

TOPOLOGICAL GYSIN COHERENCE FOR ALGEBRAIC CHARACTERISTIC CLASSES OF SINGULAR SPACES

MARKUS BANAGL, JÖRG SCHÜRMANN, AND DOMINIK J. WRAZIDLO

ABSTRACT. Brasselet, the second author and Yokura introduced Hodge-theoretic Hirzebruch-type characteristic classes $IT_{1,*}$, and conjectured that they are equal to the Goresky-MacPherson L -classes for pure-dimensional compact complex algebraic varieties. In this paper, we show that the framework of Gysin coherent characteristic classes of singular complex algebraic varieties developed by the first and third author in previous work applies to the characteristic classes $IT_{1,*}$. In doing so, we prove the ambient version of the above conjecture for subvarieties in a Grassmannian. Since the homology of Schubert subvarieties injects into the homology of the ambient Grassmannian, this implies the conjecture for all Schubert varieties in a Grassmannian. We also study other algebraic characteristic classes such as Chern classes and Todd classes (or their variants for the intersection cohomology sheaves) within the framework of Gysin coherent characteristic classes.

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1. INTRODUCTION

We show that various algebraic characteristic classes for singular spaces fit into the framework of Gysin coherent characteristic classes of singular complex algebraic varieties that was developed by the first and the third author in [11]. This framework formalizes Verdier-Riemann-Roch type formulae with respect to Gysin restriction arising from transverse intersection with smooth varieties. In particular, we show that the framework applies to the Hodge-theoretic intersection Hirzebruch characteristic classes $IT_{1,*}$ introduced by the second author jointly with Brasselet and Yokura in [16]. The classes $IT_{1,*}$ are conjectured in [16, Remark 5.4] to be equal to the topological L -classes of Goresky and MacPherson for pure-dimensional compact complex algebraic varieties. By applying the uniqueness theorem for Gysin coherent characteristic classes of [11] to both the classes $IT_{1,*}$ and the Goresky-MacPherson L -classes, we conclude that both classes coincide on subvarieties of Grassmannians after push-forward into the homology of the ambient Grassmannian. Since the homology of Schubert subvarieties injects into the homology of an ambient Grassmannian, this implies the conjecture for all Schubert varieties in a Grassmannian.

The notion of Gysin coherent characteristic classes is recalled in Section 6. This notion was introduced in [11] with respect to a family \mathcal{X} of inclusions $i: X \rightarrow W$ of compact irreducible subvarieties of smooth pure-dimensional projective complex algebraic varieties W which satisfy primarily an analog of the Kleiman-Bertini transversality theorem with respect to a suitable notion of transversality. A Gysin coherent characteristic class is a pair $cl = (cl^*, cl_*)$ consisting of a function cl^* that assigns to every inclusion $f: M \rightarrow W$ of a smooth closed subvariety $M \subset W$ in a smooth variety W a normalized element $cl^*(f) \in H^*(M; \mathbb{Q})$, and a function cl_* that assigns to every inclusion $i: X \rightarrow W$ of a compact possibly singular subvariety of a smooth variety an element $cl_*(i) \in H_*(W; \mathbb{Q})$ whose highest non-trivial homogeneous component is the ambient fundamental class of X in W such that the Gysin restriction formula

$$f^! cl_*(i) = cl^*(f) \cap cl_*(M \cap X \subset M)$$

holds for all i contained in \mathcal{X} . Here, $f^!$ denotes the topological Gysin homomorphism on singular rational homology. Furthermore, we require that cl_* is multiplicative under products, that cl^* and cl_* transform naturally under isomorphisms of ambient smooth varieties, and that cl_* is natural with respect to inclusions in larger ambient smooth varieties. It was shown in [11] that the Goresky-MacPherson L -class gives rise to a Gysin coherent characteristic class.

In this paper, the focus lies on application of the Gysin coherence framework to the Hodge-theoretic intersection Hirzebruch characteristic classes $IT_{1,*}$ introduced by the second author jointly with Brasselet and Yokura in [16]. Section 2 provides a brief outline of the theory of Hodge-theoretic characteristic classes on singular algebraic varieties. For an introduction to algebraic characteristic classes of singular spaces via mixed Hodge theory in the complex algebraic context, see the second author's expository paper [59]. For an introduction to topological characteristic classes of singular spaces, see [6].

Let $L^*(\mathbf{v})$ denote the Hirzebruch L -class of a topological vector bundle \mathbf{v} . The main result of the present paper is Theorem 7.3, which can be stated as follows.

Theorem. *The pair $\mathcal{L} = (\mathcal{L}^*, \mathcal{L}_*)$ defined by $\mathcal{L}^*(f) = L^*(\mathbf{v}_f)$ for every inclusion $f: M \rightarrow W$ of a smooth closed subvariety $M \subset W$ in a smooth complex algebraic variety W with normal bundle \mathbf{v}_f , and by $\mathcal{L}_*(i) = i_* IT_{1,*}(X)$ for every inclusion $i: X \rightarrow W$ of a compact irreducible possibly singular subvariety $X \subset W$ in a smooth variety W , is a Gysin coherent characteristic class.*

In Section 4, we discuss the main ingredient for the proof, which is a Verdier-Riemann-Roch type formula derived by the first author in [8] for the behavior of the Hodge-theoretic intersection Hirzebruch characteristic classes $IT_{1,*}$ under algebraic Gysin restriction with respect to normally nonsingular embeddings of singular spaces. In order for this formula to fit the axioms of Gysin coherent characteristic classes, we need to apply it in a transverse setup (see Theorem 4.15), where Tor-independence turns out to be the correct notion of transversality for subvarieties of an ambient smooth variety. To achieve Tor-independence generically, we use the Kleiman-Bertini type theorem of Miller and Speyer [47]. For the Gysin restriction formula to be valid in the transverse setup, we need to handle some technical issues regarding the behavior of Whitney stratifications under blow-up of complex manifolds along transverse submanifolds, see Section 3. Finally, to establish the topological Gysin coherence axiom for the algebraic characteristic class $IT_{1,*}$, we compare the algebraic Gysin map with the topological Gysin map in Section 5, and find that they coincide at least on algebraic cycles (see Theorem 5.10).

In [16, Remark 5.4], it was conjectured that the equality $L_*(X) = IT_{1,*}(X)$ holds for all pure-dimensional compact complex algebraic varieties X . Since the Goresky-MacPherson L -class is a Gysin coherent characteristic class, the uniqueness theorem for such classes implies that both the classes $IT_{1,*}$ and the Goresky-MacPherson L -classes coincide on irreducible subvarieties of Grassmannians after push-forward into the homology of the ambient Grassmannian (see Theorem 7.5). That is, the ambient version of the Brasselet-Schürmann-Yokura conjecture holds:

Theorem. (*Ambient Brasselet-Schürmann-Yokura Conjecture.*) *We have*

$$i_*L_*(X) = i_*IT_{1,*}(X)$$

for all inclusions $i: X \rightarrow G$ of compact irreducible subvarieties in Grassmannians G .

Since the homology of Schubert varieties injects into the homology of the Grassmannian, we obtain Corollary 7.6, which states the following.

Corollary. *The equality $L_*(X) = IT_{1,*}(X)$ holds for all Schubert varieties X in a Grassmannian. More generally, the conjecture holds for all compact irreducible varieties X that admit an H_* -injective embedding into a Grassmannian.*

In joint work with Cappell, Maxim, and Shaneson, the second author proved the conjecture in [22, Cor. 1.2] for orbit spaces $X = Y/G$, with Y a projective G -manifold and G a finite group of algebraic automorphisms. They also showed the conjecture for certain complex hypersurfaces with isolated singularities which are assumed rational homology manifolds [21, Theorem 4.3]. The conjecture holds for simplicial projective toric varieties as shown by Maxim and the second author [46, Corollary 1.2(iii)]. In [7], the first author proved that his extension of the Goresky-MacPherson L -class to oriented pseudomanifolds that possess Lagrangian structures along strata of odd codimension, introduced in [5], transfers to the L -class of any finite covering space (see Theorem 3.14 in [7]). He deduced from this that the conjecture holds for normal connected complex projective 3-folds X that have at worst canonical singularities, trivial canonical divisor, and $\dim H^1(X; \mathcal{O}_X) > 0$. If ξ is an oriented PL block bundle over a closed PL Witt space B (e.g. a pure-dimensional complex algebraic variety) with closed d -dimensional PL manifold fiber and total space X , then $\xi^!L_*(B) = L^*(\nu_\xi) \cap L_*(X)$ under the block bundle transfer $\xi^!: H_*(B; \mathbb{Q}) \rightarrow H_{*+d}(X; \mathbb{Q})$ associated to ξ , see [9]. Here, ν_ξ is the oriented stable vertical normal PL microbundle of ξ . In fact, it is shown in [10] that the KO-theoretic block bundle transfer $\xi^!: KO_*(B) \otimes \mathbb{Z}[\frac{1}{2}] \rightarrow KO_{*+d}(X) \otimes \mathbb{Z}[\frac{1}{2}]$ sends the Siegel-Sullivan orientation $\Delta(B)$ to $\Delta(X)$. The Siegel-Goresky-MacPherson

L -class can be recovered from $\Delta(-)$ by applying the Pontrjagin character. Note that transfer does not generally commute with localization of homotopy theoretic spectra if the Thom classes defining these transfers are not aligned. The aforementioned cases of the conjecture concern rational homology manifolds. Fernández de Bobadilla and Pallarés [28] proved the conjecture for all projective complex algebraic varieties that are rational homology manifolds. In joint work with Saito [29], they also gave a different proof for compact instead of projective complex varieties, using the theory of mixed Hodge modules. As Schubert varieties in a Grassmannian are generally singular enough so as not to be rational homology manifolds, the results of the present paper, at the time of writing in Oct. 2023, were the first known cases of the conjecture for varieties that are not rational homology manifolds. Very recently (Nov. 2025), many new cases of the conjecture, including all toric and Schubert varieties, have been verified by completely different methods in the preprint [1].

Furthermore, we shall clarify how other algebraic characteristic classes such as Chern classes (see Section 8) and Todd classes (see Section 9) (or their variants for the intersection cohomology sheaves) fit into the framework of Gysin coherence. In Theorem 9.1, we show that Todd classes are Gysin coherent. The proof is similar to that of our main result, but is based on the Verdier-Riemann-Roch formula for the Todd class transformation τ_* , which was conjectured by Baum-Fulton-MacPherson in [12, p. 137], and proved by Verdier [63, p. 214, Theorem 7.1]. On the other hand, Theorem 8.6 shows that Chern classes are Gysin coherent, which exploits the second author's Verdier-Riemann-Roch type theorem (see [60]) for the behavior of the Chow homology Chern class transformation $c_* : F(X) \rightarrow A_*(X)$ on complex algebraically constructible functions under refined Gysin maps associated to transverse intersections in a microlocal context.

We conclude by mentioning that the normally nonsingular expansions derived in [11, Theorem 7.1] provide a systematic method for the recursive computation of Gysin coherent characteristic classes in ambient Grassmannians in terms of genera of explicitly constructed characteristic subvarieties. For the classes discussed in this paper, the computational consequences e.g. in the case of Schubert varieties in a Grassmannian remain open for future study. Chern classes of Schubert varieties in Grassmannians were computed by Aluffi and Mihalcea in [2]. Moreover, an algorithm for the computation of Chern classes for Schubert cells in a generalized flag manifold is provided in [3]. For an extension to the equivariant setting, as well as applications to positivity of non-equivariant Chern classes and related classes, we refer the reader to [4].

Notation. Regarding singular cohomology, singular homology and Borel-Moore homology, we work with rational coefficients and will write these groups as $H^*(-)$, $H_*(-)$, and $H_*^{\text{BM}}(-)$.

2. HODGE-THEORETIC CHARACTERISTIC CLASSES

This section provides a brief outline of the theory of Hodge-theoretic characteristic classes on singular algebraic varieties (compare also Section 5 in [8]). After recalling the motivic Hodge Chern class transformation in Definition 2.1 and the twisted Todd transformation of Baum, Fulton, MacPherson in Definition 2.7, we define the motivic Hirzebruch class transformation in Definition 2.9, and finally the intersection Hirzebruch characteristic class IT_{y*} in Definition 2.11. For a more detailed exposition of the topic, we refer e.g. to the expository paper [59].

For an algebraic variety X , let $K_0^{\text{alg}}(X)$ denote the Grothendieck group of the abelian category of coherent sheaves of \mathcal{O}_X -modules. When there is no danger of confusion with other K -homology groups, we shall also write $K_0(X) = K_0^{\text{alg}}(X)$. Let $K^0(X) = K_{\text{alg}}^0(X)$ denote the

Grothendieck group of the exact category of algebraic vector bundles over X . The tensor product $\otimes_{\mathcal{O}_X}$ induces a *cap product*

$$\cap : K^0(X) \otimes K_0(X) \longrightarrow K_0(X), [E] \cap [\mathcal{F}] = [E \otimes_{\mathcal{O}_X} \mathcal{F}].$$

Thus,

$$(1) \quad - \cap [\mathcal{O}_X] : K^0(X) \longrightarrow K_0(X)$$

sends a vector bundle $[E]$ to its associated (locally free) sheaf of germs of local sections $[E \otimes_{\mathcal{O}_X}]$. If X is smooth, then $- \cap [\mathcal{O}_X]$ is an isomorphism.

Let X be a complex algebraic variety and E an algebraic vector bundle over X . For a nonnegative integer p , let $\Lambda^p(E)$ denote the p -th exterior power of E . The *total λ -class* of E is by definition

$$\lambda_y(E) = \sum_{p \geq 0} \Lambda^p(E) \cdot y^p,$$

where y is an indeterminate functioning as a bookkeeping device. This construction induces a homomorphism $\lambda_y(-) : K_{\text{alg}}^0(X) \longrightarrow K_{\text{alg}}^0(X)[[y]]$ from the additive group of $K^0(X)$ to the multiplicative monoid of the power series ring $K^0(X)[[y]]$. Now let X be a smooth variety, let TX denote its holomorphic tangent bundle and T^*X its holomorphic cotangent bundle. Then $\Lambda^p(T^*X)$ is the vector bundle of holomorphic p -forms on X . Its associated sheaf of sections is denoted by Ω_X^p . Thus

$$[\Lambda^p(T^*X)] \cap [\mathcal{O}_X] = [\Omega_X^p]$$

and hence

$$\lambda_y(T^*X) \cap [\mathcal{O}_X] = \sum_{p=0}^{\dim X} [\Omega_X^p] y^p.$$

Let X be a complex algebraic variety and let $MHM(X)$ denote the abelian category of M. Saito's algebraic mixed Hodge modules on X . Totaro observed in [62] that Saito's construction of a pure Hodge structure on the intersection homology of compact varieties implicitly contains a definition of certain characteristic homology classes for singular algebraic varieties. The following definition is based on this observation and due to Brasselet, Schürmann and Yokura, [16], see also the expository paper [59].

Definition 2.1. *The motivic Hodge Chern class transformation*

$$MHC_y : K_0(MHM(X)) \rightarrow K_0^{\text{alg}}(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

is defined by

$$MHC_y[M] = \sum_{i,p} (-1)^i [\mathcal{H}^i(Gr_{-p}^F DR[M])] (-y)^p.$$

Here, $Gr_p^F DR : D^b MHM(X) \rightarrow D_{\text{coh}}^b(X)$, with $D_{\text{coh}}^b(X)$ the bounded derived category of sheaves of \mathcal{O}_X -modules with coherent cohomology sheaves, denotes the functor of triangulated categories constructed by M. Saito, see [53, §2.3], [57, §1], [54, §3.10], obtained by taking a suitable filtered de Rham complex of the filtered holonomic D -module underlying a mixed Hodge module. The functor $Gr_p^F DR$ is first defined for mixed Hodge modules on smooth X and then extended using locally defined closed embeddings into smooth varieties, the key point being that the graded quotient cohomology sheaves $\mathcal{H}^i Gr_p^F DR(-)$ are independent of the local embeddings and are \mathcal{O}_X -modules. For every $p \in \mathbb{Z}$, these functors induce functors between the associated Grothendieck groups, with $Gr_p^F DR[M] \simeq 0$ for a given M and almost all p .

A flat morphism $f : X \rightarrow Y$ gives rise to a flat pullback $f^* : \text{Coh}(Y) \rightarrow \text{Coh}(X)$ on coherent sheaves, which is exact and hence induces a flat pullback $f_K^* : K_0^{\text{alg}}(Y) \rightarrow K_0^{\text{alg}}(X)$. This applies in particular to smooth morphisms and is then often called smooth pullback. An arbitrary algebraic morphism $f : X \rightarrow Y$ (not necessarily flat) induces a homomorphism

$$f^* : K_0(\text{MHM}(Y)) \longrightarrow K_0(\text{MHM}(X))$$

which corresponds under the forgetful functor $\text{rat} : D^b\text{MHM}(-) \rightarrow D_c^b(-; \mathbb{Q})$ to f^{-1} on constructible complexes of sheaves. (Additional remarks on rat are to be found further below.) We record Schürmann's [59, Cor. 5.11, p. 459]:

Proposition 2.2. (*Verdier-Riemann-Roch for smooth pullbacks.*) *For a smooth morphism $f : X \rightarrow Y$ of complex algebraic varieties, the Verdier Riemann-Roch formula*

$$\lambda_y(T_{X/Y}^*) \cap f_K^* \text{MHC}_y[M] = \text{MHC}_y(f^*[M]) = \text{MHC}_y[f^*M]$$

holds for $M \in D^b\text{MHM}(Y)$, where $T_{X/Y}^*$ denotes the relative cotangent bundle of f .

