-1}^{\bar{m}}(S_x^{4k-1}; \mathcal{L}_x) \longrightarrow IH_{2k-1}^{\bar{l}}(S_x^{4k-1}; \mathcal{L}_x))$$

at $x \in V$. The link pair at x has the form (S_x^{4k-1}, F_x) with F_x a PL pseudomanifold of dimension $4k-3$. The pairing β_S is Hermitian as $2k-1$ is odd. By assumption, S is a 2-sphere, in particular simply connected. This implies that $\mathcal{B}_S$ is constant (untwisted) on S. Thus the corresponding Poincaré local system $\mathcal{B}_S^{\mathbb{R}}$ is constant and $\beta_S^{\mathbb{R}} : \mathcal{B}_S^{\mathbb{R}} \otimes \mathcal{B}_S^{\mathbb{R}} \to \mathbb{R}$ is skew-symmetric. It follows that the signature of any stalk $(\mathcal{B}_S^{\mathbb{R}})_x$ is zero, $\sigma((\mathcal{B}_S^{\mathbb{R}})_x) = 0$. Since $\mathcal{B}_S^{\mathbb{R}}$ is constant on S, the twisted signature factors as

$$\sigma(S; \mathcal{B}_S^{\mathbb{R}}) = \sigma((\mathcal{B}_S^{\mathbb{R}})_x) \cdot \sigma(S) = 0.$$

Assembling the above information using the Cappell-Shaneson L-class formula (1.7) for singular embeddings, we obtain

$$\begin{aligned}\sigma(X) &= L_0(X) \\ &= L_0(M, X) - \sum_{V \subset X_4 - X_2} \sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) - \sum_{S \subset X_2} \sigma(S; \mathcal{B}_S^{\mathbb{R}}) \\ &= L_0(M, X) - \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}).\end{aligned}$$

□

Corollary 3 *Let $i : X^n \hookrightarrow S^{n+2}$, $n \equiv 0(4), n > 0$, be a PL embedding of a compact oriented PL pseudomanifold X in a sphere satisfying the hypotheses of Theorem 7. Then*

$$\sigma(X) + \sum_{V \subset X_4 - X_2} \epsilon_V r_V \sigma(\overline{V}) = 0.$$

Proof Observe that $L_0(S^{n+2}, X) = 0$ according to (1.8). □

Although not always explicitly stated, similar corollaries for embeddings in spheres can be deduced in the contexts of the subsequent results as well.

For a space Y, let $Sh_{\text{lch}}(Y)$ denote the collection of isomorphism classes of locally constant perfect complex Hermitian sheaves of finite rank on Y. The following theorem extends Theorem 7 to the case of an indefinite structure group $U(p,q)$. Its conclusion reduces to the conclusion of Theorem 7 when $p = 0$ or $q = 0$. We do maintain the zero self-intersection assumption for now.

Theorem 8 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X = X_n \supset X_4 \supset X_2 \supset X_{-1} = \varnothing,$$

such that

(i) *for every connected component V of $X_4 - X_2$, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the complexified Blanchfield system $\mathcal{B}_V^{\mathbb{C}}$ of the link pair of every such V has structure group $U(p_V, q_V)$,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2 \subset \overline{V}$, the latter embedding is locally flat with zero self-intersection number.*

Then there exists an integral characteristic class

$$2(c_1^2 - 2c_2) : Sh_{\text{lch}}(V) \longrightarrow H^4(\overline{V}; \mathbb{Z})$$

such that

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} (p_V - q_V)\sigma(\overline{V}) - \sum_{V \subset X_4 - X_2} \langle 2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}]\rangle,$$

where the two sums range over all connected components V of $X_4 - X_2$.

