THE TENSOR PRODUCT OF FUNCTION SEMIMODULES

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ABSTRACT. Given two domains of functions with values in a field, the canonical map from the algebraic tensor product of the vector spaces of functions on the two domains to the vector space of functions on the product of the two domains is well-known to be injective, but not generally surjective. By constructing explicit examples, we show that the corresponding map for semimodules of semiring-valued functions is in general not even injective anymore. This impacts the formulation of topological quantum field theories over semirings. We also confirm the failure of surjectivity for functions with values in complete, additively idempotent semirings by describing a large family of functions that do not lie in the image.

1. INTRODUCTION

The question whether a product decomposition $D = A \times B$ of some domain D implies some kind of decomposition of spaces of functions on D is a classical and important one in functional analysis. More precisely, one frequently wants to know whether a function on D can be approximated by finite sums of products of functions on A and functions on B. For instance, in solving a partial differential equation one may first seek separable solutions and then approximate the general solution by sums of separable solutions. The notion of a tensor product is central to this issue, for the following reason: Given a set X and a field K, let K^X denote the K-vector space of functions $X \to K$. The map $\beta : K^A \times K^B \to K^{A\times B} = K^D$ sending (f,g) to $(a,b) \mapsto f(a)g(b)$ is bilinear and hence induces a linear map $\mu : K^A \otimes K^B \to K^{A\times B}$, where $\otimes$ denotes the algebraic tensor product of K-vector spaces. If A and B are finite sets, then μ is an isomorphism and the above question is answered. In the case of actual interest where A and B are infinite, μ is generally not surjective anymore. This is the reason why the functional analyst completes the tensor product $\otimes$ using various topologies available, arriving at products $\widehat{\otimes}$. For example, for compact Hausdorff spaces A and B, let $C(A), C(B)$ denote the Banach spaces of all complex-valued continuous functions on A, B, respectively, endowed with the supremum-norm, yielding the topology of uniform convergence. Then the image of $\mu : C(A) \otimes C(B) \to C(A \times B)$, while not all of $C(A\times B)$, is however dense in $C(A\times B)$ by the Stone-Weierstraß theorem. After completion, μ induces an isomorphism $C(A)\widehat{\otimes}_\varepsilon C(B) \cong C(A\times B)$ of Banach spaces, where $\widehat{\otimes}_\varepsilon$ denotes the so-called ε-tensor product or injective tensor product of two locally convex topological vector spaces. For n-dimensional Euclidean space $\mathbb{R}^n$, let $L^2(\mathbb{R}^n)$ denote the Hilbert space of square integrable functions on $\mathbb{R}^n$. Then μ induces an isomorphism $L^2(\mathbb{R}^n)\widehat{\otimes}L^2(\mathbb{R}^m) \cong L^2(\mathbb{R}^{n+m}) = L^2(\mathbb{R}^n \times \mathbb{R}^m)$, where $\widehat{\otimes}$ denotes the Hilbert space tensor product, a completion of the algebraic tensor product $\otimes$ of two Hilbert spaces. For more information on topological tensor products see [Sch50], [Gro55], [Tre67].

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What underlies all of the above results implicitly is of course the fact that even when A and B are infinite, the map μ still remains *injective*. We recall the argument for injectivity in the proof of Proposition 3.3. The purpose of the present paper is to observe that, in marked contrast to the situation over fields, the canonical map μ ceases to be injective in general when one studies functions with values in a (commutative) semiring S. Given two infinite sets A and B, we construct explicitly a commutative, additively idempotent semiring $S = S(A,B)$ such that $\mu : S^A \otimes S^B \to S^{A\times B}$ is not injective (Theorem 3.10). Here, $\otimes$ denotes the (algebraic) tensor product of S-semimodules satisfying the standard universal property expected of such a product. This tensor product was constructed by Y. Katsov in [Kat97]. The infinite cardinality of A,B has to be used in an essential way to obtain the noninjectivity result, since μ for semirings is certainly injective (in fact an isomorphism) when A and B are finite. Our purely algebraic result has the immediate consequence that in performing functional analysis over a semiring which is not a field, one cannot identify the function $(a,b) \mapsto f(a)g(b)$ on $A\times B$ with $f\otimes g$ for $f \in S^A$, $g \in S^B$.

The failure of surjectivity for μ over a semiring which is not a ring, though expected, cannot logically be deduced from the corresponding failure over rings. Our other main result, Theorem 3.1, shows that even over the smallest complete (in particular zerosumfree) and additively idempotent semiring, namely the Boolean semiring $\mathbb{B}$, and for the smallest infinite cardinal $\aleph_0$, modeled by a countably infinite set A, there is a bilinear map $\phi : \mathbb{B}^A \times \mathbb{B}^A \to \mathbb{B}$ which permits two different linear maps $\psi_1, \psi_2 : \mathbb{B}^{A\times A} \to \mathbb{B}$ such that $\psi_1\beta = \phi = \psi_2\beta$. This implies in particular that μ is not surjective, but it actually yields a large class of functions that are not in the image of μ, namely all functions in $\mathbb{B}^{A\times A}$ which are distinguished by ψ_1 and ψ_2.

