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Stratified Spaces: Joining Analysis, Topology and Geometry

Organised by
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December 11th – December 17th, 2011

Abstract. For manifolds, topological properties such as Poincaré duality and invariants such as the signature and characteristic classes, results and techniques from complex algebraic geometry such as the Hirzebruch-Riemann-Roch theorem, and results from global analysis such as the Atiyah-Singer index theorem, worked hand in hand in the past to weave a tight web of knowledge. Individually, many of the above results are in the meantime available for singular stratified spaces as well. The 2011 Oberwolfach workshop "Stratified Spaces: Joining Analysis, Topology and Geometry" discussed these with the specific aim of cross-fertilization in the three contributing fields.

Mathematics Subject Classification (2000): 57N80, 58A35, 32S60, 55N33, 57R20.

Introduction by the Organisers

The workshop *Stratified Spaces: Joining Analysis, Topology and Geometry*, organised by Markus Banagl (Heidelberg), Ulrich Bunke (Regensburg) and Shmuel Weinberger (Chicago) was held December 11th – 17th, 2011. It had three main components: 1) Three special introductory lectures by Jonathan Woolf (Liverpool), Shoji Yokura (Kagoshima) and Eric Leichtnam (Paris); 2) 20 research talks, each 60 minutes; and 3) a problem session, led by Shmuel Weinberger.

In total, this international meeting was attended by 45 participants from Canada, China, England, France, Germany, Italy, Japan, the Netherlands, Spain and the USA. The "Oberwolfach Leibniz Graduate Students" grants enabled five advanced doctoral students from Germany and the USA to attend the meeting. One of these students, Florian Gaisendrees (Heidelberg), presented his thesis results. Both established senior mathematicians and postdocs were well represented.

The workshop pursued a twofold aim. While the attendees received an overview of the current state-of-the-art in stratified space theory, we hoped that in addition the representatives of the three major fields algebraic geometry, topology and global analysis, which contribute each in their own way to the study of singular stratified spaces, would move closer together and learn from each others' viewpoints. Ample time between lunch and the afternoon session provided the opportunity for discussions among the participants. In Germany, this was the first such meeting on stratified spaces in decades and underscores the recent surge of interest in this area, due on one hand to the realization that results and frameworks capable of handling singularities are not limited to internal applications, but also shed light on nonsingular spaces, and on the other hand to prodigious progress that the area has made in the last dozen years. Naturally, the aforementioned goal is an ambitious one that is not expected to be achieved by a single meeting. Partly, the techniques in the three respective fields are very different from each other, and to master them requires substantial effort from someone not already in the field. But, as the meeting has shown, there is a lot of common ground as well, and this is where one can start to interact: Characteristic classes of singular spaces are a good example, as became clear in Yokura's survey. The organizers feel that a first step in reaching the goal has been made in view of the fact that participants expressed vivid interest in the results presented by proponents of adjacent fields and are now certainly aware of the kind of contribution each of the fields is now making, or will make in the near future, to the advancement of the theory of stratified spaces.

In his expository lecture, Jonathan Woolf gave an introduction to intersection cohomology, choosing as his vehicle perverse sheaves. This approach is not new, but more familiar to topologists and algebraic geometers than to analysts. This theme was later in the week continued by Jörg Schürmann, who studied decompositions in the Witt group of perverse sheaves (jointly with Woolf). Yokura provided an overview of his joint work with Brasselet and Schürmann, which unifies for singular varieties several genera and characteristic classes arising in algebraic geometry and topology. Eric Leichtnam's special lecture reviewed the higher signature index class first for manifolds, then for stratified (Witt) spaces.

