POSITIVE TOPOLOGICAL QUANTUM FIELD THEORIES

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ABSTRACT. We propose a new notion of positivity for topological field theories (TFTs), based on S. Eilenberg's concept of completeness for semirings. We show that a complete ground semiring, a system of fields on manifolds and a system of action functionals on these fields determine a positive TFT. The main feature of such a theory is a semiring-valued topologically invariant state sum that satisfies a gluing formula. The abstract framework has been carefully designed to cover a wide range of phenomena. We indicate how to employ the framework presented here in constructing a new differential topological invariant that detects exotic smooth structures on spheres.

CONTENTS

1. INTRODUCTION

We propose a mathematically rigorous new method for constructing topological field theories (TFT), which allows for action functionals that take values in any monoidal category, not just real (or complex) values. The second novelty of our approach is that state

Date: October, 2013.

2010 *Mathematics Subject Classification.* 57R56, 81T45, 16Y60.

Key words and phrases. Topological quantum field theory, complete semirings, function semialgebras, topological manifolds, smooth manifolds, exotic spheres, singularities of smooth maps, fold maps.

The author was in part supported by a research grant of the Deutsche Forschungsgemeinschaft.

sums (path integrals) are based on an algebraic notion of completeness in semirings, introduced by S. Eilenberg in his work [Eil74] on formal languages and automata, informed by ideas of J. H. Conway [Con71]. This enables us to bypass the usual measure theoretic problems associated with traditional path integrals over spaces of fields. On the other hand, the oscillatory nature of the classical path integral, and consequently, associated asymptotic expansions via the method of stationary phase, are not necessarily retained in our approach. Instead, a principle of topological permanence tends to hold: Once the action functional detects a topological feature, this feature will remain present in the state sum invariant. Roughly, a semiring is a ring that has only "positive" elements, that is, has no negatives (and is thus sometimes also called a "rig"). A key insight of Eilenberg was that the absence of negatives allows for the notion of completeness, reviewed in Section 2 of the present paper. As our state sums will have values in semirings, we call the resulting theories *positive* TFTs. In this context one may also recall that every additively idempotent semiring has a canonical partial order, and in this order every element is nonnegative. In their work [BD95], J. C. Baez and J. Dolan remark on page 6100: "In Physics, linear algebra is usually done over $\mathbb{R}$ or $\mathbb{C}$, but for higher-dimensional linear algebra, it is useful to start more generally with any commutative rig" and "one reason we insist on such generality is to begin grappling with the remarkable fact that many of the important vector spaces in physics are really defined over the natural numbers, meaning that they contain a canonical lattice with a basis of 'positive' elements. Examples include the weight spaces of semisimple Lie algebras, fusion algebras in conformal field theory, and thanks to the work of Kashiwara and Lusztig on canonical bases, the finite-dimensional representations of quantum groups." Thus this emerging theme of positivity and semirings in physics is also reflected in the present paper.

A positive TFT possesses the following features, which may be taken as axioms. These axioms are close to Atiyah's axioms [Ati88], but differ in some respects, as we will discuss shortly. As far as the algebraic environment is concerned, instead of working with vector spaces over a field such as the real or complex numbers, a positive TFT is defined over a pair (Q^c, Q^m) of semirings, which have the same underlying additive monoid Q. Furthermore, a functional tensor product $E\widehat{\otimes}F$ of two function semimodules E,F is used, which in the idempotent regime is due to G. L. Litvinov, V. P. Maslov and G. B. Shpiz [LMS99]. It comes with a canonical map $E \otimes F \to E\widehat{\otimes}F$ (where $\otimes$ is the algebraic tensor product), which is generally neither surjective nor even injective (contrary to the analogous map over fields such as the complex numbers). It can be conceptualized as a completion of the image of the map which sends an algebraic tensor product $f \otimes g$ to the function $(x,y) \mapsto f(x)g(y)$. This completion is necessary because Example 6.9 shows that state sums need not lie in the image of the algebraic tensor product. The two multiplications on Q induce two generally different tensor products $f\widehat{\otimes}_c g$ and $f\widehat{\otimes}_m g$ of two Q-valued functions f,g.

An $(n+1)$-dimensional *positive TFT* Z assigns to every closed topological n-manifold M a semialgebra $Z(M)$ over both Q^c and Q^m, called the *state module* of M, and to every $(n+1)$-dimensional topological bordism (W,M,N) with boundary $\partial W = M \sqcup N$ (where M is the incoming and N the outgoing boundary) an element

$$Z_W \in Z(\partial W),$$

called the *state sum* of W. We adopt the viewpoint that the latter is the primary invariant, while the state module itself is of lesser importance. This assignment satisfies the following properties:

(1) If M and N are closed n-manifolds and $M \sqcup N$ their (ordered) disjoint union, then there is an isomorphism

$$Z(M \sqcup N) \cong Z(M) \widehat{\otimes} Z(N)$$

of Q^c-semialgebras and of Q^m-semialgebras.

(2) The state module $Z(-)$ is a covariant functor on the category of closed topological n-manifolds and homeomorphisms. In particular, the group $\mathrm{Homeo}(M)$ of self-homeomorphisms $M \to M$ acts on $Z(M)$.

(3) *Pseudo-Isotopy Invariance*: Pseudo-isotopic homeomorphisms $\phi, \psi : M \to N$ induce equal isomorphisms $\phi_* = \psi_* : Z(M) \to Z(N)$. In particular, the action of $\mathrm{Homeo}(M)$ on $Z(M)$ factors through the mapping class group.

(4) The state sum Z_W is a topological invariant: If $\phi : W \to W'$ is a homeomorphism and ϕ_∂ its restriction to the boundary, then $\phi_{\partial *}(Z_W) = Z_{W'}$, where $\phi_{\partial *} : Z(\partial W) \to Z(\partial W')$ denotes the isomorphism induced by ϕ_∂.

(5) The state sum $Z_{W \sqcup W'} \in Z(\partial W \sqcup \partial W')$ of a disjoint union of bordisms W and W' is the tensor product

$$Z_{W \sqcup W'} = Z_W \widehat{\otimes}_m Z_{W'} \in Z(\partial W) \widehat{\otimes} Z(\partial W') \cong Z(\partial W \sqcup \partial W').$$

(6) *Gluing*: For n-manifolds M, N, P, and vectors $z \in Z(M) \widehat{\otimes} Z(N)$, $z' \in Z(N) \widehat{\otimes} Z(P)$, there is a contraction product

$$\langle z, z' \rangle \in Z(M) \widehat{\otimes} Z(P),$$

which involves the product $\widehat{\otimes}_c$. Let W' be a bordism from M to N and let W'' be a bordism from N to P. Let $W = W' \cup_N W''$ be the bordism from M to P obtained by gluing W' and W'' along N. Then the state sum of W can be calculated as the contraction product

$$Z_W = \langle Z_{W'}, Z_{W''} \rangle \in Z(M) \widehat{\otimes} Z(P) \cong Z(M \sqcup P).$$

(7) The state modules $Z(M)$ have the additional structure of a noncommutative Frobenius semialgebra over Q^m and over Q^c. This means that there is a counit functional $\varepsilon_M : Z(M) \to Q$, which is a Q-bisemimodule homomorphism such that the bilinear form

$$Z(M) \times Z(M) \to Q, \ (z, z') \mapsto \varepsilon_M(zz')$$

is nondegenerate. Maps induced on state modules by homeomorphisms respect the Frobenius counit.

(8) For every closed n-manifold, there is an element $\mathfrak{A}(M) \in Z(M)$, the *coboundary aggregate*, which is topologically invariant: If $\phi : M \to N$ is a homeomorphism, then $\phi_* \mathfrak{A}(M) = \mathfrak{A}(N)$.

The coboundary aggregate of an n-manifold has no counterpart in classical TFTs and is a somewhat surprising new kind of topological invariant. In a classical *unitary* theory, the $Z(M)$ have natural nondegenerate Hermitian structures. What may substitute to some extent for this in positive TFTs is the Frobenius counit, constructed in Section 7. The counit is available in every dimension n and has nothing to do with pairs of pants. Atiyah's axioms imply that state modules are always finite dimensional, but on p. 181 of [Ati88], he indicates that allowing state modules to be infinite dimensional may be necessary in interesting examples of TFTs. The state modules of a positive TFT are indeed usually infinitely generated, though finitely generated modules can result from using very small systems of fields on manifolds. Such systems do in fact arise in practice, for example in TFTs with a view towards finite group theory, such as the Freed-Quinn theory [QF93], [Qui95], [Fre92]. Frobenius algebras, usually assumed to be finite dimensional over a field, have been generalized by Jans [Jan59] to infinite dimensions. Atiyah's classical axioms also demand that the state sum $Z_{M\times I}$ of a cylinder $W = M \times I$ be the identity when viewed as an endomorphism. The map $Z_{M\times I}$ on $Z(M)$ should be the "imaginary time" evolution operator e^{-tH} (where t is the length of the interval I), so Atiyah's axiom means that the Hamiltonian $H = 0$ and there is no dynamics along a cylinder. We do not require this for positive TFTs and will allow interesting propagation along the cylinder. In particular, we do not phrase positive TFTs as monoidal functors on bordism categories, as this would imply that the cylinder, which is an identity morphism in the bordism category, would have to be mapped to an identity morphism. For various future applications, this is not desirable. Since $Z_{M\times I}$ need not be the identity, it can also generally not be deduced in a positive TFT that $Z_{M\times S^1} = \dim Z(M)$, an identity that would not make sense in the first place, as $Z(M)$ need not have finite dimension. The present paper does not make substantial use of the bordism category. Nor do we (yet) consider n-categories or manifolds with corners in this paper. Our manifolds will usually be topological, but we will indicate the modifications necessary to deal with smooth manifolds. Only Section 10 is specifically concerned with smooth manifolds.

The main result of the present paper is that any system of fields $\mathcal{F}$ on manifolds together with a system of action functionals $\mathbb{T}$ on these fields gives rise in a natural way to a positive topological field theory Z. Our framework is not limited to particular dimensions and will produce a TFT in any dimension. Systems of fields are axiomatized in Definition 5.1. As in [Kir10] and [Fre92], they are to satisfy the usual properties with respect to restrictions and action of homeomorphisms. The key properties are that they must decompose on disjoint unions as a cartesian product with factors associated to the components and it must be possible to glue two fields along a common boundary component on which they agree. Every field on the glued space must be of this form. Rather than axiomatizing actions on fields (which would lead to additive axioms), we prefer to axiomatize the exponential of an action directly, since it is the exponential that enters into the Feynman path integral. Also, this yields multiplicative axioms, which is closer to the nature of TFTs, as TFTs are multiplicative, rather than additive (the former corresponding to the quantum nature of TFTs and the latter corresponding to homology theories). Let $\mathbf{C}$ be a (strict, small) monoidal category. We axiomatize systems $\mathbb{T}$ of $\mathbf{C}$-valued action exponentials in Definition 5.8. Given a system $\mathcal{F}$ of fields, $\mathbb{T}$ consists of functions $\mathbb{T}_W$ that associate to every field on an $(n+1)$-dimensional bordism W a morphism in $\mathbf{C}$. For a disjoint union, one requires $\mathbb{T}_{W\sqcup W'}(f) = \mathbb{T}_W(f|_W) \otimes \mathbb{T}_{W'}(f|_{W'})$ for all fields $f \in \mathcal{F}(W \sqcup W')$. If $W = W' \cup_N W''$ is obtained by gluing a bordism W' with outgoing boundary N to a bordism W'' with incoming boundary N, then $\mathbb{T}_W(f) = \mathbb{T}_{W''}(f|_{W''}) \circ \mathbb{T}_{W'}(f|_{W'})$ for all fields $f \in \mathcal{F}(W)$. Also, $\mathbb{T}$ behaves as expected under the action of homeomorphisms on fields. These axioms express that the action ought to be local to a certain extent.

In Section 4, motivated by the group algebra $L^1(G)$ in harmonic analysis and by the categorical algebra $R[\mathbf{C}]$ over a ring R, we show that $\mathbf{C}$, together with an arbitrary choice of an Eilenberg-complete ground semiring S, determines a pair (Q^c, Q^m) of complete semirings with the same underlying additive monoid $Q = Q_S(\mathbf{C})$, using certain convolution formulae. The composition law $\circ$ in $\mathbf{C}$ determines the multiplication $\cdot$ for Q^c, while the tensor functor $\otimes$ in $\mathbf{C}$ (i.e. the monoidal structure) determines the multiplication $\times$ for Q^m.

We then construct a positive TFT Z for any given system $\mathcal{F}$ of fields and $\mathbf{C}$-valued action exponentials $\mathbb{T}$ in Section 6. For a closed manifold M, the state vectors are Q-valued functions on the set of fields $\mathcal{F}(M)$ on M, solving a certain constraint equation. Fields on closed n-manifolds act as boundary conditions for state sums. The state sum of a bordism W is the vector given on a boundary field $f \in \mathcal{F}(\partial W)$ by

$$Z_W(f) = \sum_{F \in \mathcal{F}(W,f)} T_W(F) \in Q,$$

where we sum over all fields F on W which restrict to f on the boundary. The terms $T_W(F)$ correspond to the exponential of the action evaluated on F and are characteristic functions determined by $\mathbb{T}_W$. The key technical point is that this sum uses the infinite summation law on a complete semiring and thus yields a well-defined element of Q, despite the fact that the sum may well involve uncountably many nonzero terms. It is precisely at this point, where Eilenberg's ideas are brought to bear and where using semirings that are not rings is crucial. The formalism developed here thus has strong ties to lattice theory as well as to areas of logic and computer science such as automata theory and formal languages. It may therefore be viewed as a contribution to implementing the program envisioned by Baez and Stay in [BS11].

Theorem 6.4 establishes the topological invariance of the state sum. Pseudo-isotopy invariance of induced maps is provided by Theorem 6.7. The state sum of a disjoint union is calculated in Theorem 6.8, while the gluing formula is the content of Theorem 6.10. In Section 8, we show how matrix-valued positive TFTs arise from monoidal functors into linear categories. An important technical role is played by the Schauenburg tensor product. Section 9 carries out a study of cylindrical state sums. We observe that such a state sum is idempotent as an immediate consequence of the gluing theorem. Given closed manifolds M and N, we use the state sum of the cylinder on M (alternatively N) to construct a projection operator $\pi_{M,N} : Z(M \sqcup N) \to Z(M \sqcup N)$, which acts as the identity on all state sums of bordisms W from M to N. There is a formal analogy to integral transforms given by an integral kernel: The kernel corresponds to the state sum of the cylinder. Furthermore, we analyze to what extent the projection of a tensor product of states breaks up into a tensor product of projections of these states.

In Section 10, we sketch our main application of the framework of positive topological field theories presented here. We indicate how to construct a concrete, new high-dimensional TFT defined on smooth manifolds, which detects exotic smooth structures on spheres. The details of this construction are involved and will appear elsewhere.

The final Section 11 gives additional concrete examples and application patterns. In Section 11.1, we derive Pólya's theory of counting, using positive TFT methods. Moreover, we construct examples based on the signature of manifolds and bundle spaces, and conclude with some remarks on multiplicative arithmetic functions arising in number theory.

A remark on notation: Hoping that no confusion will ensue as a consequence, we use the letter I for unit objects of monoidal categories and sometimes also for the unit interval $[0,1]$.

Acknowledgements: We would like to thank two anonymous referees for their valuable remarks on a preliminary version of this article, leading to several improvements.

2. Monoids, Semirings, and Semimodules

We recall some foundational material on monoids, semirings and semimodules over semirings. Such structures seem to have appeared first in Dedekind's study of ideals in a commutative ring: one can add and multiply two ideals, but one cannot subtract them. The theory has been further developed by H. S. Vandiver, S. Eilenberg, A. Salomaa, J. H. Conway, J. S. Golan and many others. Roughly, a semiring is a ring without general additive inverses. More precisely, a *semiring* is a set S together with two binary operations $+$ and $\cdot$ and two elements $0,1 \in S$ such that $(S,+,0)$ is a commutative monoid, $(S,\cdot,1)$ is a monoid, the multiplication $\cdot$ distributes over the addition from either side, and 0 is absorbing, i.e. $0 \cdot s = 0 = s \cdot 0$ for every $s \in S$. If the monoid $(S,\cdot,1)$ is commutative, the semiring S is called commutative. The addition on the *Boolean monoid* $(\mathbb{B},+,0)$, $\mathbb{B} = \{0,1\}$, is the unique operation such that 0 is the neutral element and $1+1=1$. The Boolean monoid becomes a commutative semiring by defining $1 \cdot 1 = 1$. (Actually, the multiplication on $\mathbb{B}$ is completely forced by the axioms.) A morphism of semirings sends 0 to 0, 1 to 1 and respects addition and multiplication.

Let S be a semiring. A *left S-semimodule* is a commutative monoid $(M,+,0_M)$ together with a scalar multiplication $S \times M \to M$, $(s,m) \mapsto sm$, such that for all $r,s \in S$, $m,n \in M$, we have $(rs)m = r(sm)$, $r(m+n) = rm+rn$, $(r+s)m = rm+sm$, $1m = m$, and $r0_M = 0_M = 0m$. Right semimodules are defined similarly using scalar multiplications $M \times S \to M$, $(m,s) \mapsto ms$. Given semirings R and S, an *R-S-bisemimodule* is a commutative monoid $(M,+,0)$, which is both a left R-semimodule and a right S-semimodule such that $(rm)s = r(ms)$ for all $r \in R$, $s \in S$, $m \in M$. (Thus the notation rms is unambiguous.) An *R-S-bisemimodule homomorphism* is a homomorphism $f : M \to N$ of the underlying additive monoids such that $f(rms) = rf(m)s$ for all r,m,s. If $R = S$, we shall also speak of an S-bisemimodule. Every semimodule M over a commutative semiring S can and will be assumed to be both a left and right semimodule with $sm = ms$. In fact, M is then a bisemimodule, as for all $r,s \in S$, $m \in M$,

$$(rm)s = s(rm) = (sr)m = (rs)m = r(sm) = r(ms).$$

In this paper, we will often refer to elements of a semimodule as "vectors".

Let S be any semiring, not necessarily commutative. Regarding the tensor product of a right S-semimodule M and a left S-semimodule N, one has to exercise caution because even when S is commutative, two nonisomorphic tensor products, both called *the* tensor product of M and N and both written $M \otimes_S N$, exist in the literature. A map $\phi : M \times N \to A$ into a commutative additive monoid A is called *middle S-linear*, if it is biadditive, $\phi(ms,n) = \phi(m,sn)$ for all m,s,n, and $\phi(0,0) = 0$. For us, an (algebraic) *tensor product of M and N* is a commutative monoid $M \otimes_S N$ (written additively) satisfying the following (standard) universal property: $M \otimes_S N$ comes equipped with a middle S-linear map $M \times N \to M \otimes_S N$ such that given any commutative monoid A and middle S-linear map $\phi : M \times N \to A$, there exists a unique monoid homomorphism $\psi : M \otimes_S N \to A$ such that

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[d] & A \\ M \otimes_S N \arrow[ur, dotted, "\psi"'] & \end{tikzcd} \tag{1}$$

commutes. The existence of such a tensor product is shown for example in [Kat97], [Kat04]. To construct it, take $M \otimes_S N$ to be the quotient monoid $F/\sim$, where F is the free commutative monoid generated by the set $M \times N$ and $\sim$ is the congruence relation on F generated by all pairs of the form

$$((m+m',n),(m,n)+(m',n)),\ ((m,n+n'),(m,n)+(m,n')),\ ((ms,n),(m,sn)),$$

$m,m' \in M$, $n,n' \in N$, $s \in S$. If M is an R-S-bisemimodule and N an S-T-bisemimodule, then the monoid $M \otimes_S N$ as constructed above becomes an R-T-bisemimodule by declaring

$$r \cdot (m \otimes n) = (rm) \otimes n,\ (m \otimes n) \cdot t = m \otimes (nt).$$

If in diagram (1), the monoid A is an R-T-semimodule and $M \times N \to A$ satisfies

$$\phi(rm,n) = r\phi(m,n),\ \phi(m,nt) = \phi(m,n)t$$

(in addition to being middle S-linear; let us call such a map $R_S T$*-linear*), then the uniquely determined monoid map $\psi : M \otimes_S N \to A$ is an R-T-bisemimodule homomorphism, for

$$\psi(r(m \otimes n)) = \psi((rm) \otimes n) = \phi(rm,n) = r\phi(m,n) = r\psi(m \otimes n)$$

and similarly for the right action of T. If $R = S = T$ and S is commutative, the above means that the commutative monoid $M \otimes_S N$ is an S-semimodule with $s(m \otimes n) = (sm) \otimes n = m \otimes (sn)$ and the diagram (1) takes place in the category of S-semimodules.

The tensor product of [Tak82] and [Gol99] — let us here write it as $\otimes'_S$ — satisfies a different universal property. A semimodule C is called *cancellative* if $a+c = b+c$ implies $a = b$ for all $a,b,c \in C$. A monoid $(M,+,0)$ is *idempotent* if $m+m = m$ for all elements $m \in M$. For example, the Boolean monoid $\mathbb{B}$ is idempotent. A nontrivial idempotent semimodule is never cancellative. Given an arbitrary right S-semimodule M and an arbitrary left S-semimodule N, the product $M \otimes'_S N$ is always cancellative. If one of the two semimodules, say N, is idempotent, then $M \otimes'_S N$ is idempotent as well, since $m \otimes' n + m \otimes' n = m \otimes' (n+n) = m \otimes' n$. Thus if one of M,N is idempotent, then $M \otimes'_S N$ is trivial, being both idempotent and cancellative. Since in our applications, we desire nontrivial tensor products of idempotent semimodules, the product $\otimes'_S$ will neither be defined nor used in this paper.

The key feature of the constituent algebraic structures of state modules that will allow us to form well-defined state sums is their completeness. Thus let us recall the notion of a complete monoid, semiring, etc. as introduced by S. Eilenberg on p. 125 of [Eil74]; see also [Kar92], [Kro88]. A *complete monoid* is a commutative monoid $(M,+,0)$ together with an assignment $\sum$, called a *summation law*, which assigns to every family $(m_i)_{i \in I}$ of elements $m_i \in M$, indexed by an arbitrary set I, an element $\sum_{i \in I} m_i$ of M (called the *sum* of the m_i), such that

$$\sum_{i \in \varnothing} m_i = 0,\ \sum_{i \in \{1\}} m_i = m_1,\ \sum_{i \in \{1,2\}} m_i = m_1 + m_2,$$

and for every partition $I = \bigcup_{j \in J} I_j$,

$$\sum_{j \in J} \left(\sum_{i \in I_j} m_i \right) = \sum_{i \in I} m_i.$$

Note that these axioms imply that if $\sigma : J \to I$ is a bijection, then

$$\sum_{i \in I} m_i = \sum_{j \in J} \sum_{i \in \{\sigma(j)\}} m_i = \sum_{j \in J} m_{\sigma(j)}.$$

Also, since a cartesian product $I \times J$ comes with two canonical partitions, namely $I \times J = \bigcup_{i \in I} \{i\} \times J = \bigcup_{j \in J} I \times \{j\}$, one has

$$\sum_{(i,j) \in I \times J} m_{ij} = \sum_{i \in I} \sum_{j \in J} m_{ij} = \sum_{j \in J} \sum_{i \in I} m_{ij} \tag{2}$$

for any family $(m_{ij})_{(i,j) \in I \times J}$. This is the analog of Fubini's theorem in the theory of integration. Given $(M, +, 0)$, the summation law $\sum$, if it exists, is not in general uniquely determined by the addition, as examples in [Gol85] show. For a semiring S to be complete one requires that $(S, +, 0, \sum)$ be a complete monoid and adds the infinite distributivity requirements

$$\sum_{i \in I} s s_i = s\big(\sum_{i \in I} s_i\big), \; \sum_{i \in I} s_i s = \big(\sum_{i \in I} s_i\big) s.$$

Note that in a complete semiring, the sum over any family of zeros must be zero, as $\sum_{i \in I} 0 = \sum_i (0 \cdot 0) = 0 \cdot \sum_i 0 = 0$. Complete left, right and bisemimodules are defined analogously. If $(s_{ij})_{(i,j) \in I \times J}$ is a family in a complete semiring of the form $s_{ij} = s_i t_j$, then, using (2) together with the infinite distributivity requirements,

$$\sum_{(i,j) \in I \times J} s_i t_j = \sum_{i \in I} \sum_{j \in J} s_i t_j = \big(\sum_{i \in I} s_i\big)\big(\sum_{j \in J} t_j\big).$$

A semiring is *zerosumfree*, if $s + t = 0$ implies $s = t = 0$ for all s, t in the semiring. The so-called "Eilenberg-swindle" shows that every complete semiring is zerosumfree: If $s + t = 0$, then

$$\begin{aligned} 0 &= 0 + 0 + \cdots = (s+t) + (s+t) + \cdots = s + (t+s) + (t+s) + \cdots \\ &= s + (s+t) + (s+t) + \cdots = s + 0 = s. \end{aligned}$$

In a ring we have additive inverses, which means that a nontrivial ring is never zerosumfree and so cannot be endowed with an infinite summation law that makes it complete. This shows that giving up additive inverses, thereby passing to semirings that are not rings, is an essential prerequisite for completeness and in turn essential for the construction of our topological field theories.

Examples 2.1.

(1) The Boolean semiring $\mathbb{B}$ is complete with respect to the summation law

$$\sum_{i \in I} b_i = \begin{cases} 0, & \text{if } b_i = 0 \text{ for all } i, \\ 1, & \text{otherwise.} \end{cases}$$

(2) Let $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the natural numbers. Then $(\mathbb{N}^\infty, +, \cdot, 0, 1)$ is a semiring by extending addition and multiplication on $\mathbb{N}$ via the rules

$$n + \infty = \infty + n = \infty + \infty = \infty, \; n \in \mathbb{N},$$

$$n \cdot \infty = \infty \cdot n = \infty \text{ for } n \in \mathbb{N}, \; n > 0,$$

$$\infty \cdot \infty = \infty, \; 0 \cdot \infty = \infty \cdot 0 = 0.$$

It is complete with $\sum_{i \in I} n_i = \sup\{\sum_{i \in J} n_i \mid J \subset I, \; J \text{ finite}\}$.

(3) Let $\mathbb{R}_+$ denote the nonnegative real numbers and $\mathbb{R}_+^\infty = \mathbb{R}_+ \cup \{\infty\}$. Extending addition and multiplication as in the case of natural numbers, one obtains the complete semiring $(\mathbb{R}_+^\infty, +, \cdot, 0, 1)$.

(4) The tropical semirings $(\mathbb{N}^\infty, \min, +, \infty, 0)$ and $(\mathbb{R}_+^\infty, \min, +, \infty, 0)$, that is, the sum is the minimum of two numbers and the product is the ordinary addition of numbers, are complete.

(5) The arctic semirings $(\overline{\mathbb{N}}, \max, +, -\infty, 0)$ and $(\overline{\mathbb{R}}_+, \max, +, -\infty, 0)$ are complete, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{-\infty, \infty\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{-\infty, \infty\}$.
(6) Let Σ be a finite alphabet and let Σ^* be the free monoid generated by Σ. Its neutral element is the empty word ε. A subset of Σ^* is called a *formal language over* Σ. Define the product of two formal languages $L, L' \subset \Sigma^*$ by
$$L \cdot L' = \{ww' \mid w \in L,\ w' \in L'\}.$$
Then $(2^{\Sigma^*}, \cup, \cdot, \varnothing, \{\varepsilon\})$ is a semiring, the semiring of formal languages over Σ. It is complete.
(7) For a set A, the power set $2^{A\times A}$ is the set of binary relations R over A. Define the product of two relations R, R' over A by
$$R \cdot R' = \{(a,a') \mid \exists a_0 \in A : (a,a_0) \in R,\ (a_0,a') \in R'\}.$$
Let $\Delta = \{(a,a) \mid a \in A\}$ be the diagonal. Then $(2^{A\times A}, \cup, \cdot, \varnothing, \Delta)$ is a semiring, the semiring of binary relations over A. It is complete.
(8) Any bounded distributive lattice defines a semiring $(L, \vee, \wedge, 0, 1)$. It is complete if L is a join-continuous complete lattice, that is, every subset of L has a supremum in L and $a \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \wedge a_i)$ for any subset $\{a_i \mid i \in I\} \subset L$. For example, the ideals of a ring form a complete lattice.
(9) If S is a complete semiring, then the semiring $S[[q]]$ of formal power series over S becomes a complete semiring by transferring the summation law on S pointwise to $S[[q]]$, see [Kar92].
(10) Completeness of S also implies the completeness of the semirings of square matrices over S.
(11) If $(M_i)_{i\in I}$ is a family of complete S-semimodules, then their product $\prod_{i\in I} M_i$ is a complete S-semimodule.
(12) A semiring S is *additively idempotent* if $s+s=s$ for all $s \in S$. Every additively idempotent semiring can be embedded in a complete semiring, [GW96].

Let R, S be any semirings, not necessarily commutative. An (associative, unital) *R-S-semialgebra* is a semiring A which is in addition an R-S-bisemimodule such that for all $a, b \in A$, $r \in R$, $s \in S$, one has $r(ab) = (ra)b,\ (ab)s = a(bs)$. (Note that we refrain from using the term "R-S-bisemialgebra" for such a structure, since "bialgebra" refers to something completely different, namely a structure with both multiplication and comultiplication.) If $R = S$, we shall also use the term *two-sided S-semialgebra* for an S-S-semialgebra. If S is commutative, then a two-sided S-semialgebra A with $sa = as$ is simply a semialgebra over S in the usual sense, as
$$(sa)b = s(ab) = (ab)s = a(bs) = a(sb).$$
A morphism of R-S-semialgebras is a morphism of semirings which is in addition a R-S-bisemimodule homomorphism.

3. Function Semialgebras

Let S be a semiring. Given a set A, let $\mathrm{Fun}_S(A) = \{f : A \to S\}$ be the set of all S-valued functions on A. If S is understood, we will also write $\mathrm{Fun}(A)$ for $\mathrm{Fun}_S(A)$.

