The L-Class of Singular Spaces: Old and New Results

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July 2018

Self-Dual Perverse Sheaves on General Pseudomanifolds

- X: oriented n-dimensional topological pseudomanifold with a locally cone-like topological stratification.
- Define category $SD(X)$: full subcategory of derived category $D_c^b(X)$ satisfying axioms: top stratum normalization, lower bound, $\bar{n}$-stalk vanishing condition, self-duality.
- $SD(X)$ may or may not be empty.
- **Def.** X is an *L-space*, if $SD(X) \neq \varnothing$.
- For $\mathbf{S}^\bullet \in SD(X)$, there exist morphisms

$$\begin{array}{ccc} \mathbf{IC}^\bullet_{\bar{m}}(X) & \longrightarrow & \mathbf{S}^\bullet \\ \cong \uparrow & & \uparrow \cong \\ \mathcal{D}\mathbf{IC}^\bullet_{\bar{n}}(X)[n] & \longrightarrow & \mathcal{D}\mathbf{S}^\bullet[n] \end{array}$$

Postnikov system of Lagrangian Structures

Def. A *Lagrangian structure* for a complex $\mathbf{S}^\bullet \in SD(U_k)$ along a stratum $U_{k+1} - U_k = X_{n-k} - X_{n-k-1}$ of odd codimension k is a monomorphism

$$\mathbf{L} \longrightarrow \mathbf{H}^{\bar{n}(k)-n}(Ri_{k*}\mathbf{S}^\bullet),\ i_k : U_k \subset U_{k+1},$$

which is stalkwise a Lagrangian subspace.

Thm.(B.) There is an equivalence of categories (say, for n even)

$$SD(X) \simeq \operatorname{Lag}(X_1 - X_0) \rtimes \operatorname{Lag}(X_3 - X_2) \rtimes \ldots$$

$$\rtimes \operatorname{Lag}(X_{n-3} - X_{n-4}) \rtimes \operatorname{Coeff}(X - \Sigma).$$

(Similarly for n odd.)

Examples

- If X is a Witt space (e.g. a complex algebraic variety), then $\mathrm{Ob}(SD(X)) = \{\mathbf{IC}^{\bullet}_{\bar{m}}(X) \cong \mathbf{IC}^{\bullet}_{\bar{n}}(X)\}$.
- $X^6 = S^1 \times \mathrm{Susp}(\mathbb{C}P^2)$: $SD(X^6) = \varnothing$.
- $X^4 = S^1 \times \mathrm{Susp}(T^2)$: $SD(X^4) \neq \varnothing$.
- Primary obstruction: Signature of the link,
  Secondary obstruction: Monodromy.
- **Thm.**(B., Kulkarni) Let $\overline{X}$ be the reductive Borel-Serre compactification of a Hilbert modular surface X. Then $\overline{X}$ an L-space, though it is not a Witt space.

Analytic Approach: Cheeger Structures

Joint work with Albin, Leichtnam, Mazzeo, Piazza.

- X: oriented smoothly Thom-Mather-stratified pseudomanifold.
- Radial Blow-up $\widetilde{X}$: manifold with corners (Melrose).
- $S \subset X$ stratum, $x \in C^\infty(X)$ a boundary defining function ($S = x^{-1}(0)$, $|dx|$ bounded away from 0). An *incomplete edge metric* takes near S the form

  $$dx^2 + x^2 g_{\mathrm{Link}} + \pi^* g_S, \ \pi : \partial\mathrm{tube} \to S.$$

  (iterate inductively.)
- Fact: (Iterated) incomplete edge metrics exist.
- ${}^{\mathrm{ie}}T^*\widetilde{X}$: sections are 1-forms whose restrictions on boundary hypersurfaces vanish on vertical vector fields.