Proof Let V be a connected component of $X_4 - X_2$ with associated real Blanchfield system $\mathcal{B}_V^{\mathbb{R}}$. The pairing $\mathcal{B}_V^{\mathbb{R}} \otimes \mathcal{B}_V^{\mathbb{R}} \to \mathbb{R}$ is symmetric. Thus the complexified form $\mathcal{B}_V^{\mathbb{C}} \otimes (\mathcal{B}_V^{\mathbb{C}})^{\text{op}} \to \mathbb{C}_V$ is Hermitian with structure group $U(p_V, q_V)$. In order to compute the twisted signature $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}})$, we modify the proof of Theorem 4.3 [Ban08] so that it applies to PL spaces. As in the proof of the previous Theorem 7, we can do surgery on $X_2 \cap \overline{V}$ to obtain a PL manifold M^4 with a perfect Hermitian local system $\overline{\mathcal{B}}_{\mathbb{C}}$ defined everywhere on M such that

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) = \sigma(M; \overline{\mathcal{B}}_{\mathbb{C}})$$

via the trace of the surgery. The manifold M is smoothable because it is 4-dimensional, and therefore

$$\begin{aligned}\sigma(M; \overline{\mathcal{B}}_{\mathbb{C}}) &= \langle \widetilde{\text{ch}}[\overline{\mathcal{B}}_{\mathbb{C}}]_K \cup L^*(M), [M]\rangle \\ &= \Big\langle ((p_V - q_V) + 2c_1[\overline{\mathcal{B}}_{\mathbb{C}}]_K + 2(c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K)\end{aligned}$$

$$\cup\left(1+\frac{1}{3}p_1(M)\right),[M]\right\rangle$$
$$= (p_V - q_V)\sigma(M) + 2\langle (c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K, [M]\rangle.$$

There is a unique isomorphism

$$\phi : H^4(M) \xrightarrow{\cong} H^4(\overline{V})$$

such that

$$\begin{tikzcd}
H_c^4(M - \bigsqcup(D^3 \times S^1)) \arrow[r, equal] \arrow[d, "\cong"'] & H_c^4(V) \arrow[d, "\cong"] \\
H^4(M) \arrow[r, "\cong", "\phi"'] & H^4(\overline{V})
\end{tikzcd}$$

commutes, where $H_c^*(-)$ denotes cohomology with compact supports and the vertical maps are given by extension by zero. We set

$$2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}) = \phi(2(c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K) \in H^4(\overline{V}; \mathbb{Z}).$$

Let γ be a 4-dimensional PL cochain on M representing the cohomology class $2(c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K$. Since extension by zero is here an isomorphism in dimension 4, we may assume that γ has compact support in $M - \bigsqcup(D^3 \times S^1)$. As

$$M - \bigsqcup(D^3 \times S^1) \cong V,$$

γ is a cochain on V with compact support and thus, by extension by zero, a cochain on $\overline{V}$. This cochain represents $2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}})$ and

$$2\langle (c_1^2 - 2c_2)[\overline{\mathcal{B}}_{\mathbb{C}}]_K, [M]\rangle = \langle (2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}]\rangle.$$

Since $\sigma(M) = \sigma(\overline{V})$ by bordism invariance, we have

$$\sigma(\overline{V}, \mathcal{B}_V^{\mathbb{R}}) = (p_V - q_V)\sigma(\overline{V}) + \langle (2(c_1^2 - 2c_2)(\mathcal{B}_V^{\mathbb{C}}), [\overline{V}]\rangle.$$

As in the proof of Theorem 7, $\sigma(S; \mathcal{B}_S^{\mathbb{R}}) = 0$ for every connected component S of X_2. The result follows from substituting the above information into the Cappell-Shaneson L-class formula

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \sigma(\overline{V}; \mathcal{B}_V^{\mathbb{R}}) - \sum_{S \subset X_2} \sigma(S; \mathcal{B}_S^{\mathbb{R}}).$$

□

Theorem 8 together with Corollary 4.4 [Ban08] and Proposition 4.5 [Ban08] imply:

Corollary 4 *Let (M^{n+2}, X^n) be stratified as in Theorem 8 and assume that $\overline{V}$ is a 4-sphere for every connected component V of $X_4 - X_2$. Then*

$$\sigma(X) - L_0(M, X)$$

is divisible by 8. If for every V, $\overline{V} - V$ is connected and $X_2 \cap \overline{V} \hookrightarrow \overline{V}$ is the Artin spin of a classical knot, then

$$\sigma(X) = L_0(M, X).$$

When a lower stratum has nonzero self-intersection inside a higher one, rho-invariants enter into signature formulae, as the next theorem illustrates. Let (p, q) be coprime integers such that $0 \le q < p$ and write $\mathbb{Z}/_p = \{1, \xi, \xi^2, \dots, \xi^{p-1}\}$, ξ a primitive p-th root of unity. For a representation $\alpha : \mathbb{Z}/_p \to U(k)$, let $\chi_\alpha : \mathbb{Z}/_p \to \mathbb{C}$ denote the character of α. Set

$$\rho_\alpha(p, q) = \frac{1}{p} \sum_{j=1}^{p-1} (k - \chi_\alpha(\xi^j)) \cot \frac{j\pi}{p} \cot \tfrac{j\pi q}{p}.$$