Proposition 3.1. *Using pointwise addition and multiplication,* $\mathrm{Fun}_S(A)$ *inherits the structure of a two-sided S-semialgebra from the operations of S. If S is complete (as a semiring), then* $\mathrm{Fun}_S(A)$ *is complete as a semiring and as an S-bisemimodule. If S is commutative, then* $\mathrm{Fun}_S(A)$ *is a commutative S-semialgebra.*

Proof. Define $0 \in \mathrm{Fun}(A)$ to be $0(a) = 0 \in S$ for all $a \in A$ and define $1 \in \mathrm{Fun}(A)$ to be $1(a) = 1 \in S$ for all $a \in A$. Given $f, g \in \mathrm{Fun}(A)$, define $f + g \in \mathrm{Fun}(A)$ and $f \cdot g \in \mathrm{Fun}(A)$ by $(f+g)(a) = f(a) + g(a)$, $(f \cdot g)(a) = f(a) \cdot g(a)$ for all $a \in A$. Then $(\mathrm{Fun}(A), +, 0)$ is a commutative monoid and $(\mathrm{Fun}(A), \cdot, 1)$ is a monoid, which is commutative if S is commutative. The distributive laws hold and the 0-function is absorbing. Thus $(\mathrm{Fun}(A), +, \cdot, 0, 1)$ is a semiring. Given $s \in S$, define $sf, fs \in \mathrm{Fun}(A)$ by $(sf)(a) = s \cdot (f(a))$, $(fs)(a) = (f(a)) \cdot s$, respectively, for all $a \in A$. This makes $\mathrm{Fun}(A)$ into a a two-sided S-semialgebra. If S is a complete semiring, then an infinite summation law in $\mathrm{Fun}_S(A)$ can be introduced by

$$\Big(\sum_{i \in I} f_i\Big)(a) = \sum_{i \in I} (f_i(a)),$$

$f_i \in \mathrm{Fun}(A)$. With this law, $(\mathrm{Fun}_S(A), +, 0, \sum)$ is a complete monoid, $\mathrm{Fun}_S(A)$ is complete as a semiring and complete as an S-bisemimodule. $\square$

Let B be another set. Then, regarding $\mathrm{Fun}_S(A)$ and $\mathrm{Fun}_S(B)$ as S-bisemimodules, the tensor product $\mathrm{Fun}_S(A) \otimes_S \mathrm{Fun}_S(B)$ is defined. It is an S-bisemimodule such that given any S-bisemimodule M and S_SS-linear map $\phi : \mathrm{Fun}(A) \times \mathrm{Fun}(B) \to M$, there exists a unique S-S-bisemimodule homomorphism $\psi : \mathrm{Fun}(A) \otimes_S \mathrm{Fun}(B) \to M$ such that

$$\begin{tikzcd} \mathrm{Fun}(A) \times \mathrm{Fun}(B) \arrow[r, "\phi"] \arrow[d] & M \\ \mathrm{Fun}(A) \otimes_S \mathrm{Fun}(B) \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. The S-bisemimodule $\mathrm{Fun}(A \times B)$ comes naturally equipped with an S_SS-linear map

$$\beta : \mathrm{Fun}(A) \times \mathrm{Fun}(B) \longrightarrow \mathrm{Fun}(A \times B),\ \beta(f, g) = ((a, b) \mapsto f(a) \cdot g(b)).$$

(If S is commutative, then β is S-bilinear.) Thus, taking $M = \mathrm{Fun}_S(A \times B)$ in the above diagram, there exists a unique S-S-bisemimodule homomorphism $\mu : \mathrm{Fun}(A) \otimes_S \mathrm{Fun}(B) \to \mathrm{Fun}(A \times B)$ such that

$$\begin{tikzcd} \mathrm{Fun}(A) \times \mathrm{Fun}(B) \arrow[r, "\beta"] \arrow[d] & \mathrm{Fun}(A \times B) \\ \mathrm{Fun}(A) \otimes_S \mathrm{Fun}(B) \arrow[ur, "\mu"'] & \end{tikzcd} \tag{3}$$

commutes. In the commutative setting, this homomorphism was studied in [Ban13], where we showed that it is generally neither surjective nor injective when S is not a field. If A and B are finite, then μ is an isomorphism. If A, B are infinite but S happens to be a field, then μ is still injective, but not generally surjective. This is the reason why the functional analyst completes the tensor product $\otimes$ using various topologies available, arriving at products $\widehat{\otimes}$. For example, for compact Hausdorff spaces A and B, let $C(A), C(B)$ denote the Banach spaces of all complex-valued continuous functions on A, B, respectively, endowed with the supremum-norm, yielding the topology of uniform convergence. Then the image of $\mu : C(A) \otimes C(B) \to C(A \times B)$, while not all of $C(A \times B)$, is however dense in $C(A \times B)$ by the Stone-Weierstraß theorem. After completion, μ induces an isomorphism $C(A) \widehat{\otimes}_\varepsilon C(B) \cong C(A \times B)$ of Banach spaces, where $\widehat{\otimes}_\varepsilon$ denotes the so-called ε-tensor product or injective tensor product of two locally convex topological vector spaces. For n-dimensional Euclidean space $\mathbb{R}^n$, let $L^2(\mathbb{R}^n)$ denote the Hilbert space of square integrable functions on $\mathbb{R}^n$.

Then μ induces an isomorphism $L^2(\mathbb{R}^n)\widehat{\otimes}L^2(\mathbb{R}^m) \cong L^2(\mathbb{R}^{n+m}) = L^2(\mathbb{R}^n \times \mathbb{R}^m)$, where $\widehat{\otimes}$ denotes the Hilbert space tensor product, a completion of the algebraic tensor product $\otimes$ of two Hilbert spaces. For more information on topological tensor products see [Sch50], [Gro55], [Tre67]. In [Ban13] we show that even over the smallest complete (in particular zerosumfree) and additively idempotent commutative semiring, namely the Boolean semiring $\mathbb{B}$, and for the smallest infinite cardinal $\aleph_0$, modeled by a countably infinite set A, the map μ is not surjective, which means that in the context of the present paper, some form of completion must be used as well. However, there is an even more serious complication which arises over semirings: In marked contrast to the situation over fields, the canonical map μ ceases to be *injective* in general when one studies functions with values in a semiring S. Given two infinite sets A and B, we construct explicitly in [Ban13] a commutative, additively idempotent semiring $S = S(A,B)$ such that $\mu : \mathrm{Fun}_S(A)\otimes \mathrm{Fun}_S(B) \to \mathrm{Fun}_S(A\times B)$ is not injective. This has the immediate consequence that in performing functional analysis over a semiring which is not a field, one cannot identify the function $(a,b) \mapsto f(a)g(b)$ on $A\times B$ with $f\otimes g$ for $f \in \mathrm{Fun}_S(A)$, $g \in \mathrm{Fun}_S(B)$. Thus the algebraic tensor product is not the correct device to formulate positive topological field theories.

In the boundedly complete idempotent setting, Litvinov, Maslov and Shpiz have constructed in [LMS99] a tensor product, let us here write it as $\widehat{\otimes}$, which for bounded functions does not exhibit the above deficiencies of the algebraic tensor product. Any idempotent semiring S is a partially ordered set with respect to the order relation $s \le t$ if and only if $s+t = t$; $s,t \in S$. Then the addition has the interpretation of a least upper bound, $s+t = \sup\{s,t\}$. The semiring S is called *boundedly complete* (b-complete) if every subset of S which is bounded above has a supremum. (The supremum of a subset, if it exists, is unique.) The above semiring $S(A,B)$ is b-complete. Given a b-complete commutative idempotent semiring S and b-complete idempotent semimodules V,W over S, Litvinov, Maslov and Shpiz define a tensor product $V\widehat{\otimes}W$, which is again idempotent and b-complete. The fundamental difference to the algebraic tensor product lies in allowing *infinite* sums of elementary tensors. A linear map $f : V \to W$ is called b-linear if $f(\sup V_0) = \sup f(V_0)$ for every bounded subset $V_0 \subset V$. The canonical map $\pi : V\times W \to V\widehat{\otimes}W$ is b-bilinear. For each b-bilinear map $f : V\times W \to U$ there exists a unique b-linear map $f_{\widehat{\otimes}} : V\widehat{\otimes}W \to U$ such that $f = f_{\widehat{\otimes}}\pi$. Given any set A, let $\mathcal{B}_S(A)$ denote the set of bounded functions $A \to S$. Then $\mathcal{B}_S(A)$ is a b-complete idempotent S-semimodule. According to [LMS99, Prop. 5], $\mathcal{B}_S(A)\widehat{\otimes}\mathcal{B}_S(B)$ and $\mathcal{B}_S(A\times B)$ are isomorphic for arbitrary sets A and B. Note that when S is complete, then every S-valued function is bounded and thus

$$\mathrm{Fun}_S(A)\widehat{\otimes}\,\mathrm{Fun}_S(B) = \mathcal{B}_S(A)\widehat{\otimes}\mathcal{B}_S(B) \cong \mathcal{B}_S(A\times B) = \mathrm{Fun}_S(A\times B).$$

In light of the above remarks and in order to make function semialgebras into a monoidal category (cf. Proposition 3.5), we adopt the following definition.

Definition 3.2. The *functional tensor product* $\mathrm{Fun}_S(A)\widehat{\otimes}\,\mathrm{Fun}_S(B)$ of two function semimodules $\mathrm{Fun}_S(A)$ and $\mathrm{Fun}_S(B)$ is given by $\mathrm{Fun}_S(A)\widehat{\otimes}\,\mathrm{Fun}_S(B) = \mathrm{Fun}_S(A\times B)$.

Diagram (3) can be rewritten as

$$\begin{tikzcd}
\mathrm{Fun}(A)\times\mathrm{Fun}(B) \arrow[r, "\beta"] \arrow[d] & \mathrm{Fun}(A)\widehat{\otimes}\,\mathrm{Fun}(B) \\
\mathrm{Fun}(A)\otimes_S\mathrm{Fun}(B) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Remark 3.3. It might be tempting to define the functional tensor product of $\mathrm{Fun}(A)$ and $\mathrm{Fun}(B)$ in a different way, namely as the subsemimodule of $\mathrm{Fun}(A\times B)$ consisting of all functions $F: A\times B\to S$ that can be written in the form

$$F(a,b)=\sum_{i=1}^{k} f_i(a)g_i(b)$$

for some $f_i\in \mathrm{Fun}(A)$, $g_i\in \mathrm{Fun}(B)$, in other words, let the functional tensor product be the image of μ. Such a definition would be incorrect, however, since the resulting smaller semimodule would in general be too small to contain the main invariant of a TFT, the state sum, as constructed in Section 6. In Example 6.9, we construct a system of fields on manifolds and action functionals such that the resulting state sum is not in the image of μ.

Given two functions $f\in \mathrm{Fun}(A)$, $g\in \mathrm{Fun}(B)$ we call $f\widehat{\otimes}g=\beta(f,g)$ the (functional) *tensor product* of f and g. We note that $f\widehat{\otimes}g$ depends on the multiplication on S. If the underlying additive monoid of S is equipped with a different multiplication, then $f\widehat{\otimes}g$ will of course change. For the functional tensor product with S we have

$$\mathrm{Fun}(A)\widehat{\otimes}S\cong \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(\{*\})=\mathrm{Fun}(A\times\{*\})\cong \mathrm{Fun}(A).$$

For fixed S, $\mathrm{Fun}_S(-)$ is a contravariant functor $\mathrm{Fun}_S: \mathbf{Sets}\to S\text{-}S\text{-}\mathbf{SAlgs}$ from the category of sets to the category of two-sided S-semialgebras: A morphism $\phi: A\to B$ of sets induces a morphism of two-sided S-semialgebras $\mathrm{Fun}(\phi): \mathrm{Fun}(B)\to \mathrm{Fun}(A)$ by setting $\mathrm{Fun}(\phi)(f)=f\circ\phi$. Clearly, $\mathrm{Fun}(\mathrm{id}_A)=\mathrm{id}_{\mathrm{Fun}(A)}$ and $\mathrm{Fun}(\psi\circ\phi)=\mathrm{Fun}(\phi)\circ\mathrm{Fun}(\psi)$ for $\psi: B\to C$.

Proposition 3.4. *The functor* $\mathrm{Fun}_S(-)$ *is faithful.*

Proof. Given functions $\phi,\psi: A\to B$ such that $\mathrm{Fun}(\phi)=\mathrm{Fun}(\psi)$, we know that $f(\phi(a))=f(\psi(a))$ for all $f\in \mathrm{Fun}(B)$ and $a\in A$. Taking f to be the characteristic function $\chi_{\phi(a)}$, given by $\chi_{\phi(a)}(b)=\delta_{\phi(a),b}$, we find that $\chi_{\phi(a)}(\psi(a))=\chi_{\phi(a)}(\phi(a))=1$, whence $\phi(a)=\psi(a)$. □

Let $\mathbf{Fun}_S$ be the category whose objects are $\mathrm{Fun}_S(A)$ for all sets A, and whose morphisms are those morphisms of two-sided S-semialgebras $\mathrm{Fun}_S(B)\to\mathrm{Fun}_S(A)$ that have the form $\mathrm{Fun}_S(\phi)$ for some $\phi: A\to B$. The preceding remarks imply that this is indeed a category with the obvious composition law. For $\mathrm{Fun}(\phi): \mathrm{Fun}(B)\to\mathrm{Fun}(A)$ and $\mathrm{Fun}(\phi'): \mathrm{Fun}(B')\to\mathrm{Fun}(A')$, we define

$$\mathrm{Fun}(\phi)\widehat{\otimes}\mathrm{Fun}(\phi'): \mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(B')\longrightarrow \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(A')$$

to be $\mathrm{Fun}(\phi\times\phi'): \mathrm{Fun}(B\times B')\longrightarrow \mathrm{Fun}(A\times A')$, where $\phi\times\phi': A\times A'\to B\times B'$ is the cartesian product $(\phi\times\phi')(a,a')=(\phi(a),\phi'(a'))$. In this manner, the functional tensor product becomes a functor $\widehat{\otimes}: \mathbf{Fun}_S\times\mathbf{Fun}_S\to\mathbf{Fun}_S$. Define the unit object of $\mathbf{Fun}_S$ to be $I=\mathrm{Fun}_S(\{*\})=S$ and define associators a by

$$a_{A,B,C}=\mathrm{Fun}(\alpha_{A,B,C}): \mathrm{Fun}(A)\widehat{\otimes}(\mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C))\xrightarrow{\cong}(\mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B))\widehat{\otimes}\mathrm{Fun}(C),$$

where $\alpha_{A,B,C}: (A\times B)\times C\xrightarrow{\cong} A\times(B\times C)$ is the standard associator that makes $\mathbf{Sets}$ into a monoidal category, given by $\alpha_{A,B,C}((a,b),c)=(a,(b,c))$, $a\in A$, $b\in B$, $c\in C$. Define left unitors l by

$$l_A=\mathrm{Fun}(\lambda_A): I\widehat{\otimes}\mathrm{Fun}(A)=\mathrm{Fun}(\{*\}\times A)\xrightarrow{\cong}\mathrm{Fun}(A),$$

where $\lambda_A : A \to \{*\} \times A$ is the bijection $\lambda_A(a) = (*, a)$. Similarly, right unitors r are defined as

$$r_A = \mathrm{Fun}(\rho_A) : \mathrm{Fun}(A)\widehat{\otimes} I = \mathrm{Fun}(A \times \{*\}) \xrightarrow{\cong} \mathrm{Fun}(A),$$

where $\rho_A : A \to A \times \{*\}$ is the bijection $\rho_A(a) = (a, *)$. A braiding b is given by

$$b_{A,B} = \mathrm{Fun}(\beta_{B,A}) : \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B) \xrightarrow{\cong} \mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(A),$$

with $\beta_{B,A} : B \times A \xrightarrow{\cong} A \times B$, $\beta_{B,A}(b,a) = (a,b)$. Then it is a routine task to verify the following assertion:

Proposition 3.5. *The septuple* $(\mathbf{Fun}_S, \widehat{\otimes}, I, a, l, r, b)$ *is a symmetric monoidal category.*

Let S be a complete semiring and let A, B, C be sets. We put $\mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C) = \mathrm{Fun}(A \times B \times C)$ (triples), etc. A *contraction*

$$\gamma : \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C) \longrightarrow \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(C)$$

can then be defined, using the summation law in S, by

$$\gamma(f)(a,c) = \sum_{b \in B} f(a,b,b,c),$$

$f : A \times B \times B \times C \to S$, $(a,c) \in A \times C$. Given $f \in \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B)$ and $g \in \mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C)$, we shall also write $\langle f, g \rangle = \gamma(f \widehat{\otimes} g)$. This contraction appears in describing the behavior of our state sum invariant under gluing of bordisms. The proof of the following two statements is straightforward.

Proposition 3.6. *The contraction*

$$\langle -, - \rangle : (\mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B)) \times (\mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C)) \longrightarrow \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(C)$$

is $_S S$*-linear.*

Proposition 3.7. *The contraction* $\langle -, - \rangle$ *is associative, that is, given four sets* A, B, C, D *and elements* $f \in \mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(B)$, $g \in \mathrm{Fun}(B)\widehat{\otimes}\mathrm{Fun}(C)$ *and* $h \in \mathrm{Fun}(C)\widehat{\otimes}\mathrm{Fun}(D)$*, the equation*

$$\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle$$

holds in $\mathrm{Fun}(A)\widehat{\otimes}\mathrm{Fun}(D)$.

4. Convolution Semirings Associated to Monoidal Categories

Let S be a semiring and let $\mathbf{C}$ be a small category. The symbol $\mathrm{Mor}(\mathbf{C})$ will denote the set of all morphisms of $\mathbf{C}$. Given a morphism $\alpha \in \mathrm{Mor}(\mathbf{C})$, $\mathrm{dom}(\alpha)$ is its domain and $\mathrm{cod}(\alpha)$ its codomain. We shall show that if S is complete, then this data determines a (complete) semiring $Q = Q_S(\mathbf{C})$. Of course, given concrete $\mathbf{C}$, systems of fields and action functionals, one may wish to adapt the general definition of Q given here, so as to better reflect algebraically the topological structures to be studied. Our construction is motivated by similar ones in harmonic analysis. Suppose that $\mathbf{C}$ is a group, i.e. a groupoid with one object $*$. If the group $G = \mathrm{Hom}_{\mathbf{C}}(*, *)$ is locally compact Hausdorff, then one may consider $L^1(G)$, the functions on G that are integrable with respect to Haar measure. A multiplication, called the convolution product, on $L^1(G)$ is given by

$$(f * g)(t) = \int_G f(s) g(s^{-1} t) ds.$$

The resulting algebra is called the group algebra or convolution algebra. If the functions on G take values in a complete semiring S, one can drop the integrability assumption (and in fact the assumption that G be topological) and use the summation law in S instead of integration against Haar measure:

$$(f*g)(t) = \sum_{ss'=t} f(s)g(s').$$

Written like this, it now becomes clear that even the assumption that $\mathbf{C}$ be a group is obsolete. There are also close connections of our construction of $Q_S(\mathbf{C})$ to the categorical algebra $R[\mathbf{C}]$ associated to a locally finite category $\mathbf{C}$ and a ring R. A category is locally finite, if every morphism can be factored in only finitely many ways as a product of nonidentity morphisms. Elements of $R[\mathbf{C}]$ are functions $\mathrm{Mor}(\mathbf{C}) \to R$, i.e. $R[\mathbf{C}] = \mathrm{Fun}_R(\mathrm{Mor}(\mathbf{C}))$. The local finiteness of $\mathbf{C}$ ensures that the convolution product in R is well-defined. The difference to our construction is that we do not need $\mathbf{C}$ to be locally finite, but we need $R = S$ to be a complete semiring.

Let S be a semiring and $\mathbf{C}$ a small category. For every pair (X,Y) of objects in $\mathbf{C}$, we then have the commutative monoid $(\mathrm{Fun}_S(\mathrm{Hom}_{\mathbf{C}}(X,Y)),+,0)$ and can form the product

$$Q_S(\mathbf{C}) = \prod_{X,Y\in \mathrm{Ob}\,\mathbf{C}} \mathrm{Fun}_S(\mathrm{Hom}_{\mathbf{C}}(X,Y)).$$

Elements $f \in Q_S(\mathbf{C})$ are families $f = (f_{XY})$ of functions $f_{XY} : \mathrm{Hom}_{\mathbf{C}}(X,Y) \to S$. Such a family is of course the same thing as a function $f : \mathrm{Mor}(\mathbf{C}) \to S$ and $Q_S(\mathbf{C}) = \mathrm{Fun}_S(\mathrm{Mor}(\mathbf{C}))$, but we find it convenient to keep the bigrading of $Q_S(\mathbf{C})$ by pairs of objects. The addition on $Q_S(\mathbf{C})$ preserves the bigrading, i.e. $(f+g)_{XY} = f_{XY} + g_{XY}$. The neutral element $0 \in Q_S(\mathbf{C})$ is given by $0 = (0_{XY})$ with $0_{XY} = 0 : \mathrm{Hom}_{\mathbf{C}}(X,Y) \to S$ the constant map sending every morphism $X \to Y$ to $0 \in S$. Then $(Q_S(\mathbf{C}),+,0)$ is a commutative monoid.

Proposition 4.1. *If S is a complete semiring, then $Q_S(\mathbf{C})$ inherits a summation law from S, making $(Q_S(\mathbf{C}),+,0)$ into a complete monoid.*

Proof. Given an index set J and a family $\{f_j\}_{j\in J}$ of elements $f_j \in Q_S(\mathbf{C})$, declare a summation law on $Q_S(\mathbf{C})$ by

$$(\sum_{j\in J} f_j)_{XY}(\gamma) = \sum_{j\in J}((f_j)_{XY}(\gamma)) \in S,$$

using on the right hand side the summation law of S. In this formula, $\gamma : X \to Y$ is a morphism in $\mathbf{C}$. Using this summation law, all the axioms for a complete monoid are satisfied. For example, given a partition $J = \bigcup_{k\in K} J_k$, one has

$$\begin{aligned}
(\sum_{k\in K}(\sum_{j\in J_k} f_j))_{XY}(\gamma) &= \sum_{k\in K}((\sum_{j\in J_k} f_j)_{XY}(\gamma)) = \sum_{k\in K}(\sum_{j\in J_k}((f_j)_{XY}(\gamma))) \\
&= \sum_{j\in J}((f_j)_{XY}(\gamma)) = (\sum_{j\in J} f_j)_{XY}(\gamma),
\end{aligned}$$

using the partition axiom provided by the completeness of S. □

Assume that S is complete. Then we can define a (generally noncommutative) multiplication $\cdot : Q_S(\mathbf{C}) \times Q_S(\mathbf{C}) \to Q_S(\mathbf{C})$ by $(f_{XY}) \cdot (g_{XY}) = (h_{XY})$, with $h_{XY} : \mathrm{Hom}_{\mathbf{C}}(X,Y) \to S$ defined on a morphism $\gamma : X \to Y$ by the convolution formula

$$h_{XY}(\gamma) = \sum_{\beta\alpha=\gamma} g_{ZY}(\beta) \cdot f_{XZ}(\alpha),$$

where α, β range over all $\alpha \in \mathrm{Hom}_{\mathbf{C}}(X,Z)$, $\beta \in \mathrm{Hom}_{\mathbf{C}}(Z,Y)$ with $\gamma = \beta \circ \alpha$. The right hand side of this formula uses the multiplication of the semiring S. Note that the sum may well be infinite, but nevertheless yields a well-defined element of S by completeness. An element $1 \in Q_S(\mathbf{C})$ is given by (f_{XY}) with

$$f_{XY} = \begin{cases} f_{XX}, & X = Y \\ 0 & X \neq Y \end{cases}, \; f_{XX}(\alpha) = \begin{cases} 1, & \alpha = \mathrm{id}_X \\ 0, & \alpha \neq \mathrm{id}_X \end{cases}.$$

Proposition 4.2. *The quintuple $(Q_S(\mathbf{C}), +, \cdot, 0, 1)$ is a complete semiring.*

Proof. Let us verify that $(Q_S(\mathbf{C}), \cdot, 1)$ is a monoid. Given $(f_{XY}), (g_{XY})$ and $(h_{XY}) \in Q_S(\mathbf{C})$, let $(l_{XY}) = (f_{XY}) \cdot (g_{XY})$ and $(r_{XY}) = (g_{XY}) \cdot (h_{XY})$. Let $\gamma : X \to Y$ be a morphism in $\mathbf{C}$. Then

$$\begin{aligned} ((l_{XY}) \cdot (h_{XY}))_{XY}(\gamma) &= \sum_{\beta\alpha=\gamma} h_{ZY}(\beta) \cdot l_{XZ}(\alpha) = \sum_{\beta\alpha=\gamma} h_{ZY}(\beta) \cdot \sum_{\sigma\tau=\alpha} g_{UZ}(\sigma) \cdot f_{XU}(\tau) \\ &= \sum_{\beta\sigma\tau=\gamma} h_{ZY}(\beta) \cdot g_{UZ}(\sigma) \cdot f_{XU}(\tau) = \sum_{\zeta \in L} s(\zeta), \end{aligned}$$

where $L = \{\zeta = (\beta, \sigma, \tau) \mid \beta\sigma\tau = \gamma\}$, involving all possible factorizations of γ into three factors

$$X \xrightarrow{\tau} U \xrightarrow{\sigma} Z \xrightarrow{\beta} Y,$$

and the function s is given by $s(\beta, \sigma, \tau) = h_{ZY}(\beta) \cdot g_{UZ}(\sigma) \cdot f_{XU}(\tau)$. On the other hand,

$$\begin{aligned} ((f_{XY}) \cdot (r_{XY}))_{XY}(\gamma) &= \sum_{\beta\alpha=\gamma} r_{ZY}(\beta) \cdot f_{XZ}(\alpha) = \sum_{\beta\alpha=\gamma} (\sum_{\sigma\tau=\beta} h_{VY}(\sigma) \cdot g_{ZV}(\tau)) \cdot f_{XZ}(\alpha) \\ &= \sum_{\sigma\tau\alpha=\gamma} h_{VY}(\sigma) \cdot g_{ZV}(\tau) \cdot f_{XZ}(\alpha) = \sum_{\zeta \in R} s(\zeta), \end{aligned}$$

where $R = \{\zeta = (\sigma, \tau, \alpha) \mid \sigma\tau\alpha = \gamma\}$, involving all possible factorizations of γ into three factors

$$X \xrightarrow{\alpha} Z \xrightarrow{\tau} V \xrightarrow{\sigma} Y,$$

and the function s is the same as above. As $L = R$, this shows that the multiplication $\cdot$ on $Q_S(\mathbf{C})$ is associative. The element $1 \in Q_S(\mathbf{C})$ is neutral with respect to this multiplication, for

$$\begin{aligned} ((f_{XY}) \cdot 1)_{XY}(\gamma) &= \sum_{\beta\alpha=\gamma} 1_{ZY}(\beta) \cdot f_{XZ}(\alpha) \\ &= \sum_{\beta\alpha=\gamma,\ Z \neq Y} 1_{ZY}(\beta) \cdot f_{XZ}(\alpha) + \sum_{\beta\alpha=\gamma,\ Z=Y} 1_{YY}(\beta) \cdot f_{XY}(\alpha) \\ &= \sum_{\beta\alpha=\gamma} 1_{YY}(\beta) \cdot f_{XY}(\alpha) \\ &= \sum_{\beta\alpha=\gamma,\ \beta \neq \mathrm{id}_Y} 1_{YY}(\beta) \cdot f_{XY}(\alpha) + \sum_{\beta\alpha=\gamma,\ \beta=\mathrm{id}_Y} 1_{YY}(\beta) \cdot f_{XY}(\alpha) \\ &= \sum_{\alpha=\gamma} 1_{YY}(\mathrm{id}_Y) \cdot f_{XY}(\gamma) \\ &= f_{XY}(\gamma) \end{aligned}$$

and similarly $1 \cdot (f_{XY}) = (f_{XY})$. Therefore, $(Q_S(\mathbf{C}), \cdot, 1)$ is a monoid. The distribution laws are readily verified and the element $0 \in Q_S(\mathbf{C})$ is obviously absorbing.

By Proposition 4.1, $(Q_S(\mathbf{C}), +, 0)$ is a complete monoid. It remains to be shown that the summation law satisfies the infinite distributivity requirement with respect to $\cdot$ on $Q_S(\mathbf{C})$. As above, $\gamma : X \to Y$ is a morphism in $\mathbf{C}$. Given an element $g \in Q_S(\mathbf{C})$, an index set J and a family $(f_j)_{j \in J}$ of elements $f_j \in Q_S(\mathbf{C})$, let $\Gamma = \{(\beta, \alpha) \mid \beta\alpha = \gamma\}$ and let P be the cartesian product $P = J \times \Gamma$. Note that P comes with two natural partitions, namely into sets $\{j\} \times \Gamma$, $j \in J$, and into sets $J \times \{(\beta,\alpha)\}$, $(\beta,\alpha) \in \Gamma$. Using the infinite distribution axiom for S and the partition axiom, we have

$$\begin{aligned}(g \cdot \sum_{j\in J} f_j)_{XY}(\gamma) &= \sum_{(\beta,\alpha)\in\Gamma} (\sum_{j\in J} f_j)_{ZY}(\beta) \cdot g_{XZ}(\alpha) = \sum_{(\beta,\alpha)\in\Gamma} (\sum_{j\in J}((f_j)_{ZY}(\beta))) \cdot g_{XZ}(\alpha) \\ &= \sum_{(j,(\beta,\alpha))\in P} (f_j)_{ZY}(\beta) \cdot g_{XZ}(\alpha) = \sum_{j\in J}(\sum_{(\beta,\alpha)\in\Gamma} (f_j)_{ZY}(\beta) \cdot g_{XZ}(\alpha)) \\ &= \sum_{j\in J}((g \cdot f_j)_{XY}(\gamma)) = (\sum_{j\in J}(g \cdot f_j))_{XY}(\gamma).\end{aligned}$$

Similarly $(\sum f_j) \cdot g = \sum (f_j \cdot g)$. □

The above proof shows that the (strict) associativity of the composition law $\circ$ of $\mathbf{C}$ implies the associativity of the multiplication $\cdot$ in $Q_S(\mathbf{C})$. Similarly, the presence of identity morphisms in $\mathbf{C}$ implies the existence of a unit element 1 for the multiplication. It is clear that the multiplication $\cdot$ on $Q_S(\mathbf{C})$ is generally noncommutative, even if S happens to be commutative. For example, let $\mathbf{C}$ be the category given by two distinct objects X, Y and morphisms

$$\mathrm{Hom}_{\mathbf{C}}(X,X) = \{\mathrm{id}_X\},\ \mathrm{Hom}_{\mathbf{C}}(Y,Y) = \{\mathrm{id}_Y\},\ \mathrm{Hom}_{\mathbf{C}}(X,Y) = \{\gamma\},\ \mathrm{Hom}_{\mathbf{C}}(Y,X) = \varnothing.$$

The composition law is uniquely determined. Let $S = \mathbb{B}$ be the Boolean semiring, which is commutative. If $f(\gamma) = 1$, $g(\mathrm{id}_X) = 0$, $g(\gamma) = 0$, $g(\mathrm{id}_Y) = 1$, then $(f \cdot g)(\gamma) = 1$, but $(g \cdot f)(\gamma) = 0$.