In the boundedly complete idempotent setting, Litvinov, Maslov and Shpiz have constructed in [LMS99] a tensor product, let us here write it as $\widehat{\otimes}$, which for bounded functions does not exhibit the above deficiencies of the algebraic tensor product. Any idempotent semiring S is a partially ordered set with respect to the order relation $s \le t$ if and only if $s+t = t$; $s,t \in S$. Then the addition has the interpretation of a least upper bound, $s+t = \sup\{s,t\}$. The semiring S is called *boundedly complete* (b-complete) if every subset of S which is bounded above has a supremum. (The supremum of a subset, if it exists, is unique.) The above semiring $S(A,B)$ is b-complete. Given a b-complete commutative idempotent semiring S and b-complete idempotent semimodules V,W over S, Litvinov, Maslov and Shpiz define a tensor product $V\widehat{\otimes}W$, which is again idempotent and b-complete. The fundamental difference to the algebraic tensor product lies in allowing *infinite* sums of elementary tensors. A linear map $f : V \to W$ is called b-linear if $f(\sup V_0) = \sup f(V_0)$ for every bounded subset $V_0 \subset V$. The canonical map $\pi : V\times W \to V\widehat{\otimes}W$ is b-bilinear. For each b-bilinear map $f : V\times W \to U$ there exists a unique b-linear map $f_{\widehat{\otimes}} : V\widehat{\otimes}W \to U$ such that $f = f_{\widehat{\otimes}}\pi$. Given any set A, let $\mathcal{B}(A,S)$ denote the set of bounded functions $A \to S$. Then $\mathcal{B}(A,S)$ is a b-complete idempotent S-semimodule. According to [LMS99, Prop. 5], $\mathcal{B}(A,S)\widehat{\otimes}\mathcal{B}(B,S)$ and $\mathcal{B}(A\times B,S)$ are isomorphic for arbitrary sets A and B. The functions constructed in our Theorem 3.10 are indeed unbounded.

In writing the present paper, we were chiefly motivated by the problem of developing a correct formulation of topological quantum field theories over semirings, which has recently become necessary in certain constructions and intended applications. Over rings, such theories have been axiomatized by Atiyah in [Ati88]. Tensor products play a central role in these axioms and in constructions of such theories. For instance, the state module of a disjoint union of two manifolds should be the tensor product of the state modules of the two manifolds. As the results of this paper show, Atiyah's axioms cannot be transplanted indiscriminately to yield a correct foundation for TQFTs over semirings.

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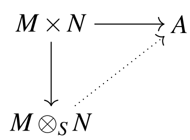
2. Monoids, Semirings, and Semimodules

We recall some fundamental material on monoids, semirings and semimodules over semirings. Such structures seem to have appeared first in Dedekind's study of ideals in a commutative ring: one can add and multiply two ideals, but one cannot subtract them. The theory has been further developed by H. S. Vandiver, S. Eilenberg, A. Salomaa, J. H. Conway and others. Roughly, a semiring is a ring without general additive inverses. More precisely, a *semiring* is a set S together with two binary operations $+$ and $\cdot$ and two elements $0,1 \in S$ such that $(S,+,0)$ is a commutative monoid, $(S,\cdot,1)$ is a monoid, the multiplication $\cdot$ distributes over the addition from either side, and 0 is absorbing, i.e. $0 \cdot s = 0 = s \cdot 0$ for every $s \in S$. If the monoid $(S,\cdot,1)$ is commutative, the semiring S is called commutative. The addition on the *Boolean monoid* $(\mathbb{B},+,0)$, $\mathbb{B} = \{0,1\}$, is the unique operation such that 0 is the neutral element and $1+1=1$. The Boolean monoid becomes a commutative semiring by defining $1 \cdot 1 = 1$. (Actually, the multiplication on $\mathbb{B}$ is completely forced by the axioms.)

Let S be a semiring. A *(left) S-semimodule* is a commutative monoid $(M,+,0_M)$ together with a scalar multiplication $S \times M \to M$, $(s,m) \mapsto sm$, such that for all $r,s \in S$, $m,n \in M$, we have $(rs)m = r(sm)$, $r(m+n) = rm+rn$, $(r+s)m = rm+sm$, $1m = m$, and $r0_M = 0_M = 0m$. Right semimodules are defined similarly using scalar multiplications $M \times S \to M$, $(m,s) \mapsto ms$. Every semimodule M over a commutative semiring S can and will be assumed to be both a left and right semimodule with $sm = ms$. In fact, M is then a bisemimodule, as for all $r,s \in S$, $m \in M$,

$$(rm)s = s(rm) = (sr)m = (rs)m = r(sm) = r(ms).$$

Let S be a commutative semiring. Regarding the tensor product of two S-semimodules M and N, one has to exercise caution because two nonisomorphic tensor products, both called *the* tensor product of M and N and both written $M \otimes_S N$, exist in the literature. For us, a *tensor product of M and N* is an S-semimodule $M \otimes_S N$ satisfying the following (standard) universal property: $M \otimes_S N$ comes equipped with an S-bilinear map $M \times N \to M \otimes_S N$ such that given any S-semimodule A and S-bilinear map $M \times N \to A$, there exists a unique S-linear map $M \otimes_S N \to A$ such that



commutes. The existence of such a tensor product is shown for example in [Kat97], [Kat04]. To construct it, take $M \otimes_S N$ to be the quotient monoid $F/\sim$, where F is the free commutative monoid generated by the set $M \times N$ and $\sim$ is the congruence relation on F generated by all pairs of the form

$$((m+m',n),(m,n)+(m',n)),\ ((m,n+n'),(m,n)+(m,n')),\ ((sm,n),(m,sn)),$$

$m,m' \in M$, $n,n' \in N$, $s \in S$. The commutative monoid $M \otimes_S N$ is an S-semimodule with $s(m \otimes n) = (sm) \otimes n = m \otimes (sn)$. If S is understood, we shall also write $M \otimes N$ for $M \otimes_S N$.