- $C_c^\infty(X_{reg}; \Lambda^j({}^{ie}T^*\widetilde{X})) \subset L^2(\widetilde{X}; \Lambda^j({}^{ie}T^*\widetilde{X}))$ dense, so exterior derivative

$$d : C_c^\infty(X_{reg}; \Lambda^j({}^{ie}T^*\widetilde{X})) \longrightarrow C_c^\infty(X_{reg}; \Lambda^{j+1}({}^{ie}T^*\widetilde{X}))$$

has 2 canonical extensions to a closed operator on L^2: min/max extension with domains

$$D_{\min}(d), D_{\max}(d) \subset L^2(\widetilde{X}; \Lambda^*({}^{ie}T^*\widetilde{X})).$$

- X Witt $\Rightarrow D_{\min}(d) = D_{\max}(d)$.
- Every closed extension $(d, D(d))$ satisfies

$$D_{\min}(d) \subset D(d) \subset D_{\max}(d).$$

- δ formal adjoint of d.
- $\omega_\delta =$ orthog. proj. of $\omega \in D_{\max}(d)$ off of $\ker(\delta, D_{\min}(\delta))$.
- Asymptotic Expansion at boundary: Leading term $\alpha(\omega_\delta) + dx \wedge \beta(\omega_\delta)$.

- A flat subbundle L of the vertical cohomology bundle defines a domain

$$D_L(d) = \{\omega \in D_{\max}(d) \mid \alpha(\omega_\delta) \text{ is a section of } L\}.$$

("Cheeger ideal boundary conditions" imposed by L).
- $(d, D_L(d))$ is a closed operator.
- **Def.** X is a *Cheeger space*, if it admits a *self-dual* subbundle L, i.e. $*_{\text{vert}} L = L^\perp$.
- Set

$$\mathbf{L}^2_L\Omega^\bullet = \text{Sheaf}(U \mapsto \{\omega \in D_L(d) \mid \text{supp}(\omega) \subset U \cap X_{\text{reg}}\}).$$

- **Thm.**(B., Albin, Leichtnam, Mazzeo, Piazza.) If X is a Cheeger space with self-dual Cheeger condition L, then $\mathbf{L}^2_L\Omega^\bullet \in SD(X)$, in particular

$$\mathcal{D}\mathbf{L}^2_L\Omega^\bullet[n] \cong \mathbf{L}^2_L\Omega^\bullet.$$

- **Cor.** Every Cheeger space is an L-space (but not conversely).

The L-class of Singular Spaces.

- Let X be a compact L-space without boundary.
- Given $\mathbf{S}^\bullet \in SD(X)$, self-duality $\rightsquigarrow$ signature $\sigma(\mathbf{S}^\bullet) \in \mathbb{Z}$, bordism invariant.
- Thom-Pontrjagin construction $\rightsquigarrow$

$$L_*(\mathbf{S}^\bullet) \in H_*(X;\mathbb{Q}).$$

- **Thm.** (B.) $L_*(\mathbf{S}^\bullet)$ is independent of the choice of $\mathbf{S}^\bullet \in SD(X)$.
- Idea of proof: Construct concordance between different choices by stratifying cylinder with cuts at $\frac{1}{2}$ to disentangle Langrangian structures. Note that cut has *even* codimension, so does not create problems.
- Thus L-spaces have a well-defined L-class $L_*(X) := L_*(\mathbf{S}^\bullet)$.

Special Cases.

- If $X = M$ is a smooth manifold, get Poincaré dual of Hirzebruch's L-class

$$L_*(X) = L^*(TM) \cap [M].$$

- If X has only even codimensional strata (e.g. a complex algebraic variety), then $L_*(X)$ is the Goresky-MacPherson L-class.
- If X is a Witt space, then $L_*(X)$ is Siegel's L-class.

Relevance in Classification Problems.

- M a closed, smooth, simply connected manifold of even dimension $n \geq 5$.
- Manifold structure set $S(M) = \{[N \xrightarrow{h.e.} M]\}/\,\text{Diffeo}$.
- The map

$$\begin{array}{ccc} S(M)\otimes\mathbb{Q} & \overset{L}{\hookrightarrow} & \bigoplus H^{4j}(M;\mathbb{Q}), \\ [h : N \simeq M] & \mapsto & (h^*)^{-1}L^*(TN) - L^*(TM), \end{array}$$

  is injective.
- In other words: M is determined, up to finite ambiguity, by *its homotopy type and its L-classes*.
- X an even dim. stratified pseudomanifold that has no strata of odd dimension. All strata S have dim ≥ 5, all strata and all links simply connected.
- Cappell-Weinberger: Difference of L-classes gives an injection

$$S(X)\otimes\mathbb{Q} \hookrightarrow \bigoplus_{S\subset X}\bigoplus_{j} H_j(\overline{S};\mathbb{Q}),$$

  where S ranges over the strata of X.