Now suppose that $(\mathbf{C}, \otimes, I)$ is a strict monoidal category. Then, using the monoidal structure, we can define a different multiplication $\times : Q_S(\mathbf{C}) \times Q_S(\mathbf{C}) \to Q_S(\mathbf{C})$ by $(f_{XY}) \times (g_{XY}) = (h_{XY})$, with $h_{XY} : \mathrm{Hom}_{\mathbf{C}}(X,Y) \to S$ defined on a morphism $\gamma : X \to Y$ as the convolution

$$h_{XY}(\gamma) = \sum_{\alpha\otimes\beta=\gamma} g_{X''Y''}(\beta) \cdot f_{X'Y'}(\alpha),$$

where α, β range over all $\alpha \in \mathrm{Hom}_{\mathbf{C}}(X',Y')$, $\beta \in \mathrm{Hom}_{\mathbf{C}}(X'',Y'')$ such that $X = X' \otimes X''$, $Y = Y' \otimes Y''$ and $\gamma = \alpha \otimes \beta$. Again, the right hand side of this formula uses the multiplication of the complete ground semiring S. An element $1^\times \in Q_S(\mathbf{C})$ is given by (f_{XY}) with

$$f_{XY} = \begin{cases} f_{II}, & X = Y = I \text{ (unit obj.)} \\ 0 & \text{otherwise} \end{cases},\ f_{II}(\alpha) = \begin{cases} 1, & \alpha = \mathrm{id}_I \\ 0, & \alpha \neq \mathrm{id}_I \end{cases}.$$

Proposition 4.3. *The quintuple $(Q_S(\mathbf{C}), +, \times, 0, 1^\times)$ is a complete semiring.*

Proof. We check that $(Q_S(\mathbf{C}), \cdot, 1^\times)$ is a monoid, knowing already that $(Q_S(\mathbf{C}), +, 0)$ is a commutative monoid. Given (f_{XY}), (g_{XY}) and $(h_{XY}) \in Q_S(\mathbf{C})$, let $(l_{XY}) = (f_{XY}) \times (g_{XY})$ and $(r_{XY}) = (g_{XY}) \times (h_{XY})$. Let $\gamma : X \to Y$ be a morphism in $\mathbf{C}$. Then

$$\begin{aligned}((l_{XY}) \times (h_{XY}))_{XY}(\gamma) &= \sum_{\alpha\otimes\beta=\gamma} h_{X''Y''}(\beta) \cdot l_{X'Y'}(\alpha) \\ &= \sum_{\alpha\otimes\beta=\gamma} h_{X''Y''}(\beta) \cdot \sum_{\sigma\otimes\tau=\alpha} g_{X_2Y_2}(\tau) \cdot f_{X_1Y_1}(\sigma) \\ &= \sum_{\sigma\otimes\tau\otimes\beta=\gamma} h_{X''Y''}(\beta) \cdot g_{X_2Y_2}(\tau) \cdot f_{X_1Y_1}(\sigma) \\ &= \sum_{\zeta\in L} s(\zeta),\end{aligned}$$

where $L = \{\zeta = (\sigma,\tau,\beta) \mid \sigma\otimes\tau\otimes\beta = \gamma\}$, involving all possible factorizations of γ into three tensor factors

$$X = X_1\otimes X_2\otimes X'' \stackrel{\sigma\otimes\tau\otimes\beta}{\longrightarrow} Y_1\otimes Y_2\otimes Y'' = Y,$$

and the function s is given by $s(\sigma,\tau,\beta) = h_{X''Y''}(\beta)\cdot g_{X_2Y_2}(\tau)\cdot f_{X_1Y_1}(\sigma)$. (Note that since $(\mathbf{C},\otimes,I)$ is strict, we do not have to indicate parentheses. As is customary in strict monoidal categories, we write $X_1\otimes X_2\otimes X''$ for $(X_1\otimes X_2)\otimes X'' = X_1\otimes(X_2\otimes X'')$. Similarly for morphisms.) On the other hand,

$$\begin{aligned}
((f_{XY})\times(r_{XY}))_{XY}(\gamma) &= \sum_{\alpha\otimes\beta=\gamma} r_{X''Y''}(\beta)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\beta=\gamma}\Big(\sum_{\sigma\otimes\tau=\beta} h_{X_2Y_2}(\tau)\cdot g_{X_1Y_1}(\sigma)\Big)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\sigma\otimes\tau=\gamma} h_{X_2Y_2}(\tau)\cdot g_{X_1Y_1}(\sigma)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\zeta\in R} s(\zeta),
\end{aligned}$$

where $R = \{\zeta = (\alpha,\sigma,\tau) \mid \alpha\otimes\sigma\otimes\tau = \gamma\}$, involving all possible factorizations of γ into three tensor factors

$$X = X'\otimes X_1\otimes X_2 \stackrel{\alpha\otimes\sigma\otimes\tau}{\longrightarrow} Y'\otimes Y_1\otimes Y_2 = Y,$$

and the function s is the same as above. As $\mathbf{C}$ is strict, we have $L = R$, which shows that the multiplication $\times$ on $Q_S(\mathbf{C})$ is associative. The element $1^\times \in Q_S(\mathbf{C})$ is neutral with respect to this multiplication, for

$$\begin{aligned}
((f_{XY})\times 1^\times)_{XY}(\gamma) &= \sum_{\alpha\otimes\beta=\gamma} 1^\times_{X''Y''}(\beta)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\beta=\gamma,\ X''\neq I \text{ or } Y''\neq I} 1^\times_{X''Y''}(\beta)\cdot f_{X'Y'}(\alpha)\\
&\qquad + \sum_{\alpha\otimes\beta=\gamma,\ X''=Y''=I} 1^\times_{X''Y''}(\beta)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\beta=\gamma} 1^\times_{II}(\beta)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\beta=\gamma,\ \beta\neq \mathrm{id}_I} 1^\times_{II}(\beta)\cdot f_{X'Y'}(\alpha) + \sum_{\alpha\otimes\beta=\gamma,\ \beta=\mathrm{id}_I} 1^\times_{II}(\beta)\cdot f_{X'Y'}(\alpha)\\
&= \sum_{\alpha\otimes\mathrm{id}_I=\gamma} f_{XY}(\gamma)\\
&= f_{XY}(\gamma)
\end{aligned}$$

and similarly $1^\times\times(f_{XY}) = (f_{XY})$. (In this calculation, we have used $X'\otimes I = X'$, $\alpha\otimes\mathrm{id}_I = \alpha$, valid in a strict monoidal category such as $\mathbf{C}$.) Therefore, $(Q_S(\mathbf{C}),\cdot,1^\times)$ is a monoid. The distribution laws are satisfied and the element $0\in Q_S(\mathbf{C})$ is absorbing.

By Proposition 4.1, $(Q_S(\mathbf{C}),+,0)$ is a complete monoid. It remains to be shown that the summation law satisfies the infinite distributivity requirement with respect to $\times$ on $Q_S(\mathbf{C})$. As above $\gamma : X\to Y$ is a morphism in $\mathbf{C}$. Given an element $g\in Q_S(\mathbf{C})$, an index set J and a family $(f_j)_{j\in J}$ of elements $f_j\in Q_S(\mathbf{C})$, let $\Gamma = \{(\alpha,\beta)\mid \alpha\otimes\beta = \gamma\}$ and let P be the cartesian product $P = J\times\Gamma$. Note that P comes with two natural partitions, namely into sets $\{j\}\times\Gamma$, $j\in J$, and into sets $J\times\{(\alpha,\beta)\}$, $(\alpha,\beta)\in\Gamma$. Using the infinite distribution axiom for S and the partition axiom, we have

$$\begin{aligned}
(g\times\sum_{j\in J} f_j)_{XY}(\gamma) &= \sum_{(\alpha,\beta)\in\Gamma}(\sum_{j\in J} f_j)_{X''Y''}(\beta)\cdot g_{X'Y'}(\alpha)\\
&= \sum_{(\alpha,\beta)\in\Gamma}(\sum_{j\in J}((f_j)_{X''Y''}(\beta)))\cdot g_{X'Y'}(\alpha)\\
&= \sum_{(j,(\alpha,\beta))\in P}(f_j)_{X''Y''}(\beta)\cdot g_{X'Y'}(\alpha)\\
&= \sum_{j\in J}(\sum_{(\alpha,\beta)\in\Gamma}(f_j)_{X''Y''}(\beta)\cdot g_{X'Y'}(\alpha))\\
&= \sum_{j\in J}((g\times f_j)_{XY}(\gamma))\\
&= (\sum_{j\in J}(g\times f_j))_{XY}(\gamma).
\end{aligned}$$

□

It is crucial in the above proof to know that **C** is strict. In order for the multiplication $\times$ to be associative, one must know that the sets of factorizations $L_\gamma=\{(\sigma,\tau,\beta)\mid(\sigma\otimes\tau)\otimes\beta=\gamma\}$ and $R_\gamma=\{(\sigma,\tau,\beta)\mid\sigma\otimes(\tau\otimes\beta)=\gamma\}$ are equal. This holds when **C** is strict, but may fail when **C** is not strict. Similarly, we used the property $\alpha\otimes\mathrm{id}_I=\alpha$, which holds in a strict category but may fail to do so in a nonstrict one, to prove that $1^\times$ is a unit element for the multiplication $\times$.

We shall refer to the semiring $Q^c=(Q_S(\mathbf{C}),+,\cdot,0,1)$ as the *composition semiring* of **C** (with ground semiring S), and to $Q^m=(Q_S(\mathbf{C}),+,\times,0,1^\times)$ as the *monoidal semiring* of **C**.

Given morphisms

$$X'\xrightarrow{\xi'}Z'\xrightarrow{\eta'}Y',\ X''\xrightarrow{\xi''}Z''\xrightarrow{\eta''}Y'',$$

the identity

$$(\eta'\circ\xi')\otimes(\eta''\circ\xi'')=(\eta'\otimes\eta'')\circ(\xi'\otimes\xi'') \tag{4}$$

holds. This shows that the *composition-tensor-composition (CTC) set* of a morphism $\gamma: X\to Y$ in **C**,

$$CTC(\gamma)=\{(\xi',\xi'',\eta',\eta'')\in\mathrm{Mor}(\mathbf{C})^4\mid(\eta'\circ\xi')\otimes(\eta''\circ\xi'')=\gamma\}$$

is a subset of the *tensor-composition-tensor (TCT) set* of γ,

$$TCT(\gamma)=\{(\xi',\xi'',\eta',\eta'')\in\mathrm{Mor}(\mathbf{C})^4\mid(\eta'\otimes\eta'')\circ(\xi'\otimes\xi'')=\gamma\},$$

since the equation $(\eta'\circ\xi')\otimes(\eta''\circ\xi'')=\gamma$ implies that $\operatorname{cod}\xi'=\operatorname{dom}\eta'$ and $\operatorname{cod}\xi''=\operatorname{dom}\eta''$, so that (4) is applicable. However, knowing only $(\eta'\otimes\eta'')\circ(\xi'\otimes\xi'')=\gamma$, one can infer $\operatorname{cod}(\xi'\otimes\xi'')=\operatorname{dom}(\eta'\otimes\eta'')$, but *not* the individual statements $\operatorname{cod}\xi'=\operatorname{dom}\eta'$ and $\operatorname{cod}\xi''=\operatorname{dom}\eta''$. Thus (4) is not necessarily applicable and $TCT(\gamma)$ is in general strictly larger than $CTC(\gamma)$.

Proposition 4.4. *Let* **C** *be a strict monoidal category. Then $TCT(\gamma)=CTC(\gamma)$ for all morphisms γ in* **C** *if and only if* **C** *is a monoid, i.e. has only one object.*

Proof. If **C** has only one object, then this object must be the unit object I and $\operatorname{cod}\xi'=I=\operatorname{dom}\eta'$ and $\operatorname{cod}\xi''=I=\operatorname{dom}\eta''$ for all $(\xi',\xi'',\eta',\eta'')\in TCT(\gamma)$. Thus $TCT(\gamma)=CTC(\gamma)$ for all morphisms γ. For the converse direction, suppose $TCT(\gamma)=CTC(\gamma)$ for all morphisms γ, and let X be any object of $\mathbf{C}$. Write $\gamma = \mathrm{id}_X = (\mathrm{id}_X \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes \mathrm{id}_X)$. Then $(\mathrm{id}_I, \mathrm{id}_X, \mathrm{id}_X, \mathrm{id}_I) \in TCT(\mathrm{id}_X) = CTC(\mathrm{id}_X)$ and hence $\mathrm{id}_X = (\mathrm{id}_X \circ \mathrm{id}_I) \otimes (\mathrm{id}_I \circ \mathrm{id}_X)$. It follows that $X = \mathrm{dom}(\mathrm{id}_X) = \mathrm{cod}(\mathrm{id}_I) = I$. □

Let us translate the equivalent statements of the preceding proposition into a statement about the algebraic structure of $Q_S(\mathbf{C})$.

Proposition 4.5. *If* $\mathbf{C}$ *is a monoid (i.e. has only one object), then for any elements* $a,b,c,d \in Q_S(\mathbf{C})$ *such that* b *or* c *maps entirely into the center of* S*, the multiplicative compatibility relation*

$$(a \times b) \cdot (c \times d) = (a \cdot c) \times (b \cdot d)$$

holds.

Proof. On an endomorphism $\gamma : I \to I$ in $\mathbf{C}$,

$$((a\times b)\cdot(c\times d))_{II}(\gamma) = \sum_{\eta\circ\xi=\gamma} (c\times d)_{II}(\eta)\cdot(a\times b)_{II}(\xi)$$

$$\begin{aligned}
&= \sum_{\eta\circ\xi=\gamma} \Big\{ \sum_{\eta'\otimes\eta''=\eta} d_{II}(\eta'')\cdot c_{II}(\eta') \Big\} \cdot \Big\{ \sum_{\xi'\otimes\xi''=\xi} b_{II}(\xi'')\cdot a_{II}(\xi') \Big\} \\
&= \sum_{(\xi',\xi'',\eta',\eta'')\in TCT(\gamma)} d_{II}(\eta'')\cdot c_{II}(\eta')\cdot b_{II}(\xi'')\cdot a_{II}(\xi') \\
&= \sum_{(\xi',\xi'',\eta',\eta'')\in CTC(\gamma)} d_{II}(\eta'')\cdot b_{II}(\xi'')\cdot c_{II}(\eta')\cdot a_{II}(\xi') \\
&= \sum_{\gamma'\otimes\gamma''=\gamma} \Big\{ \sum_{\eta''\circ\xi''=\gamma''} d_{II}(\eta'')\cdot b_{II}(\xi'') \Big\} \cdot \Big\{ \sum_{\eta'\circ\xi'=\gamma'} c_{II}(\eta')\cdot a_{II}(\xi') \Big\} \\
&= \sum_{\gamma'\otimes\gamma''=\gamma} (b\cdot d)_{II}(\gamma'')\cdot(a\cdot c)_{II}(\gamma') \\
&= ((a\cdot c)\times(b\cdot d))_{II}(\gamma).
\end{aligned}$$

In this calculation, we were able to commute $c_{II}(\eta')$ and $b_{II}(\xi'')$ because one of these two commutes with every element of S. □

When ξ' and ξ'' are fixed, we shall also write

$$CTC(\gamma;\xi',\xi'') = \{(\eta',\eta'') \in \mathrm{Mor}(\mathbf{C})^2 \mid (\eta'\circ\xi')\otimes(\eta''\circ\xi'') = \gamma\},$$

$$TCT(\gamma;\xi',\xi'') = \{(\eta',\eta'') \in \mathrm{Mor}(\mathbf{C})^2 \mid (\eta'\otimes\eta'')\circ(\xi'\otimes\xi'') = \gamma\}.$$

For certain applications, let us record the following simple observation.

Lemma 4.6. *A commutative monoid* $(C,\cdot,1_C)$ *determines a small strict monoidal category* $\mathbf{C} = \mathbf{C}(C)$ *by*

$$\mathrm{Ob}\,\mathbf{C} = \{I\},\ \mathrm{End}_{\mathbf{C}}(I) = C,\ I\otimes I = I,\ \alpha\circ\beta = \alpha\cdot\beta = \alpha\otimes\beta$$

for all $\alpha,\beta \in C$.

5. Fields and Category-Valued Actions

The two ingredients needed to form a field theory are the fields and an action functional on these fields. Both have to satisfy certain natural axioms. Regarding the fields, our axioms will not deviate essentially from the usual axioms as employed in [Kir10], [Fre92], for example. We emphasize, however, that our axiomatization assigns fields only in codimensions 0 and 1, and not in higher codimensions. Closed n-dimensional topological manifolds will be denoted by M, N, P, M_0, etc. Our manifolds need not be orientable. The empty set $\varnothing$ is a manifold of any dimension. The symbol $\sqcup$ denotes the ordinary ordered disjoint union of manifolds. It is not commutative and not associative (see the Remark on p. 72 of [MR94]), but there are obvious canonical homeomorphisms $M \sqcup N \cong N \sqcup M$, $(M \sqcup N) \sqcup P \cong M \sqcup (N \sqcup P)$, $M \sqcup \varnothing \cong M \cong \varnothing \sqcup M$. Note that the triple union $M \sqcup N \sqcup P$ is well-defined and canonically homeomorphic to both $(M \sqcup N) \sqcup P$ and $M \sqcup (N \sqcup P)$. An $(n+1)$-dimensional *bordism* (sometimes also called spacetime in the literature) is a triple (W, M, N), where W is a compact $(n+1)$-dimensional topological manifold with boundary $\partial W = M \sqcup N$. The closed n-manifold M is called the *incoming boundary* of W and N is called the *outgoing boundary* of W. (Strictly speaking, recording the outgoing boundary is redundant since $N = \partial W - M$; nevertheless we find it convenient to include N in the notation as well.) We shall also say that W is a bordism from M to N. Setting $\partial W^{\text{in}} = M$, $\partial W^{\text{out}} = N$, we may simply write W for the bordism $(W, \partial W^{\text{in}}, \partial W^{\text{out}})$. For example, the cylinder $M \times [0,1]$ on a connected M gives rise to three distinct bordisms, namely $(M \times [0,1], M \times \{0,1\}, \varnothing)$, $(M \times [0,1], M \times 0, M \times 1)$ and $(M \times [0,1], \varnothing, M \times \{0,1\})$. The operation disjoint union is defined on bordisms by

$$(W, M, N) \sqcup (W', M', N') = (W \sqcup W', M \sqcup M', N \sqcup N').$$

If the outgoing boundary N of W is the incoming boundary of a bordism W', then we may glue along N to obtain the bordism

$$(W, M, N) \cup_N (W', N, P) = (W \sqcup_N W', M, P).$$

A *homeomorphism* $\phi : (W, M, N) \to (W', M', N')$ *of bordisms* is a homeomorphism $W \to W'$, which preserves incoming boundaries and outgoing boundaries, $\phi(M) = M'$, $\phi(N) = N'$. The bordism (W_0, M_0, N_0) is a *subbordism* of (W, M, N) if W_0 is a codimension 0 submanifold of W and the following two conditions are satisfied: For every connected component C of M_0 either $C \cap \partial W = \varnothing$ or $C \subset M$, and for every connected component C of N_0 either $C \cap \partial W = \varnothing$ or $C \subset N$. For instance, (W, M, N) is a subbordism of $(W, M, N) \sqcup (W', M', N')$ and it is a subbordism of $(W, M, N) \cup_N (W', N, P)$.

Definition 5.1. A *system $\mathcal{F}$ of fields* assigns to each $(n+1)$-dimensional bordism W a set $\mathcal{F}(W)$ (whose elements are called the *fields* on W) and to every closed n-manifold M a set $\mathcal{F}(M)$ such that $\mathcal{F}(\varnothing)$ is a set with one element and the following axioms are satisfied:

(FRES) *Restrictions*: If $W_0 \subset W$ is a subbordism, then there is a restriction map $\mathcal{F}(W) \to \mathcal{F}(W_0)$. If $M_0 \subset M$ is a codimension 0 submanifold, then there is a restriction map $\mathcal{F}(M) \to \mathcal{F}(M_0)$. If $M \subset W$ is a closed (as a manifold) codimension 1 submanifold, then there is a restriction map $\mathcal{F}(W) \to \mathcal{F}(M)$. If $f \in \mathcal{F}(W)$ is a field, we will write $f|_M$ for its restriction to M, and similarly for the other types of restriction. All these restriction maps are required to commute with each other in the obvious way, e.g. for $M_0 \subset M \subset W$, the map $\mathcal{F}(W) \to \mathcal{F}(M_0)$ is the composition $\mathcal{F}(W) \to \mathcal{F}(M) \to \mathcal{F}(M_0)$. Let M be a closed (as a manifold) codimension 0 submanifold of ∂W. A given field $f \in \mathcal{F}(M)$ may be imposed as a boundary condition by setting

$$\mathcal{F}(W,f) = \{F \in \mathcal{F}(W) \mid F|_M = f\}.$$

If W is a bordism from M to N and $f \in \mathcal{F}(M)$, $g \in \mathcal{F}(N)$, we shall also use the notation $\mathcal{F}(W,f,g) \subset \mathcal{F}(W)$ for the set of all fields on W which restrict to f on the incoming boundary M and to g on the outgoing boundary N.

(FHOMEO) *Action of homeomorphisms*: A homeomorphism $\phi : W \to W'$ of bordisms induces contravariantly a bijection $\phi^* : \mathcal{F}(W') \to \mathcal{F}(W)$ such that $(\mathrm{id}_W)^* = \mathrm{id}_{\mathcal{F}(W)}$ and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ for a homeomorphism $\psi : W' \to W''$. Similarly for n-dimensional homeomorphisms $M \to N$. These induced maps are required to commute with the restriction maps of (FRES). For example, if $M \subset W$ and $M' \subset W'$ are codimension 1 submanifolds and $\phi : W \to W'$ restricts to a homeomorphism $\phi| : M \to M'$, then the diagram

$$\begin{tikzcd} \mathcal{F}(W') \arrow[r, "\phi^*"] \arrow[d, "\mathrm{res}"'] & \mathcal{F}(W) \arrow[d, "\mathrm{res}"] \\ \mathcal{F}(M') \arrow[r, "(\phi|)^*"] & \mathcal{F}(M) \end{tikzcd}$$

is to commute.

(FDISJ) *Disjoint Unions*: The product of restrictions

$$\mathcal{F}(W \sqcup W') \longrightarrow \mathcal{F}(W) \times \mathcal{F}(W')$$

is a bijection, that is, a field on the disjoint union $W \sqcup W'$ is uniquely determined by its restrictions to W and W', and a field on W and a field on W' together give rise to a field on $W \sqcup W'$. Similarly, $\mathcal{F}(M \sqcup N) \longrightarrow \mathcal{F}(M) \times \mathcal{F}(N)$ must be a bijection in dimension n.

(FGLUE) *Gluing*: Let W' be a bordism from M to N and let W'' be a bordism from N to P. Let $W = W' \cup_N W''$ be the bordism from M to P obtained by gluing W' and W'' along N. Let $\mathcal{F}(W',W'')$ be the pullback $\mathcal{F}(W') \times_{\mathcal{F}(N)} \mathcal{F}(W'')$ fitting into a cartesian square

$$\begin{tikzcd} \mathcal{F}(W',W'') \arrow[r] \arrow[d] & \mathcal{F}(W') \arrow[d, "\mathrm{res}"] \\ \mathcal{F}(W'') \arrow[r, "\mathrm{res}"] & \mathcal{F}(N). \end{tikzcd}$$

Since

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"'] & \mathcal{F}(W') \arrow[d, "\mathrm{res}"] \\ \mathcal{F}(W'') \arrow[r, "\mathrm{res}"] & \mathcal{F}(N) \end{tikzcd}$$

commutes, there exists a unique map $\mathcal{F}(W) \to \mathcal{F}(W',W'')$ such that $\mathcal{F}(W) \to \mathcal{F}(W',W'') \to \mathcal{F}(W')$ is the restriction to W' and $\mathcal{F}(W) \to \mathcal{F}(W',W'') \to \mathcal{F}(W'')$ is the restriction to W''. We require that $\mathcal{F}(W) \to \mathcal{F}(W',W'')$ is a bijection.

Note that the convention for $\mathcal{F}(\varnothing)$ is consistent with axioms (FHOMEO) and (FDISJ): The homeomorphism $\phi : M \sqcup \varnothing \xrightarrow{\cong} M$ induces a bijection $\phi^* : \mathcal{F}(M) \xrightarrow{\cong} \mathcal{F}(M \sqcup \varnothing)$. Thus by (FDISJ),

$$\mathcal{F}(M) \cong \mathcal{F}(M \sqcup \varnothing) \cong \mathcal{F}(M) \times \mathcal{F}(\varnothing) = \mathcal{F}(M) \times \{\mathrm{pt}\}.$$

For bordisms W' with empty outgoing boundary and bordisms W'' with empty incoming boundary, axiom (FDISJ) follows from (FGLUE) by taking $N = \varnothing$. For then $W' \cup_\varnothing W'' = W' \sqcup W''$ and $\mathcal{F}(W', W'') = \mathcal{F}(W') \times \mathcal{F}(W'')$. However, since not all bordisms are of this type, the $(n+1)$-dimensional part of axiom (FDISJ) is not redundant.

Lemma 5.2. *Let $\mathcal{F}$ be a system of fields and $M \subset \partial W, M' \subset \partial W'$ closed codimension 0 submanifolds. Then axiom (FDISJ) continues to hold in the presence of boundary conditions. More precisely: If $f \in \mathcal{F}(M \sqcup M')$ is a field, then the bijection $\mathcal{F}(W \sqcup W') \to \mathcal{F}(W) \times \mathcal{F}(W')$ restricts to a bijection*

$$\mathcal{F}(W \sqcup W', f) \longrightarrow \mathcal{F}(W, f|_M) \times \mathcal{F}(W', f_{M'}).$$

Proof. The bijection $\sigma : \mathcal{F}(W \sqcup W') \to \mathcal{F}(W) \times \mathcal{F}(W')$ is given by $\sigma(G) = (G|_W, G|_{W'})$. To show that it restricts as claimed, let $G \in \mathcal{F}(W \sqcup W')$ be a field with $G|_{M \sqcup M'} = f$. Using the diagram of restrictions

$$\begin{tikzcd} \mathcal{F}(W \sqcup W') \arrow[r] \arrow[d] & \mathcal{F}(W) \arrow[d] \\ \mathcal{F}(M \sqcup M') \arrow[r] & \mathcal{F}(M), \end{tikzcd} \tag{5}$$

which commutes by axiom (FRES), (and using also the analogous diagram for M'), we have

$$(G|_W)|_M = (G|_{M \sqcup M'})|_M = f|_M, \; (G|_{W'})|_{M'} = (G|_{M \sqcup M'})|_{M'} = f|_{M'}.$$

Thus $\sigma(G) \in \mathcal{F}(W, f|_M) \times \mathcal{F}(W', f|_{M'})$ and σ restricts preserving boundary conditions.

This restriction is injective as the restriction of the injective map σ. To show that the restriction is surjective, let $F \in \mathcal{F}(W)$ and $F' \in \mathcal{F}(W')$ be fields with $F|_M = f|_M$ and $F'|_{M'} = f|_{M'}$. Since σ is surjective, there exists a field $G \in \mathcal{F}(W \sqcup W')$ such that $G|_W = F$ and $G|_{W'} = F'$. Using again diagram (5), we find

$$((G|_{M \sqcup M'})|_M, (G|_{M \sqcup M'})|_{M'}) = ((G|_W)|_M, (G|_{W'})|_{M'}) = (F|_M, F'|_{M'}) = (f|_M, f|_{M'}).$$

Since $\mathcal{F}(M \sqcup M') \to \mathcal{F}(M) \times \mathcal{F}(M')$ is a bijection by axiom (FDISJ), we conclude that $G|_{M \sqcup M'} = f$, that is, $G \in \mathcal{F}(W \sqcup W', f)$. □

Lemma 5.3. *Let $\mathcal{F}$ be a system of fields, let W' be a bordism from M to N and let W'' be a bordism from N to P. Let $W = W' \cup_N W''$ be the bordism from M to P obtained by gluing W' and W'' along N. Then axiom (FGLUE) continues to hold in the presence of boundary conditions. More precisely: Given fields $g' \in \mathcal{F}(M)$, $g'' \in \mathcal{F}(P)$, let $\mathcal{F}(W', W'', g', g'')$ be the pullback $\mathcal{F}(W', g') \times_{\mathcal{F}(N)} \mathcal{F}(W'', g'')$. Then, given a field $f \in \mathcal{F}(\partial W)$, the unique map ρ such that*

$$\begin{tikzcd} \mathcal{F}(W,f) \arrow[dr, "\rho"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{F}(W', W'', f|_M, f|_P) \arrow[r] \arrow[d] & \mathcal{F}(W', f|_M) \arrow[d] \\ & \mathcal{F}(W'', f|_P) \arrow[r] & \mathcal{F}(N) \end{tikzcd}$$

commutes is a bijection.

Proof. The bijection $\sigma : \mathcal{F}(W) \to \mathcal{F}(W', W'')$ is given by $\sigma(G) = (G|_{W'}, G|_{W''})$. Let $G \in \mathcal{F}(W)$ be a field with $G|_{\partial W} = f$. Using the diagram of restrictions

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r] \arrow[d] & \mathcal{F}(W') \arrow[d] \\ \mathcal{F}(M \sqcup P) \arrow[r] & \mathcal{F}(M), \end{tikzcd} \tag{6}$$

which commutes by axiom (FRES), (and using also the analogous diagram for P), we have

$$(G|_{W'})|_M = (G|_{M\sqcup P})|_M = f|_M,\ (G|_{W''})|_P = (G|_{M\sqcup P})|_P = f|_P.$$

Since in addition $(G|_{W'})|_N = (G|_{W''})|_N$, we conclude that $\sigma(G) \in \mathcal{F}(W', W'', f|_M, f|_P)$ and thus ρ is the restriction of σ to $\mathcal{F}(W, f) \subset \mathcal{F}(W)$.

This restriction ρ is injective as the restriction of the injective map σ. To show that ρ is surjective, let

$$(F', F'') \in \mathcal{F}(W', W'', f|_M, f|_P) \subset \mathcal{F}(W', W'').$$

Since σ is surjective, there exists a field $G \in \mathcal{F}(W)$ such that $G|_{W'} = F'$ and $G|_{W''} = F''$. Using again diagram (6), we find

$$((G|_{\partial W})|_M, (G|_{\partial W})|_P) = ((G|_{W'})|_M, (G|_{W''})|_P) = (F'|_M, F''|_P) = (f|_M, f|_P).$$

Since $\mathcal{F}(\partial W) = \mathcal{F}(M \sqcup P) \to \mathcal{F}(M) \times \mathcal{F}(P)$ is a bijection by axiom (FDISJ), we conclude that $G|_{\partial W} = f$, that is, $G \in \mathcal{F}(W, f)$. □

Remark 5.4. As with all axiomatic systems, the above axioms may need to be appropriately adapted to concrete situations. For instance, the manifolds to be considered may be decorated with additional structure, for instance orientations. If the fields interact with the additional structure, then the restrictions in axiom (FRES) will in general only be available for inclusions that preserve the additional structure. In (FHOMEO), only those homeomorphisms that preserve the structure will act on the fields. For example, in an equivariant context, one may wish to impose (FHOMEO) only on equivariant homeomorphisms. In (FDISJ), the disjoint union will be assumed to be equipped with the structure compatible to the structures on the component manifolds. Analogous provisos apply to (FGLUE). The axioms can be adapted to the category of smooth manifolds and smooth maps. The main issue there is to arrive at a correct version of (FGLUE), since gluing two smooth maps that agree on the common boundary component N only yields a map which is continuous but usually not smooth. This can be achieved by not only requiring equality of the function values (as we have done in (FGLUE)), but also equality of all higher partial derivatives. Another possibility is to require the functions to be equal on collar neighborhoods of N and then to glue the collars. See also Section 10 for a concrete solution.