The tensor product of [Tak82] and [Gol99] — let us here write it as $\otimes'_S$ — satisfies a different universal property and is thus different from $\otimes_S$. A semimodule C is called *cancellative* if $a+c=b+c$ implies $a=b$ for all $a,b,c\in C$. A monoid $(M,+,0)$ is *idempotent* if $m+m=m$ for all elements $m\in M$. For example, the Boolean monoid $\mathbb{B}$ is idempotent. A semiring is called *additively idempotent* if $1+1=1$. Note that this implies that the additive monoid of the semiring is idempotent. A semimodule is called idempotent if its additive monoid is idempotent. A nontrivial idempotent semimodule is never cancellative. Given arbitrary S-semimodules M,N, the product $M\otimes'_S N$ is always cancellative. If one of the two semimodules, say N, is idempotent, then $M\otimes'_S N$ is idempotent as well, since $m\otimes' n+m\otimes' n=m\otimes'(n+n)=m\otimes' n$. Thus if one of M,N is idempotent, then $M\otimes'_S N$ is trivial, being both idempotent and cancellative. We want to be able to apply the tensor product nontrivially also to idempotent semimodules and so will not use the product $\otimes'_S$ in this paper.

3. Function Semimodules and Their Tensor Products

Let S be a commutative semiring. Given a set A, let $S^A=\{f:A\to S\}$ be the set of all S-valued functions on A. Elements of S^A are sometimes referred to as *S-subsets* in the literature, for example in [Eil74]. Using pointwise addition and multiplication of function values by elements of S, S^A inherits the structure of an S-semimodule from the operations of S. In fact, S^A is a commutative S-semialgebra, but the semialgebra multiplication will play no role in the present paper.

Let B be another set. Then, regarding S^A and S^B as S-semimodules, the tensor product $S^A\otimes_S S^B$ is defined. It is an S-semimodule satisfying the universal property recalled in Section 2: Given any S-semimodule M and S-bilinear map $\phi: S^A\times S^B\to M$, there exists a unique S-linear map $\psi: S^A\otimes_S S^B\to M$ such that

$$\begin{array}{ccc} S^A\times S^B & \xrightarrow{\ \phi\ } & M \\ \downarrow & \nearrow_{\psi} & \\ S^A\otimes_S S^B & & \end{array}$$

commutes. The S-semimodule $S^{A\times B}$ comes naturally equipped with an S-bilinear map

$$\beta: S^A\times S^B\longrightarrow S^{A\times B},$$

namely

$$\beta(f,g)=((a,b)\mapsto f(a)\cdot g(b)).$$

Note that the commutativity of S is crucial for the property $\beta(f,sg)=s\beta(f,g)$, $s\in S$. Taking $M=S^{A\times B}$ in the universal property, there exists thus a unique S-linear map $\mu: S^A\otimes_S S^B\to S^{A\times B}$ such that

(1)
$$\begin{array}{ccc} S^A\times S^B & \xrightarrow{\ \beta\ } & S^{A\times B} \\ \downarrow & \nearrow_{\mu} & \\ S^A\otimes_S S^B & & \end{array}$$

commutes. If A and B are finite, μ is an isomorphism, as in this case S^A, S^B and $S^{A\times B}$ are finitely generated free S-semimodules. A basis of S^A is given by the characteristic functions χ_a of the elements $a\in A$, similarly for S^B. Then the elements $\chi_a\otimes\chi_b$ form a basis for

$S^A \otimes S^B$ which μ sends to the functions $(a',b') \mapsto \chi_a(a')\chi_b(b')$. But these are precisely the characteristic functions $\chi_{(a,b)}$ of the elements $(a,b) \in A \times B$, and thus form a basis of $S^{A\times B}$. The unique solution μ' of the universal problem

$$\begin{array}{ccc} S^A \times S^B & \xrightarrow{\phi} & M \\ {\scriptstyle\beta}\downarrow & \nearrow_{\mu'} & \\ S^{A\times B} & & \end{array}$$

is given by

$$\mu'(F) = \sum_{(a,b)\in A\times B} F(a,b)\phi(\chi_a,\chi_b),\ F \in S^{A\times B}.$$

For $M = S^A \otimes S^B$ and ϕ the canonical bilinear map, μ' is inverse to μ and $\mu'(\chi_{(a,b)}) = \chi_a \otimes \chi_b$. Consequently, the emphasis is to be placed on infinite sets.

Already when S is a field, μ is generally not surjective if A and B are infinite. This is a well-known phenomenon in classical functional analysis and, as discussed in the introduction, leads to various forms of completed tensor products $\widehat{\otimes}$, using a topology on the function space. For semirings that are not rings, such as zerosumfree semirings, in particular complete or additively idempotent semirings, the failure of surjectivity can of course not be logically deduced from the corresponding failure over fields. The following theorem implies that surjectivity fails also for complete and idempotent semirings. In fact, the semiring used in the theorem is the Boolean semiring $\mathbb{B}$, which satisfies both of these properties. The theorem actually describes a large class of functions that are not in the image of μ.