The *constancy rank*, c-rk$(\mathcal{S})$, of a local system $\mathcal{S}$ on a connected space with cyclic fundamental group is defined to be the rank of the 1-eigenspace of the monodromy matrix of $\mathcal{S}$.

Theorem 9 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M which induces a stratification of the form*

$$X = X_n \supset X_4 \supset X_2 \supset X_{-1} = \varnothing,$$

such that

(i) *for every connected component V of $X_4 - X_2$, the closure $\overline{V}$ is a 4-manifold,*
(ii) *the link pair of every such V is a (necessarily nontrivial but locally flat) spherical knot (S^{n-3}, S^{n-5}) with positive, say, definite complex Blanchfield form $\mathcal{B}_V^{\mathbb{C}}$ of rank r_V,*
(iii) *X_2 is a disjoint union of 2-spheres, and*
(iv) *for every such S^2 and 4-dimensional V with $S^2 \subset \overline{V}$, the latter embedding is locally flat with nonzero self-intersection number.*

Then

$$\sigma(X) = L_0(M, X) - \sum_{V \subset X_4 - X_2} \left(r_V \sigma(V) + \sum_{i=1}^{n_V} (\text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \operatorname{sign}[S_i^2]^2 - \rho_{\alpha_i}(p_i, q_i)) \right),$$

where the sum ranges over all connected components V of $X_4 - X_2$, $\sigma(V)$ denotes the (Novikov-) signature of the exterior of the link $\overline{V} \cap X_2 = \bigsqcup_{i=1}^{n_V} S_i^2 \subset \overline{V}$, $L_i =$

$L(p_i, q_i)$, a lens space, is the boundary of a regular neighborhood of S_i^2 in $\overline{V}$, and α_i is obtained by restricting $\mathcal{B}_V^{\mathbb{C}}$ to L_i.

Proof Let V be a connected component of $X_4 - X_2$. The locally flat PL-2-link

$$\overline{V} \cap X_2 = \bigsqcup S_i^2 \hookrightarrow \overline{V}$$

is isotopic to a smooth 2-link

$$\bigsqcup S_i^2 \overset{C^\infty}{\hookrightarrow} M^4,$$

where M is a smooth 4-manifold homeomorphic to $\overline{V}$. The isotopy ensures that $\mathcal{B}_V^{\mathbb{C}}$ defines a complex Blanchfield local system on the complement of the smooth 2-link and

$$\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{C}}) = \sigma(M; \mathcal{B}_V^{\mathbb{C}}).$$

The latter signature can be computed using Theorem 4.8 of [Ban08]. Let us recall the method. Let $(E^4, \partial E)$ be the exterior of the smooth 2-link. Its boundary $\partial E = \bigsqcup L_i$ is a disjoint union of lens spaces $L_i = L(p_i, q_i)$ with finite fundamental group $\mathbb{Z}/_{p_i}$, $p_i \geq 1$, since S_i^2 has nonzero self-intersection number by (iv). Let $(N_i^4, \partial N_i = L_i)$ be the total space of the disc bundle of $S_i^2 \subset M$ so that

$$\partial E = \bigsqcup \partial N_i, \qquad M = E \cup_{\partial E} \bigcup N_i.$$

By Novikov additivity,

$$\sigma(M; \mathcal{B}_V^{\mathbb{C}}) = \sigma(E; \mathcal{B}_V^{\mathbb{C}}) + \sum_{i=1}^{n_V} \sigma(N_i; \mathcal{B}_V^{\mathbb{C}}).$$