Example 5.5. Let B be a fixed space. Taking $\mathcal{F}(W)$ and $\mathcal{F}(M)$ to be the set of all continuous maps $W \to B$, $M \to B$, respectively, and using the ordinary restrictions of such maps to subspaces in (FRES), one obtains a system $\mathcal{F}$ of fields in the sense of Definition 5.1. The action of homeomorphisms on fields is given by composition of fields with a given homeomorphism. In practice, B is often the classifying space BG of some topological group G (which may be discrete), so that fields in that case have the interpretation of principal G-bundles over W and M. This has been considered for finite groups G in work of Freed and Quinn [QF93], [Qui95], [Fre92], see also [DW90]. Fields of this kind are also used in the construction of the twisted signature TFT given in Section 11.3. Let us note in passing that taking manifolds endowed with maps to a fixed space B as *objects* (and not as fields on objects), one arrives at the notion of a *homotopy quantum field theory* (HQFT), [Tur10]. Taking B to be a point, HQFTs are seen to be generalizations of TQFTs. In the smooth category, one may fix B to be a smooth manifold and consider $\mathcal{F}(W) = C^\infty(W,B)$, the space of smooth maps $W \to B$. This is roughly the setting for Chern-Simons theory.

Remark 5.6. Walker's axiomatization of fields, [Wal06], differs from ours (and from [Kir10], [Fre92]) in that he does not allow for codimension 0 restrictions and he requires the existence of an injection $\mathcal{F}(W',W'') \hookrightarrow \mathcal{F}(W)$ in the context of the gluing axiom. However, he does assume any field on W to be close to a field in the image of the injection in the sense that the field on W can be moved by a homeomorphism, which is isotopic to the identity and supported in a small neighborhood of $N \subset W$, to a field coming from $\mathcal{F}(W',W'')$ under gluing. Walker does not require a bijection because he wants to allow for the following application: Fields could be embedded submanifolds, or even more intricate "designs" on manifolds, which are transverse to the boundary. Given any submanifold of W, there is no way of guaranteeing that it is transverse to N (though it can be made so by an arbitrarily small movement). Thus there is no restriction map from such fields on W to fields on W', and not every field on W comes from one on W' and one on W'' by gluing.

Given a system $\mathcal{F}$ of fields, the second ingredient necessary for a field theory is an action functional defined on $\mathcal{F}(W)$ for bordisms W. In classical quantum field theory, the action is usually a system of real-valued functions $S_W : \mathcal{F}(W) \to \mathbb{R}$ such that the additivity axiom

$$S_{W \sqcup W'}(f) = S_W(f|_W) + S_{W'}(f|_{W'}),\ f \in \mathcal{F}(W \sqcup W'), \tag{7}$$

is satisfied for disjoint unions, and the additivity axiom

$$S_W(f) = S_{W'}(f|_{W'}) + S_{W''}(f|_{W''}),\ f \in \mathcal{F}(W), \tag{8}$$

is satisfied for $W = W' \cup_N W''$, the result of gluing a bordism W' with outgoing boundary N to a bordism W'' with incoming boundary N. Moreover, the action should be topologically invariant: if $\phi : W \to W'$ is a homeomorphism, then for any field $f \in \mathcal{F}(W')$, one requires that under the bijection $\phi^* : \mathcal{F}(W') \to \mathcal{F}(W)$ of (FHOMEO), the action is preserved,

$$S_W(\phi^* f) = S_{W'}(f). \tag{9}$$

Sometimes, for example in Chern-Simons theory, the action is only well-defined up to an integer, that is, takes values in $\mathbb{R}/\mathbb{Z}$. Thus it is better to exponentiate and consider the complex-valued function $T_W = e^{2\pi i S_W} : \mathcal{F}(W) \to \mathbb{C}$ whose image lies in the unit circle. The above two additivity axioms are then transformed into the multiplicativity axioms

$$T_{W \sqcup W'}(f) = T_W(f|_W) \cdot T_{W'}(f|_{W'}), \tag{10}$$

and

$$T_W(f) = T_{W'}(f|_{W'}) \cdot T_{W''}(f|_{W''}). \tag{11}$$

These axioms express that the action should be local to a certain extent.

Example 5.7. In the smooth oriented category, for the system of fields $\mathcal{F}(W) = C^\infty(W,B)$, B a fixed smooth manifold, fix a differential $(n+1)$-form $\omega \in \Omega^{n+1}(B)$ on B. Setting

$$S_W(f) = \int_W f^* \omega,$$

the axioms (7) and (8) are satisfied. For an orientation preserving diffeomorphism $\phi : W \to W'$, (9) holds. If ω is closed and W has no boundary, then $S_W(f)$ only depends on the homotopy class of f. For if $f,g : W \to B$ are homotopic, then a homotopy between them gives rise to a homotopy operator $h : \Omega^*(B) \to \Omega^{*-1}(W)$, $dh + hd = f^* - g^*$, so that for closed ω one has $f^*(\omega) - g^*(\omega) = dh(\omega)$. By Stokes theorem,

$$\int f^*\omega - \int g^*\omega = \int dh(\omega) = 0.$$

The Chern-Simons action is roughly of this type.

For a number of purposes, remembering only a real number for a given field is too restrictive and it is desirable to retain more information about the field. The present paper thus introduces category valued actions. We will in fact directly axiomatize the analog of the exponential T of an action. Let $(\mathbf{C}, \otimes, I)$ be a strict monoidal category. (The strictness is not a very serious assumption, as a well-known process turns any monoidal category into a monoidally equivalent strict one, see [Kas95]). Since in a monoidal context of bordisms, disjoint union corresponds to the tensor product, while gluing of bordisms corresponds to the composition of morphisms, it is natural to modify the classical axioms (10) and (11) as follows:

Definition 5.8. Given a system $\mathcal{F}$ of fields, a *system* $\mathbb{T}$ *of* $\mathbf{C}$*-valued action exponentials* consists of functions $\mathbb{T}_W : \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{C})$, for all bordisms W, such that for the empty manifold, $\mathbb{T}_\varnothing(p) = \mathrm{id}_I$, where p is the unique element of $\mathcal{F}(\varnothing)$, and the following three axioms are satisfied:

(TDISJ) If $W \sqcup W'$ is the ordered disjoint union of two bordisms W, W', then

$$\mathbb{T}_{W \sqcup W'}(f) = \mathbb{T}_W(f|_W) \otimes \mathbb{T}_{W'}(f|_{W'})$$

for all $f \in \mathcal{F}(W \sqcup W')$,

(TGLUE) If $W = W' \cup_N W''$ is obtained by gluing a bordism W' with outgoing boundary N to a bordism W'' with incoming boundary N, then

$$\mathbb{T}_W(f) = \mathbb{T}_{W''}(f|_{W''}) \circ \mathbb{T}_{W'}(f|_{W'})$$

for all $f \in \mathcal{F}(W)$, and

(THOMEO) If $\phi : W \to W'$ is a homeomorphism of bordisms, then for any field $f \in \mathcal{F}(W')$, we require that under the bijection $\phi^* : \mathcal{F}(W') \to \mathcal{F}(W)$ of (FHOMEO),

$$\mathbb{T}_W(\phi^* f) = \mathbb{T}_{W'}(f).$$

If $W = M \times I$ is the cylindrical bordism from M to M, then we do *not* require that $\mathbb{T}_{M\times I}$ is an identity morphism. In particular, $\mathbb{T}$ cannot be rephrased as a monoidal functor. There is a simple test that shows that both the tensor product and the composition product of $\mathbf{C}$ must enter into the axioms for a category-valued action, underlining the correctness of the above definition:

Example 5.9. *(The tautological action.)* Let $\mathbf{C} = \mathbf{Bord}(n+1)^{\mathrm{str}}$ a strict version of the $(n+1)$-dimensional bordism category. A bordism W defines a morphism $[W] \in \mathrm{Mor}(\mathbf{Bord}(n+1)^{\mathrm{str}})$. Then one has the tautological action exponential $\mathbb{T}_W(f) = [W]$. It satisfies

$$\mathbb{T}_{W \sqcup W'}(f) = [W \sqcup W'] = [W] \otimes [W'] = \mathbb{T}_W(f|) \otimes \mathbb{T}_{W'}(f|)$$

and

$$\mathbb{T}_{W' \cup_N W''}(f) = [W' \cup_N W''] = [W''] \circ [W'] = \mathbb{T}_{W''}(f|) \circ \mathbb{T}_{W'}(f|).$$

This forces the above axioms (TDISJ) and (TGLUE).

If the manifolds W are equipped with some extra structure, then one will in practice usually modify (THOMEO) to apply only to those homeomorphisms that preserve the extra structure. For example, if the W are oriented, one will usually require ϕ to preserve orientations. Note that under the canonical homeomorphism $\phi : W \cong W \sqcup \varnothing$,

$$\mathbb{T}_W(\phi^* f) = \mathbb{T}_{W\sqcup\varnothing}(f) = \mathbb{T}_W(f|_W) \otimes \mathbb{T}_\varnothing(f|_\varnothing) = \mathbb{T}_W(f|_W) \otimes \mathrm{id}_I = \mathbb{T}_W(f|_W),$$

using axioms (TDISJ) and (THOMEO). If W' has empty outgoing boundary and W'' empty incoming boundary, then we can "glue" along the empty set and get $W' \cup_\varnothing W'' = W' \sqcup W''$. Thus (TGLUE) and (TDISJ) apply simultaneously and yield

$$\mathbb{T}_{W'}(f) \otimes \mathbb{T}_{W''}(g) = \mathbb{T}_{W''}(g) \circ \mathbb{T}_{W'}(f).$$

In particular, the domain of any $\mathbb{T}_{W'}(f)$ must be the tensor product of the domain of $\mathbb{T}_{W'}(f)$ with the domain of any $\mathbb{T}_{W''}(g)$. In practice, this usually means that the domains of all $\mathbb{T}_{W''}(g)$ are the unit object I of $\mathbf{C}$. But the domain of $\mathbb{T}_{W''}(g)$ equals the codomain of $\mathbb{T}_{W'}(f)$. So in practice, the codomains of the $\mathbb{T}_{W'}(f)$ are usually I as well. We would like to emphasize again that these remarks apply only to bordisms whose incoming or outgoing boundary is empty.

Let W'_1 be a bordism with empty outgoing boundary, W'_2 a bordism from $\varnothing$ to N, W''_2 a bordism from N to $\varnothing$ and let W''_1 be a bordism with empty incoming boundary. Then we can form the bordism

$$W = (W'_2 \sqcup W'_1) \cup_N (W''_2 \sqcup W''_1),$$

which we can also think of as

$$W = (W'_2 \cup_N W''_2) \sqcup (W'_1 \cup_\varnothing W''_1).$$

These two representations of W allow us to calculate the action associated with W in two different ways:

$$\mathbb{T}_W(f) = \mathbb{T}_{W''_2 \sqcup W''_1}(f|) \circ \mathbb{T}_{W'_2 \sqcup W'_1}(f|) = (\mathbb{T}_{W''_2}(f|) \otimes \mathbb{T}_{W''_1}(f|)) \circ (\mathbb{T}_{W'_2}(f|) \otimes \mathbb{T}_{W'_1}(f|))$$

and

$$\mathbb{T}_W(f) = \mathbb{T}_{W'_2 \cup_N W''_2}(f|) \otimes \mathbb{T}_{W'_1 \cup_\varnothing W''_1}(f|) = (\mathbb{T}_{W''_2}(f|) \circ \mathbb{T}_{W'_2}(f|)) \otimes (\mathbb{T}_{W''_1}(f|) \circ \mathbb{T}_{W'_1}(f|)).$$

This implies the equation

$$(\mathbb{T}_{W''_2}(f|) \circ \mathbb{T}_{W'_2}(f|)) \otimes (\mathbb{T}_{W''_1}(f|) \circ \mathbb{T}_{W'_1}(f|)) = (\mathbb{T}_{W''_2}(f|) \otimes \mathbb{T}_{W''_1}(f|)) \circ (\mathbb{T}_{W'_2}(f|) \otimes \mathbb{T}_{W'_1}(f|)),$$

which indeed holds automatically in any monoidal category $\mathbf{C}$.

The result of gluing two copies $W' = M \times [0,1]$ and $W'' = M \times [0,1]$ of the unit cylinder on M, identifying $M \times 1 \subset W'$ with $M \times 0 \subset W''$, is $W = M \times [0,2]$. For $(F', F'') \in \mathcal{F}(W', W'')$, axioms (TGLUE) and (THOMEO) imply the formula

$$\mathbb{T}_{M\times[0,1]}(2^*\sigma^{-1}(F', F'')) = \mathbb{T}_{M\times[0,1]}(F'') \circ \mathbb{T}_{M\times[0,1]}(F'), \tag{12}$$

where $\sigma : \mathcal{F}(W) \to \mathcal{F}(W', W'')$ is the bijection of axiom (FGLUE) and $2 : M \times [0,1] \to M \times [0,2]$ is the stretching homeomorphism $2(x,t) = (x, 2t)$, $x \in M$, $t \in [0,1]$.

Remark 5.10. The classical axioms (10) and (11) do fit into the framework of Definition 5.8. From the perspective of this definition, the fact that in both (10) and (11) the ordinary multiplication of complex numbers appears is just a reflection of the coincidence that under the standard isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$, the tensor product of two $\mathbb{C}$-linear maps $\alpha, \beta : \mathbb{C} \to \mathbb{C}$ is given by multiplication $\alpha \cdot \beta$, and the composition of two linear maps $\alpha, \beta : \mathbb{C} \to \mathbb{C}$ *also* happens to be given by multiplication, $\alpha \cdot \beta$. More precisely, let $\widehat{\mathbb{C}}$ be the category which has $\mathbb{C}$ as its single object and $\mathrm{Hom}_{\widehat{\mathbb{C}}}(\mathbb{C}, \mathbb{C}) = \{\alpha : \mathbb{C} \to \mathbb{C} \mid \alpha \text{ is } \mathbb{C}\text{-linear}\}$. Such an

α is of course determined by $\alpha(1)$, whence $\mathrm{Hom}_{\widehat{\mathbb{C}}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}$. In $\widehat{\mathbb{C}}$, define $\mathbb{C}\otimes\mathbb{C} := \mathbb{C}$ and define $\alpha\otimes\beta : \mathbb{C}\otimes\mathbb{C} = \mathbb{C} \to \mathbb{C} = \mathbb{C}\otimes\mathbb{C}$ by $(\alpha\otimes\beta)(1) = \alpha(1)\cdot\beta(1)$. Taking $I = \mathbb{C}$, $(\widehat{\mathbb{C}},\otimes,I)$ is a strict monoidal category. If a classical action exponential $T_W : \mathcal{F}(W) \to \mathbb{C}$ is interpreted as a $\widehat{\mathbb{C}}$-valued action exponential $\mathbb{T}_W : \mathcal{F}(W) \to \mathbb{C} \cong \mathrm{Mor}(\widehat{\mathbb{C}})$, then (TDISJ) translates to (10) and (TGLUE) translates to (11).

Remark 5.11. *(On cutting.)* Suppose that W is an oriented bordism and $M \hookrightarrow W$ a closed oriented codimension 1 submanifold situated in the interior of W. Let W^{cut} be the compact manifold with boundary $\partial W^{\mathrm{cut}} = \partial W \sqcup M \sqcup M$ obtained from W by cutting along M. The problem is that, contrary to the operations of disjoint union and gluing, this construction is *not* well-defined on bordisms because there is no canonical way to define the incoming and outgoing boundary of W^{cut}. (Should $M \sqcup M$ belong to the incoming or outgoing boundary? Should one of them belong to the incoming and the other to the outgoing boundary? If so, which of the two copies is incoming and which outgoing?) Our field axioms do not provide for an equalizer diagram $\mathcal{F}(W) \to \mathcal{F}(W^{\mathrm{cut}}) \rightrightarrows \mathcal{F}(M)$ and our action axioms do *not* stipulate

$$\mathbb{T}_{W^{\mathrm{cut}}}(f^{\mathrm{cut}}) = \mathbb{T}_W(f), \tag{13}$$

where f^{cut} is the image of f under a putative $\mathcal{F}(W) \to \mathcal{F}(W^{\mathrm{cut}})$. Such axioms are classically sometimes adopted, for example in [Fre92], and are strongly motivated by thinking of actions as being given by integrals of pullbacks of differential forms. In the setting of the present paper, we wish to think of actions in much more general terms. For instance, actions might be certain subspaces of W associated to fields. But if W is cut, then these subspaces are also cut and consequently (13) cannot hold.

The next definition will be used in Section 9, when we discuss the behavior of a certain projection operator on tensor products of states. The projection is associated to the state sum of cylinders.

Definition 5.12. A system $\mathbb{T}$ of $\mathbf{C}$-valued action exponentials is called *cylindrically firm*, if

$$CTC(\gamma;\mathbb{T}_{M\times[0,1]}(F_M),\mathbb{T}_{N\times[0,1]}(F_N)) = TCT(\gamma;\mathbb{T}_{M\times[0,1]}(F_M),\mathbb{T}_{N\times[0,1]}(F_N))$$

for all morphisms γ in $\mathbf{C}$, closed n-manifolds M,N and fields $F_M \in \mathcal{F}(M\times[0,1])$, $F_N \in \mathcal{F}(N\times[0,1])$. Here, $M\times[0,1]$ is to be read as the bordism from $M\times 0$ to $M\times 1$, similarly for $N\times[0,1]$.

For instance, by Proposition 4.4, $\mathbb{T}$ is cylindrically firm if $\mathbf{C}$ is a monoid.

Proposition 5.13. *Let $\mathbb{T}$ be cylindrically firm. Then the codomain of every $\mathbb{T}_{M\times[0,1]}(F)$ is the unit object I of $\mathbf{C}$. Furthermore, if $F|_{M\times 0} = F|_{M\times 1}$, then $\mathbb{T}_{M\times[0,1]}(F)$ is an endomorphism of the unit object.*

Proof. For $M = \varnothing$, we have $\mathbb{T}_{M\times[0,1]}(F_M) = \mathbb{T}_\varnothing(p) = \mathrm{id}_I$. Taking $\gamma = \mathbb{T}_{N\times[0,1]}(F_N)$, the equation

$$(\mathrm{id}_X\otimes\mathrm{id}_I)\circ(\mathbb{T}_\varnothing(p)\otimes\mathbb{T}_{N\times[0,1]}(F_N)) = \mathbb{T}_{N\times[0,1]}(F_N)$$

holds, where X is the codomain of $\mathbb{T}_{N\times[0,1]}(F_N)$. This places $(\eta',\eta'') = (\mathrm{id}_X,\mathrm{id}_I)$ into

$$TCT(\mathbb{T}_{N\times[0,1]}(F_N);\mathbb{T}_\varnothing(p),\mathbb{T}_{N\times[0,1]}(F_N)) = CTC(\mathbb{T}_{N\times[0,1]}(F_N);\mathbb{T}_\varnothing(p),\mathbb{T}_{N\times[0,1]}(F_N)).$$

Therefore,

$$(\mathrm{id}_X\circ\mathrm{id}_I)\otimes(\mathrm{id}_I\circ\mathbb{T}_{N\times[0,1]}(F_N)) = \mathbb{T}_{N\times[0,1]}(F_N)$$

so that in particular $X = I$. Let M be any closed n-manifold. If $F \in \mathcal{F}(M \times [0,1])$ satisfies $F|_{M\times 0} = F|_{M\times 1}$, then the diagonal element (F,F) lies in the pullback $\mathcal{F}(M \times [0,1], M \times [0,1])$ and thus Equation (12) shows that $\operatorname{dom} \mathbb{T}_{M\times[0,1]}(F) = \operatorname{cod} \mathbb{T}_{M\times[0,1]}(F) = I$. □

It follows from the proposition that for a cylindrically firm system of action exponentials, the above TCT and CTC sets can be nonempty only for γ that factor through the unit object.

6. Quantization

We shall define our positive TFT Z in this section. We will specify the state module $Z(M)$ for a closed n-manifold M as well as an element $Z_W \in Z(\partial W)$, the *Zustandssumme*, for a bordism W, which may have a nonempty boundary ∂W. Neither M nor W have to be oriented; thus Z will be a nonunitary theory. In [Wit89], Witten starts out with the phase space $\mathcal{M}_0$ of all connections on a trivial G-bundle over $\Sigma \times \mathbb{R}^1$, where Σ is a Riemann surface and G a compact simple gauge group. In Section 3 of *loc. cit.*, he carries out the quantization of Chern-Simons theory on Σ in two steps: First, constraint equations are imposed, which reduce $\mathcal{M}_0$ to the finite dimensional moduli space $\mathcal{M}$ of flat connections, where two flat connections are identified if they differ by a gauge transformation. Second, Witten's quantum Hilbert space (state module) $\mathcal{H}_\Sigma$ is obtained by taking functions on $\mathcal{M}$, more precisely, global holomorphic sections of a certain line bundle on $\mathcal{M}$. This provides a model for our construction of the state module $Z(M)$.

Fix a complete semiring S, which will play the role of a ground semiring for the theory to be constructed. To any given system $\mathcal{F}$ of fields, strict monoidal small category $(\mathbf{C}, \otimes, I)$, and system $\mathbb{T}$ of $\mathbf{C}$-valued action exponentials, we shall now associate a positive topological field theory Z. In Section 4, we have seen that $\mathbf{C}$ determines two complete semirings: the composition semiring $Q^c = (Q_S(\mathbf{C}), +, \cdot, 0, 1)$, whose multiplication $\cdot$ encodes the composition law of $\mathbf{C}$, and the monoidal semiring $Q^m = (Q_S(\mathbf{C}), +, \times, 0, 1^\times)$, whose multiplication $\times$ encodes the monoidal structure on $\mathbf{C}$, i.e. the tensor functor $\otimes$. Both of these semirings have the same underlying (complete) additive monoid $(Q_S(\mathbf{C}), +, 0)$. For a closed n-dimensional manifold M, we define its *pre-state module* to be

$$E(M) = \operatorname{Fun}_Q(\mathcal{F}(M)),$$

where we have abbreviated $Q = Q_S(\mathbf{C})$. By Proposition 3.1, $E(M)$ is a two-sided Q^c-semialgebra and a two-sided Q^m-semialgebra. We observe that for the empty manifold,

$$E(\varnothing) = \operatorname{Fun}_Q(\mathcal{F}(\varnothing)) = \operatorname{Fun}_Q(\{\mathrm{pt}\}) \cong Q.$$

We recall that a *pseudo-isotopy* is a homeomorphism $H : M \times [0,1] \to N \times [0,1]$ such that $H(M \times \{0\}) = N \times \{0\}$ and $H(M \times \{1\}) = N \times \{1\}$. Except for 0 and 1, such a homeomorphism need not preserve the levels of the cylinders. The homeomorphisms $H|_{M\times\{0\}}$ and $H|_{M\times\{1\}}$ are then said to be pseudo-isotopic to each other. We now impose the constraint equation

$$z(\phi^* f) = z(f) \tag{14}$$

on pre-states $z \in E(M)$, where $f \in \mathcal{F}(M)$ and $\phi : M \to M$ is a homeomorphism, which is pseudo-isotopic to the identity. In other words, call two fields $f, g \in \mathcal{F}(M)$ equivalent, if there is a ϕ, pseudo-isotopic to the identity, such that $g = \phi^*(f)$. Let $\overline{\mathcal{F}}(M)$ be the set of equivalence classes. If M is empty, $\overline{\mathcal{F}}(M)$ consists of a single element. We define the *state module* (or *quantum Hilbert space*) of M to be

$$Z(M) = \mathrm{Fun}_Q(\overline{\mathcal{F}}(M)) = \{z : \mathcal{F}(M) \to Q \mid z(\phi^* f) = z(f) \text{ for all } \phi \in \mathrm{Homeo}_0(M)\} \subset E(M),$$

where $\mathrm{Homeo}_0(M)$ denotes all homeomorphisms $\phi : M \to M$ pseudo-isotopic to the identity. Then $Z(M)$ is a two-sided Q^c-semialgebra and a two-sided Q^m-semialgebra. It is in general infinitely generated as a semimodule. Again, for the empty manifold $Z(\varnothing) \cong Q$.

Remark 6.1. Attempting to define the (pre-)state module of a manifold M with connected components $M_1, \dots, M_k$ as the subsemimodule of $\mathrm{Fun}_Q(\mathcal{F}(M_1 \sqcup \cdots \sqcup M_k))$ consisting of all z that can be written as

$$z(f) = \sum_{i=1}^{l} z_{i1}(f|_{M_1}) \times z_{i2}(f|_{M_2}) \times \cdots \times z_{ik}(f|_{M_k})$$

for suitable functions $z_{ij} \in \mathrm{Fun}_Q(\mathcal{F}(M_j))$, leads to an incorrect state module. The reason is that it will generally not contain the state sum of an $(n+1)$-manifold W with $\partial W = M$, as Example 6.9 below shows. See also Remark 3.3.

Proposition 6.2. *If M and N are closed n-manifolds and $M \sqcup N$ their ordered disjoint union, then the restrictions to M and N induce an isomorphism*

$$Z(M \sqcup N) \cong Z(M) \widehat{\otimes} Z(N)$$

of two-sided Q^c-semialgebras and of two-sided Q^m-semialgebras.

Proof. We define a map

$$\overline{\rho} : \overline{\mathcal{F}}(M \sqcup N) \longrightarrow \overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N)$$

by $[f] \mapsto ([f|_M], [f|_N])$. We need to prove that this is well-defined. Suppose that $\phi \in \mathrm{Homeo}(M \sqcup N)$ is pseudo-isotopic to the identity, so that $[\phi^* f] = [f]$. Then ϕ induces the identity map on $\pi_0(M \sqcup N)$ and thus restricts to homeomorphisms $\phi|_M \in \mathrm{Homeo}(M)$, $\phi|_N \in \mathrm{Homeo}(N)$. Similarly, a pseudo-isotopy $H : (M \sqcup N) \times [0,1] \to (M \sqcup N) \times [0,1]$ from ϕ to the identity restricts to pseudo-isotopies $H| : M \times [0,1] \to M \times [0,1]$, $H| : N \times [0,1] \to N \times [0,1]$. Hence $\phi|_M$ is pseudo-isotopic to id_M and $\phi|_N$ is pseudo-isotopic to id_N. Using the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(M \sqcup N) & \xrightarrow[\cong]{\phi^*} & \mathcal{F}(M \sqcup N) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{F}(M) & \xrightarrow[\cong]{(\phi|_M)^*} & \mathcal{F}(M) \end{array}$$

provided by axiom (FHOMEO), we arrive at $[(\phi^* f)|_M] = [(\phi|_M)^*(f|_M)] = [f|_M]$, and similarly $[(\phi^* f)|_N] = [f|_N]$. A map in the other direction

$$\overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N) \longrightarrow \overline{\mathcal{F}}(M \sqcup N) \tag{15}$$

is given as follows: By axiom (FDISJ), the product of restrictions $\rho : \mathcal{F}(M \sqcup N) \to \mathcal{F}(M) \times \mathcal{F}(N)$ is a bijection. Thus given a pair of fields $(f, f') \in \mathcal{F}(M) \times \mathcal{F}(N)$, there exists a unique field $F \in \mathcal{F}(M \sqcup N)$ such that $F|_M = f$ and $F|_N = f'$. Then (15) is defined as $([f],[f']) \mapsto [F]$. Again, it must be checked that this is well-defined. Suppose $\phi \in \mathrm{Homeo}(M), \psi \in \mathrm{Homeo}(N)$ are pseudo-isotopic to id_M, id_N, respectively, so that $[\phi^* f] = [f]$ and $[\psi^* f'] = [f']$. Let $G \in \mathcal{F}(M \sqcup N)$ be the unique field with $G|_M = \phi^* f$ and $G|_N = \psi^* f'$. We have to show that $[G] = [F]$ on the disjoint union. The disjoint union $\Phi = \phi \sqcup \psi$ defines a homeomorphism $\Phi : M \sqcup N \to M \sqcup N$. It is pseudo-isotopic to the identity via the disjoint union $H \sqcup H' : (M \sqcup N) \times [0,1] \to (M \sqcup N) \times [0,1]$ of pseudo-isotopies H from ϕ to id_M and H' from ψ to id_N. Since $(\Phi^* F)|_M = \phi^*(F|_M) = \phi^* f = G|_M$ and similarly $(\Phi^* F)|_N = G|_N$, we have $\Phi^* F = G$ by uniqueness. Therefore, $[G] = [\Phi^* F] = [F]$ as required.

The two maps $\overline{\rho}$ and (15) are inverse to each other, and thus are both bijections. As discussed in Section 3, $\overline{\rho}$ induces a morphism of two-sided semialgebras (over Q^c and over Q^m)

$$\mathrm{Fun}(\overline{\rho}) : Z(M) \widehat{\otimes} Z(N) = \mathrm{Fun}_Q(\overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N)) \longrightarrow \mathrm{Fun}_Q(\overline{\mathcal{F}}(M \sqcup N)) = Z(M \sqcup N)$$

and $\overline{\rho}^{-1}$ induces a morphism of two-sided semialgebras

$$\mathrm{Fun}(\overline{\rho}^{-1}) : Z(M \sqcup N) \longrightarrow Z(M) \widehat{\otimes} Z(N).$$

These two morphisms are inverse to each other by the functoriality of Fun_Q. □

Let W be a bordism. A field $F \in \mathcal{F}(W)$ determines an element $T_W(F) \in Q_S(\mathbf{C})$ by

$$T_W(F)_{XY}(\gamma) = \begin{cases} 1, & \text{if } \gamma = \mathbb{T}_W(F) \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

where γ ranges over morphisms $\gamma : X \to Y$ of $\mathbf{C}$ and 1 is the 1-element of S. In other words, $T_W(F) = \chi_{\mathbb{T}_W(F)}$ is the characteristic function of $\mathbb{T}_W(F)$. Suppose that W is a bordism from M to N and $f \in \mathcal{F}(M \sqcup N) = \mathcal{F}(\partial W)$. Then we define the state sum (or partition function) Z_W of W on f by

$$Z_W(f) = \sum_{F \in \mathcal{F}(W,f)} T_W(F) \in Q_S(\mathbf{C}),$$

using the summation law of the complete monoid $(Q_S(\mathbf{C}), +, 0)$. This is a well-defined element of $Q_S(\mathbf{C})$ that only depends on W and f.

Remark 6.3. This sum replaces in our context the notional path integral

$$\int_{F \in \mathcal{F}(W,f)} e^{iS_W(F)} d\mu_W$$

used in classical quantum field theory. As a mathematical object, this path integral is problematic, since in many situations of interest, an appropriate measure μ_W has not been defined or is known not to exist. The present paper utilizes the notion of completeness in semirings to bypass measure theoretic questions on spaces of fields. The appearance of the 1-element of S in formula (16) can be interpreted as a reflection of the fact that the amplitude of the integrand in the Feynman path integral is always 1, $|e^{iS_W(F)}| = 1$.