- Even in the manifold case, many mysteries remain concerning L_* (Novikov conj., effective computation, local formulae,...).
- Novikov, 1966: "In those cases in which the preceding question (homotopy invariance of higher signatures) has been answered affirmatively, there arises the problem of **computing** the classes L_* in terms of homotopy invariants. This problem has not been solved (...)."
- Much less is known about L_* in the singular situation. Effective computation? Perhaps for algebraic varieties?
- Here, will discuss **transformational properties** of L_*, adopting the following point of view: Frequently, laws are easier to discern for fundamental classes on bordism, rather than directly for L_*.

Intersection Homology Poincaré Spaces

- To implement this philosophy, need morphism of spectra

$$(\text{singular bordism spectra}) \longrightarrow \mathbb{L}^\bullet,$$

$\mathbb{L}^\bullet = \mathbb{L}^\bullet\langle 0\rangle(\mathbb{Z})$ Ranicki's symmetric L-spectrum,
$\pi_n(\mathbb{L}^\bullet) = L^n(\mathbb{Z})$.

- joint work with Gerd Laures, Jim McClure.
- Can even work integrally, not just rationally.

Def. (Goresky, Siegel) An n-dimensional *Intersection homology Poincaré (IP-) space* is an n-dimensional PL pseudomanifold X such that:

1. $IH_k^{\bar{m}}(L^{2k};\mathbb{Z}) = 0$ for links L^{2k} and
2. $IH_k^{\bar{m}}(L^{2k+1};\mathbb{Z})$ is torsion free for links L^{2k+1}.

- **Thm.** (Goresky-Siegel.) If $(X^n, \partial X)$ is an oriented compact IP-space, then

$$\mathbf{IC}^\bullet_{\bar{m}}(X - \partial X; \mathbb{Z}) \cong \mathrm{R}\mathbf{Hom}^\bullet(\mathbf{IC}^\bullet_{\bar{m}}(X - \partial X; \mathbb{Z}), \mathbb{D}^\bullet_{X - \partial X})[n]$$

(Verdier self-duality over $\mathbb{Z}$ in the derived category of sheaf complexes) and intersection of cycles induces a nonsingular pairing

$$IH_i(X, \partial X; \mathbb{Z})/\operatorname{Tors} \times IH_{n-i}(X; \mathbb{Z})/\operatorname{Tors} \longrightarrow \mathbb{Z}.$$

- W. Pardon: IP-bordism $\Omega^{\mathrm{IP}}_*(-)$, is a gen. homology theory,

$$\Omega^{\mathrm{IP}}_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ \mathbb{Z}/_2, & n \geq 5, n \equiv 1(4), \\ 0 & \text{otherwise.} \end{cases}$$

Note: very close to $L^n(\mathbb{Z})$.

Ad Theories (Quinn; Buoncristiano-Rourke-Sanderson; Laures-McClure).

Target categories $\mathcal{A}$ of an ad-theory:
$\mathbb{Z}$-graded categories $\mathcal{A}$ (no morphisms that decrease dimension), with involution (will suppress). (Have inclusions of cells $\tau \subset \sigma$ only when $\dim \tau \leq \dim \sigma$.)

Def. An *ad-theory* ad with target category $\mathcal{A}$ is an assignment

$$k \in \mathbb{Z}, \text{ ball complex pairs } (K, L) \mapsto \mathrm{ad}^k(K, L),$$

$$\mathrm{ad}^k(K, L) \subset \{\text{functors } F : K - L \to \mathcal{A} \mid F \text{ decr. dim. by } k\}$$

satisfying axioms regulating reindexing of ball pairs, gluing of subdivisions, extension to cylinders.