Let us first discuss the terms $\sigma(N_i; \mathcal{B}_V^{\mathbb{C}})$, where $\mathcal{B}_V^{\mathbb{C}}$ is only given on the complement of the zero-section. This complement deformation retracts onto $\partial N_i = L_i$, whence $\mathcal{B}_V^{\mathbb{C}}$ is determined by a unitary representation $\alpha_i : \pi_1(L_i) = \mathbb{Z}/_{p_i} \to U(r_V)$, given by a monodromy matrix $A \in U(r_V)$. Diagonalizing A, we obtain a decomposition $\mathcal{B}_V^{\mathbb{C}}|_{L_i} \cong \mathcal{B}_{(1)} \oplus \mathcal{B}'$, where $\mathcal{B}_{(1)}$ is a constant sheaf of rank c-rk$(\mathcal{B}_V^{\mathbb{C}}|_{L_i})$, corresponding to the eigenvalue 1 (if it is present) of A, and the monodromy matrix of $\mathcal{B}'$ does not have 1 among its eigenvalues. Since $\sigma(N_i) = \operatorname{sign}[S_i^2]^2$, where S_i^2 is the zero-section of N_i, we have

$$\sigma(N_i; \mathcal{B}_{(1)}) = \text{c-rk}(\mathcal{B}_V^{\mathbb{C}}|_{L_i}) \cdot \operatorname{sign}[S_i^2]^2.$$

Since 1 is not an eigenvalue of $\mathcal{B}'$, the intersection chain sheaf $\mathbf{IC}^{\bullet}_{\bar{m}}(\overset{\circ}{N}_i; \mathcal{B}')$ is zero over $S_i^2 \subset \overset{\circ}{N}_i$ and thus for the middle hypercohomology,

$$\mathcal{H}_c^{-2}(\overset{\circ}{N}_i; \mathbf{IC}^{\bullet}_{\bar{m}}(\mathcal{B}')) \cong H_c^2(\overset{\circ}{N}_i - S_i^2; \mathcal{B}').$$

From a transfer-map argument involving the universal cover $S^3 \to L(p_i, q_i)$, one infers $H_c^2(\overset{\circ}{N}_i - S_i^2; \mathcal{B}') = 0$. In particular $\sigma(N_i; \mathcal{B}') = 0$, and therefore

$$\sigma(N_i; \mathcal{B}_V^{\mathbb{C}}) = \sigma(N_i; \mathcal{B}_{(1)}).$$

By [APS75], the difference between the untwisted and the twisted signature of E is a differential invariant of ∂E, the rho-invariant

$$\rho(\partial E; \mathcal{B}_V^{\mathbb{C}}) = r_V \sigma(E) - \sigma(E; \mathcal{B}_V^{\mathbb{C}}).$$

Thus

$$\begin{aligned} \sigma(E; \mathcal{B}_V^{\mathbb{C}}) &= r_V \sigma(E) - \rho\left(\bigsqcup_i L_i; \bigsqcup_i \mathcal{B}_V^{\mathbb{C}}|_{L_i}\right) \\ &= r_V \sigma(V) - \sum_i \rho(L_i; \mathcal{B}_V^{\mathbb{C}}|_{L_i}) \\ &= r_V \sigma(V) - \sum_i \rho_{\alpha_i}(p_i, q_i). \end{aligned}$$

□

We shall now turn our attention to fibered embeddings of strata; the dimension of the singular set is arbitrary.

Theorem 10 *Let $i : X^n \hookrightarrow M^{n+2}$, $n \equiv 0(4)$, be a PL embedding of a compact oriented PL pseudomanifold X in a closed oriented PL manifold M with stratification*

$$X = X_n \supset X_{n-2} \supset X_{n-4} \supset \cdots \supset X_{-1} = \varnothing.$$