If $\partial W = \varnothing$, then $Z_W \in Q \cong Z(\varnothing)$. If W is empty, then $\mathcal{F}(\partial W) = \mathcal{F}(\varnothing) = \{p\}$ is a singleton and $\mathcal{F}(W, p) = \mathcal{F}(W) = \{p\}$ so that

$$Z_\varnothing(p) = T_\varnothing(p) = \chi_{\mathbb{T}_\varnothing(p)} = \chi_{\mathrm{id}_I} = 1^\times,$$

the unit element of the semiring Q^m. This accords with Atiyah's requirement (4b) [Ati88, p. 179]. Given a morphism $\gamma : X \to Y$ in $\mathbf{C}$, we have the formula

$$Z_W(f)_{XY}(\gamma) = \sum_{F \in \mathcal{F}(W,f),\ \mathbb{T}_W(F) = \gamma} 1,$$

which exhibits $Z_W(f)$ as an elaborate counting device: On a morphism γ, it "counts" for how many fields F on W, which restrict to f on the boundary, γ appears as the action exponential of F. This is a hint that certain kinds of counting functions in number theory and combinatorics may be expressible as state sums of suitable positive TFTs. Our theorems, such as e.g. the gluing theorem, will then yield identities for such functions. In Section

11.1, we illustrate this by deriving Pólya's counting theory using positive TFT methods. In Example 11.4, we consider arithmetic functions arising in number theory. In practice, it is often most important to know whether $Z_W(f)_{XY}(\gamma)$ is zero or nonzero. If the ground semiring is the Boolean semiring, then $Z_W(f)_{XY}(\gamma) = 1$ if and only if there exists a field $F \in \mathcal{F}(W, f)$ such that $\mathbb{T}_W(F) = \gamma$. In this case, $Z_W(f)$ admits the interpretation as a subset $Z_W(f) \subset \mathrm{Mor}(\mathbf{C})$, namely $Z_W(f) = \{\mathbb{T}_W(F) \mid F \in \mathcal{F}(W, f)\}$. The main results below then show that this system of subsets transforms like a topological quantum field theory.

Returning to the general discussion and letting f vary, we have a pre-state vector

$$Z_W \in E(\partial W) = E(M \sqcup N) \cong E(M) \widehat{\otimes} E(N).$$

Let us discuss the topological invariance of the state sum. A homeomorphism $\phi : M \to N$ induces as follows covariantly a pre-state map

$$\phi_* : E(M) \to E(N),$$

which is an isomorphism of both two-sided Q^c-semialgebras and two-sided Q^m-semialgebras: By axiom (FHOMEO), ϕ induces a bijection $\phi^* : \mathcal{F}(N) \to \mathcal{F}(M)$. As shown in Section 3, this bijection in turn induces a morphism

$$\phi_* = \mathrm{Fun}_Q(\phi^*) : E(M) = \mathrm{Fun}_Q(\mathcal{F}(M)) \longrightarrow \mathrm{Fun}_Q(\mathcal{F}(N)) = E(N)$$

of two-sided Q^c- and Q^m-semialgebras. Since ϕ^* is a bijection, ϕ_* is indeed an isomorphism. Moreover, if $\psi : N \to P$ is another homeomorphism, then $\psi_* \circ \phi_* = (\psi \circ \phi)_* : E(M) \to E(P)$ and $(\mathrm{id}_M)_* = \mathrm{id}_{E(M)} : E(M) \to E(M)$, that is, the pre-state module $E(-)$ is a functor on the category of closed n-manifolds and homeomorphisms. In particular the group $\mathrm{Homeo}(M)$ of self-homeomorphisms $M \to M$ acts on $E(M)$.

Let $\phi : W \to W'$ be a homeomorphism of bordisms. Then ϕ restricts to a homeomorphism $\phi_\partial = \phi| : \partial W \to \partial W'$ which induces an isomorphism $\phi_{\partial *} : E(\partial W) \to E(\partial W')$.

Theorem 6.4. *(Topological Invariance.) If $\phi : W \to W'$ is a homeomorphism of bordisms, then $\phi_{\partial *}(Z_W) = Z_{W'}$. If W and W' are closed, then $\phi_{\partial *} = \mathrm{id} : Q \to Q$ and thus $Z_W = Z_{W'}$.*

Proof. Let $f \in \mathcal{F}(\partial W')$ be a field. We claim first that the bijection $\phi^* : \mathcal{F}(W') \to \mathcal{F}(W)$ restricts to a bijection $\phi^*_{\mathrm{rel}} : \mathcal{F}(W', f) \to \mathcal{F}(W, \phi^*_\partial f)$. To see this, suppose that $F' \in \mathcal{F}(W', f)$. Then the field $\phi^* F' \in \mathcal{F}(W)$ satisfies $(\phi^* F')|_{\partial W} = \phi^*_\partial (F'|_{\partial W'}) = \phi^*_\partial(f)$, where we have used the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(W') & \xrightarrow[\cong]{\phi^*} & \mathcal{F}(W) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{F}(\partial W') & \xrightarrow[\cong]{\phi^*_\partial} & \mathcal{F}(\partial W) \end{array}$$

provided by axiom (FHOMEO). Thus $\phi^* F' \in \mathcal{F}(W, \phi^*_\partial f)$ and the desired restriction ϕ^*_{rel} exists. As ϕ^* is injective, ϕ^*_{rel} is injective as well. Given $F \in \mathcal{F}(W, \phi^*_\partial f)$, the field $(\phi^{-1})^*(F)$ lies in $\mathcal{F}(W', f)$ and

$$\phi^*_{\mathrm{rel}}((\phi^{-1})^*(F)) = \phi^*(\phi^{-1})^*(F) = (\phi^{-1}\phi)^*(F) = F.$$

This shows that ϕ^*_{rel} is surjective, too, and proves the claim.

Axiom (THOMEO) for the system $\mathbb{T}$ of action exponentials asserts that $\mathbb{T}_W(\phi^* F') = \mathbb{T}_{W'}(F')$. Consequently, for a morphism $\gamma : X \to Y$ in $\mathbf{C}$,

$$T_{W'}(F')_{XY}(\gamma) = \begin{cases} 1, & \text{if } \gamma = \mathbb{T}_{W'}(F') \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } \gamma = \mathbb{T}_W(\phi^* F') \\ 0, & \text{otherwise} \end{cases} = T_W(\phi^*_{\mathrm{rel}}(F'))_{XY}(\gamma),$$

that is,

$$T_W(\phi^*_{\mathrm{rel}}(F')) = T_{W'}(F').$$

The bijection $\phi^*_{\mathrm{rel}} : \mathcal{F}(W', f) \to \mathcal{F}(W, \phi^*_\partial f)$ implies the identity

$$\sum_{F \in \mathcal{F}(W, \phi^*_\partial f)} T_W(F) = \sum_{F' \in \mathcal{F}(W', f)} T_W(\phi^*_{\mathrm{rel}} F').$$

Hence, the pushforward of the state sum of W, subject to the boundary condition f, can be calculated as

$$\begin{aligned} (\phi_{\partial *}(Z_W))(f) &= (\mathrm{Fun}_Q(\phi^*_\partial)(Z_W))(f) = Z_W(\phi^*_\partial(f)) \\ &= \sum_{F \in \mathcal{F}(W, \phi^*_\partial f)} T_W(F) = \sum_{F' \in \mathcal{F}(W', f)} T_W(\phi^*_{\mathrm{rel}} F') \\ &= \sum_{F' \in \mathcal{F}(W', f)} T_{W'}(F') = Z_{W'}(f). \end{aligned}$$

If W and W' are closed, then $\phi^*_\partial = \mathrm{id} : \mathcal{F}(\varnothing) = \{p\} \to \{p\} = \mathcal{F}(\varnothing)$. Hence,

$$\phi_{\partial *} = \mathrm{Fun}_Q(\phi^*_\partial) = \mathrm{Fun}_Q(\mathrm{id}) = \mathrm{id} : E(\varnothing) = \mathrm{Fun}_Q(\mathcal{F}(\varnothing)) = Q \longrightarrow Q = E(\varnothing).$$

$\square$

We will now use topological invariance to show that the state sum is really a state, not just a pre-state.

Proposition 6.5. *The state sum $Z_W \in E(\partial W)$ solves the constraint equation (14). Thus Z_W lies in the state module $Z(\partial W) \subset E(\partial W)$.*

Proof. Given a field $f \in \mathcal{F}(\partial W)$ and $\phi \in \mathrm{Homeo}(\partial W)$ pseudo-isotopic to the identity, we need to show that $Z_W(\phi^* f) = Z_W(f)$. Let $H : \partial W \times [0,1] \to \partial W \times [0,1]$ be a pseudo-isotopy, $H(x,0) = x$, $H(x,1) = \phi(x)$, for all $x \in \partial W$. This pseudo-isotopy fits into a commutative diagram

$$\begin{tikzcd} W \arrow[d, equal] & \partial W \arrow[l, hook'] \arrow[r, hook, "(\mathrm{id}_{\partial W}, 0)"] \arrow[d, equal] & \partial W \times [0,1] \arrow[d, "H", "\cong"'] \\ W & \partial W \arrow[l, hook'] \arrow[r, hook, "(\mathrm{id}_{\partial W}, 0)"] & \partial W \times [0,1] \end{tikzcd}$$

The pushout of the rows is homeomorphic to W (using collars, which are available in the topological category by Marston Brown's collar neighborhood theorem [Bro62]) via a homeomorphism which is the identity on the boundary. Thus the universal property of pushouts applied to the above diagram yields a homeomorphism $\Phi : W \to W$ of bordisms, whose restriction to the boundary Φ_∂ is ϕ. By Theorem 6.4 on topological invariance, we have $\Phi_{\partial *}(Z_W) = Z_W$. Consequently, $Z_W(f) = \Phi_{\partial *}(Z_W)(f) = \phi_*(Z_W)(f) = Z_W(\phi^* f)$. $\square$

Homeomorphisms between closed n-manifolds induce isomorphisms on the associated state modules, as we will now explain.

Lemma 6.6. *Let $\phi : M \to N$ be any homeomorphism of closed n-manifolds. Then the induced isomorphism $\phi_* : E(M) \to E(N)$ of pre-state modules restricts to an isomorphism $\phi_* : Z(M) \to Z(N)$ of state modules.*

Proof. Let $z \in Z(M)$ be a state, that is, $z : \mathcal{F}(M) \to Q$ is a function with $z(\psi^* f) = z(f)$ for all $\psi \in \mathrm{Homeo}(M)$ pseudo-isotopic to the identity. Let $g \in \mathcal{F}(N)$ be any field on N and $\xi \in \mathrm{Homeo}(N)$ pseudo-isotopic to id_N. Let $\psi \in \mathrm{Homeo}(M)$ be the homeomorphism $\psi = \phi^{-1}\xi\phi$. If $H : N \times [0,1] \to N \times [0,1]$ is a pseudo-isotopy from ξ to id_N, then

$$M \times [0,1] \overset{\phi\times \mathrm{id}_{[0,1]}}{\longrightarrow} N \times [0,1] \overset{H}{\longrightarrow} N \times [0,1] \overset{\phi^{-1}\times \mathrm{id}_{[0,1]}}{\longrightarrow} M \times [0,1]$$

is a pseudo-isotopy from ψ to id_M. Thus

$$\begin{aligned}\phi_*(z)(\xi^* g) &= (z \circ \phi^*)(\xi^* g) = z((\xi\phi)^* g) = z((\phi\psi)^* g) \\ &= z(\psi^*(\phi^* g)) = z(\phi^* g) = \phi_*(z)(g).\end{aligned}$$

Hence $\phi_*(z)$ solves the constraint equation on N and so $\phi_*(z) \in Z(N)$. Its inverse is given by $(\phi^{-1})_* : Z(N) \to Z(M)$. □

By the above lemma, any homeomorphism $\phi : M \to N$ induces an isomorphism $\phi_* : Z(M) \to Z(N)$. If $\psi : N \to P$ is another homeomorphism, then $\psi_* \circ \phi_* = (\psi \circ \phi)_* : Z(M) \to Z(P)$ and $(\mathrm{id}_M)_* = \mathrm{id}_{Z(M)} : Z(M) \to Z(M)$, that is, the state module $Z(-)$ is a functor on the category of closed n-manifolds and homeomorphisms. In particular the group $\mathrm{Homeo}(M)$ of self-homeomorphisms $M \to M$ acts on $Z(M)$.

Theorem 6.7. *(Pseudo-Isotopy Invariance.) Pseudo-isotopic homeomorphisms $\phi, \psi : M \to N$ induce equal isomorphisms $\phi_* = \psi_* : Z(M) \to Z(N)$ on state modules. In particular, the action of* $\mathrm{Homeo}(M)$ *on $Z(M)$ factors through the mapping class group.*

Proof. Let $H : M \times [0,1] \to N \times [0,1]$ be a pseudo-isotopy from ϕ to ψ. Then $(\psi^{-1} \times \mathrm{id}_{[0,1]}) \circ H : M \times [0,1] \to M \times [0,1]$ is a pseudo-isotopy from $\psi^{-1}\phi$ to id_M. Hence for $z \in Z(M)$, $(\psi^{-1}\phi)_*(z)(f) = z((\psi^{-1}\phi)^* f) = z(f)$. It follows that

$$(\psi_*)^{-1} \circ \phi_* = (\psi^{-1})_* \circ \phi_* = (\psi^{-1}\phi)_* = \mathrm{id}_{Z(M)}$$

and therefore $\phi_* = \psi_*$. □

As there are two multiplications available on $Q_S(\mathbf{C})$, there are also two corresponding $Q_Q Q$-linear maps $\beta^c, \beta^m : E(M) \times E(N) \to E(M) \widehat{\otimes} E(N)$, given by $\beta^c(z,z')(f,g) = z(f) \cdot z(g)$, using the multiplication $\cdot$ of the composition semiring Q^c, and $\beta^m(z,z')(f,g) = z(f) \times z(g)$, using the multiplication $\times$ of the monoidal semiring Q^m. If $(z,z') \in Z(M) \times Z(N)$, and $\phi \in \mathrm{Homeo}(M)$, $\psi \in \mathrm{Homeo}(N)$ are pseudo-isotopic to the respective identities, then

$$\beta^c(z,z')(\phi^* f, \psi^* g) = z(\phi^* f) \cdot z'(\psi^* g) = z(f) \cdot z'(g) = \beta^c(z,z')(f,g).$$

Therefore, $\beta^c(z,z') \in Z(M) \widehat{\otimes} Z(N)$. Similarly, we have a $Q_Q Q$-linear map $\beta^m : Z(M) \times Z(N) \to Z(M) \widehat{\otimes} Z(N)$. Thus a pair of vectors $(z,z') \in Z(M) \times Z(N)$ determines two generally different tensor products in $Z(M) \widehat{\otimes} Z(N)$, namely $z \widehat{\otimes}_m z' = \beta^m(z,z')$ and $z \widehat{\otimes}_c z' = \beta^c(z,z')$.

Theorem 6.8. *The state sum $Z_{W \sqcup W'} \in Z(\partial W \sqcup \partial W')$ of an ordered disjoint union of bordisms W and W' is the tensor product*

$$Z_{W \sqcup W'} = Z_W \widehat{\otimes}_m Z_{W'} \in Z(\partial W) \widehat{\otimes} Z(\partial W') \cong Z(\partial W \sqcup \partial W').$$

Proof. Let $F \in \mathcal{F}(W \sqcup W')$ be a field on the disjoint union. Then, regarding $Q_S(\mathbf{C})$ as the monoidal semiring Q^m, we have on a morphism $\gamma : X \to Y$ of $\mathbf{C}$,

$$(T_W(F|_W) \times T_{W'}(F|_{W'}))_{XY}(\gamma) = \sum_{\alpha \otimes \beta = \gamma} T_{W'}(F|_{W'})_{X''Y''}(\beta) \cdot T_W(F|_W)_{X'Y'}(\alpha).$$

The element $T_W(F|_W)_{X'Y'}(\alpha) \in S$ is 1 when $\alpha = \mathbb{T}_W(F|_W)$ and 0 otherwise. Similarly, $T_{W'}(F|_{W'})_{X''Y''}(\beta) \in S$ is 1 when $\beta = \mathbb{T}_{W'}(F|_{W'})$ and 0 otherwise. Thus

$$\begin{aligned}(T_W(F|_W) \times T_{W'}(F|_{W'}))_{XY}(\gamma) &= \sum_{\mathbb{T}_W(F|_W) \otimes \mathbb{T}_{W'}(F|_{W'}) = \gamma} 1 \cdot 1 \\ &= \begin{cases} 1, & \text{if } \gamma = \mathbb{T}_W(F|_W) \otimes \mathbb{T}_{W'}(F|_{W'}) \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Now by axiom (TDISJ), $\mathbb{T}_W(F|_W) \otimes \mathbb{T}_{W'}(F|_{W'}) = \mathbb{T}_{W \sqcup W'}(F)$, which implies

$$T_W(F|_W) \times T_{W'}(F|_{W'}) = T_{W \sqcup W'}(F)$$

in Q^m. We apply this identity in calculating the state sum of the disjoint union subject to the boundary condition $f \in \mathcal{F}(\partial W \sqcup \partial W')$:

$$Z_{W \sqcup W'}(f) = \sum_{F \in \mathcal{F}(W \sqcup W', f)} T_{W \sqcup W'}(F) = \sum_{F \in \mathcal{F}(W \sqcup W', f)} T_W(F|_W) \times T_{W'}(F|_{W'}).$$

As pointed out in Section 2, if $(m_i)_{i \in I}$, $(n_j)_{j \in J}$ are families of elements in a complete monoid and $\sigma : J \to I$ is a bijection such that $m_{\sigma(j)} = n_j$, then $\sum_{i \in I} m_i = \sum_{j \in J} n_j$. Applying this principle to the families

$$(T_W(G) \times T_{W'}(G'))_{(G,G') \in \mathcal{F}(W, f|_{\partial W}) \times \mathcal{F}(W', f|_{\partial W'})},$$

$$(T_W(F|_W) \times T_{W'}(F|_{W'}))_{F \in \mathcal{F}(W \sqcup W', f)}$$

and to the bijection

$$\sigma : \mathcal{F}(W \sqcup W', f) \longrightarrow \mathcal{F}(W, f|_{\partial W}) \times \mathcal{F}(W', f|_{\partial W'})$$

given by Lemma 5.2, we obtain that

$$\begin{aligned} Z_{W \sqcup W'}(f) &= \sum_{(G,G') \in \mathcal{F}(W, f|_{\partial W}) \times \mathcal{F}(W', f|_{\partial W'})} T_W(G) \times T_{W'}(G') \\ &= \left\{ \sum_{G \in \mathcal{F}(W, f|_{\partial W})} T_W(G) \right\} \times \left\{ \sum_{G' \in \mathcal{F}(W', f|_{\partial W'})} T_{W'}(G') \right\} \\ &= Z_W(f|_{\partial W}) \times Z_{W'}(f|_{\partial W'}) \\ &= \beta^m(Z_W, Z_{W'})(f|_{\partial W}, f|_{\partial W'}) \\ &= (Z_W \widehat{\otimes}_m Z_{W'})(f|_{\partial W}, f|_{\partial W'}). \end{aligned}$$

□

Especially in light of the decomposition of the state sum of a disjoint union provided by Theorem 6.8, one may wonder whether the state sum $Z_W \in Z(M_1) \widehat{\otimes} \cdots \widehat{\otimes} Z(M_k)$ of any compact manifold W whose boundary has the connected components $M_1, \ldots, M_k$, can always be written as

$$Z_W(f) = \sum_{i=1}^{l} z_{i1}(f|_{M_1}) \times z_{i2}(f|_{M_2}) \times \cdots \times z_{ik}(f|_{M_k})$$

for suitable functions $z_{ij} \in \mathrm{Fun}_Q(\mathcal{F}(M_j))$, see Remarks 3.3 and 6.1. The following example shows that this is not the case.

Example 6.9. Take $n = 0$ and the Boolean semiring $\mathbb{B}$ as the ground semiring. Let $\mathbf{C}$ be the category with one object I and one morphism, $\mathrm{id}_I : I \to I$. This category has a unique monoidal structure, which is strict. Then

$$Q_{\mathbb{B}}(\mathbf{C}) = \mathrm{Fun}_{\mathbb{B}}(\mathrm{Hom}_{\mathbf{C}}(I,I)) = \mathrm{Fun}_{\mathbb{B}}(\{\mathrm{id}_I\}) \cong \mathbb{B}$$

is the Boolean semiring. For a 1-dimensional bordism W, let $\mathcal{F}(W)$ be the locally constant functions on W with values in the natural numbers $\mathbb{N} = \{0,1,2,\ldots\}$, that is, if W has k connected components, then $\mathcal{F}(W) = \mathbb{N}^k$. Fields $\mathcal{F}(M)$ on a closed n-manifold M are defined in the same way. The restrictions are given by the ordinary restriction of functions to subspaces. Homeomorphisms $W \cong W'$ and $M \cong N$ act on fields by permuting function values in a manner consistent with the permutation which the homeomorphism induces on the connected components. Since two locally constant functions that agree on a common boundary component glue to give a locally constant function again, $\mathcal{F}$ is a system of fields. The action exponential $\mathbb{T}_W : \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{C})$, $\mathbb{T}_W(F) = \mathrm{id}_I$ is uniquely determined. Then $\mathbb{T}$ is indeed a valid system of action exponentials. Now let W be the unit interval $[0,1]$, whose boundary $\partial W = M \sqcup N$ is given by the incoming boundary $M = \{0\}$ and the outgoing boundary $N = \{1\}$. Let $f = (m,n) \in \mathcal{F}(\partial W) \cong \mathcal{F}(M) \times \mathcal{F}(N) = \mathbb{N} \times \mathbb{N}$ be a boundary condition. If $m \neq n$, then $\mathcal{F}(W,f)$ is empty, for there is no constant function on the unit interval that restricts to two distinct numbers on the endpoints. Consequently,

$$Z_W(m,n) = \sum_{F \in \mathcal{F}(W,(m,n))} T_W(F) = 0 \in Q \cong \mathbb{B}.$$

On the other hand, if $m = n$, then $\mathcal{F}(W,(m,n))$ consists of a single element, namely $m = n$, so that $Z_W(m,n) = T_W(m) = 1 \in \mathbb{B}$. Therefore, $Z_W(m,n) = \delta_{mn}$, that is, viewed as a countably infinite Boolean value matrix, Z_W is the identity matrix. But it follows from results of [Ban13] that the identity matrix is not in the image of

$$\mu : \mathrm{Fun}_{\mathbb{B}}(\mathbb{N}) \otimes \mathrm{Fun}_{\mathbb{B}}(\mathbb{N}) \longrightarrow \mathrm{Fun}_{\mathbb{B}}(\mathbb{N} \times \mathbb{N}).$$

This can also be seen directly by observing that a representation $\delta_{mn} = \sum_{i=1}^{l} z_i(m) z'_i(n)$ (where we assume that δ_{mn} cannot be expressed in such a form with fewer than l terms), implies that $z_i(m) = \delta_{mn_i}$, $z'_i(n) = \delta_{n_i n}$ for certain $n_1, \ldots, n_l$. But the finite sum $\sum_{i=1}^{l} \delta_{mn_i}\delta_{n_i n}$ can certainly not equal δ_{mn} for all m,n as follows by taking $m = n \notin \{n_1, \ldots, n_l\}$.

Let M,N,P be closed n-manifolds. The contraction γ of Section 3 defines a map

$$\gamma : E(M) \widehat{\otimes} E(N) \widehat{\otimes} E(N) \widehat{\otimes} E(P) \longrightarrow E(M) \widehat{\otimes} E(P).$$

This contraction allows us to define an inner product $\langle z, z' \rangle$ of a vector $z \in E(M) \widehat{\otimes} E(N)$ and a vector $z' \in E(N) \widehat{\otimes} E(P)$ by setting

$$\langle z, z' \rangle = \gamma(z \widehat{\otimes}_c z') \in E(M) \widehat{\otimes} E(P),$$

where we have used the composition product $\widehat{\otimes}_c$. By Proposition 3.6, this inner product

$$\langle -, - \rangle : (E(M) \widehat{\otimes} E(N)) \times (E(N) \widehat{\otimes} E(P)) \longrightarrow E(M) \widehat{\otimes} E(P)$$

is $Q^c_{Q^c} Q^c$-linear. By Proposition 3.7, the inner product is associative, i.e. for a fourth n-manifold R and a vector $z'' \in E(P) \widehat{\otimes} E(R)$, one has $\langle \langle z, z' \rangle, z'' \rangle = \langle z, \langle z', z'' \rangle \rangle$ in $E(M) \widehat{\otimes} E(R)$. An injection $Z(M) \widehat{\otimes} Z(N) \hookrightarrow E(M) \widehat{\otimes} E(N)$ is given by sending $z : \overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N) \to Q$ to

$$\mathcal{F}(M) \times \mathcal{F}(N) \xrightarrow{\mathrm{quot} \times \mathrm{quot}} \overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N) \xrightarrow{\quad z \quad} Q.$$

Then the inner product $\langle z,z'\rangle$ of a state vector $z \in Z(M)\widehat{\otimes}Z(N)$ and a state vector $z' \in Z(N)\widehat{\otimes}Z(P)$ satisfies the constraint equation and is thus a state $\langle z,z'\rangle \in Z(M)\widehat{\otimes}Z(P)$. Therefore, restriction defines an inner product

$$\langle -,-\rangle : (Z(M)\widehat{\otimes}Z(N)) \times (Z(N)\widehat{\otimes}Z(P)) \longrightarrow Z(M)\widehat{\otimes}Z(P).$$

Theorem 6.10. *(Gluing Formula.) Let W' be a bordism from M to N and let W'' be a bordism from N to P. Let $W = W' \cup_N W''$ be the bordism from M to P obtained by gluing W' and W'' along N. Then the state sum of W can be calculated as the contraction inner product*

$$Z_W = \langle Z_{W'}, Z_{W''}\rangle \in Z(M)\widehat{\otimes}Z(P) \cong Z(M \sqcup P).$$

Proof. Let $F \in \mathcal{F}(W)$ be a field on the glued bordism. Then, regarding $Q_S(\mathbf{C})$ as the composition semiring Q^c, we have on a morphism $\gamma : X \to Y$ of $\mathbf{C}$,

$$(T_{W'}(F|_{W'}) \cdot T_{W''}(F|_{W''}))_{XY}(\gamma) = \sum_{\beta\circ\alpha=\gamma} T_{W''}(F|_{W''})_{ZY}(\beta) \cdot T_{W'}(F|_{W'})_{XZ}(\alpha).$$

(Here, Z is of course an object of $\mathbf{C}$ and not a state sum.) The element $T_{W'}(F|_{W'})_{XZ}(\alpha) \in S$ is 1 when $\alpha = \mathbb{T}_{W'}(F|_{W'})$ and 0 otherwise. Similarly, $T_{W''}(F|_{W''})_{ZY}(\beta) \in S$ is 1 when $\beta = \mathbb{T}_{W''}(F|_{W''})$ and 0 otherwise. Thus

$$\begin{aligned}(T_{W'}(F|_{W'}) \cdot T_{W''}(F|_{W''}))_{XY}(\gamma) &= \sum_{\mathbb{T}_{W''}(F|_{W''})\circ\mathbb{T}_{W'}(F|_{W'})=\gamma} 1\cdot 1 \\ &= \begin{cases} 1, & \text{if } \gamma = \mathbb{T}_{W''}(F|_{W''}) \circ \mathbb{T}_{W'}(F|_{W'}) \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Now by axiom (TGLUE), $\mathbb{T}_{W''}(F|_{W''}) \circ \mathbb{T}_{W'}(F|_{W'}) = \mathbb{T}_W(F)$, which implies

$$T_{W'}(F|_{W'}) \cdot T_{W''}(F|_{W''}) = T_W(F)$$

in Q^c. Let $f \in \mathcal{F}(\partial W) = \mathcal{F}(M \sqcup P)$. Lemma 5.3 asserts that the unique map ρ such that

$$\begin{tikzcd} \mathcal{F}(W,f) \arrow[dr, "\rho"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathcal{F}(W',W'',f|_M,f|_P) \arrow[r] \arrow[d] & \mathcal{F}(W',f|_M) \arrow[d] \\ & \mathcal{F}(W'',f|_P) \arrow[r] & \mathcal{F}(N) \end{tikzcd}$$

commutes, is a bijection. Therefore,

$$\sum_{F\in\mathcal{F}(W,f)} T_{W'}(F|_{W'}) \cdot T_{W''}(F|_{W''}) = \sum_{(G',G'')\in\mathcal{F}(W',W'',f|_M,f|_P)} T_{W'}(G') \cdot T_{W''}(G'').$$

The pullback $\mathcal{F}(W',W'',f|_M,f|_P)$ possesses the natural partition

$$\mathcal{F}(W',W'',f|_M,f|_P) = \bigcup_{g\in\mathcal{F}(N)} \mathcal{F}(W',f|_M,g) \times \mathcal{F}(W'',g,f|_P),$$

which enables us to rewrite the sum over the pullback as

$$\begin{aligned}&\sum_{(G',G'')\in\mathcal{F}(W',W'',f|_M,f|_P)} T_{W'}(G') \cdot T_{W''}(G'') = \\ &\qquad = \sum_{g\in\mathcal{F}(N)} \sum_{(G',G'')\in\mathcal{F}(W',f|_M,g)\times\mathcal{F}(W'',g,f|_P)} T_{W'}(G') \cdot T_{W''}(G'')\end{aligned}$$

according to the partition property that the summation law in a complete monoid satisfies. Thus for the state sum of the glued bordism, subject to the boundary condition f,

$$\begin{aligned}
Z_W(f) &= \sum_{F\in\mathcal{F}(W,f)} T_W(F) \\
&= \sum_{F\in\mathcal{F}(W,f)} T_{W'}(F|_{W'})\cdot T_{W''}(F|_{W''}) \\
&= \sum_{(G',G'')\in\mathcal{F}(W',W'',f|_M,f|_P)} T_{W'}(G')\cdot T_{W''}(G'') \\
&= \sum_{g\in\mathcal{F}(N)} \sum_{G'\in\mathcal{F}(W',f|_M,g)} \sum_{G''\in\mathcal{F}(W'',g,f|_P)} T_{W'}(G')\cdot T_{W''}(G'') \\
&= \sum_{g\in\mathcal{F}(N)} \Big(\sum_{G'\in\mathcal{F}(W',f|_M,g)} T_{W'}(G')\Big)\cdot\Big(\sum_{G''\in\mathcal{F}(W'',g,f|_P)} T_{W''}(G'')\Big) \\
&= \sum_{g\in\mathcal{F}(N)} Z_{W'}(f|_M\sqcup g)\cdot Z_{W''}(g\sqcup f|_P) \\
&= \sum_{g\in\mathcal{F}(N)} \beta^c(Z_{W'},Z_{W''})(f|_M\sqcup g, g\sqcup f|_P) \\
&= \gamma(Z_{W'}\widehat{\otimes}_c Z_{W''})(f) \\
&= \langle Z_{W'},Z_{W''}\rangle(f).
\end{aligned}$$

(Note that via the identification $\mathcal{F}(M\sqcup P)\cong\mathcal{F}(M)\times\mathcal{F}(P)$, we may also think of the boundary condition f as the pair $(f|_M, f|_P)$.) $\square$

Taking bordisms with $N=\varnothing$ is allowed in the above gluing theorem. In such a situation, $W=W'\cup_\varnothing W''=W'\sqcup W''$ so that by using the gluing theorem in conjunction with Theorem 6.8 on disjoint unions, we obtain $\langle Z_{W'},Z_{W''}\rangle = Z_{W'}\widehat{\otimes}_m Z_{W''}$.