By blow-up constructions, manifolds with fibered corners or cusps are closely linked to stratified spaces. While the latter are hard to study directly analytically, the former are more amenable to such methods. This was reflected in a number of talks on analysis on manifolds with (fibered) corners/cusps (Daniel Grieser, Jean-Marie Lescure, Paul Loya). Lie groupoids occurred in several talks; Hessel Posthuma discussed localized index theory on them and Markus Pflaum studied the stratified geometry of their orbit spaces. That the stratified point of view has its merits in solving PDEs was proven by Bernd Amman's talk, who introduced Lie manifolds, equipped Euclidean space with such a structure, observed that the Schrödinger operator defines a stratification, and obtained regularity results for eigenfunctions of the operator that way. Lie manifolds also appeared in Victor Nistor's lecture, who focused on the well-posedness of the Laplace equation on nonsmooth but suitably stratified domains. On a space with isolated singularities, Ursula Ludwig established a comparison theorem between the geometric Morse-Thom-Smale complex and Witten's analytic complex. On the more topological side, Jim McClure reported on his work with Greg Friedman on further clarifying the symmetric signature of Witt spaces. In collaboration with Cappell and Weinberger, Min Yan achieves a surgery-theoretic classification of multiaxial $U(n)$-manifolds. These, and their orbit spaces, carry natural stratifications and the result expresses the structure set as a sum over relative structure sets associated to the strata. Matthias Kreck discussed a geometric construction of a new equivariant cohomology theory which, for manifolds, is Poincaré dual to ordinary equivariant homology, based on interpreting the latter as a bordism group of stratified spaces ("stratifolds") with free group action together with an equivariant map to the manifold. Florian Gaisendrees implemented further steps in the program of intersection spaces associated to a stratified space, pioneered by Markus Banagl. Xianzhe Dai's lecture centered on the relation of analytic torsion to (intersection) Reidemeister torsion in the presence of an isolated conical singularity. For algebraic (or analytic) varieties over a field of characteristic zero, Edward Bierstone and his collaborators Lairez, Milman and Vera Pacheco, investigate a natural stratification determined by the desingularization invariant, which can be used to compute local normal forms for the singularities at every point of the variety. Manuel Villa, together with Budur and González-Pérez, explained that the motivic zeta function, the motivic Milnor fiber, the Hodge-Steenbrink spectrum, and the log canonical threshold of an irreducible quasi-ordinary hypersurface singularity (which is generally not isolated) are determined by its embedded topological type.

Workshop: Stratified Spaces: Joining Analysis, Topology and Geometry

Table of Contents

Abstracts

Regularity of eigenfunctions of Schrödinger type equations

Bernd Ammann

(joint work with Catarina Carvalho, Victor Nistor)

In this talk we present some regularity statements for elliptic differential operators with singular potential, as for instance Schrödinger operators. The talk summarizes results from [1] which is work in collaboration with Catarina Carvalho and Victor Nistor. It is also intensively connected to collaborations with A. Ionescu and R. Lauter.

At first, we describe Lie manifolds, which were introduced in [3]. Roughly speaking the concept of a Lie manifold is a tool to compactify non-compact complete manifolds. A Lie manifold is a manifold with corners M (always with a fixed differentiable structure) and a fixed Lie algebra of vector fields which satisfies some axiomatic properties, for example we require that the Lie algebra of vector fields be a projective module over the ring $C^\infty(M)$. This Lie algebra of vector fields is used to describe a complete metric on the interior M_0 of M.

Examples of Lie manifolds are b-manifolds in the sense of Melrose, see e.g. [4], asymptotically hyperbolic manifolds and many similarly constructed manifolds. Euclidean space $\mathbb{R}^n$ is obtained from the scattering calculus [5].

We then describe a systematic way to conformally blow-up a Lie-manifold M along a given submanifold N. The blow-up construction consists of two parts: at first we obtain a new manifold with corners $[M : N]$. This part is quite standard, and also similar to [6]. We assume transversality at the boundary and $N \cap M_0 \neq \emptyset$. The second part is to specify a Lie algebra of vector fields on this blownn-up manifold $[M : N]$. A subtle step is to prove that the Lie algebra of vector fields is a projective module, and one of the most amazing aspects is that no condition on the behavior of the Lie algebra of vector fields on M is required for our construction. Associated to this new Lie algebra of vector fields there is a complete metric on the interior $[M : N]_0$ of $[M : N]$. This new metric on $[M : N]_0 \cong M_0 \setminus N$ is conformal to the original metric on M_0.

We apply this blow-up construction to the study of the eigenvalue equation for Schrödinger type operators $\mathcal{H}$. For example we study the classical Schrödinger operator $\mathcal{H} := -\Delta + V$ of a k-electron atom with

$$V(x) = \sum_{1 \leq j \leq k} \frac{b_j(x)}{|x_j|} + \sum_{1 \leq i < j \leq k} \frac{c_{ij}(x)}{|x_i - x_j|}, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^{3k}$, $x_j \in \mathbb{R}^3$, and b_j and c_{ij} are suitable smooth functions. Another example would be a molecule, where we consider the nuclei as very heavy and thus with a fixed position.

The blow-up construction described above will be applied now. We start with euclidean space $\mathbb{R}^{3k}$ equipped with the Lie manifold structure from the scattering calculus. The Schrödinger operator defines a structure of a stratified space on $\mathbb{R}^{3k}$.