Theorem 3.1. *Given any countably infinite set A, there is a $\mathbb{B}$-bilinear map $\phi : \mathbb{B}^A \times \mathbb{B}^A \longrightarrow \mathbb{B}$ and two different $\mathbb{B}$-linear maps $\psi_1, \psi_2 : \mathbb{B}^{A\times A} \to \mathbb{B}$ such that*

(2)
$$\begin{array}{ccc} \mathbb{B}^A \times \mathbb{B}^A & \xrightarrow{\phi} & \mathbb{B} \\ {\scriptstyle\beta}\downarrow & \nearrow_{\psi_i} & \\ \mathbb{B}^{A\times A} & & \end{array}$$

commutes for $i = 1,2$. In particular, $\mathbb{B}^{A\times A}$ is not a tensor square of $\mathbb{B}^A$ and any element of the nonempty set

$$\{h \in \mathbb{B}^{A\times A} \mid \psi_1(h) \neq \psi_2(h)\}$$

is not in the image of the map

$$\mu : \mathbb{B}^A \otimes \mathbb{B}^A \longrightarrow \mathbb{B}^{A\times A}.$$

Proof. We may take the natural numbers $\mathbb{N}$ as our model for the countably infinite set A. Then $S^A = \mathbb{B}^A$ is the set $\{(s_n)_{n\in\mathbb{N}}\}$ of $\mathbb{B}$-valued sequences (s_n). We define a map

$$\phi : S^A \times S^A \longrightarrow \mathbb{B}$$

by

$$\phi((s_n),(t_n)) = \begin{cases} 1, & \text{if } s_n = 1 \text{ for infinitely many } n \text{ and} \\ & t_n = 1 \text{ for infinitely many } n, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly ϕ is symmetric. It is also $\mathbb{B}$-bilinear: We have

$$\phi(0\cdot(s_n),(t_n)) = \phi((0),(t_n)) = 0 = 0\cdot\phi((s_n),(t_n)),$$

as the zero sequence (0) has not a single 1, let alone infinitely many. Furthermore,

$$\phi(1\cdot(s_n),(t_n)) = \phi((s_n),(t_n)) = 1\cdot\phi((s_n),(t_n)).$$

The following table establishes biadditivity, where we write "∞1" for the assumption that a sequence have infinitely many 1-entries and "f1" for the assumption that a sequence have only finitely many 1-entries.

s	t	t'	$t+t'$	$\phi(s,t+t')$	$\phi(s,t)$	$\phi(s,t')$	$\phi(s,t)+\phi(s,t')$
$\infty 1$	$\infty 1$	$\infty 1$	$\infty 1$	1	1	1	1
$\infty 1$	$\infty 1$	$f1$	$\infty 1$	1	1	0	1
$\infty 1$	$f1$	$\infty 1$	$\infty 1$	1	0	1	1
$\infty 1$	$f1$	$f1$	$f1$	0	0	0	0
$f1$	$\infty 1$	$\infty 1$	$\infty 1$	0	0	0	0
$f1$	$\infty 1$	$f1$	$\infty 1$	0	0	0	0
$f1$	$f1$	$\infty 1$	$\infty 1$	0	0	0	0
$f1$	$f1$	$f1$	$f1$	0	0	0	0

We shall now construct *distinct* $\psi_1,\psi_2 : S^{A\times A}\to\mathbb{B}$ such that (2) commutes for $i=1,2$. The set $S^{A\times A}$ is the set of infinite matrices

$$S^{A\times A} = \{(h_{n,m})_{(n,m)\in\mathbb{N}\times\mathbb{N}} \mid h_{n,m}\in\mathbb{B}\}.$$

Call a row of $h=(h_{n,m})$ *long* if it contains infinitely many 1-entries, otherwise call it *short*; similarly for columns. Call h *row-expansive* if it has infinitely many long rows and *column-expansive* if it has infinitely many long columns. Set

$$\psi_1(h) = \begin{cases} 1, & \text{if } h \text{ is row- or column-expansive,} \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $\psi_1 : S^{A\times A}\to\mathbb{B}$ is linear. To see this, let $h,h'\in S^{A\times A}$ and consider the case where one of h,h', say h, is row-expansive. Let $n_1,n_2,\ldots$ be the indices of an infinite set of long rows of h. Adding h' to h can only increase the number of 1-entries in each of these rows, so each of the rows of $h+h'$ corresponding to the indices $n_1,n_2,\ldots$ is still long. Thus $h+h'$ is row-expansive and