As an application of our theorems, let us derive the well-known zig-zag equation. Given a closed n-manifold, we write $M_2=M\sqcup M$ and $M_3=M\sqcup M\sqcup M$. The cylinder $M\times[0,1]$ can be interpreted as a bordism in three different ways:

$$C=(M\times[0,1],M,M),\ C_\subset=(M\times[0,1],\varnothing,M_2),\ C_\supset=(M\times[0,1],M_2,\varnothing).$$

Using these, we can form the bordisms

$$W'=C_\subset\sqcup C,\ W''=C\sqcup C_\supset,\ W=W'\cup_{M_3}W''.$$

Then W is homeomorphic to C by a homeomorphism ϕ which is the identity on the boundary, $\phi_\partial=\mathrm{id}_{M_2}$. Thus, by topological invariance of the state sum, $Z_W=Z_C$. On the other hand, using Theorems 6.8 and 6.10,

$$Z_W=\langle Z_{W'},Z_{W''}\rangle=\langle Z_{C_\subset\sqcup C},Z_{C\sqcup C_\supset}\rangle=\langle Z_{C_\subset}\widehat{\otimes}_m Z_C, Z_C\widehat{\otimes}_m Z_{C_\supset}\rangle.$$

Thus we arrive at the zig-zag equation

$$\langle Z_{C_\subset}\widehat{\otimes}_m Z_C, Z_C\widehat{\otimes}_m Z_{C_\supset}\rangle=Z_C,$$

relating the state sums of the three bordisms associated to $M\times[0,1]$.

Finally, we introduce a new kind of invariant $\mathfrak{A}(M)\in Z(M)$, the *coboundary aggregate* of a closed n-manifold M, which has no counterpart in classical topological field theories. Call two bordisms $(W_1,\varnothing,M)$ and $(W_2,\varnothing,M)$ equivalent, if there exists a homeomorphism

$W_1 \cong W_2$ whose restriction to the boundary is the identity on M. (This is indeed an equivalence relation.) Let $\mathrm{Cob}(M)$ be the collection of all equivalence classes of bordisms ("M-coboundaries") $(W, \varnothing, M)$. If W_1 and W_2 are equivalent, then Theorem 6.4 implies that they have equal state sums, $Z_{W_1} = Z_{W_2} \in Z(M)$. Therefore, the state sum can be viewed as a well-defined function $Z : \mathrm{Cob}(M) \to Z(M)$.

Definition 6.11. The *coboundary aggregate* $\mathfrak{A}(M) \in Z(M)$ of a closed topological n-manifold is the state vector

$$\mathfrak{A}(M) = \sum_{[W] \in \mathrm{Cob}(M)} Z_{[W]}.$$

Note that this is a well-defined element of $Z(M)$, since the completeness of Q together with Proposition 3.1 implies that $Z(M)$ is complete as well.

Theorem 6.12. *If $\phi : M \to N$ is a homeomorphism, then $\phi_* \mathfrak{A}(M) = \mathfrak{A}(N)$. That is, the coboundary aggregate is a topological invariant.*

Proof. Given a bordism W with $\partial W^{\mathrm{in}} = \varnothing$ and $\partial W^{\mathrm{out}} = M$, let $\psi(W) = W \cup_\phi N \times [0,1]$, that is, let $\psi(W)$ be the mapping cylinder of ϕ^{-1} followed by the inclusion $M \subset W$. Then $\psi(W)$ is a bordism with $\partial \psi(W)^{\mathrm{in}} = \varnothing$ and $\partial \psi(W)^{\mathrm{out}} = N$. Suppose that W_1 and W_2 represent the same element of $\mathrm{Cob}(M)$. Then there exists a homeomorphism $\Phi : W_1 \to W_2$ whose restriction to the boundary is the identity on M. The pushout of the top row of the commutative diagram

$$\begin{tikzcd}
W_1 \arrow[d, "\Phi"', "\cong"] & M \arrow[l, hook'] \arrow[r, hook, "(\phi,0)"] \arrow[d, equal] & N \times [0,1] \arrow[d, equal] \\
W_2 & M \arrow[l, hook'] \arrow[r, hook, "(\phi,0)"] & N \times [0,1]
\end{tikzcd}$$

is $\psi(W_1)$, while the pushout of the bottom row is $\psi(W_2)$. Thus the universal property of pushouts applied to the above diagram yields a homeomorphism $\psi(W_1) \cong \psi(W_2)$, whose restriction to the boundary is the identity on N. Hence $\psi(W_1)$ and $\psi(W_2)$ are equivalent and represent the same element of $\mathrm{Cob}(N)$. Consequently, ψ induces a well-defined map $\psi : \mathrm{Cob}(M) \to \mathrm{Cob}(N)$. Reversing the roles of M and N (using ϕ^{-1}), we also obtain a map $\psi' : \mathrm{Cob}(N) \to \mathrm{Cob}(M)$. We claim that ψ and ψ' are inverse to each other: The colimit of the top row of the commutative diagram

$$\begin{tikzcd}
W \arrow[d, equal] & M \arrow[l, hook'] \arrow[r, hook, "(\mathrm{id}_M,0)"] \arrow[d, equal] & M \times [0,1] \arrow[d, "\cong"', "\phi \times \mathrm{id}_{[0,1]}"] & M \arrow[l, hook', "(\mathrm{id}_M,1)"'] \arrow[r, hook, "(\mathrm{id}_M,0)"] \arrow[d, "\cong"', "\phi"] & M \times [0,1] \arrow[d, equal] \\
W & M \arrow[l, hook'] \arrow[r, hook, "(\phi,0)"] & N \times [0,1] & N \arrow[l, hook', "(\mathrm{id}_N,1)"'] \arrow[r, hook, "(\phi^{-1},0)"] & M \times [0,1]
\end{tikzcd}$$

is homeomorphic to W (using the collar neighborhood theorem [Bro62]) via a homeomorphism which is the identity on the boundary, while the colimit of the bottom row is $\psi'(\psi(W))$. Thus the universal property of colimits applied to the above diagram yields a homeomorphism $W \cong \psi'(\psi(W))$, whose restriction to the boundary is the identity. We conclude that $[W] = \psi' \psi [W] \in \mathrm{Cob}(M)$. By symmetry we also have $[W'] = \psi \psi' [W'] \in \mathrm{Cob}(N)$. This shows that $\psi' = \psi^{-1}$ and ψ is a bijection.

Given a field $f \in \mathcal{F}(N)$, let us show that

$$Z_W(\phi^* f) = Z_{\psi(W)}(f). \tag{17}$$

The pushout of the top row of the commutative diagram

$$\begin{tikzcd} W & M \arrow[l, hook'] \arrow[r, hook, "(\mathrm{id}_M,0)"] & M\times[0,1] \arrow[d, "\phi\times\mathrm{id}_{[0,1]}", "\cong"'] \\ W \arrow[u, equal] & M \arrow[u, equal] \arrow[l, hook'] \arrow[r, hook, "(\phi,0)"] & N\times[0,1] \end{tikzcd}$$

is homeomorphic to W (using the collar neighborhood theorem) via a homeomorphism which is the identity on the boundary, while the pushout of the bottom row is $\psi(W)$. Thus the universal property of pushouts applied to the above diagram yields a homeomorphism of bordisms $W \cong \psi(W)$, whose restriction to the boundary is ϕ. By topological invariance of the state sum (Theorem 6.4), $\phi_*(Z_W) = Z_{\psi(W)}$, proving equation (17). Consequently,

$$\begin{aligned}(\phi_*\mathfrak{A}(M))(f) &= \mathfrak{A}(M)(\phi^* f) = \sum_{[W]\in\mathrm{Cob}(M)} Z_{[W]}(\phi^* f) = \sum_{[W]\in\mathrm{Cob}(M)} Z_{\psi[W]}(f) \\ &= \sum_{[W']\in\psi(\mathrm{Cob}(M))} Z_{[W']}(f) = \mathfrak{A}(N)(f).\end{aligned}$$

□

7. The Frobenius Structure

We shall show that the state modules $Z(M)$, in any dimension n, come naturally equipped with the structure of a Frobenius semialgebra. Let S be any semiring, not necessarily commutative.

Definition 7.1. A *Frobenius semialgebra* over S is a two-sided S-semialgebra A together with an S-bisemimodule homomorphism $\varepsilon : A \to S$, called the *counit* functional (or sometimes *trace*), such that the $S_S S$-linear form

$$A\times A \to S,\ (a,b)\mapsto \varepsilon(a\cdot b)$$

is nondegenerate. A *morphism* $\phi : A \to B$ *of Frobenius* S*-semialgebras* is a morphism of two-sided S-semialgebras which commutes with the counits, i.e. $\varepsilon_A = \varepsilon_B \circ \phi$.

Proposition 7.2. *The state module* $Z(M)$ *of a closed, n-dimensional manifold* M *becomes a Frobenius semialgebra over* Q^m *and over* Q^c *when endowed with the counit functional*

$$\varepsilon = \varepsilon_M : Z(M) \to Q,\ \varepsilon(z) = \sum_{f\in\mathfrak{F}(M)} z(f).$$

Proof. The functional is clearly a Q^c-bisemimodule homomorphism and a Q^m-bisemimodule homomorphism. To show that $Z(M)$ is indeed Frobenius over the monoidal semiring $Q^m = (Q_S(\mathbf{C}), +, \times, 0, 1^\times)$, we shall, to a given nonzero $z \in Z(M)$, construct a nonzero $z' \in Z(M)$ such that $\varepsilon(z\times z')$ is not zero. If $z \neq 0$, then there exists a class $[f_0] \in \overline{\mathfrak{F}}(M)$, represented by a field f_0, such that $z[f_0] \neq 0$. Setting

$$z'[f] = \begin{cases} 1^\times, & \text{if } [f] = [f_0] \\ 0, & \text{otherwise,}\end{cases}$$

we obtain a state $z' \in Z(M)$. Using the partition $\mathcal{F}(M) = \bigcup_{c \in \overline{\mathcal{F}}(M)} c$, the trace of the product is

$$\begin{aligned} \varepsilon(z \times z') &= \sum_{f \in \mathcal{F}(M)} z(f) \times z'(f) = \sum_{c \in \overline{\mathcal{F}}(M)} \sum_{f \in c} z(f) \times z'(f) = \sum_{f \in [f_0]} z(f) \times z'(f) \\ &= \sum_{f \in [f_0]} z(f) \times z'(f_0) = \sum_{f \in [f_0]} z(f) = \sum_{f \in [f_0]} z(f_0). \end{aligned}$$

Now if R is a complete semiring, $r \in R$ a nonzero element and J an arbitrary nonempty index set, then $\sum_{j \in J} r$ cannot be zero. For if it were zero, then $0 = \sum_{j \in J} r = r + \sum_{j \in J - \{j_0\}} r$. But as R is complete, it is in particular zerosumfree, which would imply $r = 0$, a contradiction. Thus, since $z(f_0) \neq 0$, we have $\sum_{f \in [f_0]} z(f_0) \neq 0$. Similarly, to prove that $Z(M)$ is Frobenius over the composition semiring $Q^c = (Q_S(\mathbf{C}), +, \cdot, 0, 1)$, take

$$z'[f] = \begin{cases} 1, & \text{if } [f] = [f_0] \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Remark 7.3. As every state is constant over classes $[f] \in \overline{\mathcal{F}}(M)$, it might at first seem more natural to construct the counit functional by summing over classes, rather than individual fields. However, summing over classes would invalidate Proposition 7.5, and thus would be less compatible with the existing formalism.

More generally, we can also define counits of the form $\varepsilon_{M,-} : Z(M) \widehat{\otimes} Z(N) \to Z(N)$ on the tensor product of the state modules of two closed n-manifolds M and N by

$$\varepsilon_{M,-}(z)(g) = \sum_{f \in \mathcal{F}(M)} z(f, g),$$

$z : \overline{\mathcal{F}}(M) \times \overline{\mathcal{F}}(N) \to Q$, $g \in \mathcal{F}(N)$; similarly for $\varepsilon_{-,N}$. The next proposition shows that the Frobenius counit interacts multiplicatively with the tensor product of vectors.

Proposition 7.4. *Given two vectors $z \in Z(M)$, $z' \in Z(N)$, we have the formula*

$$\varepsilon_{M \sqcup N}(z \widehat{\otimes}_c z') = \varepsilon_M(z) \cdot \varepsilon_N(z')$$

in the composition semiring Q^c, and

$$\varepsilon_{M \sqcup N}(z \widehat{\otimes}_m z') = \varepsilon_M(z) \times \varepsilon_N(z')$$

in the monoidal semiring Q^m.

Proof. Viewing Q as Q^c,

$$\begin{aligned} \varepsilon_{M \sqcup N}(z \widehat{\otimes}_c z') &= \sum_{f \in \mathcal{F}(M \sqcup N)} (z \widehat{\otimes}_c z')(f) = \sum_{f \in \mathcal{F}(M \sqcup N)} \beta^c(z, z')(f|_M, f|_N) \\ &= \sum_{(g,g') \in \mathcal{F}(M) \times \mathcal{F}(N)} \beta^c(z, z')(g, g') = \sum_{g \in \mathcal{F}(M)} \sum_{g' \in \mathcal{F}(N)} z(g) \cdot z'(g') \\ &= \left\{ \sum_{g \in \mathcal{F}(M)} z(g) \right\} \cdot \left\{ \sum_{g' \in \mathcal{F}(N)} z'(g') \right\} = \varepsilon_M(z) \cdot \varepsilon_N(z'), \end{aligned}$$

using axiom (FDISJ) for systems of fields. If one replaces $\cdot$ by $\times$ and c by m in this calculation, one obtains the corresponding formula in Q^m. $\square$

Let $W \sqcup W'$ be a disjoint union of two bordisms W, W'. By Theorem 6.8, the state sum of the disjoint union decomposes as $Z_{W\sqcup W'} = Z_W \widehat{\otimes}_m Z_{W'}$ and thus by the above Proposition

$$\varepsilon_{\partial(W\sqcup W')}(Z_{W\sqcup W'}) = \varepsilon_{\partial W}(Z_W) \times \varepsilon_{\partial W'}(Z_{W'}).$$

Given three n-manifolds M, N, P and two vectors $z \in Z(M)\widehat{\otimes}Z(N)$, $z' \in Z(N)\widehat{\otimes}Z(P)$, we may form their contraction product $\langle z, z'\rangle \in Z(M)\widehat{\otimes}Z(P)$. The following proposition expresses the counit image of $\langle z, z'\rangle$ in terms of the counits $\varepsilon_{M,-} : Z(M)\widehat{\otimes}Z(N) \to Z(N)$, $\varepsilon_{-,P} : Z(N)\widehat{\otimes}Z(P) \to Z(N)$ and $\varepsilon_N : Z(N) \to Q$.

Proposition 7.5. *Given vectors $z \in Z(M)\widehat{\otimes}Z(N)$, $z' \in Z(N)\widehat{\otimes}Z(P)$, the identity*

$$\varepsilon_{M\sqcup P}\langle z, z'\rangle = \varepsilon_N(\varepsilon_{M,-}(z) \cdot \varepsilon_{-,P}(z'))$$

holds in Q, using on the right hand side the composition semiring-multiplication on $Z(N)$.

Proof. In the following calculation, γ is the contraction defined in Section 3.

$$\begin{aligned}
\varepsilon_{M\sqcup P}\langle z, z'\rangle &= \varepsilon_{M\sqcup P}\gamma(z\widehat{\otimes}_c z') = \sum_{f\in\mathcal{F}(M\sqcup P)} \gamma(\beta^c(z,z'))(f|_M, f|_P) \\
&= \sum_{f\in\mathcal{F}(M)} \sum_{h\in\mathcal{F}(P)} \gamma(\beta^c(z,z'))(f,h) \\
&= \sum_{f\in\mathcal{F}(M)} \sum_{h\in\mathcal{F}(P)} \sum_{g\in\mathcal{F}(N)} \beta^c(z,z')(f,g,g,h) \\
&= \sum_{g\in\mathcal{F}(N)} \sum_{f\in\mathcal{F}(M)} \sum_{h\in\mathcal{F}(P)} z(f,g)\cdot z'(g,h) \\
&= \sum_{g\in\mathcal{F}(N)} \left\{ \sum_{f\in\mathcal{F}(M)} z(f,g) \right\} \cdot \left\{ \sum_{h\in\mathcal{F}(P)} z'(g,h) \right\} \\
&= \sum_{g\in\mathcal{F}(N)} \varepsilon_{M,-}(z)(g)\cdot\varepsilon_{-,P}(z')(g) \\
&= \sum_{g\in\mathcal{F}(N)} (\varepsilon_{M,-}(z)\cdot\varepsilon_{-,P}(z'))(g) \\
&= \varepsilon_N(\varepsilon_{M,-}(z)\cdot\varepsilon_{-,P}(z')).
\end{aligned}$$

$\square$

Let W' be a bordism from M to N and let W'' be a bordism from N to P. By Theorem 6.10, the state sum Z_W of the bordism $W = W' \cup_N W''$ obtained by gluing W' and W'' along N is given by the contraction product $Z_W = \langle Z_{W'}, Z_{W''}\rangle$. Thus by Proposition 7.5,

$$\varepsilon_{M\sqcup P}(Z_W) = \varepsilon_N(\varepsilon_{M,-}(Z_{W'})\cdot\varepsilon_{-,P}(Z_{W''})).$$

A homeomorphism $\phi : M \to N$ induces covariantly an isomorphism $\phi_* : Z(M) \to Z(N)$ of both two-sided Q^c-semialgebras and Q^m-semialgebras. This is even an isomorphism of Frobenius semialgebras, as the calculation

$$\begin{aligned}
\varepsilon_N\phi_*(z) &= \sum_{g\in\mathcal{F}(N)} \phi_*(z)(g) = \sum_{g\in\mathcal{F}(N)} \mathrm{Fun}_Q(\phi^*)(z)(g) \\
&= \sum_{g\in\mathcal{F}(N)} z(\phi^* g) = \sum_{f\in\mathcal{F}(M)} z(f) = \varepsilon_M(z),
\end{aligned}$$

$z \in Z(M)$, shows, where we have used the bijection $\phi^* : \mathcal{F}(N) \to \mathcal{F}(M)$ provided by axiom (FHOMEO). If $\phi : W \to W'$ is a homeomorphism of bordisms with restriction $\phi_\partial : \partial W \to \partial W'$, then by Theorem 6.4,

$$\varepsilon_{\partial W'}(Z_{W'}) = \varepsilon_{\partial W'}(\phi_{\partial *} Z_W) = \varepsilon_{\partial W}(Z_W).$$

Thus state sums of homeomorphic bordisms have equal Frobenius traces.

8. Linear Representations

Let **Vect** denote the category of vector spaces over some fixed field, with morphisms the linear maps. While category-valued systems $\mathbb{T}$ of action exponentials, as formulated in Definition 5.8, provide a lot of flexibility, one is often ultimately interested in linear categories, as those are thoroughly understood and possess a rich, well developed theory of associated invariants. Thus in practice, the process of constructing a useful positive TFT will consist of two steps: First, find a (small) strict monoidal category **C** and a system of fields that possesses an interesting system of action exponentials into **C**. The morphisms of **C** may still be geometric or topological objects associated to the fields in a monoidal way. In the second step, construct a linear representation of **C**, that is, construct a monoidal functor $R : \mathbf{C} \to \mathbf{Vect}$. This converts the morphisms of **C** into linear maps, which can then be analyzed using tools from linear algebra. From this perspective, the category **C** plays an intermediate role in the construction of a TFT: it ought to be large enough to be able to record interesting information of the fields on a manifold, but small enough so as to allow for manageable linear representations. We will verify in this section (Proposition 8.1) that the composition of the **C**-valued action exponentials with the representation R yields a system of **Vect**-valued action exponentials, which then have their associated positive TFT Z, provided that **Vect** is endowed with the structure of a strict monoidal category. At this point, we face a formal problem: The ordinary tensor product of vector spaces is not strictly associative and the unit is not strict either. This problem can be solved by endowing **Vect**, without changing its objects and morphisms, with a strict (symmetric) monoidal structure, which is monoidally equivalent to the usual monoidal structure on **Vect**.

8.1. **The Schauenburg Tensor Product.** The ordinary tensor product of vector spaces is well-known not to be associative, though it is associative up to natural isomorphism. Thus, if we endowed **Vect** with the ordinary tensor product and took the unit object I to be the one-dimensional vector space given by the ground field, then, using obvious associators and unitors, **Vect** would become a monoidal category, but not a strict one. There is an abstract process of turning a monoidal category **C** into a monoidally equivalent strict monoidal category $\mathbf{C}^{\mathrm{str}}$. However, this process changes the category considerably and is thus not always practical. Instead, we base our monoidal structure on the Schauenburg tensor product $\odot$ introduced in [Sch01], which does not change the category **Vect** at all. The product $\odot$ satisfies the strict associativity

$$(U \odot V) \odot W = U \odot (V \odot W).$$

We shall thus simply write $U \odot V \odot W$ for this vector space. The unit object I remains the same as in the usual nonstrict monoidal structure, and one has

$$V \odot I = V, \; I \odot V = V.$$

The strict monoidal category $(\mathbf{Vect}, \odot, I)$ thus obtained is monoidally equivalent to the usual nonstrict monoidal category of vector spaces. The underlying functor of this monoidal equivalence is the identity. In particular, there is a natural isomorphism $\xi : \otimes \to \odot$,

$\xi_{VW} : V \otimes W \to V \odot W$, where $\otimes$ denotes the standard tensor product of vector spaces. Naturality means that for every pair of linear maps $f : V \to V'$, $g : W \to W'$, the square

$$\begin{array}{ccc} V \otimes W & \xrightarrow[\cong]{\xi_{VW}} & V \odot W \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle f\odot g} \\ V' \otimes W' & \xrightarrow[\cong]{\xi_{V'W'}} & V' \odot W' \end{array}$$

commutes. Note that via ξ we can speak of elements $v \odot w \in V \odot W$, $v \odot w = \xi_{VW}(v \otimes w)$, $v \in V$, $w \in W$. As the diagram

$$\begin{array}{ccccc} (U \otimes V) \otimes W & \xrightarrow{\xi\otimes \mathrm{id}} & (U \odot V) \otimes W & \xrightarrow{\xi} & (U \odot V) \odot W \\ {\scriptstyle a}\downarrow & & & & \| \\ U \otimes (V \otimes W) & \xrightarrow{\mathrm{id}\otimes\xi} & U \otimes (V \odot W) & \xrightarrow{\xi} & U \odot (V \odot W) \end{array}$$

commutes, the identity

$$(u \odot v) \odot w = u \odot (v \odot w)$$

holds for elements $u \in U, v \in V$ and $w \in W$.

The basic idea behind the construction of $\odot$ is to build a specific new equivalence of categories $L : \mathbf{Vect} \rightleftarrows \mathbf{Vect}^{\mathrm{str}} : R$ such that LR is the identity and then setting $V \odot W = R(LV * LW)$, where $*$ is the strictly associative tensor product in $\mathbf{Vect}^{\mathrm{str}}$. Then

$$\begin{aligned} (U \odot V) \odot W &= R(L(U \odot V) * LW) = R(LR(LU * LV) * LW) = R((LU * LV) * LW) \\ &= R(LU * (LV * LW)) = R(LU * LR(LV * LW)) = R(LU * L(V \odot W)) \\ &= U \odot (V \odot W). \end{aligned}$$

For the rest of this section we will always use the Schauenburg tensor product on **Vect** and thus will from now on write $\otimes_{\mathbf{Vect}}$ or simply $\otimes$ for $\odot$.

A braiding on the strict monoidal category **Vect** is defined by taking $b : V \otimes W \cong W \otimes V$ to be $v \otimes w \mapsto w \otimes v$. The hexagon equations are satisfied. As $b^2 = 1$, **Vect** endowed with b is thus a symmetric strict monoidal category.

8.2. **Linear Positive TFTs.** Let $(\mathbf{C}, \otimes, I)$ be a strict monoidal small category and let $R : \mathbf{C} \to \mathbf{Vect}$ be any strict monoidal functor, that is, a linear representation of $\mathbf{C}$. Let $\mathcal{F}$ be a system of fields and $\mathbb{T}$ a system of $\mathbf{C}$-valued action exponentials. The *R-linearization* $\mathbb{L}$ of $\mathbb{T}$ is given on a bordism W by the composition

$$\mathbb{L}_W : \mathcal{F}(W) \xrightarrow{\mathbb{T}_W} \mathrm{Mor}(\mathbf{C}) \xrightarrow{R} \mathrm{Mor}(\mathbf{Vect}).$$

Proposition 8.1. *The R-linearization $\mathbb{L}$ of $\mathbb{T}$ is a system of* **Vect***-valued action exponentials.*

Proof. Using axiom (TDISJ) for $\mathbb{T}$, we have for a disjoint union $W \sqcup W'$ and a field $f \in \mathcal{F}(W \sqcup W')$:

$$\begin{aligned} \mathbb{L}_{W \sqcup W'}(f) &= R(\mathbb{T}_{W \sqcup W'}(f)) = R(\mathbb{T}_W(f|_W) \otimes_{\mathbf{C}} \mathbb{T}_{W'}(f|_{W'})) \\ &= R\mathbb{T}_W(f|_W) \otimes_{\mathbf{Vect}} R\mathbb{T}_{W'}(f|_{W'}) = \mathbb{L}_W(f|_W) \otimes_{\mathbf{Vect}} \mathbb{L}_{W'}(f|_{W'}). \end{aligned}$$

This proves (TDISJ) for $\mathbb{L}$. If $W = W' \cup_N W''$ is obtained by gluing a bordism W' with outgoing boundary N to a bordism W'' with incoming boundary N, then on $f \in \mathcal{F}(W)$,

$$\begin{aligned} \mathbb{L}_W(f) &= R(\mathbb{T}_W(f)) = R(\mathbb{T}_{W''}(f|_{W''}) \circ_{\mathbf{C}} \mathbb{T}_{W'}(f|_{W'})) \\ &= R\mathbb{T}_{W''}(f|_{W''}) \circ_{\mathbf{Vect}} R\mathbb{T}_{W'}(f|_{W'}) = \mathbb{L}_{W''}(f|_{W''}) \circ_{\mathbf{Vect}} \mathbb{L}_{W'}(f|_{W'}), \end{aligned}$$

using (TGLUE) for $\mathbb{T}$. This establishes (TGLUE) for $\mathbb{L}$. Lastly, for a homeomorphism $\phi : W \to W'$ and a field $f \in \mathcal{F}(W')$, $\mathbb{L}_W(\phi^* f) = R\mathbb{T}_W(\phi^* f) = R\mathbb{T}_{W'}(f) = \mathbb{L}_{W'}(f)$, using (THOMEO) for $\mathbb{T}$ and proving this axiom for $\mathbb{L}$. □

Using the quantization of Section 6, the linearization $\mathbb{L}$ thus determines a positive TFT with state modules $Z(M)$, which are Frobenius semialgebras over the semiring $Q_S(\mathbf{Vect})^c$ and over the semiring $Q_S(\mathbf{Vect})^m$. If W is a closed $(n+1)$-manifold, then its state sum Z_W lies in $Z(\varnothing) = Q_S(\mathbf{Vect})$. Thus given any two vector spaces V, V' and a linear operator $A : V \to V'$, the state sum yields topologically invariant values $(Z_W)_{VV'}(A)$ in S. If W is not closed, then we may apply the Frobenius counit to $Z_W \in Z(\partial W)$ to get topological invariants $(\varepsilon_W(Z_W))_{VV'}(A) \in S$.

9. The Cylinder, Idempotency, and Projections

Let M be a closed n-manifold. The cylinder $W = M \times [0,1]$, viewed as a bordism from $M = M \times \{0\}$ to $M = M \times \{1\}$, plays a special role in any topological quantum field theory, since its homeomorphism class functions as the identity morphism $M \to M$ in cobordism categories. Thus in such categories, the cylinder is in particular idempotent and in light of the Gluing Theorem 6.10 it is reasonable to expect the state sum $Z_{M\times[0,1]}$ to be idempotent as well. We shall show below that this can indeed be deduced from our axioms. It does not, however, follow from these axioms that $Z_{M\times[0,1]}$ must be a predetermined universal element in $Z(M)\widehat{\otimes}Z(M)$ which only depends on M and not on the action. This creates problems for any attempt to recast positive TFTs as functors on cobordism categories. One could of course add axioms to the definition of category valued action exponentials $\mathbb{T}$ that would force $Z_{M\times[0,1]}$ to be a "canonical" element. Practical experience indicates that this is undesirable, as it would suppress a range of naturally arising, interesting actions. We shall write $M_2 = M \sqcup M$.

Proposition 9.1. *The state sum $Z_{M\times[0,1]}$ of a cylinder is idempotent, that is,*

$$\langle Z_{M\times[0,1]}, Z_{M\times[0,1]} \rangle = Z_{M\times[0,1]} \in Z(M_2).$$

Proof. Let $\phi : M \times [0,1] \to M \times [0, \frac{1}{2}]$ be the homeomorphism $\phi(x,t) = (x, t/2)$, $x \in M$, $t \in [0,1]$. Then the restriction $\phi_\partial : M_2 \to M_2$ of ϕ to the boundary is the identity map, $\phi_\partial = \mathrm{id}_{M_2}$. Consequently, $\phi_{\partial *} : Z(M_2) \to Z(M_2)$ is the identity as well. By Theorem 6.4,

$$Z_{M\times[0,1]} = \phi_{\partial *}(Z_{M\times[0,1]}) = Z_{M\times[0,\frac{1}{2}]}.$$

Similarly, $Z_{M\times[0,1]} = Z_{M\times[\frac{1}{2},1]}$. Let $W' = M \times [0, \frac{1}{2}]$, $W'' = M \times [\frac{1}{2}, 1]$ and $N = M \times \{\frac{1}{2}\}$. Then $M \times [0,1] = W' \cup_N W''$ and we deduce from the Gluing Theorem 6.10 that

$$\langle Z_{M\times[0,1]}, Z_{M\times[0,1]} \rangle = \langle Z_{W'}, Z_{W''} \rangle = Z_{M\times[0,1]}.$$

□

Let M and N be two closed n-manifolds. Define a map $\pi_{M,N} : Z(M \sqcup N) \to Z(M \sqcup N)$ by $\pi_{M,N}(z) = \langle Z_{M\times[0,1]}, z\rangle$. By Proposition 3.6, $\pi_{M,N}$ is right Q^c-linear. The contraction involved here is

$$\gamma : E(M\times 0)\widehat{\otimes}E(M\times 1)\widehat{\otimes}E(M\times 1)\widehat{\otimes}E(N) \longrightarrow E(M\times 0)\widehat{\otimes}E(N).$$

Thus, technically, $\pi_{M,N}$ is a map $\pi_{M,N} : Z(M\times 1)\widehat{\otimes}Z(N) \to Z(M\times 0)\widehat{\otimes}Z(N)$ given explicitly by

$$\pi_{M,N}(z)(f,g) = \sum_{h\in\mathcal{F}(M\times 1)} Z_{M\times[0,1]}(f,h)\cdot z(h,g),$$

$f \in \mathcal{F}(M\times 0)$, $g \in \mathcal{F}(N)$.