The highest dimensional strata are those where exactly two particles meet, and these strata have codimension 3. The lowest stratum is where the number of meeting particles is maximal, and it is typically a single point. The Schrödinger operator has a singular potential along the strata, in the sense that the potential is unbounded in the neighborhood of the strata. Our blow-up construction now blows up the lowest-dimensional stratum first, and then iteratively the strata of higher and higher dimension. During this blow-up-construction the eigenvalue equation for the Schrödinger operator is translated into an equation which remains "bounded with respect to the blown-up metric" in a neighborhood of the strata.

This allows us to apply regularity statements for Lie manifolds, developed in [2]. In this way we obtain the following regularity result for the eigenfunctions of the Schrödinger equation. To express this result we define the Babuška-Kondratiev spaces .

$$\mathcal{K}_a^m(\mathbb{R}^{3k}, r_S) := \{u : \mathbb{R}^{3k} \to \mathbb{C} \,|\, r_S^{|\alpha|-a}\partial^\alpha u \in L^2(\mathbb{R}^{3k}),\ |\alpha| \le m\}, \tag{2}$$

where $a \in \mathbb{R}$ and $m \in \mathbb{N}$. The weight function $r_S(x)$ is the smoothed distance from x to the singular strata, however the distance r_S is not measured with respect to euclidean distance, but with respect to a metric on the ball compactification of $\mathbb{R}^{3k}$. This modified choice does not effect $\mathcal{K}_a^m(\mathbb{R}^{3k}, r_S)$ on closed balls, but it does globally.

Theorem 1 ([1, Theorem 4.3]). *Assume $u \in L^2(\mathbb{R}^{3k})$ is an eigenfunction of the single-nucleus k-electron Schrödinger operator, then*

$$u \in \mathcal{K}_a^m(\mathbb{R}^{3k}, r_S)$$

for all $m \in \mathbb{N}$ and for all $a \le 0$.

This results is mainly interesting close to the strata of non-maximal dimension. The result is already known for the classical Schrödinger operator locally along the top-dimensional strata, in fact it follows from recent research by Fournais [7] in which it is shown that the eigenfunction satisfies some modified version of analyticity. Our result is stronger close to those places where at least three particles meet. The approach in [7] is also more retrictive than ours, in particular as it requires analyticity of the potential, which we do not assume.

In current work we modify the blow-up construction, and we expect to obtain a version of the above theorem in which r_S is replaced by the euclidean distance to the singular stratum. It also seems likely to us that the coefficient a in the above theorem can be improved to all $a < 3/2$. One motivation to conjecture this is that the corresponding statement holds for the Schrödinger operator associated to several nuclei and a single electron.

Theorem 2 ([1, Theorem 4.6]). *Let $u \in L^2(\mathbb{R}^3)$ be such that $\mathcal{H}u = \lambda u$, in distribution sense. Then $u \in \mathcal{K}_a^m(\mathbb{R}^3, r_S)$ for all $m \in \mathbb{N}$ and all $a < 3/2$.*

Finally we also want to mention interesting developments by Flad, Harutyunyan,
Schneider, and Schulze, see [8, 9]. Their approach is partially related to ours, but

has different goals, in particular they obtain precise asymptotic developments for systems with few particles.

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Stratification by the desingularization invariant

Edward Bierstone

(joint work with P. Lairez, P. Milman, F. Vera Pacheco)

The philosophy of the talk is that the desingularization invariant [3] deterimines a natural stratification of an algebraic or analytic variety X (over a field of characteristic zero), and can be used to compute local normal forms for the singularity of X at every point of a stratum. The idea will be illustrated in the following problem.

The objects of study are (reduced) algebraic (or analytic) varieties X, together with birational (or bimeromorphic) morphisms $\sigma : X' \to X$.

Question. Can we find the smallest class of singularities $\mathcal{S}$ such that:

(1) $\mathcal{S}$ includes all normal crossings (nc) singularities;

(2) Given X, there is a proper birational (or bimeromorphic) morphism $\sigma : X' \to X$ such that X' has only singularities in $\mathcal{S}$, and σ is an isomorphism over the normal crossings locus of X?

The question has interesting variants; e.g., in (2) we can require that σ is an isomorphism over the $\mathcal{S}$-locus of X (the points of X having only singularities in $\mathcal{S}$).