$$\psi_1(h)+\psi_1(h') = 1+\psi_1(h') = 1 = \psi_1(h+h').$$

If one of h,h', say h, is column-expansive, then by an analogous argument $h+h'$ is column-expansive and again $\psi_1(h)+\psi_1(h') = 1 = \psi_1(h+h')$. The last case to consider is that both h and h' are neither row- nor column-expansive, in which case $\psi_1(h)+\psi_1(h') = 0+0 = 0$. So we must show that $h+h'$ is neither row- nor column-expansive. We do this by contradiction — suppose that $h+h'$ were row-expansive and let $n_1,n_2,\ldots$ be an infinite set of indices indexing long rows of $h+h'$. Since the sum of two short rows is short, not both of $(h_{n_1,m})_m$, $(h'_{n_1,m})_m$ can be short. Color n_1 white if $(h_{n_1,m})_m$ is long and color it black otherwise. So if n_1 is black, then $(h'_{n_1,m})_m$ is long. Carrying out this coloring for $n_2,n_3,\ldots$, yields a partition of the infinite set $n_1,n_2,\ldots$ into two subsets, the white and the black indices. As one cannot partition an infinite set into two finite subsets, one of these two subsets must be infinite. If the white subset is infinite, then h has an infinite set of long rows, a contradiction to the assumption that h is not row-expansive. Thus the black subset is infinite. But then h' would be row-expansive, again a contradiction. Therefore, $h+h'$ is not row-expansive. A transposed argument for the columns shows that $h+h'$ is not column-expansive. Consequently, $\psi_1(h+h')=0$ as was to be shown. Furthermore, $\psi_1(0\cdot h)=\psi_1(0)=0=0\cdot\psi_1(h)$ and $\psi_1(1\cdot h)=\psi_1(h)=1\cdot\psi_1(h)$. This proves the claimed linearity of ψ_1. Let us verify that (2) commutes for ψ_1, i.e. that the equation

$$\phi((s_n),(t_n))=\psi_1((s_n\cdot t_m)) \tag{3}$$

holds. If both (s_n) and (t_n) contain infinitely many 1-entries, then $h=(s_n\cdot t_m)$ is row- and column-expansive. Hence $\phi((s_n),(t_n))=1=\psi_1((s_n\cdot t_m))$. Suppose that at least one of (s_n), (t_n) has only finitely many 1-entries. Then the product $h=(s_n\cdot t_m)$ is neither row- nor column-expansive. Thus in this case $\phi((s_n),(t_n))=0=\psi_1((s_n\cdot t_m))$. This shows that ψ_1 solves (3).

We say that $h=(h_{n,m})$ contains a *large block of zeros* if there exist numbers N and M such that $h_{n,m}=0$ for $n\geq N$ and $m\geq M$. Set

$$\psi_2(h)=\begin{cases}0, & \text{if } h \text{ contains a large block of zeros,}\\ 1, & \text{otherwise.}\end{cases}$$

We claim that $\psi_2: S^{A\times A}\to\mathbb{B}$ is linear. To see this, let $h,h'\in S^{A\times A}$ and consider the case where both h and h' contain a large block of zeros. Then there are N,M,N',M' such that $h_{n,m}=0$ for $n\geq N$ and $m\geq M$, and $h'_{n,m}=0$ for $n\geq N'$ and $m\geq M'$. Let $N^+=\max(N,N')$, $M^+=\max(M,M')$. Then, as $h_{n,m}+h'_{n,m}=0+0=0$ for $n\geq N^+$ and $m\geq M^+$, the sum $h+h'$ contains a large block of zeros. Consequently,

$$\psi_2(h)+\psi_2(h')=0+0=0=\psi_2(h+h').$$

Now assume that at least one of h,h', say h, does not contain a large block of zeros. This means that for every $k=1,2,\ldots$ there are $n_k,m_k\geq k$ with $h_{n_k,m_k}=1$. But then $h_{n_k,m_k}+h'_{n_k,m_k}=1$ for all k and thus $h+h'$ does not contain a large block of zeros. We conclude that

$$\psi_2(h)+\psi_2(h')=1+\psi_2(h')=1=\psi_2(h+h').$$

Furthermore, $\psi_2(0\cdot h)=\psi_2(0)=0=0\cdot\psi_2(h)$ and $\psi_2(1\cdot h)=\psi_2(h)=1\cdot\psi_2(h)$. This proves that ψ_2 is linear. We shall next check that the equation

$$\phi((s_n),(t_n))=\psi_2((s_n\cdot t_m)) \tag{4}$$

holds. If both (s_n) and (t_n) contain infinitely many 1-entries, then $h=(s_nt_m)$ does not contain a large block of zeros: Given N,M, let $n\geq N$ be an index with $s_n=1$ and $m\geq M$ an index with $t_m=1$. Then $s_n\cdot t_m=1$ for this $n\geq N$ and this $m\geq M$. Thus,

$$\phi((s_n),(t_n))=1=\psi_2((s_nt_m)).$$

Suppose that at least one of (s_n), (t_n), say (s_n), has only finitely many 1-entries. Let N be such that $s_n=0$ whenever $n\geq N$. Then $h_{n,m}=s_nt_m=0$ for $n\geq N$ and $m\geq 1$. This is a large block of zeros for h, whence

$$\phi((s_n),(t_n))=0=\psi_2((s_nt_m)).$$

We have shown that ψ_2 solves (4).

Now consider the identity matrix $h=(h_{n,m})$ with $h_{n,n}=1$ and $h_{n,m}=0$ for $n\neq m$. Since it contains neither a single long row nor a single long column, h is neither row- nor column-expansive, which implies $\psi_1(h)=0$. However, the identity matrix does not contain a large block of zeros, so $\psi_2(h)=1$. This shows that (2) has two distinct solutions ψ_1 and ψ_2.