Proposition 9.2.
(1) The state sum Z_W of any bordism W from M to N is in the image of $\pi_{M,N}$. In fact $Z_W = \pi_{M,N}(Z_W)$, i.e. $\pi_{M,N}$ acts as the identity on the set of all state sums.
(2) The map $\pi_{M,N}$ is a projection, that is, $\pi^2_{M,N} = \pi_{M,N}$.

Proof. We prove (1): Let $\widehat{W} = M\times[0,1]\cup_{M\times\{1\}} W$ be the topological manifold obtained from attaching the cylinder along $M\times\{1\}$ to the incoming boundary M of W. By Brown's collar neighborhood theorem [Bro62], the boundary ∂W of the topological manifold W possesses a collar. Using this collar, there exists a homeomorphism $\phi : \widehat{W} \xrightarrow{\cong} W$, which is the identity on the boundary, $\phi_\partial = \mathrm{id}_{M\sqcup N} : \partial\widehat{W} = M\sqcup N \to M\sqcup N = \partial W$. By Theorem 6.4, $Z_{\widehat{W}} = \phi_{\partial *}(Z_{\widehat{W}}) = Z_W$. By the Gluing Theorem 6.10,

$$Z_W = Z_{\widehat{W}} = \langle Z_{M\times[0,1]}, Z_W\rangle = \pi_{M,N}(Z_W).$$

We prove (2): By Proposition 3.7, the contraction $\langle -,-\rangle$ is associative. Therefore, using the idempotency of $Z_{M\times[0,1]}$ (Proposition 9.1),

$$\begin{aligned}\pi^2_{M,N}(z) &= \langle Z_{M\times[0,1]}, \langle Z_{M\times[0,1]}, z\rangle\rangle = \langle\langle Z_{M\times[0,1]}, Z_{M\times[0,1]}\rangle, z\rangle \\ &= \langle Z_{M\times[0,1]}, z\rangle = \pi_{M,N}(z).\end{aligned}$$

□

Taking N to be the empty manifold, we obtain a projection

$$\pi_M = \pi_{M,\varnothing} : Z(M) = Z(M\times 1) \longrightarrow Z(M) = Z(M\times 0).$$

(In the case $N = \varnothing$, the map γ involved in the inner product defining $\pi_{M,N}$ becomes

$$\gamma : E(M\times 0)\widehat{\otimes}E(M\times 1)\widehat{\otimes}E(M\times 1) \longrightarrow E(M\times 0)$$

under the identifications $E(\varnothing) \cong Q$, $E\widehat{\otimes}Q \cong E$.) For $z \in Z(M)$, this projection is given by the explicit formula

$$\pi_M(z)(f) = \sum_{h\in\mathcal{F}(M\times 1)} Z_{M\times[0,1]}(f,h)\cdot z(h),$$

$f \in \mathcal{F}(M\times 0)$. In passing, let us observe the formal analogy to integral transforms

$$(Tg)(x) = \int K(x,\xi)g(\xi)d\xi,$$

given by an integral kernel K. Thus the state sum $Z_{M\times[0,1]}$ of the cylinder can be interpreted as such a kernel. We shall now pursue the question how to compute the projection $\pi_{M\sqcup N}$ of a tensor product of state vectors. Is the image again a tensor product?

Lemma 9.3. *Let $\mathbb{T}$ be a cylindrically firm system of* $\mathbf{C}$*-valued action exponentials. Given fields $f_M \in \mathcal{F}(M \times 0)$, $g_M \in \mathcal{F}(M \times 1)$, $f_N \in \mathcal{F}(N \times 0)$, $g_N \in \mathcal{F}(N \times 1)$ and state vectors $z_M \in Z(M \times 1)$, $z_N \in Z(N \times 1)$, the identity*

$$\big(Z_{M\times[0,1]}(f_M,g_M) \times Z_{N\times[0,1]}(f_N,g_N)\big) \cdot \big(z_M(g_M) \times z_N(g_N)\big)$$
$$= \big(Z_{M\times[0,1]}(f_M,g_M) \cdot z_M(g_M)\big) \times \big(Z_{N\times[0,1]}(f_N,g_N) \cdot z_N(g_N)\big)$$

holds in $Q_S(\mathbf{C})$.

Proof. We put $a = Z_{M\times[0,1]}(f_M,g_M)$, $b = Z_{N\times[0,1]}(f_N,g_N)$, $c = z_M(g_M)$ and $d = z_N(g_N)$. On a morphism ξ',

$$a(\xi') = \sum_{F_M \in \mathcal{F}(M\times[0,1], f_M \sqcup g_M)} T_{M\times[0,1]}(F_M)(\xi'),$$

with

$$T_{M\times[0,1]}(F_M)(\xi') = \begin{cases} 1, & \xi' = \mathbb{T}_{M\times[0,1]}(F_M) \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, on a morphism ξ'',

$$b(\xi'') = \sum_{F_N \in \mathcal{F}(N\times[0,1], f_N \sqcup g_N)} T_{N\times[0,1]}(F_N)(\xi''),$$

with

$$T_{N\times[0,1]}(F_N)(\xi'') = \begin{cases} 1, & \xi'' = \mathbb{T}_{N\times[0,1]}(F_N) \\ 0, & \text{otherwise.} \end{cases}$$

Thus (see also the proof of Proposition 4.5),

$$\begin{aligned}
((a\times b)\cdot(c\times d))_{XY}(\gamma) &= \sum_{(\xi',\xi'',\eta',\eta'')\in TCT(\gamma)} d(\eta'')c(\eta')b(\xi'')a(\xi') \\
&= \sum_{F_M,F_N} \left(\sum_{(\xi',\xi'',\eta',\eta'')\in TCT(\gamma)} d(\eta'')c(\eta')T_{N\times[0,1]}(F_N)(\xi'')T_{M\times[0,1]}(F_M)(\xi') \right) \\
&= \sum_{F_M,F_N} \left(\sum_{(\eta',\eta'')\in TCT(\gamma;\mathbb{T}_{M\times[0,1]}(F_M),\mathbb{T}_{N\times[0,1]}(F_N))} d(\eta'')c(\eta') \right) \\
&= \sum_{F_M,F_N} \left(\sum_{(\eta',\eta'')\in CTC(\gamma;\mathbb{T}_{M\times[0,1]}(F_M),\mathbb{T}_{N\times[0,1]}(F_N))} d(\eta'')c(\eta') \right) \\
&= \sum_{F_M,F_N} \left(\sum_{(\xi',\xi'',\eta',\eta'')\in CTC(\gamma)} d(\eta'')T_{N\times[0,1]}(F_N)(\xi'')c(\eta')T_{M\times[0,1]}(F_M)(\xi') \right) \\
&= \sum_{(\xi',\xi'',\eta',\eta'')\in CTC(\gamma)} d(\eta'')b(\xi'')c(\eta')a(\xi') \\
&= ((a\cdot c)\times(b\cdot d))_{XY}(\gamma).
\end{aligned}$$

Here, the F_M range over $\mathcal{F}(M\times[0,1], f_M \sqcup g_M)$ and the F_N over $\mathcal{F}(N\times[0,1], f_N \sqcup g_N)$. The sets CTC and TCT have been defined near the end of Section 4. Note that the element $T_{N\times[0,1]}(F_N)(\xi'')$ commutes with any element of S, since it is either 1 or 0. $\square$

Theorem 9.4. *Let M, N be closed n-manifolds and $z_M \in Z(M)$, $z_N \in Z(N)$. If $\mathbb{T}$ is a cylindrically firm system of $\mathbf{C}$-valued action exponentials, then*

$$\pi_{M \sqcup N}(z_M \widehat{\otimes}_m z_N) = \pi_M(z_M) \widehat{\otimes}_m \pi_N(z_N).$$

Proof. Using Theorem 6.8 to decompose $Z_{(M \sqcup N)\times[0,1]} = Z_{M\times[0,1]} \widehat{\otimes}_m Z_{N\times[0,1]}$, axiom (FDISJ) to decompose $\mathcal{F}((M \sqcup N)\times 1) \cong \mathcal{F}(M \times 1) \times \mathcal{F}(N \times 1)$ and Lemma 9.3, we compute the value of $\pi_{M\sqcup N}(z_M \widehat{\otimes}_m z_N)$ on a field $f \in \mathcal{F}((M \sqcup N)\times 0)$:

$$\begin{aligned}
\pi_{M\sqcup N}(z_M \widehat{\otimes}_m z_N)(f) &= \sum_{g \in \mathcal{F}((M\sqcup N)\times 1)} Z_{(M\sqcup N)\times[0,1]}(f,g) \cdot (z_M \widehat{\otimes}_m z_N)(g) \\
&= \sum_g (Z_{M\times[0,1]} \widehat{\otimes}_m Z_{N\times[0,1]})(f,g) \cdot (z_M \widehat{\otimes}_m z_N)(g) \\
&= \sum_g \Big(Z_{M\times[0,1]}(f|_{M\times 0}, g|_{M\times 1}) \times Z_{N\times[0,1]}(f|_{N\times 0}, g|_{N\times 1}) \Big) \cdot \Big(z_M(g|_{M\times 1}) \times z_N(g|_{N\times 1}) \Big) \\
&= \sum_{(g_M,g_N)\in \mathcal{F}(M)\times\mathcal{F}(N)} \Big(Z_{M\times[0,1]}(f|_M, g_M) \times Z_{N\times[0,1]}(f|_N, g_N) \Big) \cdot \Big(z_M(g_M) \times z_N(g_N) \Big) \\
&= \sum_{g_M \in \mathcal{F}(M)} \sum_{g_N \in \mathcal{F}(N)} \Big(Z_{M\times[0,1]}(f|_M, g_M) \cdot z_M(g_M) \Big) \times \Big(Z_{N\times[0,1]}(f|_N, g_N) \cdot z_N(g_N) \Big) \\
&= \Big\{ \sum_{g_M \in \mathcal{F}(M)} Z_{M\times[0,1]}(f|_M, g_M) \cdot z_M(g_M) \Big\} \times \Big\{ \sum_{g_N \in \mathcal{F}(N)} Z_{N\times[0,1]}(f|_N, g_N) \cdot z_N(g_N) \Big\} \\
&= \pi_M(z_M)(f|_{M\times 0}) \times \pi_N(z_N)(f|_{N\times 0}) \\
&= (\pi_M(z_M) \widehat{\otimes}_m \pi_N(z_N))(f).
\end{aligned}$$

□

Corollary 9.5. *If $\mathbf{C}$ is a monoid, then*

$$\pi_{M \sqcup N}(z_M \widehat{\otimes}_m z_N) = \pi_M(z_M) \widehat{\otimes}_m \pi_N(z_N).$$

(Note that if $\mathbf{C}$ is a monoid, then Proposition 4.5 is available.)

Remark 9.6. Following classical topological field theory, and informed by Proposition 9.2, one might now attempt to set $Z'(M) = \pi_M Z(M)$. In the present framework of positive topological field theory, this smaller state module is not serviceable, for at least the following reason: In order to obtain an analog of Proposition 6.2 for Z', i.e. a decomposition $Z'(M \sqcup N) \cong Z'(M) \widehat{\otimes} Z'(N)$, one would have to rely on results such as Theorem 9.4, which breaks up a projection of a tensor product on a disjoint union into a tensor product of projections on the components. But for many interesting actions that one wishes to apply the present framework to, the assumption of cylindrical firmness is not germane. So we refrain from passing to the images of the projections π_M.

10. Smooth Manifolds and Positive TFTs

We sketch an application of the positive TFT method to constructing a concrete new TFT defined on smooth bordisms of any dimension ≥ 2. Detailed statements and proofs will appear in a separate publication. As has been pointed out before, the smooth situation necessitates some small adaptations of the general framework. We shall indicate most of these during the construction below. The main idea is to take (certain) fold maps into the plane as the system of fields and certain linear operators assigned to the singular set of such fold maps as the action functional.

The Brauer algebras D_m arose in the representation theory of the orthogonal group $O(n)$, see [Bra37], [Wen88], and have since played an important role in knot theory. We shall require a categorification, which we denote by **Br**, of Brauer's algebras. Let V be a finite dimensional real vector space. A *duality structure* on V is a pair (i,e) whose components are a symmetric copairing $i:\mathbb{R}\to V\otimes V$ and a symmetric pairing $e:V\otimes V\to\mathbb{R}$, called *unit* and *counit*, respectively, satisfying the zig-zag equation, i.e. the composition

$$V = V\otimes I \xrightarrow{1_V\otimes i} V^{\otimes 3} \xrightarrow{e\otimes 1_V} I\otimes V = V$$

is the identity. Constructing concrete duality structures on a given V involves solving a system of quadratic equations. The *trace* of the duality structure (i,e) is $\operatorname{Tr}(i,e) = e\circ i = \dim V$. We use such duality structures to construct linear monoidal representations of a natural categorification **Br** of the Brauer category. But first let us introduce this categorification. Loosely speaking, the morphisms will be 1-dimensional unoriented tangles in a high-dimensional Euclidean space. As those can always be disentangled, **Br** is very close, but not equal, to the category of 1-dimensional cobordisms. One difference is that the objects of the latter, being 0-manifolds, are unordered (finite) sets, whereas the objects of **Br** will be *ordered* tuples of points. Another difference is that the cobordism category has a huge number of objects (though few isomorphism types), whereas the Brauer category has very few objects to begin with and has the property that two objects are isomorphic if and only if they are equal. Let us detail the formal definition of **Br**. Given $n=1,2,\ldots,$ we write $[n]$ for the set $\{1,\ldots,n\}$. We write $[0]$ for the empty set. The objects of **Br** are $[0],[1],[2],\ldots$. Each object $[n]$ determines a 0-submanifold $M[n]$ of $\mathbb{R}^1$ by taking $M[n]=\{1,\ldots,n\}\subset\mathbb{R}^1$. Morphisms $[m]\to[n]$ in **Br** are represented by compact smooth 1-manifolds W, smoothly embedded in $[0,1]\times\mathbb{R}^3$, such that $\partial W = W\cap(\{0,1\}\times\mathbb{R}^3)$ with

$$\partial W\cap\{0\}\times\mathbb{R}^3 = 0\times M[m]\times 0\times 0,\ \partial W\cap\{1\}\times\mathbb{R}^3 = 1\times M[n]\times 0\times 0.$$

We require that near the boundary, the embedding of W is the product embedding

$$[0,\varepsilon]\times M[m]\times 0\times 0 \sqcup [1-\varepsilon,1]\times M[n]\times 0\times 0,$$

for some small $\varepsilon>0$. Two such W for fixed $[m],[n]$ define the same morphism in **Br**, if they are smoothly isotopic in $[0,1]\times\mathbb{R}^3$ by an isotopy that is the identity near $\{0,1\}\times\mathbb{R}^3$. The composition of two morphisms $\phi:[m]\to[n]$, $\psi:[n]\to[p]$ is defined in the most natural manner: If ϕ is represented by the cobordism V and ψ by W, then we translate W from $[0,1]\times\mathbb{R}^3$ to $[1,2]\times\mathbb{R}^3$ and define a cobordism U as the union along $\{1\}\times M[n]\times 0\times 0$ of V and the translated copy of W. Then we reparametrize the embedding of U from $[0,2]\times\mathbb{R}^3$ to $[0,1]\times\mathbb{R}^3$. The resulting cobordism represents $\psi\circ\phi$; its isotopy class depends clearly only on ϕ and ψ, not on the particular choice of representatives V and W. The identity $1_{[0]}:[0]\to[0]$ is represented by the empty cobordism $W=\varnothing$. For $n>0$, the identity $1_{[n]}:[n]\to[n]$ is represented by the product $[0,1]\times M[n]\times 0\times 0$. Then **Br** is indeed a category. Note that $\operatorname{Hom}_{\mathbf{Br}}([m],[n])$ is empty for $m+n$ odd and nonempty for $m+n$ even. We make **Br** into a strict monoidal category by defining a tensor product $\otimes:\mathbf{Br}\times\mathbf{Br}\to\mathbf{Br}$ on objects by $[m]\otimes[n]=[m+n]$. Let the unit object I be $[0]$. The tensor product $\phi\otimes\phi'$ of two morphisms $\phi:[m]\to[n]$ and $\phi':[m']\to[n']$ is defined by "stacking" a representative W' of ϕ' on top of a representative W of ϕ. There is precisely one endomorphism $\lambda:[0]\to[0]$, $\lambda\neq 1_{[0]}$, such that λ is represented by a connected, nonempty manifold. This morphism is represented by the embedding of a circle in $(0,1)\times\mathbb{R}^3$. We will call this endomorphism λ the *loop*. Given two objects $[m]$ and $[n]$ in **Br**, we define the braiding $b_{m,n} : [m] \otimes [n] \to [n] \otimes [m]$ to be the isomorphism represented by a Brauer diagram which is loop-free and connects $i \in M([m] \otimes [n]) = \{1,\ldots,m,m+1,\ldots,m+n\}$, $1 \le i \le m$, to $n+i \in M([n] \otimes [m]) = \{1,\ldots,n,n+1,\ldots,n+m\}$ and $m+i \in M([m] \otimes [n])$, $1 \le i \le n$, to $i \in M([n] \otimes [m])$. Since we are in codimension 3, b is symmetric, $b_{m,n} = b^{-1}_{n,m}$. Particularly important is the elementary braiding $b_{1,1}$. Then the structure $(\mathbf{Br},\otimes,I,b)$ is a symmetric strict monoidal category. A unit $i_n : I \to [n] \otimes [n]$ is given by interpreting the cylinder $[0,1] \times M[n] \times 0 \times 0$ as a morphism $I \to [n] \otimes [n]$. In particular, we have the elementary unit i_1. Similarly, a counit $e_n : [n] \otimes [n] \to I$ is given by this time interpreting the cylinder $[0,1] \times M[n] \times 0 \times 0$ as a morphism $[n] \otimes [n] \to I$. The elementary counit is e_1. We summarize: The structure $(\mathbf{Br},\otimes,I,b,i,e)$ is a compact (i.e. every object is dualizable), symmetric, strict monoidal category.

We use duality structures on vector spaces to construct linear representations of $\mathbf{Br}$, i.e. symmetric strict monoidal functors $Y : \mathbf{Br} \to \mathbf{Vect}$ which preserve duality. Let V be a finite dimensional real vector space and (i,e) a duality structure on V. Then there exists a unique symmetric strict monoidal functor $Y : \mathbf{Br} \to \mathbf{Vect}$ which satisfies $Y([1]) = V$ and preserves duality, that is, $Y(i_1) = i$, $Y(e_1) = e$. This can be proven by finding an explicit presentation of $\mathbf{Br}$ by generators and relations. Such a presentation can either be found directly, or can be derived from the presentation of oriented tangles in $[0,1] \times \mathbb{R}^2$ given in [Tur89], by forgetting orientation and adding one relation enabling strands to cross through each other. One particular example of a duality structure on $V = \mathbb{R}^2$ is

$$e(e_{11}) = 0,\ e(e_{12}) = 1,\ e(e_{21}) = 1,\ e(e_{22}) = -1,\ i(1) = e_{11} + e_{12} + e_{21}, \tag{18}$$

where $e_{ij} = e_i \otimes e_j$, and e_1, e_2 is the standard basis of $\mathbb{R}^2$. The symmetric monoidal representation $Y : \mathbf{Br} \to \mathbf{Vect}$ determined by the duality structure (18) is faithful on loops, that is, if ϕ and ψ are any two morphisms in $\mathbf{Br}$ such that $Y(\phi) = Y(\psi)$, then ϕ and ψ have the same number of loops.

We discuss next an algebraic process of profinite idempotent completion. Given objects $[m],[n]$ of $\mathbf{Br}$, we define a subset $H_{m,n}$ of the vector space $\mathrm{Hom}_{\mathbf{Vect}}(V^{\otimes m}, V^{\otimes n})$ by

$$H_{m,n} = Y(\mathrm{Hom}_{\mathbf{Br}}([m],[n])).$$

This set is nonempty only if $m+n$ is even. From now on, assume that the duality structure (i,e) is given on a vector space V with $\dim V \ge 2$ and is such that Y is faithful on loops. This is for instance the case for the duality structure (18). Applying Y to the loop λ, we obtain a scalar $\widehat{\lambda} = Y(\lambda) \in \mathrm{Hom}_{\mathbf{Vect}}(Y[0],Y[0]) = \mathbb{R}$, which is just the trace of the duality structure (i,e). If S is any complete semiring, then the semiring $S[[q]]$ of formal power series in q becomes a complete semiring by transferring the summation law on S pointwise to $S[[q]]$, see [Kar92]. Hence, $\mathbb{B}[[q]]$ is a complete idempotent semiring. The Boolean monoid $\mathbb{B}$ is an $\mathbb{N}$-semimodule in a natural way. Thus, by letting τ act as multiplication by q, $\mathbb{B}[[q]]$ becomes a semimodule over the polynomial semiring $\mathbb{N}[\tau]$. If A is a set, let $\mathrm{FM}(A)$ denote the free commutative monoid generated by A. Suppose that A is equipped with an action $\mathbb{N} \times A \to A$ of the monoid $(\mathbb{N},+)$, $(\tau^i,a) \mapsto \tau^i a$, where we have written the additive monoid $\mathbb{N}$ multiplicatively as $\mathbb{N} = \{\tau^i \mid i \in \mathbb{N}\}$. Then

$$\sum_i m_i \tau^i \cdot \sum_j \alpha_j a_j = \sum_{i,j} (m_i \alpha_j)(\tau^i a_j),\ m_i, \alpha_j \in \mathbb{N}, a_j \in A,$$

makes $\mathrm{FM}(A)$ into an $\mathbb{N}[\tau]$-semimodule. Now the sets $H_{m,n}$ are naturally equipped with the action $\tau^i h = \widehat{\lambda}^i h \in H_{m,n}$ of the monoid $\mathbb{N}$. Hence, $\mathrm{FM}(H_{m,n})$ is an $\mathbb{N}[\tau]$-semimodule and consequently the $\mathbb{N}[\tau]$-semimodule

$$Q(H_{m,n}) = \mathrm{FM}(H_{m,n}) \otimes_{\mathbb{N}[\tau]} \mathbb{B}[[q]]$$

is defined. It is idempotent since $\mathbb{B}[[q]]$ is. The fact that the Boolean semiring, whose only nonzero value is 1, appears here is a reflection of the fact that the modulus of the integrand e^{iS} appearing in the classical Feynman path integral is always $|e^{iS}| = 1$ and only the phase is relevant. Roughly, the terms in $\mathrm{FM}(H_{m,n})$ play the role of the phase. One can prove that the $Q(H_{m,n})$ are complete $\mathbb{N}[\tau]$-semimodules. We set

$$Q = Q(i,e) = \prod_{m,n\in\mathbb{N}} Q(H_{m,n})$$

in the category of $\mathbb{N}[\tau]$-semimodules. A product

$$Q(H_{m,p}) \times Q(H_{p,n}) \longrightarrow Q(H_{m,n}),\ (f, f') \mapsto f \cdot f'$$

is given on two elements $\sum_{i=1}^{k} h_i \otimes b_i \in Q(H_{m,p})$ and $\sum_{j=1}^{l} h'_j \otimes b'_j \in Q(H_{p,n})$ with $h_i \in H_{m,p}$, $h'_j \in H_{p,n}$ by $\sum_{i,j}(h'_j \circ h_i) \otimes (b_i b'_j)$. We define the product of two elements $(f_{m,n}), (f'_{m,n}) \in Q$ to be $(f_{m,n}) \cdot (f'_{m,n}) = (f''_{m,n})$, where the component $f''_{m,n} \in Q(H_{m,n})$ is given by the convolution $f''_{m,n} = \sum_{p\in\mathbb{N}} f_{m,p} \cdot f'_{p,n}$, using the completeness of $Q(H_{m,n})$, as well as the products declared above. An element $1 \in Q$ is given by the family $1 = (f_{m,n})$ with

$$f_{m,n} = \begin{cases} 1_{V^{\otimes m}} \otimes 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

The tuple $(Q, +, \cdot, 0, 1)$ is a complete idempotent semiring, an adaptation to the present context of the composition semiring Q^c of Section 4. The cross-product

$$Q(H_{m,n}) \times Q(H_{r,s}) \longrightarrow Q(H_{m+r,n+s}),\ (f, f') \mapsto f \times f'$$

of two elements $\sum_{i=1}^{k} h_i \otimes b_i \in Q(H_{m,n})$ and $\sum_{j=1}^{l} h'_j \otimes b'_j \in Q(H_{r,s})$ with $h_i \in H_{m,n}$, $h'_j \in H_{r,s}$ is given by $\sum_{i,j}(h_i \otimes h'_j) \otimes (b_i b'_j)$, where $h_i \otimes h'_j$ is the Schauenburg tensor product. We define the cross-product of two elements $(f_{m,n}), (f'_{m,n}) \in Q$ to be $(f_{m,n}) \times (f'_{m,n}) = (f''_{m,n})$, where the component $f''_{m,n} \in Q(H_{m,n})$ is given by $f''_{m,n} = \sum_{\substack{p+r=m \\ q+s=n}} f_{p,q} \times f'_{r,s}$, using the products $\times$ declared above. (It should be pointed out that in order to prove associativity of this product, the strict associativity of the Schauenburg tensor product enters crucially.) An element $1^\times \in Q$ is given by

$$1^\times_{m,n} = \begin{cases} 1_I \otimes 1, & \text{if } m = n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the tuple $(Q, +, \times, 0, 1^\times)$ is a complete idempotent semiring, an adaptation to the present context of the monoidal semiring Q^m of Section 4. We note in passing that these two semirings Q can be shown to be so-called *continuous* semirings. This can then be used to link the corresponding Eilenberg summation law to V. P. Maslov's idempotent integration theory [LMS01]. Thus all invariants developed here can equivalently be phrased as Maslov idempotent integrals.

On the topological side, a key concept in our smooth TFT is the notion of a *fold map*. Let M and N be smooth manifolds. Let $J^k(M,N)$ be the space of k-jets of smooth maps $M \to N$ and let $j^k f : M \to J^k(M,N)$ denote the k-jet extension of a smooth map $f : M \to N$. The manifold $J^1(M,N)$ is canonically isomorphic to $\mathrm{Hom}(TM,TN)$. The subsets

$$S_r = \{\sigma \in J^1(M,N) \mid \operatorname{corank} \sigma = r\},\ r = 0,1,2,\ldots,$$

are submanifolds of $J^1(M,N)$. The full-rank set S_0 is open in $J^1(M,N)$. Let W be an n-manifold, $n \geq 2$, and let $S(F) \subset W$ denote the singular set of a smooth map $F : W \to \mathbb{R}^2 = \mathbb{C}$. The jet manifold is now given by

$$J^1(W,\mathbb{R}^2) = S_0 \cup S_1 \cup S_2.$$

A smooth map $F : W \to \mathbb{R}^2$ is called a *fold map*, if j^1F is transverse to S_1, $S(F) = (j^1F)^{-1}(S_1)$, and for all $x \in S(F)$, $T_xS(F) + \ker D_xF = T_xW$. One refers to its singular set $S(F)$, a smoothly embedded 1-dimensional submanifold, also as the *fold locus* or *fold lines*. Fold points possess an invariantly defined absolute index, which is constant along connected components of $S(F)$.

Fix an integer $D \geq 2n+1$. A closed $(n-1)$-dimensional manifold can be embedded in $\mathbb{R}^{D-1}$ and then, after having made a choice of $k \in \mathbb{N}$, into a slice $\{k\} \times \mathbb{R}^{D-1} \subset \mathbb{R} \times \mathbb{R}^{D-1} = \mathbb{R}^D$. It is convenient to assume that closed $(n-1)$-manifolds M are always embedded in $\mathbb{R}^D$ in such a way that every connected component of M lies entirely in some slice $\{k_0\} \times \mathbb{R}^{D-1}$, for suitable $k_0 \in \mathbb{N}$. Given such an embedding, we let $M(k) = M \cap \{k\} \times \mathbb{R}^{D-1}$ be the part of M that lies in the k-slice. A compact smooth n-dimensional manifold W with boundary can be smoothly embedded into a closed halfspace of $\mathbb{R}^{2n+1}$ in such a way that the boundary of W lies in the bounding hyperplane and the interior lies in the interior of the halfspace, see e.g. [Hir76, Theorem 1.4.3]. Let $M,N \subset \mathbb{R}^D$ be closed, smoothly embedded, $(n-1)$-dimensional manifolds, not necessarily orientable. A *cobordism* from M to N is a compact, smoothly embedded n-dimensional manifold $W \subset [0,1] \times \mathbb{R}^D$ with boundary $\partial W = M \sqcup N$, such that

- $M \subset \{0\} \times \mathbb{R}^D, N \subset \{1\} \times \mathbb{R}^D$, $W - \partial W \subset (0,1) \times \mathbb{R}^D$,
- near the boundary of $[0,1] \times \mathbb{R}^D$, the embedding is the product embedding, that is, there exists $0 < \varepsilon < \frac{1}{2}$ such that $W \cap [0,\varepsilon] \times \mathbb{R}^D = [0,\varepsilon] \times M$ and $W \cap [1-\varepsilon,1] \times \mathbb{R}^D = [1-\varepsilon,1] \times N$, and
- every connected component of W lies entirely in some slice $[0,1] \times \{k_0\} \times \mathbb{R}^{D-1}$, $k_0 \in \mathbb{N}$.

We will see later that our invariant depends only very weakly on the embedding of W. We refer to any ε with the above properties as a *cylinder scale* of W. The manifold M, i.e. the part of ∂W that is contained in the hyperplane $0 \times \mathbb{R}^D$, is the *incoming* boundary and N, i.e. the part of ∂W that is contained in the hyperplane $1 \times \mathbb{R}^D$, is the *outgoing* boundary. Let $W(k) = W \cap [0,1] \times \{k\} \times \mathbb{R}^{D-1}$ be the part of W that lies in the k-slice. The embedding also enables us to chop W into the slices $W_t = W \cap (\{t\} \times \mathbb{R}^D)$. The first coordinate of $\mathbb{R}^{D+1}$ defines a smooth *time-function* $\omega : W \to [0,1]$, i.e. ω is the composition

$$W \hookrightarrow [0,1] \times \mathbb{R}^D \xrightarrow{\text{proj}_1} [0,1].$$

Using this function, W_t can alternatively be described as the preimage $W_t = \omega^{-1}(t)$. Let $\operatorname{Reg}(W)$ be the set of regular values of ω.