Why ask these questions? In characteristic zero, every variety is birationally equivalent to a smooth variety. But birational models with mild singularities have to be admitted in natural situations. For example, we cannot simultaneously resolve the singularities of a family of curves without allowing special fibres that have normal crossings singularities.

In general, $\mathcal{S}$ is larger than the class of normal crossings singularities:

Example. The *Whitney umbrella* or *pinch point* pp, given by $z^2 + xy^2 = 0$, has a pp singularity at the origin, but double normal crossings singularities nc2 along the nonzero x-axis. There is no proper bimeromorphic mapping that eliminates pp without modifying nc points.

Normal crossings at a point means that X is defined, in suitable local analytic coordinates $(x_1, \dots, x_n)$ at the point, by a monomial equation $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0$.

Philosophy. The desingularization invariant [3] deterimines a natural stratification of an algebraic or analytic variety X, and can be used together with natural geometric information to compute local normal forms for the singularity of X at every point of a stratum. In general, the invariant is a finite sequence $\mathrm{inv} = (\mathrm{HS}, \dots)$, beginning with the *Hilbert-Samuel function* HS of X at a point. If X is *hypersurface* (i.e., defined by a principal ideal), then HS is determined by the order of the ideal at a point. As an example of the philosophy, we recall that stratification by HS distinguishes the class of subanalytic sets on which one can do classical local analysis:

Theorem 0.1. [2] *The following are equivalent:*

(1) *HS is upper-semicontinuous on X (in the subanalytic Zariski topology).*

(2) *X is semicoherent, i.e., X has a locally finite stratification $X = \cup X_i$ such that $\mathcal{F}_a(X)$ is generated on X_i by*

$$\sum_{\alpha \in \mathbb{N}^n} f_{ij,\alpha}(a)(x-a)^\alpha \in \mathbb{R}[\![x-a]\!], \quad j = 1, \dots, q,$$

where each $f_{ij,\alpha}$ is analytic on X_i and subanalytic.

(3) *$f : X \to \mathbb{R}$ is $\mathcal{C}^\infty$ (i.e., the restriction of a $\mathcal{C}^\infty$ function) if and only if f is $\mathcal{C}^k$, for all k.*

(4) *Composite function property. Consider a proper analytic map $\varphi : M \twoheadrightarrow X \subset \mathbb{R}^n$. Then $f \in \mathcal{C}^\infty(M)$ is a $\mathcal{C}^\infty$ composite with φ (i.e., $f = g \circ \varphi$, where $g \in \mathcal{C}^\infty(\mathbb{R}^n)$) if and only if f is formally composite over every $a \in X$.*

Resolution of singularities. Resolution of singularities of a variety X is given by a sequence of blowings-up $X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t = X'$. The centre of each blowing-up is the maximum locus of an upper-semicontinuous invariant inv defined recursively over a sequence of *admissible* blowings-up.

Examples.
$$\begin{array}{lll} \mathrm{nc2} & z^2 + y^2 = 0 & \mathrm{inv}(\mathrm{nc2}) = (2,0,1,0,\infty) \\ \mathrm{pp} & z^2 + xy^2 = 0 & \mathrm{inv}(\mathrm{pp}) = (2,0,3/2,0,1,0,\infty) \end{array}$$

Lemma 0.2.
$$\begin{array}{lcl} \textit{nc2} & \iff & \mathrm{inv} = \mathrm{inv}(\mathrm{nc2}) \\ \textit{pp} & \iff & \mathrm{inv} = \mathrm{inv}(\mathrm{pp}) \ \textit{and} \ \operatorname{codim} \operatorname{Sing} X = 2 \end{array}$$

This is in *year zero* (i.e., before any blowings-up). In general, the invariant depends on the history of blowings-up. For example, in 3 variables, the locus (inv = inv(nc2)) is a smooth curve. It is generically nc2. But, at a special point, $z^2 + w^\alpha y^2 = 0$, where $(w = 0)$ is the exceptional divisor. This equation can be simplfied by *cleaning*: Blow up $(z = w = 0)$. In the coordinate chart (w, y, wz), we get $w^2(z^2 + w^{\alpha-2}y^2) = 0$. Eventually, we get $w^*(z^2 + y^2) = 0$, if al is even, or $w^*(z^2 + wy^2) = 0$, α odd.