From this it follows easily that μ is not surjective. Indeed, let $h \in \mathbb{B}^{A\times A}$ be any element such that $\psi_1(h) \neq \psi_2(h)$, for instance the identity matrix. If μ were surjective, then

$$h = \mu(\sum_i s^i \otimes t^i) = \sum_i \beta(s^i, t^i)$$

and hence

$$\psi_1(h) = \sum \psi_1\beta(s^i, t^i) = \sum \phi(s^i, t^i) = \sum \psi_2\beta(s^i, t^i) = \psi_2(h),$$

a contradiction. We conclude that μ is not surjective. (In particular, the identity matrix is not in the image of μ.) □

Remark 3.2. The above proof shows that the identity matrix is not in the image of μ, which is of course easy to verify directly and suffices to establish the failure of surjectivity. But the proof yields much more: It defines the notions *row-expansive*, *column-expansive* and the property of possessing a *large block of zeros*. Using these, we can now describe a large family of functions that does not intersect the image of μ: Any $h \in \mathbb{B}^{A\times A}$ which is neither row- nor column-expansive but does not contain a large block of zeros is not in the image of μ. On the other hand, any row- or column-expansive h cannot have a large block of zeros.

We turn to the injectivity of $\mu : S^A \otimes S^B \to S^{A\times B}$. Over vector spaces, the injectivity of μ is well-known to functional analysts and is used implicitly throughout the subject, cf. [Tre67]. Let us recall the simple proof of this fact.

Proposition 3.3. *Let K be a field and A, B sets. Then the map $\mu : K^A \otimes K^B \to K^{A\times B}$ is injective.*

Proof. Let $h \in K^A \otimes K^B$ be a nonzero element. Then h can be written as

$$h = \sum_{i=1}^{n} e_i \otimes f_i, \qquad e_i \in K^A,\ f_i \in K^B,$$

with $e_1, \ldots, e_n$ linearly independent. Since $h \neq 0$, there is an i_0 with $f_{i_0} \neq 0$. Thus $f_{i_0}(b_0) \neq 0$ for some $b_0 \in B$. By the linear independence of the e_i, the linear combination $\sum f_i(b_0)e_i \in K^A$ cannot be zero. So there is an $a_0 \in A$ with $\sum e_i(a_0)f_i(b_0) \neq 0$. This shows that $\mu(h) = \sum \mu(e_i \otimes f_i) = \sum \beta(e_i, f_i) = \sum e_i(-)f_i(-)$ is not zero in $K^{A\times B}$. □

By contrast, for semirings S that are not fields, the canonical map μ is generally not injective, as we shall now explain. Let A, B be two infinite sets.To each element $a \in A$, we associate an indeterminate f_a and to each element $b \in B$ an indeterminate g_b. Let x be an additional independent indeterminate. Let $S = S(A, B)$ be the set of all finite formal linear combinations

$$\omega + \xi x + \sum_{a\in A} \alpha_a f_a + \sum_{b\in B} \beta_b g_b,$$

where $\omega, \xi, \alpha_a, \beta_b \in \mathbb{B}$ are elements of the Boolean semiring and only finitely many α_a, β_b are nonzero. Two such linear combinations are added by adding the coefficients in $\mathbb{B}$. The product of any two indeterminates is to vanish, that is, the indeterminates are multiplied according to the rules

$$f_a g_b = f_a f_{a'} = g_b g_{b'} = f_a x = g_b x = x^2 = 0,$$

together with commutativity. Using the distributive law and the multiplication in $\mathbb{B}$, this determines a multiplication $\cdot$ on S so that $(S, +, \cdot, 0, 1)$ is a commutative semiring. (The zero-element has $\omega = \xi = \alpha_a = \beta_b = 0$ and the one-element has $\omega = 1$, $\xi = \alpha_a = \beta_b = 0$.) An explicit formula for the product of two elements $s, s' \in S$ is

$$s \cdot s' = \omega\omega' + (\omega\xi' + \omega'\xi)x + \sum_a (\omega\alpha'_a + \omega'\alpha_a) f_a + \sum_b (\omega\beta'_b + \omega'\beta_b) g_b.$$

As $\mathbb{B}$ is additively idempotent, the semiring S is additively idempotent as well.

Definition 3.4. Let R be any semiring and let r, r_0 be elements in R. We say that r_0 is a *summand of* r if $r = r_0 + s$ for some $s \in R$.

Every element is a summand of itself, and every element has 0 as a summand. If R happens to be a ring, then every r_0 is a summand of any r as $r = r_0 + (r - r_0)$. Recall that an idempotent semiring has a canonical partial order given by $r_0 \leq r$ if and only if $r_0 + r = r$. If R is idempotent, then r_0 is a summand of r precisely when $r_0 \leq r$, as can be seen from

$$r_0 + r = r_0 + (r_0 + s) = r_0 + s = r.$$

In [LMS99], an idempotent semiring R is called *a-complete*, if every subset of R has a supremum and an infimum. Furthermore, R is *b-complete* if every subset which is bounded above has a supremum.

Proposition 3.5. *The commutative idempotent semiring $S(A,B)$ is b-complete, but not a-complete.*

Proof. The infinite subset $\{f_a \mid a \in A\} \subset S$ has no upper bound: If s were an upper bound, then $f_a + s = s$ for all $a \in A$, i.e. f_a would be a summand of s for all $a \in A$. But in

$$s = \omega + \xi x + \sum_{a \in A} \alpha_a f_a + \sum_{b \in B} \beta_b g_b,$$

only finitely many α_a can be 1. Thus S is not a-complete.

