Preface

The homology of manifolds enjoys a remarkable symmetry: Poincaré duality. If the manifold is triangulated, then this duality can be established by associating to a simplex its dual block in the barycentric subdivision. In a manifold, the dual block is a cell, so the chain complex based on the dual blocks computes the homology of the manifold. Poincaré duality then serves as a cornerstone of manifold classification theory. One reason is that it enables the definition of a fundamental bordism invariant, the signature. Classifying manifolds via the surgery program relies on modifying a manifold by executing geometric surgeries. The trace of the surgery is a bordism between the original manifold and the result of surgery. Since the signature is a bordism invariant, it does not change under surgery and is thus a basic obstruction to performing surgery. Inspired by Hirzebruch's signature theorem, a method of Thom constructs characteristic homology classes using the bordism invariance of the signature. These classes are not in general homotopy invariants and consequently are fine enough to distinguish manifolds within the same homotopy type.

Singular spaces do not enjoy Poincaré duality in ordinary homology. After all, the dual blocks are not cells anymore, but cones on spaces that may not be spheres. This book discusses when, and how, the invariants for manifolds described above can be established for singular spaces. By singular space we mean here a space which has points whose neighborhoods are not Euclidean, but which can still be decomposed into subsets, each of which *is* a manifold. Such a decomposition is called a stratification, provided certain conditions are satisfied where the subsets (the "strata") meet. Thus we will not be concerned with spaces that possess an intricate local structure, such as fractals. Spaces that can be stratified include all triangulated spaces, as well as algebraic varieties. Once these invariants have been constructed, we will furthermore describe tools for their computation. For instance, we will see how the invariants behave under maps.

The germ for this book was a topology seminar on sheaf theory and intersection homology which I gave in the spring of 2000 at the University of Wisconsin, Madison, and my spring 2001 special topics course on characteristic classes of stratified spaces and maps, held at the same institution, whose syllabus included Witt space bordism, the Goresky-MacPherson L-class, Verdier self-dual complexes of sheaves, perverse sheaves and t-structures, algebraic bordism of self-dual sheaves and the Cappell-Shaneson L-class formulae for stratified maps. I would like to take this opportunity to thank the Mathematics Department of the University of Wisconsin, Madison, for giving me the chance to teach these courses. The original lecture notes were subsequently expanded and, to some extent, enriched with detail.

The text has then been used as the basis for a seminar on stratified spaces and intersection homology which I taught in the winter quarter 2004 at the University of Cincinnati, as well as for the Oberseminar Topologie at the University of Heidelberg, which in the fall of 2004 was devoted to intersection homology and topological invariants of singular spaces. My colleagues' and students' feedback during these seminars led to several improvements of the text.

I have made an effort to make the book accessible to a second year graduate student in topology. The prerequisites are thus modest: (Co)homology theory, simplicial complexes, basic differential topology such as regular values and transversality, the language of categories and functors. As far as intersection homology theory is concerned, I start *ab initio* — no prior knowledge of sheaf theory, triangulated categories, derived categories or Verdier duality is required. In addition, I hope that the book may be useful for the research mathematician who wishes to learn about intersection homology and the invariants of singular spaces it effectuates. The pace of the book is intentionally *not* uniform: I change gears by omitting some details precisely when there is a danger that a flood of technicalities may prevent the reader from seeing the forest for the trees.

Chapter 1 introduces sheaves, cohomology with coefficients in a sheaf (via injective resolutions and via the Čech resolution), complexes of sheaves (i.e. differential graded sheaves) and cohomology with coefficients in a complex of sheaves (the so-called hypercohomology). One way to construct the intersection homology groups is as the hypercohomology groups of a certain complex of sheaves, the Deligne-Goresky-MacPherson intersection chain complex. This chapter also contains the definition of truncation functors that are so important in the definition of the intersection chain complex.

Chapter 2 discusses concepts that are applicable to any abelian category, hence it does not depend on the previous chapter. We give a self-contained treatment of triangulated categories and localization of categories. In particular, localization with respect to quasi-isomorphisms of complexes will give rise to derived categories. Derived functors are constructed pragmatically.