Let E be a complex vector bundle and let a_i denote the Chern roots of E . In [37], Hirzebruch introduced a cohomological characteristic class

$$T_y^*(E) = \prod_{i=1}^{\text{rk} E} Q_y(a_i),$$

where y is an indeterminate, coming from the power series

$$Q_y(a) = \frac{a(1+y)}{1 - e^{-a(1+y)}} - ay \in \mathbb{Q}[y][[a]].$$

If R is an integral domain over \mathbb{Q} , then a power series $Q(a) \in R[[a]]$ is called *normalized* if it starts with 1, i.e. $Q(0) = 1$. With $R = \mathbb{Q}[y]$, we have $Q_y(0) = 1$, so $Q_y(a)$ is normalized. For $y = 0$,

$$(2) \quad T_0^*(E) = \prod_{i=1}^{\text{rk} E} \frac{a_i}{1 - e^{-a_i}} = \text{td}^*(E)$$

is the classical Todd class of E , while for $y = 1$,

$$(3) \quad T_1^*(E) = \prod_{i=1}^{\text{rk} E} \frac{a_i}{\tanh a_i} = L^*(E)$$

is the Hirzebruch L -class of the vector bundle E . We shall also need a certain unnormalized version of $Q_y(a)$: Let

$$\tilde{Q}_y(a) = \frac{a(1 + ye^{-a})}{1 - e^{-a}} \in \mathbb{Q}[y][[a]]$$

and set

$$\tilde{T}_y^*(E) = \prod_{i=1}^{\text{rk} E} \tilde{Q}_y(a_i).$$

Note that $\tilde{Q}_y(0) = 1 + y \neq 1$, whence $\tilde{Q}_y(a)$ is unnormalized. The relation

$$(1+y)Q_y(a) = \tilde{Q}_y((1+y)a)$$

implies:

Proposition 2.3. *If E is a complex vector bundle of complex rank r , then for the degree $2i$ components:*

$$\tilde{T}_y^i(E) = (1+y)^{r-i} T_y^i(E).$$

More conceptually, we have the following formula for the unnormalized class:

Proposition 2.4. ([8, Prop. 5.4, p. 1307].) *For any complex vector bundle E , we have*

$$\widetilde{T}_y^*(E) = \text{td}^*(E) \cup \text{ch}^*(\lambda_y(E^*)).$$

Let $\tau_* : K_0(X) \longrightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}$ denote the Todd class transformation of Baum, Fulton, MacPherson. We review, to some extent, construction and properties of this transformation. Let

$$\alpha^* : K_{\text{alg}}^0(X) \longrightarrow K_{\text{top}}^0(X)$$

be the forget map which takes an algebraic vector bundle to its underlying topological vector bundle. Composing with the Chern character, one obtains a transformation

$$\tau^* = \text{ch}^* \circ \alpha^* : K_{\text{alg}}^0(X) \longrightarrow H^{2*}(X; \mathbb{Q}),$$

see [13, p. 180]. Baum, Fulton and MacPherson construct with the use of Bott periodicity a corresponding homological version

$$\alpha_* : K_0^{\text{alg}}(X) \longrightarrow K_0^{\text{top}}(X)$$

for quasi-projective varieties X . Composing with the homological Chern character

$$\text{ch}_* : K_0^{\text{top}}(X) \longrightarrow H_{2*}^{\text{BM}}(X; \mathbb{Q}),$$

where H_*^{BM} denotes Borel-Moore homology, they obtain a transformation

$$\tau_* = \text{ch}_* \circ \alpha_* : K_0^{\text{alg}}(X) \longrightarrow H_{2*}^{\text{BM}}(X; \mathbb{Q}).$$

An algebraic version of this transformation is in fact available for any algebraic scheme over a field and generalizes the Grothendieck Riemann-Roch theorem to singular varieties.

Remark 2.5. Let $A_*(V)$ denote Chow homology of a variety V , i.e. algebraic cycles in V modulo rational equivalence. Then there is a transformation

$$\tau_* : K_0^{\text{alg}}(X) \longrightarrow A_*(X) \otimes \mathbb{Q}$$

such that for a complex algebraic variety X , the diagram

$$\begin{array}{ccc} K_0^{\text{alg}}(X) & & \\ \tau_* \downarrow & \searrow \tau_* & \\ A_*(X) \otimes \mathbb{Q} & \xrightarrow{\text{cl}} & H_{2*}^{\text{BM}}(X; \mathbb{Q}) \end{array}$$

commutes, where cl is the cycle map; see the first commutative diagram on p. 106 of [12, (0.8)]. The construction of τ_* to Chow homology is described in Fulton's book [30, p. 349]. Thus Todd classes are algebraic cycles with rational coefficients that are well-defined up to rational equivalence.

According to [13, Theorem, p. 180], τ_* and τ^* are compatible with respect to cap products, i.e. the diagram

$$\begin{array}{ccc} K^0(X) \otimes K_0(X) & \xrightarrow{\tau^* \otimes \tau_*} & H^*(X; \mathbb{Q}) \otimes H_*^{\text{BM}}(X; \mathbb{Q}) \\ \cap \downarrow & & \downarrow \cap \\ K_0(X) & \xrightarrow{\tau_*} & H_*^{\text{BM}}(X; \mathbb{Q}) \end{array}$$

commutes. Thus, if E is a vector bundle and \mathcal{F} a coherent sheaf on X , then

$$(4) \quad \tau_*([E] \cap [\mathcal{F}]) = \text{ch}^*(E) \cap \tau_*[\mathcal{F}].$$

For smooth X ,

$$\tau_*[\mathcal{O}_X] = \text{td}^*(TX) \cap [X] = T_0^*(TX) \cap [X].$$

So if E is a vector bundle on a smooth variety, then

$$(5) \quad \tau_*([E] \cap [\mathcal{O}_X]) = (\text{ch}^*(E) \cup \text{td}^*(TX)) \cap [X].$$

For locally complete intersection morphisms $f : X \rightarrow Y$, Gysin maps

$$f_{\text{BM}}^* : H_*^{\text{BM}}(Y) \longrightarrow H_{*-2d}^{\text{BM}}(X)$$

have been defined by Verdier [63, §10], and Baum, Fulton and MacPherson [12, Ch. IV, §4], where d denotes the (complex) virtual codimension of f . Thus for a regular closed embedding g , there is a Gysin map g_{BM}^* on Borel-Moore homology, which we shall also write as $g_{\text{alg}}^!$, and for a smooth morphism f of relative dimension r , there is a smooth pullback $f_{\text{BM}}^* : H_*^{\text{BM}}(Y) \rightarrow H_{*+2r}^{\text{BM}}(X)$. Baum, Fulton and MacPherson conjectured and Verdier showed:

Proposition 2.6. (*Verdier-Riemann-Roch for smooth pullbacks.*) *For a smooth morphism $f : X \rightarrow Y$ of complex algebraic varieties and $[\mathcal{F}] \in K_0^{\text{alg}}(Y)$,*

$$\text{td}^*(T_{X/Y}) \cap f_{\text{BM}}^* \tau_*[\mathcal{F}] = \tau_*(f_K^*[\mathcal{F}]).$$

Yokura [64] twisted τ_* by a Hirzebruch-type variable y :

Definition 2.7. *The twisted Todd transformation*

$$\text{td}_{1+y} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \longrightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]$$

is given by

$$\text{td}_{1+y}[\mathcal{F}] := \sum_{k \geq 0} \tau_k[\mathcal{F}] \cdot \frac{1}{(1+y)^k},$$

where the Baum-Fulton-MacPherson transformation τ_* is extended linearly over $\mathbb{Z}[y^{\pm 1}]$, and τ_k denotes the degree $2k$ -component of τ_* .

Remark 2.8. Regarding the transformation τ_* as taking values in Chow groups $A_*(-) \otimes \mathbb{Q}$ (cf. Remark 2.5), the above definition yields a twisted Todd transformation

$$\text{td}_{1+y} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \longrightarrow A_*(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}],$$

which commutes with the Borel-Moore twisted Todd transformation under the cycle map.

The definition of the motivic Hirzebruch class transformation below is due to Brasselet, Schürmann and Yokura [16], see also Schürmann's expository paper [59].

Definition 2.9. *The motivic Hirzebruch class transformation is*

$$\text{MHT}_{y*} := \text{td}_{1+y} \circ \text{MHC}_y : K_0(\text{MHM}(X)) \longrightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}].$$

For the intersection Hodge module IC_X^H on a complex purely n -dimensional variety X , we use the convention

$$IC_X^H := j_{1*}(\mathbb{Q}_U^H[n]),$$

which agrees with [59, p. 444] and [51, p. 345]. Here, $U \subset X$ is smooth, of pure dimension n , Zariski-open and dense, and j_{1*} denotes the intermediate extension of mixed Hodge modules associated to the open inclusion $j : U \hookrightarrow X$. The underlying perverse sheaf is $\text{rat}(IC_X^H) = IC_X$, the intersection chain sheaf, where $\text{rat} : \text{MHM}(X) \rightarrow \text{Per}(X) = \text{Per}(X; \mathbb{Q})$ is the faithful and exact functor that sends a mixed Hodge module to its underlying perverse sheaf. Here,

$\text{Per}(X)$ denotes perverse sheaves on X which are constructible with respect to *some* algebraic stratification of X . This functor extends to a functor $\text{rat} : D^b\text{MHM}(X) \rightarrow D_c^b(X) = D_c^b(X; \mathbb{Q})$ between bounded derived categories. For every object of $D_c^b(X)$ there exists *some* algebraic stratification with respect to which the object is constructible, and these stratifications will generally vary with the object. Recall that a functor F is *conservative*, if for every morphism ϕ such that $F(\phi)$ is an isomorphism, ϕ is already an isomorphism. Faithful functors on balanced categories (such as abelian or triangulated categories) are conservative. According to [55, p. 218, Remark (i)], $\text{rat} : D^b\text{MHM}(X) \rightarrow D_c^b(X)$ is not faithful. We recall Lemma 5.10 from [8]:

Lemma 2.10. *The functor $\text{rat} : D^b\text{MHM}(X) \rightarrow D_c^b(X)$ is conservative.*

Using cones, this lemma also appears embedded in the proof of [22, Lemma 5.3, p. 1752], see also Exercise 11.2.1 in Maxim's book [44]. The module IC_X^H is the unique simple object in the category $\text{MHM}(X)$ which restricts to $\mathbb{Q}_U^H[n]$ over U . As U is smooth and pure n -dimensional, $\mathbb{Q}_U^H[n]$ is pure of weight n . Since the intermediate extension $j_{!*}$ preserves weights, IC_X^H is pure of weight n . There is a duality isomorphism (polarization) $\mathbb{D}_X^H IC_X^H \cong IC_X^H(n)$. Taking rat , this isomorphism induces a self-duality isomorphism

$$\mathbb{D}_X IC_X = \mathbb{D}_X \text{rat} IC_X^H \cong \text{rat} \mathbb{D}_X^H IC_X^H \cong \text{rat} IC_X^H(n) \cong IC_X,$$

if an isomorphism $\mathbb{Q}_U(n) \cong \mathbb{Q}_U$ is chosen.

Definition 2.11. ([16].) The *intersection generalized Todd class* (or *intersection Hirzebruch characteristic class*) is

$$IT_{y*}(X) := MHT_{y*}[IC_X^H[-n]] \in H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y].$$

Remark 2.12. In this definition, we have used that MHT_{y*} contains no negative powers of $1+y$ as shown in [59, p. 465, Proposition 5.21], and that $IT_{y*}(X)$ contains no negative powers of y , as follows from [59, p. 451, Example 5.2].

Later on in the present paper, we will be interested in the specializations $IT_{1,*}(X)$ for $y = 1$, $IT_{0,*}(X)$ for $y = 0$ and $IT_{-1,*}(X)$ for $y = -1$ in Definition 2.11. Taking up expectations formulated by Cappell and Shaneson in [24], $IT_{y*}(X)$ started to be investigated in detail by Cappell, Maxim and Shaneson in [23].

Remark 2.13. The intersection characteristic class $IT_{y*}(X)$ is represented by an algebraic cycle by Remark 2.8.

Remark 2.14. Let $K_0(\text{var}/X)$ denote the motivic relative Grothendieck group of complex algebraic varieties over X . By definition, $K_0(\text{var}/X)$ is the free abelian group generated by isomorphism classes $[f] = [f : Z \rightarrow X]$ of morphisms f to X modulo the additivity relation (or scissor relation)

$$[f] = [f \circ i] + [f \circ j]$$

for a closed inclusion $i : Y \rightarrow X$ with complement $j : U = Z \setminus Y \rightarrow X$. As pointed out in [59, Remark 5.5], composition of the transformations MHC_y and MHT_y with the natural group homomorphism

$$\chi_{Hdg} : K_0(\text{var}/X) \rightarrow K_0(\text{MHM}(X)), \quad [f : Z \rightarrow X] \mapsto [f! \mathbb{Q}_Z^H],$$

(see [59, Corollary 4.10]) yields similar transformations

$$mC_y := MHC_y \circ \chi_{Hdg}, \quad T_{y*} := MHT_y \circ \chi_{Hdg},$$

defined on $K_0(\text{var}/X)$, which are studied in [16].

The intersection Hodge modules are multiplicative under the external product

$$\boxtimes: \mathit{MHM}(X) \times \mathit{MHM}(Y) \rightarrow \mathit{MHM}(X \times Y)$$

as follows (compare also [45, p. 471]).

Proposition 2.15. *Let X and Y be pure-dimensional complex algebraic varieties. Then there is a natural isomorphism $IC_{X \times Y}^H = IC_X^H \boxtimes IC_Y^H$ in $\mathit{MHM}(X \times Y)$.*

Proof. In the category of perverse sheaves, we have $IC_X \boxtimes IC_Y = IC_{X \times Y}$, as a standard verification of the intersection sheaf axioms shows (see [34, p. 121, Section 6.3]). The proposition follows from observing that $IC_X^H \boxtimes IC_Y^H$ is simple in $\mathit{MHM}(X \times Y)$ and restricts to $\mathbb{Q}_{U \times V}^H[\dim X + \dim Y]$, where $U \subset X$ and $V \subset Y$ are dense open smooth subsets. \square

Corollary 2.16. *The intersection generalized Todd class of the product $X \times Y$ of complex algebraic pure-dimensional varieties X, Y satisfies*

$$IT_{y*}(X \times Y) = IT_{y*}(X) \times IT_{y*}(Y) \in H_{2*}^{\text{BM}}(X \times Y) \otimes \mathbb{Q}[y].$$

Proof. By [59, Corollary 5.10, p. 458], the motivic Chern class transformation

$$\mathit{MHC}_y: K_0(\mathit{MHM}(X)) \rightarrow K_0^{\text{alg}}(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

(see Definition 2.1) commutes with the external product, that is,

$$\mathit{MHC}_y([M] \boxtimes [N]) = \mathit{MHC}_y([M]) \boxtimes \mathit{MHC}_y([N])$$

for $M \in D^b \mathit{MHM}(X)$ and $N \in D^b \mathit{MHM}(Y)$. Furthermore, according to [59, p. 462], the Todd class transformation

$$\tau_*: K_0^{\text{alg}}(X) \longrightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}$$

of Baum, Fulton, MacPherson is compatible with the external product in the sense that

$$\tau_*([\mathcal{F}] \boxtimes [\mathcal{G}]) = \tau_*([\mathcal{F}]) \times \tau_*([\mathcal{G}])$$

for $[\mathcal{F}] \in K_0(X)$ and $[\mathcal{G}] \in K_0(Y)$ (see also [30, Example 18.3.1, p. 360, and Example 19.1.9, p. 377], as well as [32, Property (i), p. 122]). The desired identity then follows from Proposition 2.15 by a straightforward computation. \square

Proposition 2.17. *Let $\Phi: X \xrightarrow{\cong} Y$ be an isomorphism of pure-dimensional complex algebraic varieties. Then, $\Phi_* IT_{y*}(X) = IT_{y*}(Y)$ in $H_{2*}^{\text{BM}}(Y) \otimes \mathbb{Q}[y]$.*

Proof. This follows from $\Phi_* IC_X^H = IC_Y^H$, and by naturality of MHT_{y*} . \square

3. BLOW-UP AND TRANSVERSALITY FOR COMPLEX MANIFOLDS

This section studies stratifications and transversality of the strict transform of Whitney stratified sets under blow-ups along smooth centers. We do this for the following reason. In order to show that $IT_{1,*}$ is a Gysin coherent class, one needs a Verdier-Riemann-Roch type formula for the Gysin restriction of $IT_{1,*}(X)$ under regular closed algebraic embeddings $g: Y \hookrightarrow X$. This has been established by the first named author in [8, Theorem 6.30]. There, one assumes that the embedding g is upwardly normally nonsingular. This condition consists of two parts: First, g has to be tight, i.e. the underlying topological embedding is normally nonsingular and the topological normal vector bundle agrees with the algebraic normal bundle. Second, the inclusion of the exceptional divisor in the blow up of $X \times \mathbb{C}$ along $Y \times 0$ is also required to be normally nonsingular. This latter part is relevant because the formula in [8, Theorem 6.30] involves Verdier's algebraic Gysin restriction, which is defined in terms of a specialization map. That map is associated to the deformation to the normal cone, which involves the aforementioned blow up. The detailed analysis, as carried out in [8], of the Hodge

nearby cycle functor applied to the Hodge intersection module on the total space of the deformation, is possible if the exceptional divisor is known to be normally nonsingular. Now, in this section we will show that in a transverse situation, the condition on the exceptional divisor is indeed satisfied (Theorem 3.6). Throughout, let M be a complex submanifold of a complex manifold W .

Lemma 3.1. *Let $S \subset W$ be a complex submanifold that is transverse to $M \subset W$ (in the sense of C^∞ manifolds). Then, the intersection $S \cap M$ is a complex submanifold of W . Moreover, the canonical map $N_{S \cap M} S \rightarrow (N_M W)|_{S \cap M}$ of complex normal bundles is an isomorphism.*

Proof. The proof of the first claim is similar to the real smooth version, but is based on the complex counterpart of the C^∞ implicit function theorem (see e.g. Griffiths-Harris [36, p. 19]). The proof of the second claim uses the definition of the complex normal bundle $N_M W$ as the quotient of the restricted complex tangent bundle $TW|_M$ by the subbundle TM (see e.g. Griffiths-Harris [36, p. 71]). \square

In the following, let $\pi : W' = \text{Bl}_M W \rightarrow W$ denote the blow-up of W along M . Thus, W' is a complex manifold, π is a holomorphic map that is an isomorphism away from M , $\pi| : W' \setminus \pi^{-1}(M) \xrightarrow{\cong} W \setminus M$, and the restriction $\pi| : E \rightarrow M$ of π to the exceptional divisor $E := \pi^{-1}(M) \subset W'$ is isomorphic over M to the projectivization $\mathbb{P}(N_M W) \rightarrow M$ of the complex normal bundle $N_M W \rightarrow M$ of M in W .