$F \in \mathrm{ad}^k(K, L)$ is called a (K, L)-*ad*.

Ad Theories: Bordism and Quinn Spectra

- A *morphism of ad theories* is a functor of target categories which takes ads to ads.
- $F, F' \in \mathrm{ad}^k(\mathrm{pt})$ are *bordant*, if exists I-ad G: $G|_0 = F$, $G|_1 = F'$. (Is an equivalence relation by axioms reindexing, gluing, cylinder.)
- bordism groups $\Omega_k :=$ bordism classes in $ad^{-k}(\mathrm{pt})$.
- Geometric realization $\mathbf{Q}_k := |Q_k|$ of semisimplicial sets Q_k with n-simplices $\mathrm{ad}^k(\Delta^n)$ gives associated *Quinn spectrum* $\mathbf{Q}$.
- $\pi_*(\mathbf{Q}) = \Omega_*$
- Morphism $\mathrm{ad}_1 \to \mathrm{ad}_2 \rightsquigarrow \mathbf{Q}_1 \to \mathbf{Q}_2$.

IP-ads and $\mathbb{L}$-ads.

- Target category $\mathcal{A}^{\mathrm{IP}}$:
  - *Objects*: pairs (X, ξ)
  - $(X, \partial X)$ compact, oriented IP-space,
  - $\xi \in IS_n^{\bar{0}}(X;\mathbb{Z})$ representative for $[X] \in IH_n^{\bar{0}}(X, \partial X;\mathbb{Z})$. (Singular intersection chains, H. King)
  - *Morphisms*: orientation-preserving PL-homeomorphisms and stratum preserving PL-embeddings $\hookrightarrow$ boundary, respecting ξ.
- $\mathrm{ad}^{\mathrm{IP},k}(K)$: all functors $F : K \to \mathcal{A}^{\mathrm{IP}}$, decr. dim. by k, s.t. for all cells $\sigma \in K$:

$$\operatorname*{colim}_{\tau\in\partial\sigma} F(\tau) \xrightarrow{\cong} \partial(F(\sigma)),\ \partial\xi_{F(\sigma)} = \sum_{\tau\in\partial\sigma} \pm\xi_{F(\tau)}.$$

- **Prop.** $\mathrm{ad}^{\mathrm{IP}}$ is an ad theory.
- Get spectrum $\mathrm{MIP} = \mathbf{Q}^{\mathrm{IP}}$ with $\pi_*(\mathbf{Q}^{\mathrm{IP}}) = \Omega_*^{\mathrm{IP}}(\mathrm{pt})$.
- Ad theory $\mathrm{ad}^{\mathbb{L}}$; get Quinn spectrum $\mathbf{Q}^{\mathbb{L}} \simeq \mathbb{L}^\bullet$.

- $\bar{n}$ upper middle perversity.
- On $X \times X$, for strata $S, T \subset X$, let

$$\bar{p}(S \times T) = \begin{cases} \bar{n}(S) + \bar{n}(T) + 2, & \text{codim } S, \text{codim } T > 0 \\ \bar{n}(S) + \bar{n}(T), & \text{otherwise} \end{cases}$$

- Diagonal $d : X \to X \times X$ induces

$$d_* : IS_*^{\bar{0}}(X) \longrightarrow IS_*^{\bar{p}}(X \times X).$$

- Have cross product

$$\beta : IS_*^{\bar{n}}(X) \otimes IS_*^{\bar{n}}(X) \xrightarrow{\simeq} IS_*^{\bar{p}}(X \times X).$$

(D. Cohen, M. Goresky, Lizhen Ji, G. Friedman)

- Functor $\mathrm{Sig} : \mathcal{A}^{\mathrm{IP}} \to \mathcal{A}^{\mathbb{L}}$:

$$(X, \xi) \mapsto (C, D, \beta, \varphi)$$

  - $C := IS_*^{\bar{n}}(X; \mathbb{Z})$,
  - $D := IS_*^{\bar{p}}(X \times X; \mathbb{Z})$,
  - $\beta :=$ cross product,
  - $\varphi := d_*(\xi)$.