Theorem 0.3. Minimal singularities [1], [4]. *The class $\mathcal{S}$ in up to four variables:*

$$\begin{aligned} xy &= 0 && \\ xyz &= 0 && \\ xyzw &= 0 && \textit{nc4} \\ z^2 + xy^2 &= 0 && (\textit{or } z^2 + (y+2x)(y-x)^2 = 0) \\ z^2 + (y+2x^2)(y-x^2)^2 &= 0 && \textit{degenerate pinch point dpp} \\ x(z^2 + wy^2) &= 0 && \textit{prod} \\ z^3 + wy^3 + w^2x^3 - 3wxyz &= 0 && \textit{cyclic point singularity cp3} \end{aligned}$$

For isomorphism over the $\mathcal{S}$-locus (second version of the main problem), $\mathcal{S}$ in four variables includes also

$$z^2 + y(wy + x^2)^2 = 0 \qquad \textit{exceptional singularity exc.}$$

The theorem depends on an understanding of limits of nc3 singularities in four variables. Consider

$$f(w, x, y, z) = z^3 + a(w, x, y)z^2 + b(w, x, y)z + c(w, x, y),$$

nc3 on the non-negative w-axis. Suppose inv(0) = inv(nc3) = $(3, 0, 1, 0, 1, 0, \infty)$. Then, at the origin:

f has 3 factors	f has 2 factors	f irreducible
	$f(w^2, x, y, z)$ splits	$f(w^3, x, y, z)$ splits

After cleaning:

$f = xyz$ nc3	$f = x(z^2 + wy^2)$	$f =$ cp3

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Analytic Torsion and Intersection R-torsion for Manifolds with Conical Singularity

Xianzhe Dai

§1. Cheeger-Müller Theorem

The Reidemeister torsion was originally introduced in the 3-dimensional setting by K. Reidemeister in 1935. It was used to give a homeomorphism classification of 3-dimensional lens spaces. The R-torsion is not a homotopy invariant, but rather a simple homotopy invariant and thus a topological invariant (under acyclicity conditions). It was generalized to arbitrary dimensions by W. Franz, de Rham, and later studied by many authors.

Ray-Singer [15] introduced analytic torsion in 1971 as an analytic analog of the Reidemeister torsion. The celebrated Ray-Singer Conjecture states that the analytic torsion equals the Reidemeister torsion on a closed manifold. Cheeger and Müller independently proved Ray-Singer conjecture, [3, 13]. Later Müller [14] generalized it to unimodular representations and Bismut-Zhang [1] dealt with general representations.

In [6] Aparna Dar introduced intersection R-torsion for a class of singular manifolds, namely the pseudomanifolds, using the intersection homology theory of Goresky-MacPherson [9, 10]. She also defined analytic torsion for manifolds with isolated conical singularity using Cheeger's theory [5]. Roughly speaking a pseudomanifold is a manifold with iterated conical singularity. Thus it is an interesting question as to what would be the generalization of Cheeger-Müller theorem for pseudomanifolds.

§2. Variations of R-torsion and Analytic Torsion

Let X be a pseudomanifold with admissible conical metric. By Cheeger [4],

$$H^*_{(2)}(X) = (IH^{\bar{m}}_*(X))^*$$

for the upper middle perversity $\bar{m} = (0, 1, 1, \cdots, [\frac{k-1}{2}], \cdots)$.

Thus one can choose the preferred bases of $IH^{\bar{m}}_*(X)$ according to the Hodge theorem. Denote by $I\tau(X)$ the intersection R-torsion of X with the upper middle perversity $\bar{m}$.

Proposition 1. *Let $g(s)$ be a family of admissible Riemannian metrics on X which are of conical type near the singularity. Then*

$$\frac{d}{ds} \ln \ I\tau(X) = \frac{1}{2} \sum_{q=0}^{m+1} (-1)^q \mathrm{Tr}(\alpha H_q)$$

where $\alpha = g^{-1}\frac{\partial}{\partial s} g$ and H_q the orthogonal projection of L^2 q-forms onto the space of harmonic q-forms.

For discussion about the analytic torsion we will restrict ourself to $(m+1)$-dimensional Riemannian manifold with isolated conical singularity satisfying the Witt condition:

$$X = C(N) \cup M,$$

where $\partial M = N$ and $C(N) = (0,1] \times N$ is the finite cone with the conical metric $dr^2 + r^2 g_N$; in addition, $H^{m/2}(N) = 0$ if m is even.