The blow-up of W along M can be characterized uniquely up to isomorphism as follows. Given a complex manifold W'_0 and a holomorphic map $\pi_0 : W'_0 \rightarrow W$ that is an isomorphism away from M , and such that the fibers of π_0 over points of M are isomorphic to complex projective $(k-1)$ -space $\mathbb{P}(\mathbb{C}^k)$, where k denotes the codimension of M in W , there is an isomorphism $\Phi : W'_0 \xrightarrow{\cong} W'$ such that $\pi_0 = \pi \circ \Phi$. See Griffiths-Harris [36, p. 604]. Consequently, we have the following

Proposition 3.2. *For every open subset $U \subset W$, the restriction $\pi| : \pi^{-1}(U) \rightarrow U$ is the blow-up of U along the complex submanifold $M \cap U \subset U$.*

The inverse image $\pi^{-1}(Z)$ of a subset $Z \subset W$ is called the *total transform* of Z under π . For any subset $Z \subset W$, the *strict transform* \tilde{Z} of Z under π is the intersection of the total transform $\pi^{-1}(Z)$ with the closure of the inverse image $\pi^{-1}(Z \setminus M)$ in W' . In particular, for a closed subset $Z \subset W$, we note that \tilde{Z} is just the closure of the inverse image $\pi^{-1}(Z \setminus M)$ in W' . In view of Proposition 3.2, it makes sense to study the behavior of the strict transform under restriction to open subsets.

Lemma 3.3. *For any subset $Z \subset W$, the strict transform of the intersection $Z \cap U$ under the blow-up $\pi| : \pi^{-1}(U) \rightarrow U$ of an open subset $U \subset W$ along $M \cap U$ is given by $\tilde{Z} \cap \pi^{-1}(U)$.*

Proposition 3.4. *If $S \subset W$ is a complex submanifold that is transverse to $M \subset W$, then the strict transform and the total transform of S under π coincide, $\tilde{S} = \pi^{-1}(S)$.*

Proof. Without loss of generality, we may assume that S is a closed subset of W . In fact, S is a closed subset of an open tubular neighborhood $U \subset W$ of S , and the restriction $\pi| : \pi^{-1}(U) \rightarrow U$ is the blow-up of U along the complex submanifold $M \cap U \subset U$ by Proposition 3.2. Hence, the strict transform of $S \cap U = S$ under $\pi| : \pi^{-1}(U) \rightarrow U$ coincides with the total transform $\pi^{-1}(S)$. On the other hand, this strict transform is given by $\tilde{S} \cap \pi^{-1}(U) = \tilde{S}$ according to Lemma 3.3.

Then, according to Griffiths-Harris [36, property 5, pp. 604-605], the intersection $\tilde{S} \cap E$ corresponds under the isomorphism $\Psi : E \cong \mathbb{P}(N_M W)$ of bundles over M to the image in

$N_M W$ of the complex tangent spaces $T_x S$ to the points $x \in S \cap M$. This image is $(N_M W)|_{S \cap M}$ because the right-hand vertical arrow in the commutative diagram

$$\begin{array}{ccc} TS|_{S \cap M} & \xrightarrow{\text{quot}} & N_{S \cap M} S \\ \text{incl} \downarrow & & \downarrow \cong \\ TW|_{S \cap M} & \xrightarrow{\text{quot}} & (N_M W)|_{S \cap M} \end{array}$$

is surjective by Lemma 3.1. Consequently,

$$\tilde{S} \cap E = \Psi^{-1} \mathbb{P}((N_M W)|_{S \cap M}) = \pi^{-1}(S \cap M).$$

All in all, we obtain

$$\tilde{S} = (\tilde{S} \cap E) \cup (\tilde{S} \cap \pi^{-1}(S \setminus M)) = \pi^{-1}(S \cap M) \cup \pi^{-1}(S \setminus M) = \pi^{-1}(S).$$

□

From now on, we assume familiarity with the notion of Whitney stratifications; for details, see e.g. [35, Section 1.2, p. 37].

Corollary 3.5. *If $X \subset W$ is a Whitney stratified subspace whose strata are complex submanifolds of W that are transverse to $M \subset W$, then the strict transform and the total transform of X coincide, $\tilde{X} = \pi^{-1}(X)$.*

Proof. Since every stratum S of X satisfies $\pi^{-1}(S) = \tilde{S}$ by Proposition 3.4, and $\tilde{S} \subset \tilde{X}$ holds by definition of the strict transform, we have

$$\pi^{-1}(X) = \bigcup_S \pi^{-1}(S) = \bigcup_S \tilde{S} \subset \tilde{X}.$$

Conversely, we have $\tilde{X} \subset \pi^{-1}(X)$ by definition of the strict transform. □

We proceed to the main result of this section.

Theorem 3.6. *Let $X \subset W$ be a Whitney stratified subspace whose strata are complex submanifolds of W that are transverse to $M \subset W$. Then, the strict transform $\tilde{X} \subset W'$ admits a Whitney stratification whose strata are transverse to the exceptional divisor $E \subset W'$ of π .*

In view of Proposition 3.4 and Corollary 3.5, Theorem 3.6 follows from the following

Theorem 3.7. *Let $X \subset W$ be a Whitney stratified subspace whose strata are transverse to $M \subset W$. Then, the total transforms $\pi^{-1}(S)$ of the strata S of X form a Whitney stratification of the total transform $\pi^{-1}(X) \subset W'$. Furthermore, every stratum of the total transform $\pi^{-1}(X)$ is transverse to $E = \pi^{-1}(M) \subset W'$.*

Proof. The claims are local in the sense that it suffices to find an open cover of W such that for every subset $U \subset W$ of this cover, the inverse images $\pi^{-1}(S \cap U)$ of the strata $S \cap U$ of $X \cap U$ form a Whitney stratification of the inverse image $\pi^{-1}(X \cap U) \subset \pi^{-1}(U)$, and every $\pi^{-1}(S \cap U)$ is transverse to $\pi^{-1}(M \cap U) \subset \pi^{-1}(U)$. Since $\pi: W' \rightarrow W$ is locally isomorphic to the blow-up of an n -dimensional open disc along a coordinate plane (see property 3 in [36, p. 604]), we may therefore assume without loss of generality that π is globally of this form. Explicitly, the blow-up $\pi: W' \rightarrow W$ of an n -dimensional open disc $W \subset \mathbb{C}^n$ with complex coordinates $z = (z_1, \dots, z_n)$ along the coordinate plane $M = W \cap \{z_i = 0; i = 1, \dots, k\}$ of codimension k is given by restricting the projection $W \times \mathbb{P}(\mathbb{C}^k) \rightarrow W$ to the first factor to

$$W' = \{(z, w = [w_1 : \dots : w_k]) \in W \times \mathbb{P}(\mathbb{C}^k); z_i w_j = z_j w_i \text{ for all } 1 \leq i, j \leq k\},$$

and we have $E = \pi^{-1}(M) = M \times \mathbb{P}(\mathbb{C}^k)$. In this situation, we have the following

Lemma 3.8. *If $S \subset W$ is a smooth submanifold that is transverse to $M \subset W$, then $S \times \mathbb{P}(\mathbb{C}^k)$ is transverse to $E = M \times \mathbb{P}(\mathbb{C}^k) \subset W \times \mathbb{P}(\mathbb{C}^k)$ and to $W' \subset W \times \mathbb{P}(\mathbb{C}^k)$.*

Proof. Since S is transverse to $M \subset W$, it follows that $S \times \mathbb{P}(\mathbb{C}^k)$ is transverse to $E = M \times \mathbb{P}(\mathbb{C}^k) \subset W \times \mathbb{P}(\mathbb{C}^k)$, which proves the first claim. To show the second claim, it remains to prove transversality of S and W' at points in $(S \times \mathbb{P}(\mathbb{C}^k)) \cap (W' \setminus E)$. For this purpose, fix a point

$$x = (y, z) \in (S \times \mathbb{P}(\mathbb{C}^k)) \cap (W' \setminus E) \subset (S \setminus M) \times \mathbb{P}(\mathbb{C}^k),$$

and let $(p, q) \in T_x(W \times \mathbb{P}(\mathbb{C}^k)) = T_y(W \setminus M) \times T_z\mathbb{P}(\mathbb{C}^k)$ be an arbitrary tangent vector. Since the differential

$$d_x\pi : T_x(W' \setminus E) \rightarrow T_y(W \setminus M)$$

is surjective, the vector $p \in T_y(W \setminus M)$ is the image under $d_x\pi$ of a vector in $T_x(W' \setminus E)$ of the form $(p, q') \in T_x(W' \setminus E) \subset T_y(W \setminus M) \times T_z\mathbb{P}(\mathbb{C}^k)$. Writing $(p, q) = (p, q') + (0, q - q')$ with $(p, q') \in T_x(W' \setminus E)$ and $(0, q - q') \in T_x(S \times \mathbb{P}(\mathbb{C}^k)) = T_yS \times T_z\mathbb{P}(\mathbb{C}^k)$, we conclude that $S \times \mathbb{P}(\mathbb{C}^k)$ and W' are transverse at x , which proves the second claim. \square

Using Lemma 3.8, we can now complete the proof of Theorem 3.7. For this purpose, let $X \subset W$ be a Whitney stratified subspace whose strata S are transverse to $M \subset W$. We have to show that the total transforms $\pi^{-1}(S)$ form a Whitney stratification of the total transform $\pi^{-1}(X) \subset W'$, and that every stratum $\pi^{-1}(S)$ of the total transform $\pi^{-1}(X)$ is transverse to $E \subset W'$. Since $\pi^{-1}(Z) = W' \cap (Z \times \mathbb{P}(\mathbb{C}^k))$ for all subsets $Z \subset W$, we obtain the claims by applying Lemma 3.9 below to the smooth submanifolds $E \subset W'$ of the smooth manifold $W \times \mathbb{P}(\mathbb{C}^k)$, and to the Whitney stratified subset $X \times \mathbb{P}(\mathbb{C}^k) \subset W \times \mathbb{P}(\mathbb{C}^k)$, whose strata $S \times \mathbb{P}(\mathbb{C}^k)$ are all transverse to $E \subset W \times \mathbb{P}(\mathbb{C}^k)$ and to $W' \subset W \times \mathbb{P}(\mathbb{C}^k)$ by Lemma 3.8.

Lemma 3.9. *Let $N \subset M$ be smooth submanifolds of a smooth manifold W . Let $X \subset W$ be a Whitney stratified subset whose strata S are transverse to $N \subset W$ and to $M \subset W$. Then, the intersection $Y := M \cap X \subset M$ is a Whitney stratified subset whose strata $M \cap S$ are transverse to $N \subset M$.*

Proof. By the theory of Whitney stratifications (see e.g. Lemma 2.2.2 in [25]), it is well-known that the intersection $Y := M \cap X \subset M$ is a Whitney stratified subset with strata $S \cap M$, where S runs through the strata of X . It remains to show that the strata of Y are transverse to $N \subset M$. For this purpose, we fix $x \in N \cap Y$, and have to show that $T_xM \subset T_xN + T_xS'$, where S' denotes the stratum of $Y = M \cap X$ that contains x . Now, $N \pitchfork_W X$ and $x \in N \cap Y \subset N \cap X$ imply that $T_xW \subset T_xN + T_xS$, where S denotes the stratum of X that contains x . Therefore, using $M \subset W$, we obtain $T_xM \subset T_xN + T_xS$. Thus, given $\mu \in T_xM$, we find $v \in T_xN$ and $\xi \in T_xS$ such that $\mu = v + \xi$. Using $T_xN \subset T_xM$, we conclude that

$$\xi = \mu - v \in T_xM \cap T_xS = T_x(M \cap S) = T_xS'.$$

All in all, we have $\mu = v + \xi \in T_xN + T_xS'$. As $\mu \in T_xM$ was arbitrary, the claim follows. \square

This completes the proof of Theorem 3.7. \square

Remark 3.10. Our proof of Theorem 3.7 is inspired by an argument of Cheniot [25, pp. 145-146]. There, one has $M = A$ of codimension $k = 2$, and thus $P(\mathbb{C}^k) = \mathbb{P}^1(\mathbb{C})$, where $\phi := \pi : Z := W' \rightarrow W$ denotes the blow-up. However, all we really need is that M is transverse to the Whitney stratification of X in W .

4. ALGEBRAIC GYSIN RESTRICTION OF $IT_{1,*}$ IN A TRANSVERSE SETUP

In Section 4.1, we review from [8] a topological notion of normally nonsingular embedding that respects a stratification of the ambient space. Then, Section 4.2 discusses along the lines of [8] the first author's Verdier-Riemann-Roch type formula for the algebraic Gysin restriction of the Hodge-theoretic characteristic classes $IT_{1,*}$ with respect to suitably normally nonsingular closed algebraic regular embeddings. Finally, in Section 4.3, we derive our algebraic Gysin restriction formula for $IT_{1,*}$ in a transverse setup (see Theorem 4.15) by invoking Theorem 3.6.

4.1. Topological Normally Nonsingular Embeddings.

Definition 4.1 (see Definition 3.1 in [8]). A *topological stratification* of a topological space X is a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_1 \supseteq X_0 \supseteq X_{-1} = \emptyset$$

by closed subsets X_i such that the difference sets $X_i - X_{i-1}$ are topological manifolds of pure dimension i , unless empty. The connected components X_α of these difference sets are called the *strata*. We will often write stratifications as $\mathcal{X} = \{X_\alpha\}$.

The following definition of locally cone-like topological stratifications, which are also known as *cs-stratifications*, is due to Siebenmann [61]; see also [58, Def. 4.2.1, p. 232].

Definition 4.2 (see Definition 3.2 in [8]). A topological stratification $\{X_i\}$ of X is called *locally cone-like* if for all $x \in X_i - X_{i-1}$ there is an open neighborhood U of x in X , a compact topological space L with filtration

$$L = L_{n-i-1} \supseteq L_{n-i-2} \supseteq \cdots \supseteq L_0 \supseteq L_{-1} = \emptyset,$$

and a filtration preserving homeomorphism $U \cong \mathbb{R}^i \times \text{cone}^\circ(L)$, where $\text{cone}^\circ(L)$ denotes the open cone on L .

We shall employ the following notion of normally nonsingular embedding of topological spaces that respects an ambient locally cone-like topological stratification.

Definition 4.3 (see Definition 3.3 in [8]). Let X be a topological space with locally cone-like topological stratification $\mathcal{X} = \{X_\alpha\}$ and let Y be any topological space. An embedding $g: Y \hookrightarrow X$ is called *normally nonsingular* (with respect to \mathcal{X}), if

- (1) $\mathcal{Y} := \{Y_\alpha := X_\alpha \cap Y\}$ is a locally cone-like topological stratification of Y ,
- (2) there exists a topological vector bundle $\pi: E \rightarrow Y$ and
- (3) there exists a (topological) embedding $j: E \rightarrow X$ such that
 - (a) $j(E)$ is open in X ,
 - (b) $j|_Y = g$, and
 - (c) the homeomorphism $j: E \xrightarrow{\cong} j(E)$ is stratum preserving, where the open set $j(E)$ is endowed with the stratification $\{X_\alpha \cap j(E)\}$ and E is endowed with the stratification $\mathcal{E} = \{\pi^{-1}Y_\alpha\}$.

Note that the above stratification \mathcal{E} of the total space E is automatically topologically locally cone-like. For example, transverse intersections give rise to normally nonsingular inclusions as follows (compare the beginning of the proof of Proposition 6.3 in [8]).

Theorem 4.4. (See Theorem 1.11 in [35, p. 47].) *Let $X \subset W$ be a Whitney stratified subset of a smooth manifold W . Suppose that $M \subset W$ is a smooth submanifold of codimension r that is transverse to every stratum of X , and that $Y = M \cap X$ is compact. Then, the inclusion $g: Y \hookrightarrow X$ is normally nonsingular of codimension r with respect to the normal bundle $\nu = \nu_{M \subset W}|_Y$ given by restriction of the normal bundle $\nu_{M \subset W}$ of M in W .*

Proof. In [50], Pati points out that one may wish to add precision to the argument given in [35, p. 48], as Thom's First Isotopy Lemma is applied there to a composition $\Psi^{-1}(X) \subset E_\varepsilon \times (-\delta, 1 + \delta) \rightarrow (-\delta, 1 + \delta)$ which is not necessarily proper. To address this, we impose the assumption that $Y = M \cap X$ be compact, and apply Thom's First Isotopy Lemma to the proper composition $\Psi^{-1}(X) \cap D_{\varepsilon/2} \subset E_\varepsilon \times (-\delta, 1 + \delta) \rightarrow (-\delta, 1 + \delta)$, where $D_{\varepsilon/2} \subset E_\varepsilon$ denotes the closed disk bundle of radius $\varepsilon/2$. To assure that the intersection $\Psi^{-1}(X) \cap D_{\varepsilon/2}$ is Whitney stratified, we choose $\varepsilon > 0$ small enough to achieve, using Whitney's condition B, that $\Psi|_{\partial D_{\varepsilon/2}}$ is transverse to $X \subset W$. \square

4.2. Algebraic Gysin Restriction of $IT_{1,*}$ for Upwardly Normally Nonsingular Embeddings. We review the Verdier-Riemann-Roch type formula for the algebraic Gysin restriction of $IT_{1,*}$ obtained in [8].

Definition 4.5 (see p. 1279 in [8]). An *algebraic stratification* of a complex algebraic variety X is a locally cone-like topological stratification $\{X_{2i}\}$ of X such that the closed subspaces X_{2i} are algebraic subsets of X .

Example 4.6. Let Z be a closed subvariety of a smooth complex algebraic variety W . A Whitney stratification of $Z \subset W$ is called *complex algebraic* if all its open strata are smooth complex algebraic subvarieties of W . It is well-known that complex algebraic Whitney stratifications always exist (see e.g. [35, Theorem, p. 43]). Furthermore, complex algebraic Whitney stratifications are algebraic stratifications in the sense of Definition 4.5. (Here, we point out that the closures of the open strata are the same in the Zariski and the complex topology; see e.g. [48, Corollary 1, p. 60].)

Definition 4.7 (see Definition 3.4 in [8]). If X and Y are complex algebraic varieties and $g: Y \hookrightarrow X$ a closed algebraic embedding whose underlying topological embedding g_{top} in the complex topology is normally nonsingular with respect to some locally cone-like stratification on X , then we will call g and g_{top} *compatibly stratifiable* if there exists an algebraic stratification \mathcal{X} of X such that g_{top} is normally nonsingular with respect to \mathcal{X} and the induced stratification \mathcal{Y} is an algebraic stratification of Y .