Suppose $T \subset S$ is a subset which is bounded above, that is, there is an $s \in S$ such that $t \leq s$ for all $t \in T$. Set

$$A(T) = \{a \in A \mid \exists t \in T : f_a \leq t\}, \; B(T) = \{b \in B \mid \exists t \in T : g_b \leq t\}.$$

For every $a \in A(T)$, we have the bound $f_a \leq s$. If $A(T)$ were infinite, then f_a would be a summand of s for infinitely many distinct f_a, which is impossible. Consequently, $A(T)$ is finite and similarly $B(T)$ is finite. This implies that T is finite, since T is contained in the finite set

$$\{\omega + \xi x + \sum_{a \in A(T)} \alpha_a f_a + \sum_{b \in B(T)} \beta_b g_b\}.$$

Then $s_0 = \sum_{t \in T} t \in S$ is the supremum of T: Clearly, $t \leq s_0$ for all $t \in T$, and if s is any upper bound for T, then $t + s = s$ for all $t \in T$ implies $s_0 + s = \sum t + s = s$, so $s_0 \leq s$. We conclude that S is b-complete. □

Definition 3.6. A function $F \in S^A$ is called *f-exhaustive* (or briefly *exhaustive*), if for infinitely many $a \in A$, the value $F(a)$ has f_a as a summand. Similarly, a function $G \in S^B$ is called *g-exhaustive* (or just *exhaustive*), if for infinitely many $b \in B$, the value $G(b)$ has g_b as a summand.

For example the function $F(a) = f_a$ is exhaustive. No function with finite support is exhaustive. Exhaustive functions are necessarily unbounded, because if there were an $s \in S$ with $F(a) \leq s$, all $a \in A$, for an exhaustive function $F : A \to S$, then for infinitely many a, $f_a \leq F(a) \leq s$, so s would contain infinitely many f_a as summands, which is impossible.

Lemma 3.7. *Let $F, F' \in S^A$ be two functions. If at least one of F, F' is f-exhaustive, then $F+F'$ is f-exhaustive. If neither F nor F' is f-exhaustive, then $F+F'$ is not f-exhaustive.*

Proof. Suppose that F is exhaustive. Thus for infinitely many $a \in A$, the value $F(a)$ has the form $F(a) = f_a + s_a$. For these a,

$$(F+F')(a) = F(a) + F'(a) = f_a + (s_a + F'(a)).$$

Hence there are infinitely many $a \in A$ such that $(F+F')(a)$ contains f_a as a summand. This shows that $F+F'$ is exhaustive. Suppose that neither F nor F' is exhaustive. Then there is a finite set $A_F \subset A$ such that for all $a \in A - A_F$,

$$F(a) = \omega_a + \xi_a x + \sum_{a' \neq a} \alpha_{a'a} f_{a'} + \sum_b \beta_{ba} g_b.$$

Similarly, there is a finite set $A_{F'} \subset A$ such that for all $a \in A - A_{F'}$,

$$F'(a) = \omega'_a + \xi'_a x + \sum_{a' \neq a} \alpha'_{a'a} f_{a'} + \sum_b \beta'_{ba} g_b.$$

The union $A_0 = A_F \cup A_{F'}$ is a finite set such that for all $a \in A - A_0$,

$$(F+F')(a) = (\omega_a + \omega'_a) + (\xi_a + \xi'_a)x + \sum_{a' \neq a} (\alpha_{a'a} + \alpha'_{a'a}) f_{a'} + \sum_b (\beta_{ba} + \beta'_{ba}) g_b,$$

which does not contain f_a as a summand. So $(F+F')(a)$ can contain f_a as a summand for at most finitely many a, which implies that $F+F'$ is not exhaustive. □

Lemma 3.8. *Let $s \in S$ be any element that does not contain* 1 *as a summand. Then for any function $F \in S^A$, the product sF is not exhaustive.*

Proof. As s does not have 1 as a summand, it can be written as

$$s = \xi x + \sum_{a' \in A'} f_{a'} + \sum_b \beta_b g_b,$$

where $A' \subset A$ is finite and only finitely many β_b are nonzero. Note that $xs = f_a s = g_b s = 0$ for all a, b. Given an $a \in A - A'$, we write

$$F(a) = \omega + \xi' x + \sum_{a \in A} \alpha'_a f_a + \sum_{b \in B} \beta'_b g_b$$

and calculate

$$\begin{aligned} sF(a) &= \omega s + \xi'(xs) + \sum_{a \in A} \alpha'_a (f_a s) + \sum_{b \in B} \beta'_b (g_b s) \\ &= \omega s \\ &= (\omega\xi)x + \sum_{a' \in A'} \omega f_{a'} + \sum_b (\omega\beta_b) g_b. \end{aligned}$$

This last expression shows that $sF(a)$ does not contain f_a as a summand whenever $a \in A - A'$. Thus $sF(a)$ can contain f_a as a summand for at most finitely many $a \in A$ and we conclude that sF is not exhaustive. □

Lemma 3.9. *Let $s \in S$ be any element that contains* 1 *as a summand. If $F \in S^A$ is not exhaustive, then the product sF is not exhaustive.*

Proof. As F is not exhaustive, there exists a finite set $A' \subset A$ such that for all $a \in A - A'$,

$$F(a) = \omega_a + \xi_a x + \sum_{a' \neq a} \alpha_{a'a} f_{a'} + \sum_b \beta_{ba} g_b.$$