Theorem 2.

$$\frac{d}{ds} \ln T(X) = \frac{1}{2} \sum_{q=0}^{m+1} (-1)^q \mathrm{Tr}(\alpha H^q) + B_{\frac{m+1}{2}},$$

where $B_{\frac{m+1}{2}}$ is the constant term in the asymptotic expansion of

$$\frac{1}{2} \sum_{q=0}^{m+1} (-1)^{q+1} \mathrm{Tr}(e^{-t\Delta_q} \alpha).$$

In fact, if the metric near the conical singularity changes only along the cross section in the variation, then $B_{\frac{m+1}{2}}$ can be computed explicitly in terms of the spectral data of the cross section.

From the variational formulas of the intersection R-torsion and the analytic torsion, the best hope for the Ray-Singer Conjecture for manifolds with conical singularity is

$$\ln T(X) = \ln I\tau(X) + \text{geometric correction term}.$$

Our variation formulas also suggest that the geometric correction term depends only on the conical singularity.

§3. Towards a Cheeger-Müller theorem for manifolds with conical singularity

For $X = C(N) \cup M$ satisfying the Witt condition, consider the smooth manifold with boundary obtained by cutting off the conical tips:

$$X_\epsilon = C_\epsilon(N) \cup M$$

where $C_\epsilon(N) = [\epsilon, 1] \times N$.

For a manifold with boundary M, the generalization of Cheeger-Müller theorem is well understood by the work of Lück [12], Dai-Fang [7], Brüning-Ma [2].

Theorem 3 (Cheeger-D)**.** *There is an explicit constant a depending on the Betti numbers of N such that difference*

$$\lim_{\epsilon \to 0} (\ln T(X_\epsilon) - a \ln \epsilon) - \ln T(X)$$

depends only on the cross section N and the normal geometry of the cross section of the conical singularity.

We are working to identify the difference in terms of geometric invariant of the cone.

In [8] we computed the intersection R-torsion of a finite cone. However, so far, we have not been able to connect it with the computation of analytic torsion of the finite cone by Hartman-Spreafico [11] and Vertman [16].

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Singular Spin Structures and Witten spinors

Anda Degeratu

The first part of the talk was based on joint work with Mark Stern, and has appeared in [1]. I ended with some work in progress with Richard Melrose.

The entire work is motivated by the following idea that Mark Stern and I had: can one adapt Witten's proof of the positive mass theorem to nonspin manifolds?

In what follows, (M^n, g) is an asymptotically flat complete Riemannian manifold of order $\tau > \frac{n-2}{2}$. This means that there exists a compact set, $K \subset M$, whose complement is a disjoint union of subsets $M_1, \dots, M_L$ – called the *ends* of M – such that for each end there exists a diffeomorphism

$$Y_l : \mathbb{R}^n \setminus B_T(0) \to M_l,$$

so that $Y_l^* g =: g_{ij} dx^i dx^j$ satisfies for $\rho = |x|$,

$$g_{ij} = \delta_{ij} + \mathcal{O}(\rho^{-\tau}), \quad \partial_k g_{ij} = \mathcal{O}(\rho^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = \mathcal{O}(\rho^{-\tau-2})$$

with δ the Euclidean metric on $\mathbb{R}^n$, and $T > 1$.

It is well-known that the obstruction to having a spin structure on a oriented differential manifold M is the second Stiefel-Whitney class $w_2(M)$. Based on the interpretation of $w_2(M)$ as the obstruction to extending $(n-1)$ linearly independent vector fields from the 1-skeleton to the 2-skeleton, we have:

Theorem 1. *Let (M, g) be an asymptotically flat Riemannian manifold which is nonspin. Then there exists a closed stratified subset V, locally quasi-isometric to an iterated cone and lying in the compact part K of M, so that the spin structure on $M \setminus V$ is maximal, in the sense that it does not extend over any of the codimension 2 strata of V. The strata of V,*

$$V = V^{k_2} \cup V^{k_3} \cup \dots V^{k_{d-1}} \cup V^{k_d},$$

have codimensions $k_b = b(b-1)$ in M. Moreover, the maximal spin structure on $M \setminus V$ is trivial on each asymptotically flat end of M.

This spin structure restricts to the trivial spin structure on each of the asymptotically flat ends of M. We denote with S the corresponding spinor bundle on $M \setminus V$.

Next we define a special type of spinors, which have exactly those properties of the spinor Witten used in his proof of the positive mass theorem in dimension 3: Let ψ_0 be a spinor on $M \setminus V$, constant near infinity. We say that a spinor ψ on $M \setminus V$ is a *Witten spinor* asymptotic to ψ_0, if the following conditions are satisfied:

(1) $\frac{\psi - \psi_0}{\rho} \in L^2(M \setminus V, S)$,
(2) ψ is strongly harmonic, i.e. $D\psi = 0$, and
(3) $\nabla(\psi - \psi_0) \in L^2(M_l, S|_{M_l})$ for each asymptotically flat end M_l of M.