We recall that a closed algebraic embedding $Y \hookrightarrow X$ is called a *regular* embedding of codimension d if every point of Y has an affine neighborhood U in X , such that if A is the coordinate ring of U , I the ideal of A defining Y , then I is generated by a regular sequence of length d . If \mathcal{J} is the ideal sheaf of Y in X , it follows that the conormal sheaf $\mathcal{J}/\mathcal{J}^2$ is a locally free sheaf on Y of rank d . The *algebraic normal bundle* $N_Y X$ to Y in X is the vector bundle on Y whose sheaf of sections is dual to $\mathcal{J}/\mathcal{J}^2$. See Fulton [30, Appendix B.7]. The algebraic normal bundle of a regular algebraic embedding does not necessarily reflect the normal topology near the subvariety. In particular, the underlying topological embedding needs not be normally nonsingular. For example, let $Y = f^{-1}(0)$ be a complex algebraic hypersurface given as the zero set of a single nontrivial algebraic function $f: X = \mathbb{C}^n \rightarrow \mathbb{C}$. Then the inclusion $Y \subset X$ is a regular embedding with trivial algebraic normal bundle $N_Y X \cong f^* N_{\{0\}} \mathbb{C}$. But this inclusion need not be topologically normally nonsingular. For example, when $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ is given by $f(x,y) = x^2 - y^2$, then Y has a nodal singularity at the origin $0 \in \mathbb{C}^2$. The link is a disjoint union of two circles. Thus Y is not normal as a pseudomanifold. If it had a topological open neighborhood U which is the total space of a vector bundle over Y , then U would be normal as an open subset of the normal space \mathbb{C}^n . But if the total space of a vector bundle over a pseudomanifold is normal, then, due to the local triviality of a vector bundle, the base pseudomanifold has to be normal as well. (Note that normality of a pseudomanifold is a topologically well-defined concept since if one choice

of link is connected, any other choice is also connected.) These observations motivate the following

Definition 4.8 (see Definition 6.1 in [8]). A closed regular algebraic embedding $Y \hookrightarrow X$ of complex algebraic varieties is called *tight*, if its underlying topological embedding (in the complex topology) is normally nonsingular and compatibly stratifiable, with topological normal bundle $\pi: E \rightarrow Y$ as in Definition 4.3, and $E \rightarrow Y$ is isomorphic (as a topological vector bundle) to the underlying topological vector bundle of the algebraic normal bundle $N_Y X$ of Y in X .

For example, closed embeddings of smooth complex varieties are tight, see Example 6.2 in [8].

Next, we recall the notion of upward normal nonsingularity for tight embeddings. For a closed regular embedding $V \hookrightarrow U$ of complex varieties, let $\pi: \text{Bl}_V U \rightarrow U$ denote the blow-up of U along V . The exceptional divisor $E = \pi^{-1}(V) \subset \text{Bl}_V U$ is the projectivization $\mathbb{P}(N)$ of the algebraic normal bundle N of V in U .

Definition 4.9 (see Definition 6.5 in [8]). A tight embedding $Y \hookrightarrow X$ is called *upwardly normally nonsingular* if the inclusion $E \subset \text{Bl}_{Y \times 0}(X \times \mathbb{C})$ of the exceptional divisor E is topologically normally nonsingular with respect to some locally cone-like stratification on $\text{Bl}_{Y \times 0}(X \times \mathbb{C})$.

According to Verdier [63], a regular closed algebraic embedding $g: Y \hookrightarrow X$ has an associated algebraic Gysin homomorphism on Borel-Moore homology

$$g_{\text{alg}}^!: H_*^{\text{BM}}(X; \mathbb{Q}) \longrightarrow H_{*-2c}^{\text{BM}}(Y; \mathbb{Q}),$$

where c is the complex codimension of Y in X .

The following result is the first author's Verdier-Riemann-Roch type formula for the algebraic Gysin restriction of the Hodge-theoretic characteristic classes $IT_{1,*}$ (see Definition 2.11) with respect to upwardly normally nonsingular embeddings.

Theorem 4.10 (see Theorem 6.30 in [8]). *Let X, Y be pure-dimensional compact complex algebraic varieties and let $g: Y \hookrightarrow X$ be an upwardly normally nonsingular embedding. Let $N = N_Y X$ be the algebraic normal bundle of g and let ν denote the topological normal bundle of the topologically normally nonsingular inclusion underlying g . Then*

$$g_{\text{alg}}^! IT_{1,*}(X) = L^*(N) \cap IT_{1,*}(Y) = L^*(\nu) \cap IT_{1,*}(Y).$$

4.3. Algebraic Gysin Restriction of $IT_{1,*}$ in a Transverse Setup. Tight embeddings arise frequently from transverse intersections in ambient smooth varieties, as we shall discuss next. For this purpose, we require the notion of Tor-independence (compare [8, p. 1312]), which can be thought of as a transversality condition in algebraic settings. For indications of this viewpoint in the literature, see for instance Section 1.6 in Baum-Fulton-MacPherson [13, p. 165], Definition 1.1.1 in Levine-Morel [42, p. 1], as well as the genericity result of Miller-Speyer [47].

Definition 4.11. Closed subschemes $X, Y \subset S$ of a scheme S are called *Tor-independent* if

$$\text{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y) = 0 \text{ for all } i > 0.$$

Proposition 4.12 (Proposition 6.3 in [8]). *Let $M \hookrightarrow W$ be a closed algebraic embedding of smooth complex algebraic varieties. Let $X \subset W$ be a (possibly singular) algebraic subvariety, equipped with an algebraic Whitney stratification and set $Y = X \cap M$. If*

- each stratum of X is transverse to M , and

- X and M are Tor-independent in W ,

then the embedding $g: Y \hookrightarrow X$ is tight.

If the embeddings $X \hookrightarrow W \leftarrow M$ in the previous result satisfy an even stronger transversality condition defined next, then the embedding $Y = X \cap M \hookrightarrow X$ is upwardly normally nonsingular according to Proposition 4.14 below.

Definition 4.13 (see Definition 6.4 in [8]). Let $X \hookrightarrow W \leftarrow M$ be closed algebraic embeddings of algebraic varieties with M, W smooth. We say that these embeddings are *upwardly transverse*, if X and M are Tor-independent in W , there exists an algebraic Whitney stratification of X which is transverse to M in W , and there exists a (possibly non-algebraic) Whitney stratification on the strict transform of $X \times \mathbb{C}$ in $\text{Bl}_{M \times 0}(W \times \mathbb{C})$ which is transverse to the exceptional divisor.

Proposition 4.14 (see Corollary 6.7 in [8]). *If $X \hookrightarrow W \leftarrow M$ are upwardly transverse embeddings, then the embedding $Y = X \cap M \hookrightarrow X$ is upwardly normally nonsingular.*

In the transverse situation, we obtain the following extension of Theorem 4.10 by eliminating the strict transform condition. This constitutes the main result of this section.

Theorem 4.15. *Let $X \hookrightarrow W \leftarrow M$ be closed algebraic embeddings of pure-dimensional complex algebraic varieties with M, W smooth, X, M irreducible, and X compact. We suppose that X and M are Tor-independent in W , that X is generically transverse to M in W (see e.g. [11, Section 3]), and that there exists a complex algebraic Whitney stratification of $X \subset W$ which is transverse to M in W . Then, the embedding $g: Y \hookrightarrow X$ of the compact pure-dimensional subvariety $Y = X \cap M \subset X$ is tight, and the algebraic normal bundle $N = N_Y X$ of g and the topological normal bundle ν of the topologically normally nonsingular inclusion underlying g satisfy*

$$g_{\text{alg}}^! IT_{1,*}(X) = L^*(\nu) \cap IT_{1,*}(Y),$$

where $g_{\text{alg}}^!: H_*^{\text{BM}}(X) \rightarrow H_{*-2d}^{\text{BM}}(Y)$ denotes the algebraic Gysin homomorphism associated to g constructed by Verdier in [63], $d = \text{codim}_{\mathbb{C}}(g)$.

Proof. Since X admits by assumption a complex algebraic Whitney stratification which is transverse to M in W , it follows that the product stratification on $X \times \mathbb{C}$ is a complex algebraic Whitney stratification which is transverse to $M \times 0$ in $W \times \mathbb{C}$. Thus, Theorem 3.6 implies that the strict transform of $X \times \mathbb{C}$ in $\text{Bl}_{M \times 0}(W \times \mathbb{C})$ admits a Whitney stratification which is transverse to the exceptional divisor. Consequently, our transversality assumptions on X and M imply that the embeddings $X \hookrightarrow W \leftarrow M$ are upwardly transverse in the sense of Definition 4.13. (Recall from Example 4.6 that the complex algebraic Whitney stratification on X is in particular an algebraic stratification.) Hence, it follows from Proposition 4.14 that the embedding $g: Y \hookrightarrow X$ of the compact subvariety $Y = X \cap M \subset X$ is upwardly normally nonsingular (and, in particular, tight). Moreover, generic transversality of X and M in W implies that Y is pure-dimensional according to [11, Corollary 3.4]. Finally, the algebraic Gysin restriction formula for $IT_{1,*}$ follows from Theorem 4.10. \square

5. ALGEBRAIC VERSUS TOPOLOGICAL GYSIN RESTRICTION

A normally nonsingular inclusion $g: Y \hookrightarrow X$ (see Definition 4.3) of a closed subset $Y \subset X$ with oriented normal bundle $\pi: E \rightarrow Y$ of rank r induces on singular homology groups a topological Gysin homomorphism

$$g_{\text{top}}^!: H_*(X; \mathbb{Q}) \rightarrow H_{*-r}(Y; \mathbb{Q})$$

given by the composition

$$(6) \quad H_*(X) \xrightarrow{\text{incl}_*} H_*(X, X \setminus Y) \xleftarrow[\cong]{e_*} H_*(E, E_0) \xrightarrow[\cong]{u \cap -} H_{*-r}(E) \xrightarrow[\cong]{\pi_*} H_{*-r}(Y),$$

where $u \in H^r(E, E_0)$ is the Thom class with $E_0 = E \setminus Y$ the complement of the zero section in E , and e_* denotes the excision isomorphism induced by the open embedding $j: E \rightarrow X$.

The next proposition interprets the topological Gysin map on singular homology in terms of transverse intersections.

Proposition 5.1 (see e.g. Proposition 2.5 in [11]). *Let W be an oriented smooth manifold, $X, K \subset W$ Whitney stratified subspaces which are oriented pseudomanifolds with $K \subset X$ and K compact. Let $M \subset W$ be an oriented smooth submanifold which is closed as a subset. Suppose that M is transverse to all strata of the Whitney stratified subspaces $X \subset W$ and $K \subset W$, and that $M \cap X$ is compact. Then the Gysin map*

$$g_{\text{top}}^! : H_*(X; \mathbb{Q}) \longrightarrow H_{*-r}(Y; \mathbb{Q})$$

associated to the normally nonsingular embedding $g: Y = M \cap X \hookrightarrow X$ (see Theorem 4.4), where r is the (real) codimension of Y in X , sends the fundamental class $[K]_X \in H_*(X; \mathbb{Q})$ of K to the fundamental class $[K \cap Y]_Y$ of the intersection $K \cap Y = M \cap K$ (which is again an oriented pseudomanifold),

$$g_{\text{top}}^![K]_X = [K \cap Y]_Y.$$

In the following, we shall discuss a variant of the topological Gysin map defined on Borel-Moore homology groups. For reasonable compact spaces, both topological Gysin maps coincide under a natural identification of Borel-Moore homology with singular homology (see Proposition 5.7 below). This variant is needed here to show that the algebraic and topological Gysin maps coincide on algebraic cycles in a transverse setup (see Theorem 5.10 below).

Following Fulton [30, p. 371, Eq. (1)], we employ here a construction of Borel-Moore homology defined for any topological space that can be embedded as a closed subset of some Euclidean space. Namely, if the space X is embedded as a closed subset of \mathbb{R}^n , then Fulton sets $H_*^{\text{BM}}(X; \mathbb{Q}) = H^{n-*}(\mathbb{R}^n, \mathbb{R}^n - X; \mathbb{Q})$. When X is a compact ENR, then the Alexander duality isomorphism $H^{n-*}(\mathbb{R}^n, \mathbb{R}^n - X; \mathbb{Q}) \cong H_*(X; \mathbb{Q})$ provides a natural identification with singular homology, $H_*^{\text{BM}}(X; \mathbb{Q}) \cong H_*(X; \mathbb{Q})$. For a detailed discussion of this viewpoint on Borel-Moore homology, we refer to [31, Appendix B.2, pp. 215 ff.].

For a normally nonsingular inclusion $g: Y \hookrightarrow X$ of a closed subset $Y \subset X$ as above with oriented normal bundle $\pi: E \rightarrow Y$ of rank r , the topological Gysin map on Borel-Moore homology,

$$g_{\text{top}}^{!!} : H_*^{\text{BM}}(X; \mathbb{Q}) \rightarrow H_{*-r}^{\text{BM}}(Y; \mathbb{Q}),$$

is given by the composition

$$(7) \quad H_*^{\text{BM}}(X) \xrightarrow{\text{res}} H_*^{\text{BM}}(U) \xrightarrow[\cong]{u \cap -} H_{*-r}^{\text{BM}}(Y),$$

where the first map restricts Borel-Moore cycles on X to the open tubular neighborhood $U := j(E) \subset X$ of $Y \subset X$, and the second map is given by cap product with the Thom class $u \in H^r(U, U_0) = H^r(E, E_0)$, where $U_0 = U \setminus Y$.

The cap product used in the definition of the topological Gysin map on Borel-Moore homology is the one mentioned by Fulton in [30, p. 371, Eq. (2)]. As explained in [30, p. 375], its construction can be found in Fulton-MacPherson [32, Eq. (2), p. 36] in a more abstract setting, and involves relative Čech cohomology groups as discussed by Dold [27, pp. 281 ff.]. For convenience, we provide here the construction of this cap product together with some of its transformational properties (see Corollary 5.5).

Remark 5.2. Recall from [27, Definition 6.1, p. 281] that the relative Čech cohomology groups $\check{H}^i(A, B)$ are defined for any pair $B \subset A$ of locally compact subspaces of an ENR. Furthermore, for any locally compact subspaces A_1, A_2 of an ENR which are separated by $A_1 \cap A_2$ (e.g. when A_1, A_2 are both open or closed in $A_1 \cup A_2$), there is an excision isomorphism $\check{H}^*(A_1 \cup A_2, A_1) \cong \check{H}^*(A_2, A_1 \cap A_2)$ by [27, p. 286, Eq. (6.16)].

Proposition 5.3. *For a topological space X that can be embedded as a closed subset of some Euclidean space, the following hold:*

- (1) *For any closed subset $Y \subset X$, there is a cap product*

$$\check{H}^i(X, X - Y) \times H_k^{\text{BM}}(X) \xrightarrow{\cap} H_{k-i}^{\text{BM}}(Y).$$

- (2) *Every open subset $X' \subset X$ can be embedded as a closed subset of some Euclidean space, and if X' is an open neighborhood of a closed subset $Y \subset X$, then we have a commutative diagram*

$$\begin{array}{ccccc} \check{H}^i(X, X - Y) & \times & H_k^{\text{BM}}(X) & \xrightarrow{\cap} & H_{k-i}^{\text{BM}}(Y) \\ \cong \downarrow \rho^* & & \downarrow r^* & & \downarrow = \\ \check{H}^i(X', X' - Y) & \times & H_k^{\text{BM}}(X') & \xrightarrow{\cap} & H_{k-i}^{\text{BM}}(Y), \end{array}$$

where ρ^* is the excision isomorphism induced by inclusion in Čech cohomology, and r^* is restriction of a Borel-Moore cycle to an open subset.

- (3) *Any cartesian diagram of closed embeddings*

$$\begin{array}{ccc} Y' = X' \cap Y & \xrightarrow{f_Y} & Y \\ g' \downarrow & & \downarrow g \\ X' & \xrightarrow{f_X} & X \end{array}$$

induces compatible cap products

$$\begin{array}{ccccc} \check{H}^i(X', X' - Y') & \times & H_k^{\text{BM}}(X') & \xrightarrow{\cap} & H_{k-i}^{\text{BM}}(Y') \\ f_X^* \uparrow & & \downarrow f_{X*} & & \downarrow f_{Y*} \\ \check{H}^i(X, X - Y) & \times & H_k^{\text{BM}}(X) & \xrightarrow{\cap} & H_{k-i}^{\text{BM}}(Y), \end{array}$$

where f_X^* is the induced map on Čech cohomology, and f_{X*}, f_{Y*} are induced on Borel-Moore homology by the proper maps f_X, f_Y .

Proof. Suppose that X is embedded as a closed subset of \mathbb{R}^n . For the construction of the cap product in claim (1), let $(U, V) \supset (X, X - Y)$ be an open neighborhood pair in \mathbb{R}^n . Since both V and $U - X$ are open in U , singular cohomology comes with a standard cup product

$$H^i(U, V) \times H^j(U, U - X) \xrightarrow{\cup} H^{i+j}(U, V \cup (U - X)).$$

Now it follows from $X - Y \subset V$ that $U - Y = (X - Y) \cup (U - X) \subset V \cup (U - X)$. Hence, there is a restriction homomorphism

$$H^{i+j}(U, V \cup (U - X)) \longrightarrow H^{i+j}(U, U - Y).$$

Composing with it, we obtain a cup product¹

$$H^i(U, V) \times H^j(U, U - X) \xrightarrow{\cup} H^{i+j}(U, U - Y).$$

Using excision, this can be rewritten as a product

$$H^i(U, V) \times H^j(\mathbb{R}^n, \mathbb{R}^n - X) \xrightarrow{\cup} H^{i+j}(\mathbb{R}^n, \mathbb{R}^n - Y).$$

Next, we note that $X - Y \subset X$ are locally compact subsets of the ENR \mathbb{R}^n by [27, Lemma 8.3, p. 80] because they are locally closed subsets by assumption. Therefore, the relative Čech cohomology groups

$$\check{H}^i(X, X - Y) := \operatorname{colim}\{H^i(U, V) \mid (U, V) \supset (X, X - Y)\}$$

are defined as in [27, Definition 6.1, p. 281]. Consequently, there is a cup product

$$\check{H}^i(X, X - Y) \times H^j(\mathbb{R}^n, \mathbb{R}^n - X) \xrightarrow{\cup} H^{i+j}(\mathbb{R}^n, \mathbb{R}^n - Y).$$

By the definition of Borel-Moore homology used by Fulton, we have

$$H^j(\mathbb{R}^n, \mathbb{R}^n - X) = H_{n-j}^{\text{BM}}(X), \quad H^{i+j}(\mathbb{R}^n, \mathbb{R}^n - Y) = H_{n-i-j}^{\text{BM}}(Y).$$

We arrive at the desired cap product

$$\check{H}^i(X, X - Y) \times H_k^{\text{BM}}(X) \xrightarrow{\cap} H_{k-i}^{\text{BM}}(Y).$$

As for claim (2), let $X' \subset X$ be an open neighborhood of the closed subset $Y \subset X$. Then, there exists an open subset $W \subset \mathbb{R}^n$ such that $X' = W \cap X$. We note that X' is a closed subset of the manifold W , which can be embedded as a closed subset of some Euclidean space. If $(U, V) \supset (X, X - Y)$ is an open neighborhood pair in \mathbb{R}^n , then $(W \cap U, W \cap V) \supset (X', X' - Y)$ is an open neighborhood pair in W , and the diagram of restrictions

$$\begin{array}{ccccc} H^i(U, V) & \times & H^j(\mathbb{R}^n, \mathbb{R}^n - X) & \xrightarrow{\cup} & H^{i+j}(\mathbb{R}^n, \mathbb{R}^n - Y) \\ \downarrow & & \downarrow & & \downarrow \\ H^i(W \cap U, W \cap V) & \times & H^j(W, W - X') & \xrightarrow{\cup} & H^{i+j}(W, W - Y) \end{array}$$

commutes by naturality of the cup product. Finally, we obtain the desired commutative diagram by passing to the colimits to obtain the induced map on Čech cohomology (see [27, Definition 6.3, p. 282]), and by definition of the restriction map in Borel-Moore homology (see [31, Eq. (30), p. 218]).