Since s contains 1 as a summand, we can write $s = 1+t$ with

$$t = \xi x + \sum_{a''\in A''} f_{a''} + \sum_b \beta'_b g_b$$

for some finite set $A'' \subset A$. Then the union $A_0 = A' \cup A''$ is finite and for $a \in A - A_0$,

$$\begin{aligned} sF(a) &= (1+t)F(a) = F(a) + \omega_a t \\ &= F(a) + (\omega_a \xi)x + \sum_{a''\in A''} \omega_a f_{a''} + \sum_b (\omega_a \beta'_b) g_b. \end{aligned}$$

From this last expression we see that $sF(a)$ does not contain f_a as a summand whenever $a \in A - A_0$. Thus $sF(a)$ can contain f_a as a summand for at most finitely many a (namely the ones in A_0). This means that sF is not exhaustive. □

Theorem 3.10. *Given any two infinite sets A, B, there exists a commutative, additively idempotent semiring $S = S(A,B)$ such that the canonical map*

$$\begin{aligned} \mu : S^A \otimes S^B &\longrightarrow S^{A\times B} \\ F \otimes G &\mapsto ((a,b) \mapsto F(a)G(b)) \end{aligned}$$

is not injective.

Proof. Given $F \in S^A$, $G \in S^B$, we set $\phi(F,G) = x \in S$ if F is f-exhaustive and G is g-exhaustive, and $\phi(F,G) = 0 \in S$ otherwise. This defines a map

$$\phi : S^A \times S^B \longrightarrow S.$$

(Note that ϕ is supported on unbounded functions.) We claim that ϕ is S-bilinear. In order to show that ϕ is biadditive, we consider functions $F, F' \in S^A$ and $G \in S^B$. Suppose that all three are exhaustive. Then by Lemma 3.7, $F+F'$ is exhaustive and

$$\phi(F+F',G) = x = x+x = \phi(F,G) + \phi(F',G).$$

If G and F are exhaustive but F' is not, then by Lemma 3.7, $F+F'$ is still exhaustive and

$$\phi(F+F',G) = x = x+0 = \phi(F,G) + \phi(F',G).$$

In the case where G is exhaustive but neither F nor F' is exhaustive, Lemma 3.7 asserts that $F+F'$ is not exhaustive and consequently

$$\phi(F+F',G) = 0 = \phi(F,G) + \phi(F',G).$$

Lastly, when G is not exhaustive,

$$\phi(F+F',G) = 0 = \phi(F,G) + \phi(F',G).$$

Similarly, one verifies $\phi(F, G+G') = \phi(F,G) + \phi(F,G')$, $F \in S^A$, $G, G' \in S^B$. Thus ϕ is biadditive.

Given $s \in S$, let us proceed to show that

$$\phi(sF,G) = s\phi(F,G). \tag{5}$$

If G is not exhaustive, then indeed $\phi(sF,G) = 0 = s\cdot 0 = s\phi(F,G)$. Let us thus assume that G is exhaustive. We break the demonstration of (5) into two cases: the element s contains 1 as a summand and s does not contain 1 as a summand. Assume first that $s = 1+t$, where t does not contain 1 as a summand. This implies that $tx = 0$. If F is exhaustive, then for infinitely many $a \in A$, $F(a) = f_a + s_a$ and so

$$sF(a) = (1+t)(f_a + s_a) = f_a + (s_a + ts_a + tf_a)$$

for these a. Therefore, sF is also exhaustive and

$$\phi(sF,G) = x = x + tx = sx = s\phi(F,G).$$

If F is not exhaustive, then by Lemma 3.9, sF is not exhaustive and

$$\phi(sF,G) = 0 = s \cdot 0 = s\phi(F,G).$$

Now assume that s does not contain 1 as a summand. Since in this case $sx = 0$ and Lemma 3.8 implies that sF is not exhaustive, we have on one hand $\phi(sF,G) = 0$ and on the other hand $s\phi(F,G) = sx = 0$ if F is exhaustive or $s\phi(F,G) = s \cdot 0 = 0$ if F is not exhaustive. Thus equation (5) holds. A similar argument establishes $\phi(F,sG) = s\phi(F,G)$. This concludes the verification that ϕ is S-bilinear.

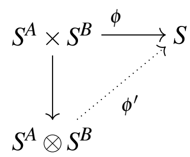
Define a function $f \in S^A$ and a function $g \in S^B$ by setting

$$f(a) = f_a, \; g(b) = g_b$$

for all $a \in A$, $b \in B$. Then f and g are exhaustive (in particular unbounded) and thus $\phi(f,g) = x$. On the other hand,

$$f(a)g(b) = f_a g_b = 0$$

for all a,b, which shows that $\mu(f \otimes g) = 0 \in S^{A\times B}$. By the universal property of the tensor product, there exists a unique S-linear $\phi' : S^A \otimes S^B \to S$ such that



commutes. Its value on $f \otimes g$ is $\phi'(f \otimes g) = \phi(f,g) = x$, whereas $\phi'(0) = 0$ by linearity. In particular,

$$\phi'(f \otimes g) \neq \phi'(0)$$

which proves that

$$f \otimes g \neq 0 \in S^A \otimes S^B.$$

But $\mu(f \otimes g) = 0 = \mu(0)$ so that μ is not injective. □

Remark 3.11. The proof shows that μ is already not injective on elementary tensors $f \otimes g$, let alone on a general tensor element $\sum f_i \otimes g_i$.

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