We show that such spinors exist on $M \setminus V$.

Theorem 2. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > \frac{n-2}{2}$ and which has nonnegative scalar curvature. Given a smooth spinor ψ_0 on $M \setminus V$ that is constant near infinity and that vanishes in a neighborhood of V, there exists a Witten spinor on $M \setminus V$ asymptotic to ψ_0.*

To implement Witten's program, one needs to use the integral form of the Lichnerowicz formula on the *incomplete* manifold $M \setminus V$. For this, one needs to analyze the behaviour of the Witten spinors near V. We study the growth of such a ψ near each of the strata V^{k_b} of V separately. Unless V^{k_b} is a closed stratum, there are no tubular neighborhoods of uniform radius over the entire stratum. Hence, for uniform estimates involving separation of variables, we formulate them in tubular neighborhoods over relatively compact subsets of V^{k_b} that do not intersect the higher codimension strata. We denote by $\mathrm{TRC}(V^{k_b})$ the set of all these good neighborhoods around points in V^{k_b}. Letting r denote the distance to V^2 the lowest codimension stratum, and r_b denote the distance to the higher codimension strata V^{k_b} of V, we have:

Theorem 3. *Let ψ be a Witten spinor constructed as in Theorem 2. Then*

(1) *for all $W \in TRC(V^2)$*

$$\frac{\psi}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}), \quad \text{for all } a > 0,$$

(2) *for all $W \in TRC(V^{k_b})$ with $k_b > 2$*

$$\frac{\psi}{r_b^{(k_b-2)/2}\ln^{1/2+a}(\frac{1}{r_b})} \in L^2(W \setminus V, S|_{W\setminus V}), \quad \text{for all } a > 0.$$

However, the decay estimates in (1) are borderline for our purposes. For any class of manifolds for which we could set $a = 0$ in (1), Witten's proof of the positive mass theorem extends.

Theorem 4. *Let (M, g) be an asymptotically flat nonspin manifold that satisfies the hypothesis of the Positive Mass Theorem (i.e. the order of decay of the metric is $\tau > \frac{n-2}{2}$ and the scalar curvature is nonnegative and integrable). If the Witten spinor constructed in Theorem 2 has the property*

$$\frac{\psi}{r^{1/2}\ln^{1/2}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}) \tag{1}$$

for all $W \in TRC(V^2)$, then the mass of (M, g) is nonnegative.

Since the spin structure on $M \setminus V$ does not extend over V^2, spinors have non-trivial holonomy around small circles normal to V^2. The L^2-harmonic spinors near V^2 have a Fourier decomposition in these normal circles whose leading order modes in polar coordinates may behave like $r^{-1/2}e^{\pm\frac{i\theta}{2}}$. Such modes prevent direct application of the Lichnerowicz formula. But, if a spinor satisfies the hypotheses of Theorem 4, these modes vanish, giving that its product with any element of $C_0^\infty(M)$ is in the minimal domain of the Dirac operator. On the other hand, the decay obtained in Theorem 3 near V^2 is not sufficient to remove them. Hence, our estimates do not suffice to extend Witten's proof of the positive mass theorem.

Next I presented some recent work in progress with Richard Melrose, in which we study APS-type boundary conditions for spinors near V.

We assume for now that V is a smooth codimension 2 oriented compact submanifold of M, connected for simplicity, and that $M \setminus V$ has a spin structure that does not extend over V. A result of Ammann-Bär (and also of Fang Wang in his MIT thesis) gives that V has an induced spin structure. Let $S(V)$ be its corresponding spinor bundle. In this context, we show:

Theorem 5. (1) *We have a well-defined boundary map* $B : Dom_{\max}(D) \to L^2(V, S(V) \oplus S(V))$ *such that the sequence*

$$0 \to Dom_{\min}(D) \to Dom_{\max}(D) \xrightarrow{B} L^2(V, S(V) \oplus S(V))$$

is exact. Moreover B *has closed range which includes the smooth sections.*

(2) *The Dirac operator* D *has a self-adjoint extension, given by an APS boundary condition.*

(3) *With the above APS boundary condition,* D *is Fredholm.*

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Poincaré Duality Groups

James Fowler

The Borel conjecture asserts that aspherical manifolds are topologically rigid; parallel to this uiniqueness conjecture, there is an existence question: which groups π are fundamental groups of closed aspherical manifolds? A necessary condition is that $K(\pi, 1)$ satisfy Poincaré duality (in which case we call π a "Poincaré duality group" following [JW72]). Wall asked whether this suffices in [Wal79], problem G2, page 391:

> Is every Poincaré duality group Γ the fundamental group of a closed $K(\Gamma, 1)$ manifold? Smooth manifold? Manifold unique up to homeomorphism?