To show claim (3), we observe that any open neighborhood pair $(U, V) \supset (X, X - Y)$ in \mathbb{R}^n induces compatible cup products

$$\begin{array}{ccccc} H^i(U, V) & \times & H^j(U, U - X') & \xrightarrow{\cup} & H^{i+j}(U, U - Y') \\ \uparrow & & \downarrow & & \downarrow \\ H^i(U, V) & \times & H^j(U, U - X) & \xrightarrow{\cup} & H^{i+j}(U, U - Y), \end{array}$$

where the vertical arrows are induced by inclusion. The claim now follows by passing to the colimits to obtain the induced map on Čech cohomology (see [27, Definition 6.3, p. 282]), and by definition of the pushforward for proper maps in Borel-Moore homology (see [31, p. 218]). \square

¹In Fulton-MacPherson [32, Eq. (2), p. 36], the additional assumption $V \cap X = X - Y$ is imposed.

Remark 5.4. In Proposition 5.1, the assumption that K is compact can be dropped when using the topological Gysin map on Borel-Moore homology instead of singular homology. The proof is very similar, but uses base change as stated in Proposition 5.3(3), as well as the fact that the topological Gysin map in Borel-Moore homology maps the fundamental class to the fundamental class.

Corollary 5.5. *For any closed subset Y of an ENR X , we have a cap product*

$$H^i(X, X - Y) \times H_k^{\text{BM}}(X) \xrightarrow{\cap} H_{k-i}^{\text{BM}}(Y)$$

with the properties (2) and (3) of Proposition 5.3 with respect to singular cohomology.

Proof. Since X is an ENR, we can embed it as a closed subset of some Euclidean space \mathbb{R}^n according to [27, Proposition 8.1 and Lemma 8.2, p. 80], so that the previous proposition yields a cap product using Čech cohomology. By assumption, X is an ENR, and hence the open subset $X - Y \subset X$ is an ENR as well. By [27, Proposition 6.12, p. 285], $\check{H}^i(X, X - Y)$ is therefore naturally isomorphic to singular cohomology $H^i(X, X - Y)$. \square

Remark 5.6. For $Y = X$ an ENR, the cap product of Corollary 5.5 specializes to a cap product of the form $H^*(X) \times H_*^{\text{BM}}(X) \rightarrow H_*^{\text{BM}}(X)$. This is the cap product that appears in Theorem 4.10 and Theorem 4.15. When X is a compact CW complex, this cap product corresponds to the ordinary cap product under the natural identification of Borel-Moore homology with singular homology. (In fact, if $\alpha \in H^i(X; \mathbb{Q}) = \check{H}^i(X; \mathbb{Q})$ is represented by $\alpha_U \in H^i(U; \mathbb{Q})$ on some open neighborhood U of $X \subset \mathbb{R}^n$, then there is a commutative diagram

$$\begin{array}{ccc} H_{n-(i+j)}(U, U - X) & \xrightarrow{\alpha_U \cap -} & H_{n-j}(U, U - X) \\ \uparrow -\cap \mu_X & & \uparrow -\cap \mu_X \\ H^{i+j}(U) & \xrightarrow{\alpha_U \cup -} & H^j(U), \end{array}$$

where $\mu_X \in H_n(U, U - X; \mathbb{Q})$ denotes the orientation class of $X \subset U$. Passing to Čech cohomology of X in the lower horizontal row induces the ordinary cup product map

$$\alpha \cup - : H^{n-(i+j)}(X) \rightarrow H^{n-j}(X),$$

and the vertical maps become Alexander duality isomorphisms $H^*(X) \cong H_{n-*}(U, U - X)$ (see e.g. Bredon [18, Theorem 8.3, p. 351]). Then, the claim follows by applying $\text{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ to the resulting commutative diagram, and identifying $H^*(-) \cong \text{Hom}_{\mathbb{Q}}(H_*(-), \mathbb{Q})$ via the Kronecker pairing. Note that the involved \mathbb{Q} -vector spaces are finite dimensional.)

Recall that the topological Gysin map $g_{\text{top}}^!$ has been defined in (6), and $g_{\text{top}}^{!!}$ in (7).

Proposition 5.7. *Let (X, Y) be a compact CW pair such that X is endowed with a topologically cone-like topological stratification. If the inclusion $g : Y \hookrightarrow X$ is normally nonsingular with oriented normal bundle $\nu : E \rightarrow Y$ of rank r , the topological Gysin maps*

$$g_{\text{top}}^!, g_{\text{top}}^{!!} : H_*(X) \longrightarrow H_{*-r}(Y)$$

coincide under the natural identification of Borel-Moore homology with singular homology.

Proof. We may identify a closed tubular neighborhood of $Y \subset X$ with the total space of the disc bundle $D\nu$ of the topological normal bundle ν of the normally nonsingular inclusion $g : Y \hookrightarrow X$. Let $S\nu \subset D\nu$ denote the sphere bundle and $D^\circ\nu = D\nu - S\nu$ the open disc bundle.

Furthermore, $Z = X - D^\circ v$ is a closed subset of X . The ENR X can be embedded a closed subset of some Euclidean space \mathbb{R}^n . Then, we obtain a commutative diagram

$$\begin{array}{ccccc} H_{n-k}(\mathbb{R}^n - Sv, \mathbb{R}^n - Dv) & \xleftarrow[\cong]{\text{incl}^*} & H_{n-k}(\mathbb{R}^n - Z, \mathbb{R}^n - X) & \xrightarrow{\text{incl}^*} & H_{n-k}(\mathbb{R}^n, \mathbb{R}^n - X) \\ \cong \uparrow -\cap \vartheta & & \cong \uparrow -\cap \vartheta & & \cong \uparrow -\cap \vartheta \\ H^k(Dv, Sv) & \xleftarrow[\text{incl}^*]{\cong} & H^k(X, Z) & \xrightarrow{\text{incl}^*} & H^k(X) \end{array}$$

and for any open neighborhood U of $Dv \subset \mathbb{R}^n$ a commutative diagram

$$\begin{array}{ccc} H_{n-k}(U - Sv, U - Dv) & \xrightarrow[\cong]{\text{incl}^*} & H_{n-k}(\mathbb{R}^n - Sv, \mathbb{R}^n - Dv) \\ \cong \uparrow -\cap \vartheta|_U & & \cong \uparrow -\cap \vartheta \\ H^k(Dv, Sv) & \xrightarrow{=} & H^k(Dv, Sv), \end{array}$$

where the vertical maps are Alexander duality isomorphisms described by cap product with an orientation ϑ of \mathbb{R}^n , or its restriction to U (see e.g. Bredon [18, Theorem 8.3, p. 351], who also provides a description of the Alexander duality isomorphisms on the (co)chain level in [18, p. 349]).

The Thom class $\tau(v) \in H^r(Dv, Sv) = \check{H}^r(Dv, Sv)$ can be represented by an element $t \in H^r(U, V)$ for some open neighborhood pair (U, V) of (Dv, Sv) in \mathbb{R}^n . By making V smaller if necessary, we may assume without loss of generality that the inclusion $(U, V) \subset (U, U - Y)$ is a homotopy equivalence of pairs. Let $u \in H^r(U, U - Y)$ be the element that restricts to t . Writing $W = U - Sv$, we obtain the commutative diagram

$$\begin{array}{ccc} H_{n-(k-r)}(W, W - Y) & \xrightarrow{u|_W \cap -} & H_{n-k}(W, W - D^\circ v) = H_{n-k}(U - Sv, U - Dv) \\ \cong \uparrow -\cap \vartheta|_W & & \cong \uparrow -\cap \vartheta|_U \\ H^{k-r}(Y) & \xleftarrow[\text{incl}^*]{\cong} H^{k-r}(Dv) \xrightarrow{\tau(v) \cup -} & H^k(Dv, Sv) \end{array}$$

where the vertical maps are Alexander duality isomorphisms described by cap product with the orientation $\vartheta|_W$ of W or the orientation $\vartheta|_U$ of U . Again, one checks commutativity of the diagram by using the description of the Alexander duality isomorphisms on the (co)chain level provided in [18, p. 349].

Finally, the claim follows by applying $\text{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ to the concatenation of the above three commutative diagrams, and identifying $H^*(-) \cong \text{Hom}_{\mathbb{Q}}(H_*(-), \mathbb{Q})$ via the Kronecker pairing. Note that the involved \mathbb{Q} -vector spaces are finite dimensional. \square

Let $A_*(Z)$ denote the Chow groups of a complex variety Z and let $\text{cl} = \text{cl}_Z : A_*(Z) \rightarrow H_{2*}^{\text{BM}}(Z)$ be the cycle map from Chow homology to rational Borel-Moore homology. Let

$$u_f \in H^{2d}(X, X - Y)$$

be the orientation class of the regular closed embedding $f : Y \hookrightarrow X$ as in [30, 19.2, p. 378].

The cap product of Corollary 5.5 can be used to evaluate the orientation class of a closed regular embedding on a cycle class in Borel-Moore homology as follows.

Theorem 5.8 (see Theorem 19.2 in Fulton [30]). *Let $f : Y \hookrightarrow X$ be a regular embedding of complex algebraic varieties of codimension d . Then, for all k -cycles α on X ,*

$$\text{cl}_Y(f_{\text{alg}}^! \alpha) = u_f \cap \text{cl}_X(\alpha)$$

in $H_{2k-2d}^{BM}(Y)$.

Remark 5.9. More generally, Theorem 19.2 in [30] uses the refined Gysin homomorphism [30, Section 6.2] associated to the fiber square formed by the regular embedding $f: Y \hookrightarrow X$ and a morphism $g: X' \rightarrow X$. However, for $g = \text{id}_X$ the refined Gysin homomorphism coincides with Verdier's algebraic Gysin homomorphism $f_{\text{alg}}^!$ for regular embeddings (see [30, p. 117], where we note that $f_{\text{alg}}^! = f^*$ in Fulton's notation).

Theorem 5.10. *Let W be a smooth complex algebraic variety, $M \subset W$ a smooth closed subvariety and $X \subset W$ a closed subvariety such that an algebraic Whitney stratification of X in W is transverse to M and X and M are Tor-independent in W . Then the algebraic and topological Gysin restriction homomorphisms associated to the inclusion $g: Y = X \cap M \hookrightarrow X$,*

$$g_{\text{alg}}^!, g_{\text{top}}^{!!}: H_*^{\text{BM}}(X) \longrightarrow H_{*-2d}^{\text{BM}}(Y)$$

coincide on algebraic cycles, where d is the complex codimension of the embedding g .

Proof. By smoothness, the closed embedding $M \subset W$ is regular with algebraic normal bundle $N_M W$. The Tor-independence of X and M ensures that the closed embedding $g: Y \hookrightarrow X$ is regular as well, and that the excess normal bundle vanishes, i.e. the canonical closed embedding $N_Y X \rightarrow j^* N_M W$ is an isomorphism of algebraic vector bundles, where j is the embedding $j: Y \hookrightarrow M$. According to Verdier [63, p. 222, 9.2.1], the algebraic Gysin map of a closed regular embedding commutes with the cycle map from Chow to Borel-Moore homology. Thus there is a commutative diagram

$$(8) \quad \begin{array}{ccc} A_*(X) & \xrightarrow{g_{\text{alg}}^!} & A_{*-d}(Y) \\ \text{cl} \downarrow & & \downarrow \text{cl} \\ H_{2*}^{\text{BM}}(X) & \xrightarrow{g_{\text{alg}}^!} & H_{2*-2d}^{\text{BM}}(Y). \end{array}$$

Since X and M are transverse and Tor-independent in W , Proposition 4.12 implies that the embedding $g: Y \hookrightarrow X$ is tight, i.e. the underlying topological embedding (in the analytic topology) is normally nonsingular with tubular neighborhood described by the disc bundle $D\nu$ of a topological normal bundle ν which is isomorphic to the underlying topological vector bundle of the algebraic normal bundle $N_Y X$. Let $S\nu \subset D\nu$ denote the sphere bundle and $D^\circ\nu = D\nu - S\nu$ the open disc bundle. Then, Proposition 5.3 yields a commutative diagram

$$(9) \quad \begin{array}{ccccc} H^i(X, X - Y) & \times & H_{p-j}^{\text{BM}}(X) & \xrightarrow{\cap} & H_{p-i-j}^{\text{BM}}(Y) \\ \cong \downarrow \rho^* & & \downarrow r^* & & \downarrow = \\ H^i(D^\circ\nu, (D^\circ\nu) - Y) & \times & H_{p-j}^{\text{BM}}(D^\circ\nu) & \xrightarrow{\cap} & H_{p-i-j}^{\text{BM}}(Y), \end{array}$$

where the left vertical arrow is the excision isomorphism induced by inclusion, and the middle vertical arrow is restriction of a Borel-Moore cycle to an open subset.

Let

$$u_g \in H^{2d}(X, X - Y)$$

be the orientation class of the regular closed embedding $g : Y \hookrightarrow X$ as in [30, 19.2, p. 378]. We shall next compute this class. Consider the cartesian diagram of closed embeddings

$$\begin{array}{ccc} Y = M \cap X & \xrightarrow{g} & X \\ j \downarrow & & \downarrow i \\ M & \xrightarrow{f} & W. \end{array}$$

By [30, Lemma 19.2 (a), p. 379], applied to the above cartesian diagram containing the regular embeddings f and g ,

$$i^*(u_f) = u_g \in H^{2d}(X, X - Y), \quad i^* : H^{2d}(W, W - M) \rightarrow H^{2d}(X, X - Y).$$

Let v_M be the underlying topological vector bundle of the algebraic normal bundle $N_M W$. By tightness of the embedding $g : Y \hookrightarrow X$, there is an isomorphism $v = j^* v_M$, $j : Y \hookrightarrow M$ inherited from the isomorphism of algebraic vector bundles. Let $i_D : Dv \rightarrow Dv_M$ denote the bundle map covering j . Then by naturality of the Thom class

$$t(v) = i_D^* t(v_M).$$

Consider the following factorization of the above cartesian diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{0} & Dv & \xrightarrow{\rho} & X \\ j \downarrow & & i_D \downarrow & & \downarrow i \\ M & \xrightarrow{0} & Dv_M & \xrightarrow{\delta} & W. \end{array}$$

By [30, p. 372, bottom],

$$\delta^*(u_f) = t(v_M),$$

since f is a closed embedding of nonsingular varieties. Therefore,

$$\rho^*(u_g) = \rho^* i^*(u_f) = i_D^* \delta^*(u_f) = i_D^* t(v_M) = t(v).$$

By Theorem 5.8,

$$(10) \quad u_g \cap \text{cl}(\alpha) = \text{cl}(g_{\text{alg}}^! \alpha)$$

for any algebraic cycle $\alpha \in A_*(X)$. Using Verdier's diagram (8), and the commutativity of the cap diagram (9), we obtain for $\alpha \in A_*(X)$,

$$\begin{aligned} g_{\text{alg}}^!(\text{cl}(\alpha)) &= \text{cl}(g_{\text{alg}}^! \alpha) \\ &= u_g \cap \text{cl}(\alpha) \\ &= \rho^*(u_g) \cap r^*(\text{cl}(\alpha)) \\ &= t(v) \cap r^*(\text{cl}(\alpha)). \end{aligned}$$

Thus for any class $\beta \in H_*^{\text{BM}}(X)$ in the image of the cycle map,

$$g_{\text{alg}}^!(\beta) = t(v) \cap r^* \beta = g_{\text{top}}^{!!}(\beta).$$

□

6. GYSIN COHERENT CHARACTERISTIC CLASSES

In this section, we recall the notion of Gysin coherent characteristic classes (see Definition 6.2 below), and state the uniqueness theorem, which is the main result of [11] (see Theorem 6.4 below). In the present paper, we shall discuss algebraic characteristic classes such as Todd classes (see Section 9), Chern classes (see Section 8), as well as motivic Hodge classes (see Section 7) within the framework of Gysin coherence.

In the following, by a variety we mean a pure-dimensional complex quasiprojective algebraic variety.

Let \mathcal{X} be a family of inclusions $i: X \rightarrow W$, where W is a smooth variety, and $X \subset W$ is a compact irreducible subvariety. We require the following properties for \mathcal{X} :

- For every Schubert subvariety $X \subset G$ of a Grassmannian G , the inclusion $X \rightarrow G$ is in \mathcal{X} .
- If $i: X \rightarrow W$ and $i': X' \rightarrow W'$ are in \mathcal{X} , then the product $i \times i': X \times X' \rightarrow W \times W'$ is in \mathcal{X} .
- Given inclusions $i: X \rightarrow W$ and $i': X' \rightarrow W'$ of compact subvarieties in smooth varieties, and an isomorphism $W \xrightarrow{\cong} W'$ that restricts to an isomorphism $X \xrightarrow{\cong} X'$, it follows from $i \in \mathcal{X}$ that $i' \in \mathcal{X}$.
- For all closed subvarieties $X \subset M \subset W$ such that X is compact and M and W are smooth, it holds that if the inclusion $X \rightarrow M$ is in \mathcal{X} , then the inclusion $X \rightarrow W$ is in \mathcal{X} .