  A morphism $(X, \xi) \to (X', \xi')$ induces maps on intersection chains.
- **Prop.** If $F \in \mathrm{ad}^{\mathrm{IP}}(K)$, then $\mathrm{Sig} \circ F \in \mathrm{ad}^{\mathbb{L}}(K)$.
- Get morphism $\mathrm{Sig} : \mathrm{ad}^{\mathrm{IP}} \to \mathrm{ad}^{\mathbb{L}}$.
- On Quinn spectra $\mathrm{Sig} : \mathbf{Q}^{\mathrm{IP}} \to \mathbf{Q}^{\mathbb{L}}$.

- In the stable category, get

$$\mathrm{Sig} : \mathrm{MIP} = \mathbf{Q}^{\mathrm{IP}} \longrightarrow \mathbf{Q}^{\mathbb{L}} \simeq \mathbb{L}^{\bullet}.$$

- Induces

$$\boxed{\Omega_*^{\mathrm{IP}}(X) \longrightarrow \mathbb{L}_*^{\bullet}(X)}.$$

- **Thm.(B., Laures, McClure**
  The map $\Omega_n^{\mathrm{IP}}(\mathrm{pt}) \to \mathbb{L}_n^{\bullet}(\mathrm{pt}) = L^n(\mathbb{Z})$ is an isomorphism for all $n \neq 1$. $(\Omega_1^{\mathrm{IP}}(\mathrm{pt}) = 0,\ L^1(\mathbb{Z}) = \mathbb{Z}/_2.)$
- This was conjectured by W. Pardon in 1990.
- For a closed IP-space $[X]_{\mathrm{IP}} := [X \xrightarrow{\mathrm{id}} X] \in \Omega_n^{\mathrm{IP}}(X)$.
- **Def.**

$$\begin{array}{ccc} \Omega_n^{\mathrm{IP}}(X) & \longrightarrow & \mathbb{L}_n^{\bullet}(X) \\ [X]_{\mathrm{IP}} & \mapsto & [X]_{\mathbb{L}}. \end{array}$$

Thm. (B., Laures, McClure)
For an n-dimensional compact oriented IP-space X there is a fundamental class $[X]_{\mathbb{L}} \in \mathbb{L}^{\bullet}_{n}(X)$ with the following properties:

1. $[X]_{\mathbb{L}}$ is an oriented PL homeomorphism invariant,
2. The image of $[X]_{\mathbb{L}}$ under assembly is the symmetric signature:

$$\begin{array}{ccc} \mathbb{L}^{\bullet}_{n}(X) & \longrightarrow & L^{n}(\mathbb{Z}\pi_1(X)) \\ [X]_{\mathbb{L}} & \mapsto & \sigma^{*}_{\mathrm{IP}}(X) \end{array}$$

3. If X is a PL manifold, then $[X]_{\mathbb{L}}$ is the fundamental class constructed by Ranicki.
4. Rationally, $[X]_{\mathbb{L}}$ agrees with the L-class of X.

Rem. Similar statements hold over $\mathbb{Q}$ for Witt spaces.

Applications: Cartesian Products (as warm up).

- The morphism
$$\mathrm{MWitt} \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})$$
is a morphism of symmetric ring spectra.
- So get *multiplicative* map
$$\Omega_*^{\mathrm{Witt}}(X) \longrightarrow (\mathbb{L}^\bullet(\mathbb{Q}))_*(X).$$
- Now $[\mathrm{id}_{X\times Y}] = [\mathrm{id}_X] \times [\mathrm{id}_Y] \in \Omega_*^{\mathrm{Witt}}(X \times Y)$, so:
- **Thm.** (J. Woolf w/ different methods.) For Witt spaces X, Y,
$$L_*(X \times Y) = L_*(X) \times L_*(Y).$$

Application: Homotopy Invariance of Higher Signatures

- $G = \pi_1(X)$, $r : X \to BG$ a classifying map for the universal cover of X.
- $r_* : H_*(X; \mathbb{Q}) \longrightarrow H_*(BG; \mathbb{Q})$.
- The *higher signatures of* X are the rational numbers

$$\langle a, r_* L(X) \rangle, \; a \in H^*(BG; \mathbb{Q}).$$

- **Thm.** (B., Laures, McClure) Let X be an n-dimensional oriented closed IP-space such that the assembly map

$$\alpha : \mathbb{L}^\bullet_n(BG) \longrightarrow L^n(\mathbb{Z}[G])$$

is rationally injective. Then the higher signatures of X are (orient. pres.) stratified homotopy invariants.