Chapter 3 on Verdier duality introduces the functors $f_!$, $f^!$ operating on sheaf complexes, where f is a continuous map. The latter functor is well-defined only on the derived category, so that this chapter depends on chapter 2. We then use $f^!$ to construct the Verdier dualizing complex of a space and the Verdier duality functor $\mathcal{D}$. Verdier self-dual sheaves, i.e. sheaves that are isomorphic to their own dual, play a central role in this book, since all invariants to be discussed are induced by self-duality isomorphisms.

Chapter 4 defines stratified pseudomanifolds and intersection homology on stratified pseudomanifolds. This is done in two different ways: In section 4.1, intersection chains are defined for a piecewise linear pseudomanifold as a subcomplex of the complex of all piecewise linear chains, by placing certain conditions on how a chain is supposed to meet strata. This is the approach that was taken by Goresky and MacPherson in their first paper [**GM80**]. Readers who wish to get a geometrically intuitive first introduction to intersection homology without having to absorb all of the material from chapters 1 through 3 may wish to peruse section 4.1, except subsection 4.1.4, before reading the rest of the book. In section 4.2, all of chapters 1 through 3 are applied in constructing intersection homology sheaf-theoretically. This is the approach that was taken by Goresky and MacPherson in their second paper [**GM83**].

Chapter 5 reviews manifold theory. Apart from setting the stage for subsequent chapters, this is done in order to place the signature and L-classes into a larger context so that from this elevated vantage point, the less advanced reader may get a better impression of their importance and applications. We compute smooth oriented bordism groups rationally and give a rudimentary introduction to surgery theory. In order to understand the construction of L-classes for singular spaces, it is helpful to be familiar with section 5.7, where Thom's method for obtaining L-classes from the signatures of submanifolds is explained.

The remaining chapters deal with the signature and L-class of singular spaces and are structured in a progression from more restricted stratifications towards less restricted ones.

Chapter 6 works on spaces that have restrictions along the strata of odd codimension. Section 6.1, for example, defines the signature, and section 6.3 the L-class, of spaces that have no strata of odd codimension. This class of spaces includes all complex algebraic varieties.

The derived categories constructed in chapter 2 are generally not abelian, but have the structure of a triangulated category. The distinguished triangles substitute for the exact sequences. Chapter 7 explains how one can obtain abelian subcategories of triangulated categories by endowing them with certain truncation-structures, or t-structures. Section 7.1 is purely algebraic and requires only chapter 2 as prerequisite. The derived category of complexes of sheaves can be endowed with a particularly important t-structure, the perverse t-structure, which is discussed in sections 7.2 and 7.3.

Chapter 8 provides tools for the computation of the characteristic classes arising from self-dual sheaves. We are mainly interested in the behavior of these classes under stratified maps. We define when two self-dual complexes of sheaves on the same space are algebraically bordant. (This is not to be confused with geometric bordism of spaces covered with self-dual sheaves.) This chapter has two highlights: In section 8.1, we give a fully detailed proof of the Cappell-Shaneson decomposition theorem for self-dual sheaves on a space with only even-codimensional strata. The theorem states that every self-dual sheaf is bordant to an orthogonal sum of intersection chain sheaves. All signature- and L-class-formulae are then implied by this. These formulae, however, contain twisted signatures and twisted L-classes. Section 8.3 shows how to calculate those when the underlying space is a Witt space.

All invariants previously constructed require the singular space to satisfy the Witt-condition. Generalized Poincaré-Verdier self-duality on singular spaces that do not necessarily satisfy the Witt condition ("non-Witt spaces") is discussed in chapter 9. We will see that self-duality on such completely general stratified spaces hinges on the existence of Lagrangian structures. If a space possesses no Lagrangian structures, then it possesses no self-dual homology groups compatible with intersection homology. The L-class of non-Witt spaces is constructed in section 9.2. The behavior of these classes under stratified maps is investigated in section 9.3, while section 9.4 once more takes up twisted characteristic classes, this time on non-Witt spaces.

Chapter 10 gives a brief introduction to Cheeger's method of recovering Poincaré duality for singular Riemannian spaces using L^2 differential forms on the top stratum. The relation to the intersection homology of Goresky and MacPherson is discussed.

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