Schubert varieties $X \subset G$ are included in the family \mathcal{X} for the following reason: The proof of the Uniqueness Theorem in [11] (recalled as Theorem 6.4 below) is based on an induction which assembles a sufficient amount of Gysin injectivity to determine the class uniquely. It turns out that a sufficient quantity of subvarieties is given by taking products of X with certain Schubert varieties, Segre embedding such products, and then intersecting with another Schubert variety. Schubert calculus can then be used to compute intersections and Gysin maps.

Next, for a given family \mathcal{X} of inclusions as above, by \mathcal{X} -*transversality*, we mean a symmetric relation for closed irreducible subvarieties of a smooth variety that satisfies the following properties:

- The intersection $Z \cap Z'$ of two \mathcal{X} -transverse closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W is *proper*, that is, $Z \cap Z'$ is pure-dimensional of codimension $c + c'$, where c and c' are the codimensions of Z and Z' in W , respectively.
- The following analog of Kleiman's transversality theorem holds for the (transitive) action of $GL_n(\mathbb{C})$ on the Grassmannians $G = G_k(\mathbb{C}^n)$. If $i: X \rightarrow G$ and $i': X' \rightarrow G$ are inclusions in \mathcal{X} , then there is a nonempty open dense subset $U \subset GL_n(\mathbb{C})$ (in the complex topology) such that X is \mathcal{X} -transverse to $g \cdot X'$ for all $g \in U$.
- *Locality*: If $Z, Z' \subset W$ are \mathcal{X} -transverse closed irreducible subvarieties of a smooth variety W and $U \subset W$ is a (Zariski) open subset that has nontrivial intersections with Z and Z' , then $Z \cap U$ and $Z' \cap U$ are \mathcal{X} -transverse in U .

Example 6.1. The family \mathcal{X}_0 consisting of all inclusions of compact irreducible subvarieties in smooth varieties satisfies the above family requirements. Furthermore, a notion of \mathcal{X}_0 -transversality is obtained by calling two closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W \mathcal{X}_0 -transverse if they admit complex algebraic Whitney stratifications (see Example 4.6) such that every stratum of Z is transverse to every stratum of Z' (as smooth submanifolds of W). This uses Kleiman's transversality theorem [39].

Recall from Section 5 that every inclusion $f: M \rightarrow W$ of a smooth closed subvariety M of (complex) codimension c in a smooth variety W induces a topological Gysin map on singular homology, $f_{\text{top}}^!: H_*(W; \mathbb{Q}) \rightarrow H_{*-2c}(M; \mathbb{Q})$.

Definition 6.2. A *Gysin coherent characteristic class* cl with respect to \mathcal{X} is a pair

$$cl = (cl^*, cl_*)$$

consisting of a function cl^* that assigns to every inclusion $f: M \rightarrow W$ of a smooth closed subvariety $M \subset W$ in a smooth variety W an element

$$cl^*(f) = cl^0(f) + cl^1(f) + cl^2(f) + \cdots \in H^*(M; \mathbb{Q}), \quad cl^p(f) \in H^p(M; \mathbb{Q}),$$

with $cl^0(f) = 1$, and a function cl_* that assigns to every inclusion $i: X \rightarrow W$ of a compact possibly singular subvariety $X \subset W$ of complex dimension d in a smooth variety W an element

$$cl_*(i) = cl_0(i) + cl_1(i) + cl_2(i) + \cdots + cl_{2d}(i) \in H_*(W; \mathbb{Q}), \quad cl_p(i) \in H_p(W; \mathbb{Q}),$$

with $cl_{2d}(i) = [X]_W$, such that the following properties hold:

- (1) (*Multiplicativity*) For every $i: X \rightarrow W$ and $i': X' \rightarrow W'$, we have

$$cl_*(i \times i') = cl_*(i) \times cl_*(i').$$

- (2) (*Isomorphism invariance*) For every $f: M \rightarrow W$ and $f': M' \rightarrow W'$, and every isomorphism $W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\phi: M \xrightarrow{\cong} M'$, we have

$$\phi^* cl^*(f') = cl^*(f).$$

Moreover, for every $i: X \rightarrow W$ and $i': X' \rightarrow W'$, and every isomorphism $\Phi: W \xrightarrow{\cong} W'$ that restricts to an isomorphism $X \xrightarrow{\cong} X'$, we have

$$\Phi_* cl_*(i) = cl_*(i').$$

- (3) (*Naturality*) For every $i: X \rightarrow W$ and $f: M \rightarrow W$ such that $X \subset M$, the inclusion $i^M := i|: X \rightarrow M$ satisfies

$$f_* cl_*(i^M) = cl_*(i).$$

- (4) (*Gysin restriction in a transverse setup*) There exists a notion of \mathcal{X} -transversality such that the following holds. For every inclusion $i: X \rightarrow W$ in \mathcal{X} and every inclusion $f: M \rightarrow W$ such that M is irreducible, and M and X are \mathcal{X} -transverse in W , the inclusion $j: Y \rightarrow M$ of the pure-dimensional compact subvariety $Y := M \cap X \subset M$ satisfies

$$f_{\text{top}}^! cl_*(i) = cl^*(f) \cap cl_*(j).$$

Such a class cl is called *Gysin coherent characteristic class* if $\mathcal{X} = \mathcal{X}_0$ is the family of all inclusions of compact irreducible subvarieties in smooth varieties (compare Example 6.1).

The genus $|cl_*|$ of a Gysin coherent characteristic class cl with respect to \mathcal{X} is defined as the composition of cl_* with the homological augmentation, $|cl_*| = \varepsilon_* cl_* \in \mathbb{Q}$.

Example 6.3. In [11, Section 9], the framework of Gysin coherence with respect to the family \mathcal{X}_0 of Example 6.1 is applied to Goresky-MacPherson L -classes (see Example 6.3 below), where \mathcal{X}_0 -transversality is chosen to mean simultaneously topological transversality with respect to appropriate Whitney stratifications and generic transversality. It is shown that the pair (cl^*, cl_*) given by $cl^*(f: M \hookrightarrow W) = L^*(\nu_{M \subset W})$ and $cl_*(i: X \hookrightarrow W) = i_* L_*(X)$, where L^* is Hirzebruch's cohomological L -class of a vector bundle and L_* is the Goresky-MacPherson L -class, forms a Gysin coherent characteristic class. The associated genus is the signature $\sigma(X)$

of the Goresky-MacPherson intersection form on middle-perversity intersection homology of X .

The main result of [11] is the following

Theorem 6.4 (Uniqueness Theorem). *Let $c\ell$ and $\tilde{c}\ell$ be Gysin coherent characteristic classes with respect to \mathcal{X} . If $c\ell^* = \tilde{c}\ell^*$ and $|c\ell_*| = |\tilde{c}\ell_*|$ for the associated genera, then we have $c\ell_*(i) = \tilde{c}\ell_*(i)$ for all inclusions $i: X \rightarrow G$ in \mathcal{X} of compact irreducible subvarieties in ambient Grassmannians.*

Remark 6.5. An inspection of the proof of the uniqueness theorem above shows that it suffices to request the assumption $c\ell_{2d}(i) = [X]_W$ in the definition of Gysin coherent classes $c\ell$ only for irreducible X . In fact, this assumption is only used in the proof of Theorem 7.1 and in the proof of Theorem 6.4 in [11], where the varieties under consideration are irreducible.

7. THE CLASS $IT_{1,*}$ AS A GYSIN COHERENT CHARACTERISTIC CLASS

The class $IT_{1,*}$ fits into the framework of Gysin coherent characteristic classes (Definition 6.2), as we shall show in Theorem 7.3 below. Our method of proof requires moving varieties into topologically and algebraically transverse position by means of suitable generalizations of Kleiman's transversality theorem. By algebraic transversality, we mean Tor-independence as formulated in Definition 4.11; this is also called homological transversality in the literature. The Tor-independence together with transversality to Whitney strata ensures tightness of the embedding (see [8, Prop. 6.3], which is recalled as Proposition 4.12 here). The notion of tightness was introduced in [8] and was recalled in Definition 4.8. Roughly, a closed regular algebraic embedding of varieties is tight, if its underlying topological embedding is normally nonsingular and the topological normal vector bundle agrees with the algebraic normal vector bundle of the regular embedding. To achieve algebraic transversality generically, we use the Kleiman-Bertini theorem for sheaf tensor products of Miller-Speyer [47].

As in Section 6, by a variety we mean a pure-dimensional complex quasiprojective algebraic variety. For later use in the proof of Theorem 7.3 below, we record the following

Proposition 7.1. *For an irreducible projective variety X of complex dimension d , the highest non-trivial homogeneous component of the class*

$$IT_{1,*}(X) = IT_{1,0}(X) + IT_{1,1}(X) + \cdots \in H_*(X; \mathbb{Q})$$

is the fundamental class, $IT_{1,2d}(X) = [X]_X$.

Proof. We have $IT_{y,2d}(X) = IT_{y,2d}(X_{\text{reg}})$ under the identification $H_{2d}(X; \mathbb{Q}) = H_{2d}^{\text{BM}}(X_{\text{reg}}; \mathbb{Q})$, where X_{reg} denotes the open smooth part of X . For the manifold $M = X_{\text{reg}}$, we observe $IT_{y,*}(M) = T_y^*(TM) \cap [M]$ with $T_y^0(TM) = 1$. \square

Remark 7.2. The statement of Proposition 7.1 holds similarly for $IT_{0,*}(X)$. Theorem 4.10 is also valid for $IT_{0,*}$ with correction factor $T_0^*(N_M W) = \text{td}^*(N_M W)$ instead of $T_1^*(v_f) = L^*(v_f)$ as an inspection of the proof of Theorem 6.30 in [8] shows. (In fact, the assumption $y = 1$ is only used in the proof of Proposition 6.26 in [8] to conclude that $T_1^*(1_X) = L^*(1_X) = 1 \in H^*(X; \mathbb{Q})$ for the trivial line bundle 1_X on a complex algebraic variety X , but we also have $T_0^*(1_X) = \text{td}^*(1_X) = 1 \in H^*(X; \mathbb{Q})$.)

In the proof of Theorem 7.3 below, let \mathcal{X}_{TI} denote the family \mathcal{X}_0 of Example 6.1. We will, however, consider \mathcal{X}_{TI} with a notion of transversality which is not the \mathcal{X}_0 -transversality of Example 6.3. The notion of \mathcal{X}_{TI} -transversality will incorporate in addition Tor-independence, which explains the subscript.

Theorem 7.3. *The pair $\mathcal{L} = (\mathcal{L}^*, \mathcal{L}_*)$ defined by $\mathcal{L}^*(f) = L^*(\nu_f)$ for every inclusion $f: M \rightarrow W$ of a smooth closed subvariety $M \subset W$ in a smooth variety W with normal bundle ν_f , and by $\mathcal{L}_*(i) = i_*IT_{1,*}(X)$ for every inclusion $i: X \rightarrow W$ of a compact irreducible possibly singular subvariety $X \subset W$ in a smooth variety W is a Gysin coherent characteristic class.*

Proof. By the properties of the cohomological Hirzebruch L -class, the class $\mathcal{L}^*(f) = L^*(\nu_f) \in H^*(M; \mathbb{Q})$ is normalized for all f . Moreover, by Proposition 7.1, the highest nontrivial homogeneous component of the class $IT_{1,*}(X) = IT_{1,0}(X) + IT_{1,1}(X) + \cdots \in H_*(X; \mathbb{Q})$ is $IT_{1,2d}(X) = [X]_X$, where d denotes the complex dimension of X , and X may be assumed to be irreducible by Remark 6.5. Consequently, the highest nontrivial homogeneous component of $\mathcal{L}_*(i) = i_*IT_{1,*}(X) \in H_*(W; \mathbb{Q})$ is the ambient fundamental class $i_*IT_{1,2d}(X) = i_*[X]_X = [X]_W$.

We proceed to check the axioms of Gysin coherent characteristic classes for the pair \mathcal{L} . As for axiom (1), we have $IT_{1,*}(X \times X') = IT_{1,*}(X) \times IT_{1,*}(X')$ in $H_*(X \times X'; \mathbb{Q})$ for all pure-dimensional compact complex algebraic varieties X and X' by Corollary 2.16. Hence, for every $i: X \rightarrow W$ and $i': X' \rightarrow W'$, the claim follows by applying $(i \times i')_*$ and using naturality of the cross product:

$$\begin{aligned} \mathcal{L}_*(i \times i') &= (i \times i')_*IT_{1,*}(X \times X') = (i \times i')_*(IT_{1,*}(X) \times IT_{1,*}(X')) \\ &= i_*IT_{1,*}(X) \times i'_*IT_{1,*}(X') = \mathcal{L}_*(i) \times \mathcal{L}_*(i'). \end{aligned}$$

Next, let us show that the pair \mathcal{L} is compatible with ambient isomorphisms as stated in axiom (2). As for \mathcal{L}^* , we consider $f: M \rightarrow W$ and $f': M' \rightarrow W'$, and an isomorphism $W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\phi: M \xrightarrow{\cong} M'$. Then, we have $\phi^*\nu_{f'} = \nu_f$, and thus

$$\phi^*\mathcal{L}^*(f') = \phi^*L^*(\nu_{f'}) = L^*(\phi^*\nu_{f'}) = L^*(\nu_f) = \mathcal{L}^*(f).$$

As for \mathcal{L}_* , we consider $i: X \rightarrow W$ and $i': X' \rightarrow W'$, and an isomorphism $\Phi: W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\Phi_0: X \xrightarrow{\cong} X'$. Then, we have $\Phi_{0*}IT_{1,*}(X) = IT_{1,*}(X')$ by Proposition 2.17. Hence, we obtain

$$\Phi_*\mathcal{L}_*(i) = \Phi_*i_*IT_{1,*}(X) = i'_*\Phi_{0*}IT_{1,*}(X) = i'_*IT_{1,*}(X') = \mathcal{L}_*(i').$$

To verify axiom (3), we consider $i: X \rightarrow W$ and $f: M \rightarrow W$ such that $X \subset M$. Then, the inclusion $i^M := i|_X: X \rightarrow M$ satisfies $f \circ i^M = i$, and we obtain

$$f_*\mathcal{L}_*(i^M) = f_*i^M_*IT_{1,*}(X) = i_*IT_{1,*}(X) = \mathcal{L}_*(i).$$

Finally, to show axiom (4), let us call closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W \mathcal{X}_{TI} -transverse if Z and Z' are simultaneously complex algebraic Whitney transverse (that is, they admit complex algebraic Whitney stratifications such that every stratum of Z is transverse to every stratum of Z' as smooth submanifolds of W), generically transverse (see e.g. [11, Section 3]), and Tor-independent (see Definition 4.11) in W . This notion of \mathcal{X}_{TI} -transversality has indeed all required properties. (In fact, properness of \mathcal{X}_{TI} -transverse intersections follows from generic transversality according to [11, Corollary 3.4]. Next, to obtain the desired analog of Kleiman's transversality theorem for the action of $GL_n(\mathbb{C})$ on the Grassmannians $G = G_k(\mathbb{C}^n)$, we consider inclusions $X \rightarrow G$ and $X' \rightarrow G$ in \mathcal{X}_{TI} . Then, we apply suitable versions of Kleiman's transversality theorem to obtain an open dense subset U of $GL_n(\mathbb{C})$ (in the complex topology) such that X is \mathcal{X}_{TI} -transverse to $g \cdot X'$ for all $g \in U$. Here, Kleiman's transversality theorem holds for (complex algebraic) Whitney transversality by [11, Theorem 2.2] since X and X' are compact, for generic transversality by [11, Theorem 3.5] since X and X' are irreducible, and for Tor-independence by the Kleiman-Bertini theorem of Miller-Speyer [47]. We also note that Zariski dense open subsets are also dense in the complex topology by [48, Theorem 1, p. 58]. Finally, locality holds evidently for our

notion of \mathcal{X}_{TI} -transversality.) Now, consider an inclusion $i: X \rightarrow W$ in \mathcal{X}_{TI} and an inclusion $f: M \rightarrow W$ (of a smooth closed subvariety $M \subset W$ in a smooth variety W) such that M is irreducible, and M and X are \mathcal{X}_{TI} -transverse in W . Then, Theorem 4.15 implies that the embedding $g: Y \hookrightarrow X$ of the compact subvariety $Y = X \cap M \subset X$ is tight, and the algebraic normal bundle $N = N_Y X$ of g and the topological normal bundle ν of the topologically normally nonsingular inclusion underlying g satisfy

$$g_{\text{alg}}^! IT_{1,*}(X) = L^*(N) \cap IT_{1,*}(Y) = L^*(\nu) \cap IT_{1,*}(Y),$$

where $g_{\text{alg}}^!: H_*(X; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ denotes the algebraic Gysin homomorphism associated to g as constructed in Verdier [63] (where we may use the natural identification of Borel-Moore homology and singular homology because X and Y are compact). Furthermore, since $i \in \mathcal{X}_{TI}$ and M and X are \mathcal{X}_{TI} -transverse in W , Theorem 5.10 and Proposition 5.7 imply that the algebraic Gysin map $g_{\text{alg}}^!: H_*(X; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ coincides with the topological Gysin map $g_{\text{top}}^!: H_*(X; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ on all fundamental classes $[Z]_X$ of closed irreducible subvarieties $Z \subset X$. As $IT_{1,*}(X)$ is an algebraic cycle according to Remark 2.13, we obtain

$$g_{\text{top}}^! IT_{1,*}(X) = g_{\text{alg}}^! IT_{1,*}(X).$$

Next, recall from Theorem 4.4 that the inclusion $g: Y \hookrightarrow X$ is normally nonsingular with topological normal bundle $\nu = j^* \nu_f$ given by the restriction under the inclusion $j: Y \rightarrow M$ of the normal bundle ν_f of M in W . Using the base change $f_{\text{top}}^! i_* = j_* g_{\text{top}}^!$ for topological Gysin maps (see [11, Proposition 2.4]), as well as $L^*(\nu) = L^*(j^* \nu_f) = j^* L^*(\nu_f)$, we conclude that

$$\begin{aligned} f_{\text{top}}^! \mathcal{L}_*(i) &= f_{\text{top}}^! i_* IT_{1,*}(X) = j_* g_{\text{top}}^! IT_{1,*}(X) = j_* g_{\text{alg}}^! IT_{1,*}(X) = j_* (L^*(\nu) \cap IT_{1,*}(Y)) \\ &= j_* (j^* L^*(\nu_f) \cap IT_{1,*}(Y)) = L^*(\nu_f) \cap j_* IT_{1,*}(Y) = \mathcal{L}^*(f) \cap \mathcal{L}_*(j). \end{aligned}$$

□

Remark 7.4. A similar proof shows that the pair $(c\ell^*, c\ell_*)$ given by $c\ell^*(f: M \hookrightarrow W) = \text{td}^*(N_M W)$ and $c\ell_*(i: X \hookrightarrow W) = i_* IT_{0,*}(X)$ is also an example of a Gysin coherent characteristic class. In fact, Theorem 4.10 (and consequently, Theorem 4.15) is also valid for $IT_{0,*}$ with correction factor $T_0^*(N_M W) = \text{td}^*(N_M W)$ instead of $T_1^*(\nu_f) = L^*(\nu_f)$ as an inspection of the proof of Theorem 6.30 in [8] shows (compare Remark 7.2). See Remark 8.7 for $IT_{-1,*}(X)$.