If $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot for all $V \in \mathcal{X}$, then

$$\sigma(X) = L_0(M, X).$$

Proof This follows from the proof of Theorem 4.7 in [Ban08], by using block bundles instead of fiber bundles. Let us recall the argument briefly. Let V be a component in $\mathcal{X}$. By assumption, the embedding $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot $S^k \hookrightarrow S^{k+2}$. The complement of this knot carries the complexified Blanchfield form $\mathcal{B}_V^{\mathbb{C}} \otimes (\mathcal{B}_V^{\mathbb{C}})^{\mathrm{op}} \to \mathbb{C}_V$. By assumption, the exterior E PL-fibers over a circle with Seifert manifold fiber F, i.e. E is a block bundle over S^1: the circle is triangulated by a finite simplicial complex K, for every simplex $\Delta \in K$, there is a block $\Delta \times F$, and E is obtained from the disjoint union of all these blocks by gluing $\Delta^0 \times F$ to $\Delta^1 \times F$ for every 1-simplex $\Delta^1 \in K$ and 0-simplex $\Delta^0 \in K$ such that Δ^0 is a face of Δ^1. The gluing is effected by a PL-homeomorphism $f(\Delta^1, \Delta^0) : F \to F$, which is the identity on $\partial F = S^k$. Set $F' = F \cup_{\partial F} D^{k+1}$. Since every $f(\Delta^1, \Delta^0)$ is the identity on ∂F, we can extend it to a PL-homeomorphism

$f'(\Delta^1, \Delta^0) : F' \to F'$ by taking f' to be the identity on D^{k+1}. Using the system $\{f'(\Delta^1, \Delta^0)\}$ to glue the blocks $\Delta \times F'$, $\Delta \in K$, we obtain the total space M^{k+2} of an F'-block bundle over S^1. This manifold M is the result of surgery on the knot. Let P be the total space of the cone-block bundle associated to the blocking of M. That is, if PL-homeomorphisms $cf'(\Delta^1, \Delta^0) : cF' \to cF'$ on the cone cF' of F' are defined by coning $f'(\Delta^1, \Delta^0)$, then P is obtained by using the system $\{cf'(\Delta^1, \Delta^0)\}$ to glue the blocks $\Delta \times cF'$, $\Delta \in K$. Let $\mathcal{B}_M^{\mathbb{C}}$ be the unique extension of $\mathcal{B}_V^{\mathbb{C}}$ to M. Then, as in the proof of Theorem 7, $\sigma(\overline{V}; \mathcal{B}_V^{\mathbb{C}}) = \sigma(M; \mathcal{B}_M^{\mathbb{C}})$. The space P is a stratified pseudomanifold-with-boundary, with stratification $P_{k+3} = P \supset P_1 = S^1$, $\partial P = M$. The singular stratum contains the cone-points of the cF'. On the interior $M \times (0, 1)$ of the top stratum, $\mathcal{B}_M^{\mathbb{C}}$ defines a perfect ± 1-Hermitian local system, which can be extended into the singular stratum P_1 by the middle-perversity Deligne step. The result is an intersection chain sheaf $\mathbf{IC}^\bullet_{\bar{m}}(P - \partial P; \mathcal{B}_M^{\mathbb{C}})$ which is self-dual, as P_1 has even codimension $k + 2$ in P. Thus

$$(P; \mathbf{IC}^\bullet_{\bar{m}}(P - \partial P; \mathcal{B}_M^{\mathbb{C}}))$$

is a null-cobordism for $(M; \mathcal{B}_M^{\mathbb{C}})$ and $\sigma(M; \mathcal{B}_M^{\mathbb{C}}) = 0$. Thus the contributions of all V in the Cappell-Shaneson L-class formula (1.7) vanish. □

Corollary 5 *Let $i : X^n \hookrightarrow S^{n+2}$, $n \equiv 0(4)$, $n > 0$, be a PL embedding of a compact oriented PL pseudomanifold X in a sphere with stratification*

$$X = X_n \supset X_{n-2} \supset X_{n-4} \supset \cdots \supset X_{-1} = \varnothing.$$

If $\overline{V} - V \hookrightarrow \overline{V}$ is a locally flat spherical fibered knot for all $V \in \mathcal{X}$, then

$$\sigma(X) = 0.$$

Proof We have $L_0(S^{n+2}, X) = 0$ by (1.8). □

Acknowledgements The authors thank Laurentiu Maxim for correcting an expository error in Sect. 1.7.

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http://www.springer.com/978-3-642-15636-6

The Mathematics of Knots
Theory and Application
(Eds.)M. Banagl; D. Vogel
2011, X, 364 p. 20 illus. in color., Hardcover
ISBN: 978-3-642-15636-6