Proof.

- $f : X' \to X$ an orient. pres. stratified homotopy equivalence.
- $r : X \to BG$, $r' = r \circ f : X' \to BG$.
- $$\begin{array}{ccc} \mathbb{L}^\bullet_n(X) & \longrightarrow & L^n(\mathbb{Z}[G]) \\ {\scriptstyle r_*}\downarrow & \nearrow{\scriptstyle \alpha} & \\ \mathbb{L}^\bullet_n(BG) & & \end{array}$$
- $\alpha r_*[X]_{\mathbb{L}} = \sigma^*_{\mathrm{IP}}(X) = \sigma^*_{\mathrm{IP}}(r) = \sigma^*_{\mathrm{IP}}(rf) = \sigma^*_{\mathrm{IP}}(X') = \alpha r'_*[X']_{\mathbb{L}}$.
- Injectivity assumption $\Rightarrow r_*[X]_{\mathbb{L}} = r'_*[X']_{\mathbb{L}} \in \mathbb{L}^\bullet_n(BG) \otimes \mathbb{Q}$.
- $$\begin{array}{ccc} \mathbb{L}^\bullet_n(X) \otimes \mathbb{Q} & \xrightarrow{r_*} & \mathbb{L}^\bullet_n(BG) \otimes \mathbb{Q} \\ {\scriptstyle S_X}\downarrow\cong & & \cong\downarrow{\scriptstyle S_{BG}} \\ \bigoplus_j H_{n-4j}(X;\mathbb{Q}) & \xrightarrow{r_*} & \bigoplus_j H_{n-4j}(BG;\mathbb{Q}) \end{array}$$
- $$\begin{aligned} r_* L_*(X) &= r_* S_X[X]_{\mathbb{L}} = S_{BG} r_*[X]_{\mathbb{L}} \\ &= S_{BG} r'_*[X']_{\mathbb{L}} = r'_* S_{X'}[X']_{\mathbb{L}} = r'_* L_*(X'). \end{aligned}$$

Complex Algebraic Geometry.

- Different Method: Decomposition Theorem (Beilinson, Bernstein, Gabber, Deligne; M. Saito; de Cataldo, Migliorini)
- If $f : Y \to X$ is a proper algebraic morphism of algebraic varieties, then

$$Rf_* \mathbf{IC}^\bullet_{\bar{m}}(Y) \cong \bigoplus_i j_* \mathbf{IC}^\bullet_{\bar{m}}(\overline{Z_i}; \mathcal{S}_i)[n_i],$$

  Z_i is a nonsingular, irreducible, locally closed subvariety of X, $\mathcal{S}_i$ a locally constant sheaf over Z_i, $n_i \in \mathbb{Z}$.
- Since $L_*(Rf_* \mathbf{S}^\bullet[-\mathrm{cod}]) = f_* L_*(\mathbf{S}^\bullet)$, $L_*(\mathbf{S}^\bullet_1 \oplus \mathbf{S}^\bullet_2) = L_*(\mathbf{S}^\bullet_1) + L_*(\mathbf{S}^\bullet_2)$, $L_*(j_* \mathbf{S}^\bullet[\mathrm{cod}]) = j_* L_*(\mathbf{S}^\bullet)$,

$$f_* L_*(Y) = \sum_i j_* L_*(\overline{Z_i}; \mathcal{S}_i).$$

Twisted L-Classes.