When X is singular, the intersection Todd class $IT_{0,*}(X)$ is generally different from the Baum-Fulton-MacPherson Todd class $\text{td}_*(X)$ studied in Section 9. Let us consider the normalization $\pi: X' \rightarrow X$ of a singular curve X such that π is a homeomorphism. Let $T_{0,*} = \tau_* \circ mC_0$, where $mC_0: K_0(\text{var}/X) \rightarrow K_0^{\text{alg}}(X)$ is the evaluation at $y = 0$ of the motivic Chern class transformation mC_y of [16], recalled here in Remark 2.14. Following [16, Example 3.1], let us show that $T_{0,*}(X) \neq \text{td}_*(X)$. The normalization π restricts to an isomorphism of regular parts. It is also an isomorphism of the singular parts since those are finite sets of points and the homeomorphism π restricts to a bijection between them. Thus, the scissor relation in the motivic group $K_0(\text{var}/X)$ yields $\pi_*[\text{id}_{X'}] = [\text{id}_X]$. Then, as the smoothness of X' implies $mC_0[\text{id}_{X'}] = [\mathcal{O}_{X'}]$ (Du Bois would suffice), we find

$$T_{0,*}(X) := T_{0,*}[\text{id}_X] = T_{0,*}\pi_*[\text{id}_{X'}] = \pi_* T_{0,*}[\text{id}_{X'}] = \pi_* \tau_* mC_0[\text{id}_{X'}] = \pi_* \tau_* [\mathcal{O}_{X'}].$$

As X is singular,

$$\pi_*[\mathcal{O}_{X'}] = [\mathcal{O}_X] + n \cdot [\mathcal{O}_{\text{pt}}], \quad n > 0.$$

Hence,

$$T_{0,*}(X) = \pi_* \tau_* [\mathcal{O}_{X'}] = \tau_* \pi_* [\mathcal{O}_{X'}] = \tau_* ([\mathcal{O}_X] + n \cdot [\mathcal{O}_{\text{pt}}]) = \text{td}_*(X) + n \cdot [\text{pt}] \neq \text{td}_*(X).$$

On the other hand, since π is a resolution of singularities, IC_X^H is a direct summand of $\pi_*\mathbb{Q}_{X'}^H[1]$ (see [59, Corollary 4.6]). As π is in fact a small resolution, we have $\pi_*(\mathbb{Q}_{X'}^H) = IC_X^H[-1]$. Moreover, in view of Remark 2.14, we have $[\mathcal{O}_{X'}] = mC_0[\text{id}_{X'}] = MHC_{0*}\chi_{Hdg}[\text{id}_{X'}] = MHC_{0*}[\mathbb{Q}_{X'}^H]$. All in all, we obtain

$$\begin{aligned} T_{0*}(X) &= \pi_*\tau_*[\mathcal{O}_{X'}] = \pi_*\tau_*MHC_{0*}[\mathbb{Q}_{X'}^H] = \pi_*MHT_{0*}[\mathbb{Q}_{X'}^H] = MHT_{0*}[\pi_*\mathbb{Q}_{X'}^H] \\ &= MHT_{0*}[IC_X^H[-1]] = IT_{0*}(X). \end{aligned}$$

Therefore, $IT_{0*}(X) = T_{0*}(X) \neq \text{td}_*(X)$. (A similar example shows that $IT_{0*}(X) \neq T_{0*}(X)$ in general.)

On the other hand, $T_0(X) = \text{td}_*(X)$ in case X has only DuBois (e.g. rational) singularities, see [16, p. 6]. For example Schubert or toric varieties have rational singularities, see [19, Corollary 1, Section 1.2] and [52, Theorem 10].

Recall from Example 6.3 that the pair $(c\ell^*, c\ell_*)$ given by $c\ell^*(f: M \hookrightarrow W) = L^*(\nu_{M \subset W})$ and $c\ell_*(i: X \hookrightarrow W) = i_*L_*(X)$, where L_* is the Goresky-MacPherson L -class, forms a Gysin coherent characteristic class. Since the L -genus, i.e. the signature, agrees with the genus of IT_{1*} on complex projective algebraic varieties by Saito's intersection cohomology Hodge index theorem (see [53], [45, Section 3.6]), the uniqueness theorem for Gysin coherent characteristic classes (Theorem 6.4) implies

Theorem 7.5. (*Ambient Brasselet-Schürmann-Yokura Conjecture.*) *We have*

$$i_*L_*(X) = i_*IT_{1,*}(X)$$

for all inclusions $i: X \rightarrow G$ of compact irreducible subvarieties in Grassmannians G .

Since the homology of Schubert subvarieties injects into the homology of the ambient Grassmannian, we obtain

Corollary 7.6. *We have $L_*(X) = IT_{1,*}(X)$ for all Schubert varieties X in a Grassmannian. More generally, the conjecture holds for all compact irreducible varieties X that admit an H_* -injective embedding into a Grassmannian.*

8. THE CHERN CLASS AS A GYSIN COHERENT CHARACTERISTIC CLASS

Let $c_*: F(X) \rightarrow H_{2*}^{BM}(X) \otimes \mathbb{Q}$ denote the (rationalized) Chern class transformation of MacPherson [43] defined on the group $F(X)$ of \mathbb{Z} -valued algebraically constructible functions on the possibly singular complex algebraic variety X . (Such a constructible function is a linear combination of indicator functions 1_Z with $Z \subset X$ a closed irreducible subvariety.) The Chern-Schwartz-MacPherson class $c_*(X)$ of a possibly singular irreducible complex algebraic variety X is defined as

$$c_*(X) := c_*(1_X) \in H_{2*}^{BM}(X; \mathbb{Q}).$$

While the Chern classes $c_*(X)$ were originally defined with integral coefficients, we need to consider them with rational coefficients to be able to study them in the framework of Gysin coherence.

In [60], the second author derives the following Verdier-Riemann-Roch type theorem for the behavior of the Chow homology Chern class transformation $c_*: F(X) \rightarrow A_*(X)$ under refined Gysin maps associated to transverse intersections in a microlocal context.

Theorem 8.1 (see Corollary 2.7 in [60]). *Let $f: M \rightarrow W$ be a closed embedding of smooth complex varieties with algebraic normal bundle $N = N_M W$. Let $X \subset W$ be a (Zariski) closed subspace, and set $Y := f^{-1}(X) = M \cap X \subset M$. Assume that $\gamma \in F(X)$ is a constructible*

function such that f is non-characteristic with respect to the support $\text{supp}(CC(\gamma))$ of the characteristic cycle $CC(\gamma)$ of γ . Then,

$$(f, i)_{\text{ref}}^!(c_*(\gamma)) = c^*(N|_Y) \cap c_*(g^*(\gamma)) \in A_*(Y),$$

where $(f, i)_{\text{ref}}^!: A_*(X) \rightarrow A_*(Y)$ denotes the refined Gysin map associated to the cartesian square of closed embeddings

$$\begin{array}{ccc} Y := M \cap X & \xrightarrow{g} & X \\ \downarrow & & \downarrow i \\ M & \xrightarrow{f} & W \end{array}$$

(see Fulton [30, Section 6.2]).

Remark 8.2. As pointed out in [60], the non-characteristic condition for $\gamma \in F(X)$ in Theorem 8.1 above is for example satisfied when M is transverse to all strata of a complex algebraic Whitney stratification of $X \subset W$, and γ is constructible with respect to this stratification, i.e., $\gamma|_S$ is locally constant for all strata S of X .

A simple consequence is that the highest non-trivial homogeneous component of the Chern class is the fundamental class (compare Proposition 7.1).

Corollary 8.3. *For an irreducible projective variety X of complex dimension d , the highest non-trivial homogeneous component of the class*

$$c_*(X) = c_0(X) + c_1(X) + \cdots \in A_*(X)$$

is the fundamental class, $c_d(X) = [X]_X$.

Proof. While this can be deduced from Theorem 8.1, an alternative route is simply to observe that the restriction of $c_d(X)$ to the regular part of X is $c_d(X_{\text{reg}})$. The restriction $A_d(X) \rightarrow A_d(X_{\text{reg}})$ is an isomorphism by [30, Proposition 1.8, p. 21] (recall that X is assumed to be irreducible). \square

Theorem 8.1 also implies the following result.

Corollary 8.4. *Let $X \hookrightarrow W \hookleftarrow M$ be closed algebraic embeddings of pure-dimensional complex quasiprojective algebraic varieties with M, W smooth. Let $V \subset X$ be an irreducible closed subvariety. We suppose that $X \subset W$ is equipped with a complex algebraic Whitney stratification such that M is transverse to all strata, and V is a union of strata. Then the refined Gysin map*

$$(f, i)_{\text{ref}}^!: A_*(X) \longrightarrow A_{*-c}(Y)$$

associated to the cartesian square of inclusions

$$\begin{array}{ccc} Y := M \cap X & \xrightarrow{g} & X \\ \downarrow & & \downarrow i \\ M & \xrightarrow{f} & W \end{array}$$

satisfies $(f, i)_{\text{ref}}^![V]_X = [V \cap Y]_Y$, where c denotes the complex codimension of M in W .

Proof. By Remark 8.2, the constructible function

$$\gamma := 1_V \in F(X)$$

satisfies the non-characteristic property of Theorem 8.1 because the given complex algebraic Whitney stratification on X is such that M is transverse to all strata, and V is a union of strata. The pull-back $g^*: F(X) \rightarrow F(Y)$ satisfies $g^*(\gamma) = g^*(1_V) = 1_{V \cap Y}$. Hence, Theorem 8.1 yields

$$(11) \quad (f, i)_{\text{ref}}^!(c_*(1_V)) = c^*(N|_Y) \cap c_*(1_{V \cap Y}) \in A_*(Y).$$

The inclusions $\alpha: V \hookrightarrow X$ and $\beta: V \cap Y \hookrightarrow Y$ are closed embeddings, and hence proper. The induced maps $\alpha_*: F(V) \rightarrow F(X)$ and $\beta_*: F(V \cap Y) \rightarrow F(Y)$ satisfy $\alpha_*(1_V) = 1_V$ and $\beta_*(1_{V \cap Y}) = 1_{V \cap Y}$, respectively. (In general, the push-forward $\varphi_*: F(U) \rightarrow F(U')$ of a proper morphism $\varphi: U \rightarrow U'$ is defined on a subvariety $Z \subset U$ by $\varphi_*(1_Z)(v) = \chi(Z \cap \varphi^{-1}(v))$, where χ denotes the topological Euler characteristic, see [30, p. 376, Example 19.1.7].) Since c_* commutes with proper push-forward according to [30, p. 377, Example 19.1.7], equation (11) becomes

$$(12) \quad (f, i)_{\text{ref}}^!(\alpha_* c_*(V)) = c^*(N|_Y) \cap \beta_* c_*(V \cap Y) \in A_*(Y).$$

Let d denote the complex dimension of V . Then, the highest non-trivial homogeneous component of $c_*(V)$ is $c_d(V) = [V]_V$ by Corollary 8.3. Similarly, since $V \cap Y = V \cap M$ is pure $(d - c)$ -dimensional, the highest non-trivial homogeneous component of $c_*(V \cap Y)$ is $c_{d-c}(V \cap Y) = [V \cap Y]_{V \cap Y}$. Consequently, evaluation of equation (12) in degree $d - c$ yields

$$(f, i)_{\text{ref}}^!([V]_X) = [V \cap Y]_Y \in A_{d-c}(Y).$$

□

Since $c_*(X) \in A_*(X)$ is an algebraic cycle, there exist a finite number of irreducible closed subvarieties $V_1, \dots, V_r \subset X$ such that

$$(13) \quad c_*(X) = \sum_{l=1}^r \lambda_l [V_l]_X, \quad \lambda_l \in \mathbb{Z}.$$

For an irreducible closed subvariety $X \subset W$ of a smooth variety W , we call a complex algebraic Whitney stratification of $X \subset W$ $c_*(X)$ -constructible if there is a representation (13) in which every V_l is a union of strata of X . It follows from [35, p. 43, Section 1.7, Theorem] that such Whitney stratifications exist on $X \subset W$.

Theorem 8.5. *Let $X \hookrightarrow W \hookleftarrow M$ be closed algebraic embeddings of pure-dimensional complex algebraic varieties with M, W smooth. Suppose that X is equipped with a complex algebraic Whitney stratification that is $c_*(X)$ -constructible, and such that M is transverse to all strata of X and $Y = M \cap X$ is compact. Then, the topologically normally nonsingular inclusion $g: Y = M \cap X \hookrightarrow X$ satisfies*

$$g_{\text{top}}^{!!}(c_*(X)) = c^*(N|_Y) \cap c_*(Y) \in H_{2*}^{\text{BM}}(Y) \otimes \mathbb{Q},$$

where $N = N_M W$ denotes the complex normal bundle of M in W .

Proof. By Remark 8.2, the constructible function

$$\gamma := 1_X \in F(X)$$

satisfies the non-characteristic property of Theorem 8.1 because M is by assumption transverse to all strata of the given complex algebraic Whitney stratification on X , and $\gamma|_S = 1_S$ is locally constant for all strata S of X . The pull-back $g^*: F(X) \rightarrow F(Y)$ induced by the closed embedding $g: Y \hookrightarrow X$ satisfies $g^*(\gamma) = g^*(1_X) = 1_{M \cap X} = 1_Y$. Hence, Theorem 8.1 yields

$$(f, i)_{\text{ref}}^!(c_*(X)) = c^*(N|_Y) \cap c_*(Y) \in A_*(Y).$$

By assumption, the Chern-Schwartz-MacPherson class $c_*(X)$ can be written in the form (13) for some irreducible closed subvarieties $V_1, \dots, V_r \subset X$ such that every V_l is a union of strata of the given complex algebraic Whitney stratification on X . Therefore, Corollary 8.4 implies that

$$(f, i)_{\text{ref}}^!(c_*(X)) = (f, i)_{\text{ref}}^! \sum_{l=1}^r \lambda_l [V_l]_X = \sum_{l=1}^r \lambda_l \cdot (f, i)_{\text{ref}}^!([V_l]_X) = \sum_{l=1}^r \lambda_l [V_l \cap Y]_Y.$$

By applying the cycle map $\text{cl} : A_*(Y) \rightarrow H_{2*}^{\text{BM}}(Y) \otimes \mathbb{Q}$ from Chow homology to Borel-Moore homology, we conclude that

$$\begin{aligned} \sum_{l=1}^r \lambda_l [V_l \cap Y]_Y &= \text{cl}(f, i)_{\text{ref}}^!(c_*(X)) = \text{cl}(c^*(N|_Y) \cap c_*(Y)) \\ &= c^*(N|_Y) \cap \text{cl}(c_*(Y)) \in H_{2*}^{\text{BM}}(Y) \otimes \mathbb{Q}, \end{aligned}$$

where the cycle map cl and the cap product with Chern classes are compatible by [30, p. 374, Prop. 19.1.2]. Furthermore, by the analog of Proposition 5.1 for the topological Gysin map in Borel-Moore homology (see Remark 5.4), we have

$$\sum_{l=1}^r \lambda_l [V_l \cap Y]_Y = \sum_{l=1}^r \lambda_l g_{\text{top}}^{\text{!!}}([V_l]_X) = g_{\text{top}}^{\text{!!}} \sum_{l=1}^r \lambda_l [V_l]_X = g_{\text{top}}^{\text{!!}}(c_*(X)) \in H_{2*}^{\text{BM}}(Y) \otimes \mathbb{Q}.$$

□

Theorem 8.6. *The pair $(c\ell^*, c\ell_*)$ given by $c\ell^*(f : M \hookrightarrow W) = c^*(N_M W)$ and $c\ell_*(i : X \hookrightarrow W) = i_* c_*(X)$ forms a Gysin coherent characteristic class.*

Proof. By the properties of the cohomological Chern class c^* , the class $c\ell^*(f) = c^*(N_f) \in H^*(M; \mathbb{Q})$ is normalized for all f . Moreover, by Corollary 8.3, the highest nontrivial homogeneous component of the class $c_*(X) = c_0(X) + c_1(X) + \dots \in H_*(X; \mathbb{Q})$ is $c_d(X) = [X]_X$, where d denotes the complex dimension of X , and X may be assumed to be irreducible by Remark 6.5. Consequently, the highest nontrivial homogeneous component of $c\ell_*(i) = i_* c_*(X) \in H_*(W; \mathbb{Q})$ is the ambient fundamental class $i_* c_d(X) = i_* [X]_X = [X]_W$.