Let W be a cobordism from M to N. Given a fold map $F : W \to \mathbb{C}$, we set $F(k) = F| : W(k) \to \mathbb{C}$ and $SF_t = S(F) \cap W_t$. We say that F has *generic imaginary parts over* $t \in [0,1]$, if $\operatorname{Im} \circ F| : SF_t \to \mathbb{R}$ is injective. We put

$$\operatorname{GenIm}(F) = \{t \in [0,1] \mid F \text{ has generic imaginary parts over } t\}$$

and

$$\pitchfork(F) = \{t \in \operatorname{Reg}(W) \mid S(F) \pitchfork W_t\}.$$

Note that for $t \in \pitchfork(F)$, SF_t is a compact 0-dimensional manifold and thus a finite set of points. The set $\pitchfork(F)$ can be expressed in terms of regular values. We are ready to define the fields $\mathcal{F}(W)$ on the bordism W.

Definition 10.1. A *fold field* on W is a fold map $F : W \to \mathbb{C}$ such that for all $k \in \mathbb{N}$,
(1) $0,1 \in \pitchfork(F(k)) \cap \operatorname{GenIm}(F(k))$, and
(2) $\operatorname{GenIm}(F(k))$ is residual in $[0,1]$.

For a nonempty cobordism W, let $\mathcal{F}(W) \subset C^\infty(W,\mathbb{C})$ be the moduli space of all fold fields on W. For W empty, we agree that $\mathcal{F}(W) = \{*\}$, a set with one element.

We shall next describe the system $\mathbb{T}$ of action exponentials on fold fields. Let $\operatorname{Mor}(\mathbf{Br})$ denote the set of all morphisms of the category $\mathbf{Br}$. There is a natural function

$$\mathbb{S} : \mathcal{F}(W) \longrightarrow \operatorname{Mor}(\mathbf{Br})$$

defined as follows. If W is empty we define $\mathbb{S}(*) = 1_I$, the identity on the unit object $I = [0]$ of $\mathbf{Br}$. Next, suppose that W is nonempty and entirely contained in a slice $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$, i.e. $W = W(k)$. Given $F \in \mathcal{F}(W)$, let m_S be the cardinality of $S(F) \cap M$ and let n_S be the cardinality of $S(F) \cap N$. The Brauer morphism $\mathbb{S}(F)$ will be a morphism $\mathbb{S}(F) : [m_S] \to [n_S]$. There is a canonical identification of points of $S(F) \cap M$ with points of $M[m_S]$ given as follows: By (1) of Definition 10.1, $\operatorname{Im} \circ F$ is injective on $S(F) \cap M = SF_0$ and therefore induces a unique ordering $p_1, p_2, \ldots, p_{m_S}$ of the points of $S(F) \cap M$ such that

$$\operatorname{Im} F(p_i) < \operatorname{Im} F(p_j) \quad \Longleftrightarrow \quad i < j.$$

This is a bijection $S(F) \cap M \cong M[m_S]$, $p_i \leftrightarrow i$. Similarly for the outgoing boundary: The function $\operatorname{Im} \circ F$ is injective on $S(F) \cap N = SF_1$ and therefore induces a unique ordering $q_1, q_2, \ldots, q_{n_S}$ of the points of $S(F) \cap N$ such that $\operatorname{Im} F(q_i) < \operatorname{Im} F(q_j) \Leftrightarrow i < j$. This gives a bijection $S(F) \cap N \cong M[n_S]$, $q_i \leftrightarrow i$. To construct the Brauer morphism $\mathbb{S}(F) : [m_S] \to [n_S]$, connect the points of $0 \times M[m_S] \times 0 \times 0$ and $1 \times M[n_S] \times 0 \times 0$ by smooth arcs in $[0,1] \times \mathbb{R}^3$ in the following manner. Let c be a connected component of the compact 1-manifold $S(F)$. We distinguish four cases. If $\partial c = \{p_i, p_j\}$, then connect $(0,i,0,0)$ to $(0,j,0,0)$ by an arc. If $\partial c = \{p_i, q_j\}$, then connect $(0,i,0,0)$ to $(1,j,0,0)$. If $\partial c = \{q_i, q_j\}$, then connect $(1,i,0,0)$ to $(1,j,0,0)$. Finally, if c is closed, i.e. $\partial c = \varnothing$, then tensor with the loop endomorphism λ. Carrying this recipe out for every connected component c of $S(F)$ completes the construction of $\mathbb{S}(F)$. Finally, if W is nonempty but otherwise arbitrary, we put $\mathbb{S}(F) = \bigotimes_{k \in \mathbb{N}} \mathbb{S}(F(k))$. (This tensor product is finite, as $W(k)$ is eventually empty.)

Let V be a real vector space of finite dimension $\dim V \geq 2$ and let (i,e) be a duality structure on V such that the induced symmetric monoidal functor Y is faithful on loops. (For instance, we may take $V = \mathbb{R}^2$ and (i,e) as in (18).) Let $Q = Q(i,e)$ be the profinite idempotent completion of $\operatorname{Mor}(\mathbf{Br})$, based on the symmetric monoidal representation $Y : \mathbf{Br} \to \mathbf{Vect}$ determined by (i,e). The composition

$$\mathbb{T}_W : \mathcal{F}(W) \xrightarrow{\mathbb{S}} \operatorname{Mor}(\mathbf{Br}) \xrightarrow{Y} \operatorname{Mor}(\mathbf{Vect})$$

is the action exponential $\mathbb{T}_W$. Thus every fold field $F \in \mathcal{F}(W)$ determines an element $Y\mathbb{S}(F) \otimes 1$ in $Q(H_{\operatorname{dom}\mathbb{S}(F), \operatorname{cod}\mathbb{S}(F)})$, and thus an element in Q, which we will denote by $T_W(F) \in Q$. This completes the construction of the action exponential.

We still have to define the fields on closed $(n-1)$-manifolds and will do so now. For a nonempty, closed, smooth $(n-1)$-manifold $M \subset \mathbb{R}^D$ (not necessarily orientable), we have the trivial cobordism from M to M, i.e. the cylinder $W = [0,1] \times M \subset [0,1] \times \mathbb{R}^D$, and we put

$$\mathcal{F}(M) = \{f \in \mathcal{F}([0,1] \times M) \mid \mathbb{S}(f) = 1 \in \mathrm{Mor}(\mathbf{Br})\},$$

where 1 denotes an identity morphism in **Br**. For $M = \varnothing$, we put $\mathcal{F}(\varnothing) = \{*\}$, the one-element set. These are the *fields* associated to a closed $(n-1)$-manifold and will act as boundary conditions.

Remark 10.2. If $g : M \to \mathbb{R}$ is an excellent Morse function on M, then $\mathrm{id} \times g : [0,1] \times M \to [0,1] \times \mathbb{R}$ is a fold field with $\mathbb{S}(\mathrm{id} \times g) = 1 \in \mathrm{Mor}(\mathbf{Br})$, hence an element in $\mathcal{F}(M)$.

We proceed to define our smooth positive TFT Z. We take the state module to be $Z(M) = \mathrm{Fun}_Q(\mathcal{F}(M))$. (We depart here slightly from the general definition as given in Section 6 and work directly with the pre-state module without imposing the constraint equation (14).) Let W^n be a cobordism from M to N. We shall define the *state sum*, $Z_W \in Z(M) \widehat{\otimes} Z(N)$. For any closed smooth manifold X, we define an equivalence relation on the collection of smooth maps of the form $[a,b] \times X \to \mathbb{C}$ for some real numbers $a < b$. Two such maps $f : [a,b] \times X \to \mathbb{C}$ and $f' : [a',b'] \times X \to \mathbb{C}$ are equivalent, written $f \approx f'$, if and only if there exists a diffeomorphism $\xi : [a,b] \to [a',b']$ with $\xi(a) = a'$ such that $f(t,x) = f'(\xi(t),x)$ for all $(t,x) \in [a,b] \times X$. Let $\varepsilon_W > 0$ be a cylinder scale of W. Given a boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, we shall write

$$\begin{aligned}\mathcal{F}(W; f_M, f_N) = \{F \in \mathcal{F}(W) \mid \exists \varepsilon(k), \varepsilon'(k) \in (0, \varepsilon_W) :\\ F|_{[0,\varepsilon(k)] \times M(k)} \approx f_M(k),\ F|_{[1-\varepsilon'(k),1] \times N(k)} \approx f_N(k), \forall k\}.\end{aligned}$$

On (f_M, f_N) the state sum Z_W is then defined as in Section 6, that is,

$$Z_W(f_M, f_N) = \sum_{F \in \mathcal{F}(W; f_M, f_N)} T_W(F).$$

Once more we stress that this sum uses the infinite summation law of the complete semiring Q and thus yields a well-defined element of Q. Using the methods developed in the present paper, as well as certain techniques from differential topology, one can prove that the assignment Z is a positive topological field theory, with the understanding that we do not claim pseudo-isotopy invariance of induced maps on the full state module, since we did not impose the corresponding constraint on states in this brief exposition. The gluing formula of Theorem 6.10 holds. Diffeomorphism invariance is understood to apply to time consistent diffeomorphisms: A diffeomorphism $\Phi : W \to W'$ is said to be *time consistent*, if it sends time slices W_t to time slices W'_τ. Thus our invariant perceives at most the time function of an embedding and therefore depends only weakly (if at all) on the embedding of W.

For cobordisms of dimension $n \geq 4$, the value of the state sum on a given boundary condition is a rational function of the loop-variable q. In fact, the denominator turns out to be universal (independent of the cobordism), whence all the information is contained in the polynomial numerator. These theorems lie well outside the scope of the present paper and will be explained elsewhere.

While the invariant constructed above is simple enough to be computable by a gluing formula, it is on the other hand subtle enough to detect exotic smooth structures, as we shall now outline. Let Σ^7 be a 7-dimensional exotic smooth sphere, for example the Milnor sphere. Thus Σ^7 is a smooth manifold homeomorphic, but not diffeomorphic, to the standard sphere S^7. On S^7, there is of course a Morse function with precisely 2 critical points, that is, a map $S^7 \to \mathbb{R}$ with precisely one maximum point and one minimum point. For $n > 4$, every n-dimensional exotic sphere is diffeomorphic to a twisted sphere and every twisted sphere has Morse number 2. Thus on Σ^7 there is also a Morse function with precisely 2 critical points, a maximum point and a minimum point. For two closed n-manifolds let $\mathrm{Cob}(M,N)$ denote the collection of all oriented (embedded) cobordisms W^{n+1} from M to N. The collection $\mathrm{Cob}(S^7,\Sigma^7)$ is not empty, since $\Omega_7^{\mathrm{SO}} = 0$. Let $f_S : S^7 \to \mathbb{R}$, $f_\Sigma : \Sigma^7 \to \mathbb{R}$ be two Morse functions, each with precisely 2 critical points. Note that f_S and f_Σ are both excellent. Let $\overline{f}_S$ and $\overline{f}_\Sigma$ be the suspensions of f_S and f_Σ, respectively, that is,

$$\overline{f}_S = \mathrm{id}_I \times f_S : I \times S^7 \longrightarrow I \times \mathbb{R} \subset \mathbb{C},\ \overline{f}_\Sigma = \mathrm{id}_I \times f_\Sigma : I \times \Sigma^7 \longrightarrow I \times \mathbb{R} \subset \mathbb{C}.$$

Then $\overline{f}_S$ and $\overline{f}_\Sigma$ are fold fields with $\mathbb{S}(\overline{f}_S) = 1$, $\mathbb{S}(\overline{f}_\Sigma) = 1$, and hence define elements $\overline{f}_S \in \mathcal{F}(S^7)$, $\overline{f}_\Sigma \in \mathcal{F}(\Sigma^7)$, see also Remark 10.2. For any closed, smooth manifold M homeomorphic to a sphere, let $C_2(M)$ denote the space of all $\overline{f}_M \in \mathcal{F}(M)$, which are the suspension of a Morse function $f_M : M \to \mathbb{R}$ with precisely two critical points. A cobordism $W \in \mathrm{Cob}(S^7,M^7)$ has a state sum $Z_W \in Z(S^7)\widehat{\otimes} Z(M^7)$, $Z_W : \mathcal{F}(S^7) \times \mathcal{F}(M^7) \to Q$, so we can evaluate on the boundary condition $(\overline{f}_S,\overline{f}_M) \in \mathcal{F}(S^7) \times \mathcal{F}(M^7)$. We get $Z_W(\overline{f}_S,\overline{f}_M) \in Q$ and shall consider the coboundary aggregate invariant

$$\mathfrak{A}(M^7) = \sum_{\overline{f}_M \in C_2(M^7)} \sum_{W \in \mathrm{Cob}(S^7,M^7)} Z_W(\overline{f}_S,\overline{f}_M) \in Q.$$

Using the concept of a Stein factorization for special generic maps and results of O. Saeki [Sae02], together with index arguments (which would break down if cusp singularities were present), we can prove:

Theorem 10.3. *If Σ^7 is an exotic sphere not diffeomorphic to S^7, then the invariant $\mathfrak{A}(\Sigma^7)$ is a multiple of q.*

For the standard sphere, it is easy to see that $\mathfrak{A}(S^7) = 1_{V\otimes V} + r(q)$ for some power series $r(q) \in Q$. Then, using that Y, induced by the duality structure (i,e), is faithful on loops, one can show that $\mathfrak{A}(\Sigma^7) \neq \mathfrak{A}(S^7)$ in the semiring Q. We conclude that the coboundary aggregate invariant $\mathfrak{A}(-)$ distinguishes Σ^7 from S^7. Dimension 7 plays no distinguished role in the above arguments and was chosen both for historical reasons and in order to make the exposition more concrete.

11. Further Examples

11.1. **The Pólya Enumeration TFT.** Not all applications of positive TFTs need be topological. For instance, Pólya's theory [Pól37] of counting colored configurations modulo symmetries can be recreated within the framework of positive topological field theories by making a judicious choice of monoidal category $\mathbf{C}$, fields $\mathcal{F}$ and action functional $\mathbb{T}$. One can then derive Pólya's enumeration formula by interpreting its terms as state sums and applying our theorems on state sums. Pólya theory has a wide range of applications, among them chemical isomer enumeration, investigation of crystal structure, applications in graph theory and statistical mechanics. This also proves Burnside's lemma on orbit counting from TFT formulae. The set $C = \{1,\chi,\mu\}$ becomes a commutative monoid $(C,\cdot,1)$ by setting

$$\chi^2 = \mu,\ \chi \cdot \mu = \mu,\ \mu^2 = \mu.$$

(Associativity is readily verified.) By Lemma 4.6, $(C,\cdot,1)$ determines a small strict monoidal category $\mathbf{C} = \mathbf{C}(C)$. Suppose that a finite group G acts on a finite set X (from the left). Let Y be another finite set, whose elements we think of as "colors". Then G acts on the set $W = \mathrm{Fun}_Y(X)$ of functions $w : X \to Y$ from the right by $(wg)(x) = w(gx)$. We interpret the elements of W as colorings of X. To form our positive TFT, we also interpret W and its subsets as $0 = (-1+1)$-dimensional bordisms (whose boundary is necessarily empty). Given a subset $W' \subset W$, i.e. a codimension 0 submanifold, we define a set $\mathcal{E}(W')$ over it by

$$\mathcal{E}(W') = \{(w,g) \in W' \times G \mid g \in G_w\},$$

where $G_w \subset G$ denotes the isotropy subgroup of G at w. There is an inclusion $\mathcal{E}(W') \subset \mathcal{E}(W)$. We define the fields on W' to be $\mathcal{F}(W') = \mathrm{Fun}_{\mathbb{B}}(\mathcal{E}(W'))$. Since we are working in an equivariant context, axiom (THOMEO) has to be restricted to equivariant homeomorphisms. Such a homeomorphism $\phi : W' \to W''$ induces a bijection $\mathcal{E}(\phi) : \mathcal{E}(W') \to \mathcal{E}(W'')$ by $\mathcal{E}(\phi)(w,g) = (\phi(w), g)$. Define the action of ϕ on fields by

$$\phi^* = \mathrm{Fun}_{\mathbb{B}}(\mathcal{E}(\phi)) : \mathcal{F}(W'') = \mathrm{Fun}_{\mathbb{B}}(\mathcal{E}(W'')) \longrightarrow \mathrm{Fun}_{\mathbb{B}}(\mathcal{E}(W')) = \mathcal{F}(W').$$

The action functional on a field $F \in \mathcal{F}(W')$ is by definition

$$\mathbb{T}_{W'}(F) = \begin{cases} \mathrm{id}_I, & \text{if } F \text{ is identically } 0, \\ \chi, & \text{if } F \text{ is the characteristic function of some element,} \\ \mu, & \text{otherwise.} \end{cases}$$

Let $S = \mathbb{N}^\infty$ be the complete semiring of Example 2.1. The cardinality of a set A is denoted $|A|$. Let Z be the positive TFT associated with $S, \mathbf{C}, \mathcal{F}$ and $\mathbb{T}$. Let W/G denote the orbit space and let $\mathcal{O} \in W/G$ be an orbit. Then the state sum of $\mathcal{O}$ evaluated at χ is $Z_{\mathcal{O}}(\chi) = \sum_{F \in \mathcal{F}(\mathcal{O})} T_{\mathcal{O}}(F)(\chi) = |\mathcal{E}(\mathcal{O})| = |G|$. The space W can be written as a disjoint union $W = \bigsqcup_{\mathcal{O} \in W/G} \mathcal{O}$. Applying Theorem 6.8 to this decomposition shows that

$$Z_W = \widehat{\bigotimes}_m \{Z_{\mathcal{O}} \mid \mathcal{O} \in W/G\}.$$

Evaluating this on χ, we get $Z_W(\chi) = |W/G| \cdot |G|$. On the other hand,

$$Z_W(\chi) = \sum_{F \in \mathcal{F}(W)} T_W(F)(\chi) = \sum_{w \in W} |G_w| = \sum_{g \in G} |W^g|,$$

where $W^g = \{w \in W \mid wg = w\}$. Thinking of g as a permutation of X, g has a unique cycle decomposition. Let $c(g)$ be the number of cycles. Then $|W^g| = |\mathrm{Fun}_Y(X)^g| = |Y|^{c(g)}$ and we arrive at

$$|\mathrm{Fun}_Y(X)/G| = \frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)},$$

which is Pólya's enumeration theorem. This application suggests that positive TFTs may be instrumental in solving other types of combinatorial problems as well.

11.2. **The Signature TFT.** Suppose that $n+1$ is divisible by 4, say $n+1 = 4k$, and assume that bordisms W are oriented. The *intersection form* of W is the symmetric bilinear form

$$H^{2k}(W, \partial W; \mathbb{R}) \times H^{2k}(W, \partial W; \mathbb{R}) \longrightarrow \mathbb{R}$$

given by evaluation of the cup product on the fundamental class $[W] \in H_{4k}(W, \partial W)$, $(x,y) \mapsto \langle x \cup y, [W] \rangle$. (If the boundary of W is empty, this form is nondegenerate.) The *signature* of W, $\sigma(W)$, is the signature of this bilinear form. Suppose that $W = W' \cup_N W''$ is obtained by gluing along N the bordism W' with outgoing boundary N to the bordism W'' whose incoming boundary is also N. The orientation of W is to restrict to the orientations of W' and W''. Then *Novikov additivity* asserts that

$$\sigma(W) = \sigma(W') + \sigma(W''),$$

see [Nov70, p. 154] and [Wal69]. (Novikov and Rohlin were actually interested in defining Pontrjagin-Hirzebruch classes modulo a prime p. The additivity property for the signature enabled them to find such a definition.) The proof of Novikov additivity is provided in [AS68, Prop. (7.1), p. 588]. It is important here that N be closed as a manifold. If W', W'' are allowed to have corners and one glues along a manifold with boundary $(N, \partial N)$, then the signature is generally non-additive but can be calculated using a formula of Wall, which contains a Maslov triple index correction term. More recently, Novikov has pointed out that his additivity property is equivalent to building a nontrivial topological quantum field theory. Let us indicate a precise construction of such a *signature TFT* Z^{sign} using the framework of the present paper. Since the signature can be negative, this example shows, prima facie paradoxically, that invariants which require additive inverses can also often be expressed by positive TFTs. To do this, one exploits that the monoidal category $\mathbf{C}$ can be quite arbitrary.

By Lemma 4.6, the additive monoid (group) $(\mathbb{Z}, +, 0)$ of integers determines a small strict monoidal category $\mathbf{Z}$. A system $\mathcal{F}$ of fields is given by admitting only a single unique field on each manifold, that is, by taking $\mathcal{F}(W) = \{\star\}$, $\mathcal{F}(M) = \{\star\}$, where $\star$ denotes a single element, which we may interpret as the unique map to a point. A system $\mathbb{T}$ of $\mathbf{Z}$-valued action exponentials on oriented bordisms W is given by

$$\mathbb{T}_W(\star) = \sigma(W).$$

Note that $\mathbb{T}_{\varnothing}(\star) = \sigma(\varnothing) = 0 = \mathrm{id}_I$. In axiom (TDISJ) it is of course now assumed that $W \sqcup W'$ is oriented in agreement with the orientations of W and W', similarly for axiom (TGLUE). By Novikov additivity, axiom (TGLUE) is satisfied. Let Z^{sign} be the positive TFT associated to $(\mathcal{F}, \mathbb{T})$. If a is any integer (a morphism in $\mathbf{Z}$), then

$$Z^{\mathrm{sign}}_W(\star)_{II}(a) = T_W(\star)_{II}(a) = \begin{cases} 1, & \text{if } a = \mathbb{T}_W(\star) = \sigma(W) \\ 0, & \text{otherwise.} \end{cases}$$

So the signature state sum on a morphism is a Kronecker delta function,

$$Z^{\mathrm{sign}}_W(\star)_{II}(a) = \delta_{a,\sigma(W)}.$$

11.3. **A Twisted Signature TFT.** The signature TFT can be twisted by allowing nontrivial fields, as we shall now explain. Let $n+1$ be the TFT-dimension and F be a closed, oriented, topological manifold whose dimension is such that $n+1+\dim F$ is divisible by 4. Let G be a topological group acting continuously on F by orientation preserving homeomorphisms. Let $EG \to BG$ be the universal principal G-bundle. The associated fiber bundle $E \to BG$ with fiber F is given by the total space $E = EG \times_G F$, i.e. the quotient of $EG \times F$ by the diagonal action of G. The projection is induced by $EG \times F \to EG \to BG$. Let W be an oriented $(n+1)$-dimensional bordism with boundary ∂W. Principal G-bundles over W have the form of a pullback f^*EG for some continuous map $f : W \to BG$. The associated F-bundle $p : f^*E \to W$ is a compact manifold with boundary $(f|_{\partial W})^*E$. (Note that the bundle f^*E is canonically isomorphic as a bundle to $(f^*EG) \times_G F$.) This manifold has a canonical orientation. We shall again use the strict monoidal category $\mathbf{Z}$ of integers, as introduced in the previous example. A system $\mathcal{F}$ of fields is given by

$$\mathcal{F}(W) = \{f : W \to BG \mid f \text{ continuous}\},$$

similarly for closed n-manifolds M. A system $\mathbb{T}$ of $\mathbf{Z}$-valued action exponentials on oriented bordisms W is given by the signature of the F-bundle pulled back from BG under

f,

$$\mathbb{T}_W(f) = \sigma(f^*E).$$

The positive TFT $\widetilde{Z}^{\mathrm{sign}}$ associated to $(\mathcal{F}, \mathbb{T})$ is the *F-twisted signature TFT*. If $f^*E \to \partial W$ is an F-bundle over the boundary, given by a map $f : \partial W \to BG$, and a any integer (a morphism in $\mathbf{Z}$), then the F-twisted signature state sum $\widetilde{Z}^{\mathrm{sign}}(f)$ is the counting function whose value on a is

$$\widetilde{Z}^{\mathrm{sign}}_W(f)_{II}(a) = \sum_A 1,$$

where A is the set of all $F : W \to BG$ extending f such that $\sigma(F^*E) = a$. So roughly, the signature state sum on a morphism a "counts" those F-bundles over W that extend $f^*E \to \partial W$ and have signature a. Naturally, since one is not summing over distinct isomorphism types of F-bundles, that is, distinct homotopy classes of maps $W \to BG$, one generally picks up either no summand or uncountably many. So, as pointed out before, what may primarily be of interest is the zero/nonzero-pattern contained in the invariant $\widetilde{Z}^{\mathrm{sign}}_W$, not the actual value in S.

11.4. **Relation to Number Theoretic Quantities.** Certain number theoretic quantities, such as arithmetic functions, can be rendered as state sums of a positive TFT. To do this, encode natural numbers as 0-dimensional manifolds roughly by viewing the former as finite multisets of prime numbers and prime numbers as points. Homeomorphisms are simply bijections of multisets, i.e. bijections of the underlying sets which preserve multiplicities. The multiplicativity of arithmetic functions on coprime integers can then be deduced from the multiplicativity of state sums on disjoint manifolds. (Disjointness is understood inside the universal multiset of all prime numbers.) The key observation is that the set of divisors of a natural number displays exactly the same characteristics as fields on manifolds according to Definition 5.1. Space constraints do not permit us to provide details.

References

[AS68] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: III*, Ann. of Math. **87** (1968), 546 – 604.

[Ati88] M. F. Atiyah, *Topological quantum field theory*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), 175 – 186.

[Ban13] M. Banagl, *The tensor product of function semimodules*, Algebra Universalis **70** (2013), DOI: 10.1007/s00012-013-0249-9.

[BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073 – 6105.

[Bra37] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Annals of Math. **38** (1937), no. 4, 857 – 872.

[Bro62] M. Brown, *Locally flat embeddings of topological manifolds*, Ann. of Math. **75** (1962), 331 – 341.

[BS11] John C. Baez and Mike Stay, *Physics, topology, logic and computation: a Rosetta stone*, in: New Structures for Physics, Lect. Notes in Physics, vol. 813, Springer, Berlin, 2011, pp. 95 – 174.

[Con71] J. H. Conway, *Regular algebra and finite machines*, Chapman and Hall, London, 1971.

[DW90] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), no. 2, 393 – 429.

[Eil74] S. Eilenberg, *Automata, languages, and machines*, Pure and Applied Mathematics, vol. A, Academic Press, 1974.

[Fre92] D. S. Freed, *Lectures on topological quantum field theory*, Lectures held at GIFT Summer School on Recent Problems in Mathematical Physics, Salamanca, Spain, 1992, 1992.

[Gol85] Martin Goldstern, *Vervollständigung von Halbringen*, Diplomarbeit, Technische Universität Wien, 1985.

[Gol99] J. S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, 1999.

[Gro55] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the Amer. Math. Soc., no. 16, Amer. Math. Soc., 1955.

[GW96] J. S. Golan and H. Wang, *On embedding in complete semirings*, Comm. Algebra **24** (1996), no. 9, 2945 – 2962.

[Hir76] M. W. Hirsch, *Differential topology*, Grad. Texts in Math., no. 33, Springer Verlag, 1976.

[Jan59] J. P. Jans, *On Frobenius algebras*, Annals of Math. **69** (1959), 392 – 407.

[Kar92] G. Karner, *On limits in complete semirings*, Semigroup Forum **45** (1992), 148 – 165.

[Kas95] Ch. Kassel, *Quantum groups*, Grad. Texts in Math., no. 155, Springer Verlag, 1995.

[Kat97] Y. Katsov, *Tensor products and injective envelopes of semimodules over additively regular semirings*, Algebra Colloquium **4** (1997), no. 2, 121 – 131.

[Kat04] ______, *Toward homological characterization of semirings: Serre's conjecture and Bass' perfectness in a semiring context*, Algebra Universalis **52** (2004), 197 – 214.

[Kir10] P. Kirk, *The impact of QFT on low-dimensional topology*, Geometric and Topological Methods for Quantum Field Theory (H. Ocampo, E. Pariguan, and S. Paycha, eds.), Cambridge University Press, 2010, pp. 1 – 53.

[Kro88] D. Krob, *Monoides et semi-anneaux continus*, Semigroup Forum **37** (1988), 59 – 78.

[LMS99] G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Tensor products of idempotent semimodules. An algebraic approach*, Math. Notes **65** (1999), no. 4, 479 – 489.

[LMS01] ______, *Idempotent functional analysis: An algebraic approach*, Math. Notes **69** (2001), no. 5, 696 – 729.

[MR94] G. Masbaum and C. Rourke, *Model categories and topological quantum field theories*, Turkish J. Math. **18** (1994), 70 – 80.

[Nov70] S. P. Novikov, *Pontrjagin classes, the fundamental group and some problems of stable algebra*, Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), Springer, New York, 1970, pp. 147 – 155.

[Pól37] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937), no. 1, 145 – 254.

[QF93] F. Quinn and D. S. Freed, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys. **156** (1993), no. 3, 435 – 472.

[Qui95] F. Quinn, *Lectures on axiomatic topological quantum field theory*, Geometry and Quantum Field Theory (D. S. Freed and K. K. Uhlenbeck, eds.), vol. 1, Amer. Math. Soc., 1995, pp. 323 – 459.

[Sae02] O. Saeki, *Cobordism groups of special generic functions and groups of homotopy spheres*, Jpn. J. Math. **28** (2002), 287 – 297.

[Sch50] R. Schatten, *A theory of cross-spaces*, Annals of Math. Studies, no. 26, Princeton University Press, 1950.

[Sch01] P. Schauenburg, *Turning monoidal categories into strict ones*, New York J. Math. **7** (2001), 257 – 265.

[Tak82] M. Takahashi, *On the bordism categories III: Functors Hom and $\otimes$ for semimodules*, Math. Sem. Notes Kobe Univ. **10** (1982), 211 – 236.

[Tre67] F. Treves, *Topological vector spaces, distributions and kernels*, Pure and Applied Mathematics, no. 25, Academic Press, New York - London, 1967.

[Tur89] V. G. Turaev, *Operator invariants of tangles, and R-matrices*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 5, 1073 – 1107, English transl.: Math. USSR-Izv. 35 (1990), no. 2, 411 - 444.

[Tur10] V. Turaev, *Homotopy quantum field theory*, EMS Tracts in Mathematics, vol. 10, European Math. Soc., Zürich, 2010.

[Wal69] C. T. C. Wall, *Non-additivity of the signature*, Invent. Math. **7** (1969), 269 – 274.

[Wal06] K. Walker, *TQFTs*, http://canyon23.net/math/tc.pdf, 2006.

[Wen88] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Annals of Math. **128** (1988), no. 1, 173 – 193.

[Wit89] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), 351 – 399.

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