- Problems with decomposition: Determine $Z_i, \mathcal{S}_i$ in practice, compute $L_*(\overline{Z_i}; \mathcal{S}_i)$. (But see Schürmann-Woolf on $W(\mathrm{Perv}(X))$.)
- Phenomenon: (Non)Multiplicativity of signature. (W. Neumann, A. Némethi,...)
- **Thm.** (B.) X closed L-space, $\mathcal{S}/X - \Sigma$ Poincaré local system constant on links. Then

$$L_*(X; \mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L_*(X).$$

- Proof: Uses Signature Homology Theory (Minatta,Kreck), results of Sullivan, Siegel, families index theorem (Atiyah, W. Meyer).
- Assumption on extendability cannot be eliminated because the formula will fail: examples of 4-dimensional orbifolds with isolated singularities.
- Special case: Witt spaces — B., Cappell, Shaneson.

Finite degree covers.

- $p : X' \to X$ cover of finite degree d, X, X' singular L-spaces, e.g. complex algebraic pseudomanifolds.
- Transfer
$$p_! : H_*(X; \mathbb{Q}) \to H_*(X'; \mathbb{Q})$$
such that $p_* p_! = d \cdot \mathrm{id}$.
- **Thm.** (B.)
$$L_*(X') = p_! L_*(X).$$
- **Cor.** Multiplicativity of L-classes for finite covers:
$$p_* L_*(X') = d \cdot L_*(X),$$
where $p_* : H_*(X'; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is induced by p.

Application: Hodge L-Class and the BSY-Conjecture.

- Let X be a complex algebraic variety.
- Looijenga, Bittner: $K_0(\mathrm{Var}/X) = \{[Y \xrightarrow{\text{alg}} X]\}$, modulo scissor relation

$$[Y \to X] = [Z \hookrightarrow Y \to X] + [Y - Z \hookrightarrow Y \to X]$$

for $Z \subset Y$ a closed algebraic subvariety of Y.
- Brasselet-Schürmann-Yokura: motivic Hirzebruch natural transformation

$$T_{y*} : K_0(\mathrm{Var}/X) \longrightarrow H_{2*}^{BM}(X) \otimes \mathbb{Q}[y],$$

- Hirzebruch class of X is

$$T_{y*}(X) := T_{y*}([\mathrm{id}_X]).$$

- $y = 1$: $T_{1*}(X)$ is called the *Hodge L-class* of X.

Application: Hodge L-Class and the BSY-Conjecture.

- If X is smooth and pure-dimensional:
  $T_{y*}(X) = T_y^*(TX) \cap [X]$. For $y = 1$, $T_1^*(TX) = L^*(X)$, so

$$T_{1*}(X) = L_*(X).$$

- Examples of singular curves show that generally $T_{1*}(X) \neq L_*(X)$.
- BSY conjecture: $T_{1*}(X) = L_*(X)$ for compact complex algebraic varieties that are rational homology manifolds.
- Cappell, Maxim, Schürmann, Shaneson: holds for $X = Y/G$, Y a projective G-manifold, G a finite group of algebraic automorphisms. Also: certain complex hypersurfaces with isolated singularities.
- In proj. case, holds in deg 0 by Saito's intersection cohomology Hodge index theorem.
- Maxim, Schürmann: for simplicial projective toric varieties.

Example: 3-Folds with Trivial Canonical Class

Thm. (B.) The Brasselet-Schürmann-Yokura conjecture holds for normal, projective, complex 3-folds X with at worst canonical singularities, trivial canonical divisor and $\dim H^1(X; \mathcal{O}_X) > 0$.
Proof.

- Have Albanese morphism $X \to \mathrm{Alb}(X)$.
- $K_X \equiv 0 \Rightarrow$ Kawamata splitting up to finite degree covering:

$$\begin{array}{ccc} F \times E & \longrightarrow & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_E} \\ X & \longrightarrow & \mathrm{Alb}(X), \end{array}$$

- Interesting case irreg. $q(X) = 1$. Then F is a surface.
- Use above multiplicativity of L_*, and of T_{1*} ($\to$ CMSS).
- Use scissor relations and ADE theory on F.
- $\sigma(F) \in \{-16, -15, \ldots, 2, 3\}$.