We proceed to check the axioms of Gysin coherent characteristic classes for the pair $c\ell$. As for axiom (1), multiplicativity $c_*(X \times X') = c_*(X) \times c_*(X')$ holds for all compact irreducible complex algebraic varieties X and X' by Kwiecinski [40] and Kwiecinski-Yokura [41]. Hence, for every $i : X \rightarrow W$ and $i' : X' \rightarrow W'$, the claim follows by applying $(i \times i')_*$ and using naturality of the cross product:

$$\begin{aligned} c\ell_*(i \times i') &= (i \times i')_* c_*(X \times X') = (i \times i')_*(c_*(X) \times c_*(X')) \\ &= i_* c_*(X) \times i'_* c_*(X') = c\ell_*(i) \times c\ell_*(i'). \end{aligned}$$

Next, let us show that the pair $c\ell$ is compatible with ambient isomorphisms as stated in axiom (2). As for c^* , we consider $f : M \rightarrow W$ and $f' : M' \rightarrow W'$, and an isomorphism $W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\phi : M \xrightarrow{\cong} M'$. Then, we have $\phi^* N_{f'} = N_f$, and thus

$$\phi^* c\ell^*(f') = \phi^* c^*(N_{f'}) = c^*(\phi^* N_{f'}) = c^*(N_f) = c\ell^*(f).$$

As for $c\ell_*$, we consider $i : X \rightarrow W$ and $i' : X' \rightarrow W'$, and an isomorphism $\Phi : W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\Phi_0 : X \xrightarrow{\cong} X'$. Then, we have

$$c_*(X') = c_*(1_{X'}) = c_*(\Phi_{0*} 1_X) = \Phi_{0*} c_*(1_X) = \Phi_{0*} c_*(X)$$

because $c_* : F(X) \rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}$ commutes with proper push-forward (see e.g. [30, p. 377, top]). Hence, we obtain

$$\Phi_* c\ell_*(i) = \Phi_* i_* c_*(X) = i'_* \Phi_{0*} c_*(X) = i'_* c_*(X') = c\ell_*(i').$$

To verify axiom (3), we consider $i: X \rightarrow W$ and $f: M \rightarrow W$ such that $X \subset M$. Then, the inclusion $i^M := i|_M: X \rightarrow M$ satisfies $f \circ i^M = i$, and we obtain

$$f_* c\ell_*(i^M) = f_* i_*^M c_*(X) = i_* c_*(X) = c\ell_*(i).$$

Finally, to show axiom (4), let us recall from Example 6.1 that $\mathcal{X} = \mathcal{X}_0$ is the family of all inclusions $i: X \hookrightarrow W$ of compact irreducible subvarieties X in smooth varieties W . Furthermore, let us call two closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W \mathcal{X}_0 -transverse if they admit complex algebraic Whitney stratifications such that every stratum of Z is transverse to every stratum of Z' , where the stratification of Z (resp. Z') can be chosen to be $c_*(Z)$ -constructible when Z is compact (resp. $c_*(Z')$ -constructible when Z' is compact). Then, the notion of \mathcal{X}_0 -transversality satisfies all required properties. (In fact, the intersection $Z \cap Z'$ of two \mathcal{X}_0 -transverse closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W is proper, which follows from the transversality of the strata of the complex algebraic Whitney stratifications of Z and Z' . Moreover, for any inclusions $i: X \rightarrow G$ and $i': X' \rightarrow G$ in \mathcal{X}_0 , there is a nonempty open dense subset $U \subset GL_n(\mathbb{C})$ (in the complex topology) such that X is \mathcal{X}_0 -transverse to $g \cdot X'$ for all $g \in U$. This can be achieved by applying a topological version of Kleiman's transversality theorem (see e.g. [35, p. 39, Theorem 1.3.6 and Example 1.3.7]) to complex algebraic Whitney stratifications on the compact varieties X and X' that are $c_*(X)$ -constructible and $c_*(X')$ -constructible, respectively. Finally, to see that our notion of \mathcal{X}_0 -transversality satisfies locality, we suppose that $Z, Z' \subset W$ are \mathcal{X}_0 -transverse. Then, the desired transverse Whitney stratifications on $Z \cap U$ and $Z' \cap U$ are obtained by intersecting those on Z and Z' with U . Moreover, if, say, $Z \cap U$ is compact, and hence a complete variety, then it is also a closed subset of Z , see e.g. [48, p. 55, property i)]. But since $Z \cap U$ is also non-empty open subset of the irreducible space Z , we conclude that $Z \cap U = Z$. Hence, Z is compact and $c_*(Z)$ -constructible, that is, $Z \cap U$ is $c_*(Z \cap U)$ -constructible.) Now, consider an inclusion $i: X \rightarrow W$ in \mathcal{X}_0 and an inclusion $f: M \rightarrow W$ (of a smooth closed subvariety $M \subset W$ in a smooth variety W) such that M is irreducible, and M and X are \mathcal{X}_0 -transverse in W . By definition of \mathcal{X}_0 -transversality, X can be equipped with a complex algebraic Whitney stratification that is $c_*(X)$ -constructible, and such that M is transverse to all strata of X . Hence, by Theorem 8.5 and Proposition 5.7 (where we may use the natural identification of Borel-Moore homology and singular homology because X and Y are compact), the topologically normally nonsingular inclusion $g: Y = M \cap X \hookrightarrow X$ satisfies

$$g_{\text{top}}^!(c_*(X)) = c^*(j^*N) \cap c_*(Y) \in H_{2*}(Y) \otimes \mathbb{Q},$$

where $N = N_M W$ denotes the complex normal bundle of M in W , and $j: Y \hookrightarrow M$ is the inclusion. Using the base change $f_{\text{top}}^! i_* = j_* g_{\text{top}}^!$ for topological Gysin maps (see [11, Proposition 2.4]), as well as $c^*(j^*N) = j^* c^*(N)$, we conclude that

$$\begin{aligned} f_{\text{top}}^! c\ell_*(i) &= f_{\text{top}}^! i_* c_*(X) = j_* g_{\text{top}}^! c_*(X) = j_* (c^*(j^*N) \cap c_*(Y)) \\ &= j_* (j^* c^*(N) \cap c_*(Y)) = c^*(N) \cap j_* c_*(Y) = c\ell^*(f) \cap c\ell_*(j). \end{aligned}$$

This completes the proof of Theorem 8.6. \square

Remark 8.7. A similar proof shows that the pair $(c\ell^*, c\ell_*)$ given by $c\ell^*(f) = c^*(N_f)$ and $c\ell_*(i) = i_* Ic_*(X)$ is also an example of a Gysin coherent characteristic class. Here, $Ic_*(X) := c_* \mathcal{X}_{\text{stalk}}(IC_X^H[-n])$, with the constructible function $\mathcal{X}_{\text{stalk}}$ given by the stalkwise Euler characteristic. (For completeness, we also point out that $Ic_*(X) = IT_{-1,*}(X)$.)

Remark 8.8. The same type of argument can be used to show that the pair $(c\ell^*, c\ell_*)$ given by $c\ell^*(f) = c^*(N_f)$ and $c\ell_*(i) = i_*c_*^M(X)$ is another example of a Gysin coherent characteristic class, where

$$c_*^M(X) := c_*(Eu_X) \in H_{2*}^{BM}(X; \mathbb{Q})$$

is the Chern-Mather class of a possibly singular irreducible complex projective algebraic variety X , and Eu_X denotes the Euler obstruction constructible function of X introduced by MacPherson [43]. In the proof, one exploits the following well known properties (see e.g. Parusiński and Pragacz [49, Lemma 1.1]), established by MacPherson in [43] and by Brasselet in [15] based on Brasselet-Schwartz [17]:

- (1) $Eu_X(x) = 1$ for $x \in X_{reg}$.
- (2) $Eu_{X \times X'}(x, x') = E_X(x) \cdot Eu_{X'}(x')$ for $x \in X, x' \in X'$.
- (3) Eu_X is constructible with respect to any complex Whitney stratification of X .
- (4) $f^*(Eu_X) = Eu_{M \cap X}$ for a complex manifold embedding $f: M \rightarrow W$ transverse to a complex Whitney stratification of X .

We have seen that the three generalizations c_* , c_*^M and Ic_* of the Chern class to singular varieties give rise to Gysin coherent classes. As these classes are generally not equal, their associated genera must already be different in light of the Uniqueness Theorem 6.4 for Gysin coherent classes. Indeed,

$$|c_*|(X) = \chi(X)$$

the topological Euler characteristic,

$$|c_*^M|(X) = \chi(X; Eu_X) = \sum_S Eu_X(S) \cdot \chi(S),$$

where the sum ranges over all connected strata S of a complex algebraic Whitney stratification of X , and

$$|Ic_*|(X) = |IT_{-1*}|(X) = I\chi_{-1}(X) = \chi(IH^*(X; \mathbb{Q})),$$

the intersection cohomology Euler characteristic.

9. THE TODD CLASS AS A GYSIN COHERENT CHARACTERISTIC CLASS

Let $\tau_*: K_0(X) \rightarrow H_{2*}^{BM}(X) \otimes \mathbb{Q}$ denote the Todd class transformation of Baum, Fulton and MacPherson [12], [32]. Recall from Remark 2.5 that this transformation is compatible with its Chow homology analog $\tau_*: K_0(X) \rightarrow A_*(X) \otimes \mathbb{Q}$ under the cycle map $cl: A_*(X) \otimes \mathbb{Q} \rightarrow H_{2*}^{BM}(X) \otimes \mathbb{Q}$. Baum, Fulton and MacPherson [12] define the Todd class of a possibly singular complex algebraic variety X as

$$td_*(X) := \tau_*([\mathcal{O}_X]) \in H_{2*}^{BM}(X; \mathbb{Q}).$$

Its genus $|td_*|$ (for X compact) is the arithmetic genus of a singular variety, i.e. the holomorphic Euler characteristic.

Todd classes can be studied within the framework of Gysin coherent characteristic classes by a similar method to that used for the class $IT_{1,*}$ in Section 7. In the following result, we are therefore using the same definition of the set $\mathcal{X}_{TI} = \mathcal{X}_0$ and the notion of \mathcal{X}_{TI} -transversality as in the proof of Theorem 7.3 to be able to replace the algebraic by the topological Gysin map in Verdier's Gysin restriction formula for the Todd class.

Theorem 9.1. *The pair $(c\ell^*, c\ell_*)$ defined by $c\ell^*(f) = td^*(N_f)$ for every inclusion $f: M \rightarrow W$ of a smooth closed subvariety $M \subset W$ in a smooth variety W with complex normal bundle N_f , and by $c\ell_{2*}(i) = i_*td_*(X)$ for every inclusion $i: X \rightarrow W$ of a compact possibly singular subvariety $X \subset W$ in a smooth variety W , is a Gysin coherent characteristic class.*

Proof. By the properties of the cohomological Todd class, the class $c\ell^*(f) = \text{td}^*(N_f) \in H^*(M; \mathbb{Q})$ is normalized for all f . Moreover, the highest non-vanishing homogeneous component of $\text{td}_*(X)$ is the fundamental class $\text{td}_d(X) = [X]_X \in H_{2d}(X; \mathbb{Q})$ according to [30, p. 353, Theorem 18.3.5(5)]. Consequently, the highest nontrivial homogeneous component of $c\ell_{2*}(i) = i_* \text{td}_*(X) \in H_*(W; \mathbb{Q})$ is the ambient fundamental class $i_* \text{td}_d(X) = i_* [X]_X = [X]_W$.

We proceed to check the axioms of Gysin coherent characteristic classes for the pair $c\ell$. As for axiom (1), the multiplicativity property $\text{td}_*(X \times X') = \text{td}_*(X) \times \text{td}_*(X')$ holds for all compact irreducible complex algebraic varieties X and X' by [30, p. 360, Example 18.3.1]. Hence, for every $i: X \rightarrow W$ and $i': X' \rightarrow W'$, the claim follows by applying $(i \times i')_*$ and using naturality of the cross product:

$$\begin{aligned} c\ell_{2*}(i \times i') &= (i \times i')_* \text{td}_*(X \times X') = (i \times i')_*(\text{td}_*(X) \times \text{td}_*(X')) \\ &= i_* \text{td}_*(X) \times i'_* \text{td}_*(X') = c\ell_{2*}(i) \times c\ell_{2*}(i'). \end{aligned}$$

Next, let us show that the pair $c\ell$ is compatible with ambient isomorphisms as stated in axiom (2). As for $c\ell^*$, we consider $f: M \rightarrow W$ and $f': M' \rightarrow W'$, and an isomorphism $W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\phi: M \xrightarrow{\cong} M'$. Then, we have $\phi^* N_{f'} = N_f$, and thus

$$\phi^* c\ell^*(f') = \phi^* \text{td}^*(N_{f'}) = \text{td}^*(\phi^* N_{f'}) = \text{td}^*(N_f) = c\ell^*(f).$$

As for $c\ell_*$, we consider $i: X \rightarrow W$ and $i': X' \rightarrow W'$, and an isomorphism $\Phi: W \xrightarrow{\cong} W'$ that restricts to an isomorphism $\Phi_0: X \xrightarrow{\cong} X'$. Invariance $\Phi_{0*} \text{td}_*(X) = \text{td}_*(X')$ under algebraic isomorphisms $\Phi_0: X \xrightarrow{\cong} X'$ follows from [30, p. 360, Example 18.3.3]. Hence, we obtain

$$\Phi_* c\ell_{2*}(i) = \Phi_* i_* \text{td}_*(X) = i'_* \Phi_{0*} \text{td}_*(X) = i'_* \text{td}_*(X') = c\ell_{2*}(i').$$

To verify axiom (3), we consider $i: X \rightarrow W$ and $f: M \rightarrow W$ such that $X \subset M$. Then, the inclusion $i^M := i|_X: X \rightarrow M$ satisfies $f \circ i^M = i$, and we obtain

$$f_* c\ell_{2*}(i^M) = f_* i_* \text{td}_*(X) = i_* \text{td}_*(X) = c\ell_{2*}(i).$$

Finally, to show axiom (4), let us proceed as in the proof of Theorem 7.3 and call closed irreducible subvarieties $Z, Z' \subset W$ of a smooth variety W \mathcal{X}_{TI} -transverse if Z and Z' are simultaneously complex algebraic Whitney transverse (that is, they admit complex algebraic Whitney stratifications such that every stratum of Z is transverse to every stratum of Z' as smooth submanifolds of W), generically transverse (see e.g. [11, Section 3]), and Tor-independent (see Definition 4.11) in W . Now, consider an inclusion $i: X \rightarrow W$ in \mathcal{X}_{TI} and an inclusion $f: M \rightarrow W$ (of a smooth closed subvariety $M \subset W$ in a smooth variety W) such that M is irreducible, and M and X are \mathcal{X}_{TI} -transverse in W . Then, Proposition 4.12 implies that the embedding $g: Y \hookrightarrow X$ of the compact subvariety $Y = X \cap M \subset X$ is tight. For the regular closed embedding $g: Y \hookrightarrow X$ with algebraic normal bundle $N = N_Y X$, we have the Gysin restriction formula $g_{\text{alg}}^1 \text{td}_*(X) = \text{td}^*(N) \cap \text{td}_*(Y)$ on Chow homology (see [30, p. 361, Example 18.3.5]). The latter is a direct consequence of the Verdier-Riemann-Roch formula for the Todd class transformation τ_* , which was conjectured by Baum-Fulton-MacPherson in [12, p. 137], and proved by Verdier [63, p. 214, Theorem 7.1] (see also [30, p. 349, Theorem 18.2(3)]). By invoking the cycle map $\text{cl}: A_*(X) \otimes \mathbb{Q} \rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}$, we obtain

$$\begin{aligned} g_{\text{alg}}^1 \text{td}_*(X) &= g_{\text{alg}}^1 \text{cl}(\text{td}_*(X)) = \text{cl} g_{\text{alg}}^1 \text{td}_*(X) \\ &= \text{cl}(\text{td}^*(N) \cap \text{td}_*(Y)) = \text{td}^*(N) \cap \text{cl}(\text{td}_*(Y)) = \text{td}^*(N) \cap \text{td}_*(Y), \end{aligned}$$

where we used that, according to Verdier [63, p. 222, 9.2.1], the algebraic Gysin map of a closed regular embedding commutes with the cycle map (see diagram (8)), and that the cycle map cl and the cap product with Chern classes are compatible by [30, p. 374, Prop. 19.1.2]

(where note that the Todd class of a complex vector bundle is a rational polynomial in Chern classes, see e.g. [30, p. 56, Example 3.2.4]). Since $i \in \mathcal{X}_{TI}$ and M and X are \mathcal{X}_{TI} -transverse in W , Theorem 5.10 and Proposition 5.7 (where we may use the natural identification of Borel-Moore homology and singular homology because X and Y are compact) imply that the algebraic Gysin map $g_{\text{alg}}^!: H_*(X; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ coincides with the topological Gysin map $g_{\text{top}}^!: H_*(X; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ on all fundamental classes $[Z]_X$ of closed irreducible subvarieties $Z \subset X$. As $\text{td}_*(X) \in H_*(X; \mathbb{Q})$ is an algebraic cycle according to Remark 2.5, we obtain

$$g_{\text{top}}^! \text{td}_*(X) = g_{\text{alg}}^! \text{td}_*(X).$$

Since the embedding $g: Y \hookrightarrow X$ is tight, we know that the underlying inclusion $Y \subset X$ is topologically normally nonsingular with topological normal bundle ν isomorphic (as a topological vector bundle) to the underlying topological vector bundle of the algebraic normal bundle $N = N_Y X$. Next, recall from Theorem 4.4 that the inclusion $g: Y \hookrightarrow X$ is normally nonsingular with topological normal bundle $\nu = j^* \nu_f$ given by the restriction under the inclusion $j: Y \rightarrow M$ of the normal bundle ν_f of M in W , which is the underlying topological vector bundle of the algebraic normal bundle $N_f = N_M W$. Using the base change $f_{\text{top}}^! i_* = j_* g_{\text{top}}^!$ for topological Gysin maps (see [11, Proposition 2.4]), as well as $\text{td}^*(N) = \text{td}^*(j^* N_f) = j^* \text{td}^*(N_f)$, we conclude that

$$\begin{aligned} f_{\text{top}}^! c\ell_{2*}(i) &= f_{\text{top}}^! i_* \text{td}_*(X) = j_* g_{\text{top}}^! \text{td}_*(X) = j_* g_{\text{alg}}^! \text{td}_*(X) = j_*(\text{td}^*(N) \cap \text{td}_*(Y)) \\ &= j_*(j^* \text{td}^*(N_f) \cap \text{td}_*(Y)) = \text{td}^*(N_f) \cap j_* \text{td}_*(Y) = c\ell^*(f) \cap c\ell_{2*}(j). \end{aligned}$$

This completes the proof of Theorem 9.1. \square

Declarations

Data availability Data sharing is not applicable to this article, as no datasets were generated or analysed in the course of the research presented herein.

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 205, 69120 HEIDELBERG, GERMANY

Email address: `banagl@mathi.uni-heidelberg.de`

MATHEMATISCHES INSTITUT, UNIVERSITÄT MÜNSTER, EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY

Email address: `jschuerm@uni-muenster.de`

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 205, 69120 HEIDELBERG, GERMANY

Email address: `dwrazidlo@mathi.uni-heidelberg.de`