Further history of this problem is discussed in [FRR95a, FRR95b, Dav00]; more recently, in light of the Bryant–Ferry–Mio–Weinberger surgery for ANR $\mathbb{Z}$-homology manifolds [BFMW96, BFMW93], it is natural to ask for an ANR $\mathbb{Z}$-homology manifold rather than a topological manifold. Moreover, the existence question for closed aspherical $\mathbb{Z}$-homology manifolds can be formulated for R-homology manifolds; Mike Davis asked whether every torsion-free finitely presented group satisfying R-Poincaré duality is the fundamental group of an aspherical closed R-homology n-manifold [Dav00]. In this talk, I produce a counterexample via the following recipe:

- Let $X = S^2 \cup_f D^3$ where $f : \partial D^3 \to S^2$ is a degree two map; note that X is $\mathbb{Q}$-acyclic but not $\mathbb{Z}$-acyclic.
- Triangulate X and apply Bestvina–Brady Morse theory [BB97] to get a group G which is not FP($\mathbb{Z}$) but which is FH($\mathbb{Q}$), so there is a finite complex K with $\pi_1 K = G$ and $\tilde{K}$ rationally acyclic.
- Apply a variant of M. Davis' reflection group trick [Dav83] to a regular neighborhood of K embedded in Euclidean space. This produces a torsion-free group Γ which is not only FH($\mathbb{Q}$) but which also satisfies rational Poincaré duality.

Since $K(\Gamma,1)$ retracts onto $K(G,1)$, if there were a finite $K(\Gamma,1)$, then $K(G,1)$ would be finitely dominated, but G is not $\mathrm{FP}(\mathbb{Z})$.

The counterexample arises from a finiteness issue. To circumvent the finiteness issue, we can modify Mike Davis' question by asking the following: given a finitely presented group satisfying $\mathbb{Q}$-Poincaré duality, is there a closed $\mathbb{Q}$-homology manifold M with $\pi_1 M = \Gamma$ and $\mathbb{Q}$-acyclic universal cover $\tilde{M}$? Note that we have now dropped the torsion-free condition, so finite Γ are obvious 0-dimensional counterexamples, but there are others.

Theorem. Let Γ be a uniform lattice in a semisimple Lie group containing p-torsion (for $p \neq 2$). Then there does not exist an ANR $\mathbb{Q}$-homology manifold X having $\pi_1 X = \Gamma$ and having $\mathbb{Q}$-acyclic universal cover $\tilde{X}$.

The proof goes by a controlled symmetric signature calculation, and finishes with a ρ invariant calculation. There is a nice relationship with stratified spaces: the locally symmetric space $K\backslash G/\Gamma$ is an orbifold, having Γ as its orbifold fundamental group. The existence of the singular object $K\backslash G/\Gamma$, along with uniqueness (i.e., the Novikov conjecture for Γ), obstructs the existence of a non-singular $\mathbb{Q}$-homology manifold with the desired properties.

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Intersection Spaces and Fiberwise Homology Truncation

Florian Gaisendrees

This talk is based on [2]. In [1] a spatial version of intersection homology is defined. A key step is fiberwise homology truncation of the link bundle of a pseudomanifold. This is done in order to construct an intersection space, the (ordinary) homology of which is then considered. A comparison with the intersection homology of the pseudomanifold reveals some differences and some similarities.

Fiberwise homology truncation of the link bundle is implemented in [1] for trivial link bundles. The difficulty of extending said results to more general link bundles is informed by two factors: firstly, the type of fiber (which is also the link of the pseudomanifold), and secondly, the base space of the bundle (which is the singular set of the pseudomanifold). We extend the methods introduced in [1] to link bundles with fibers interleaf CW-complexes (see [1, Definition 1.62]) and base space a sphere.

In this setting, truncation of the fiberwise gluing homeomorphisms yields only homotopy equivalences. Hence homotopy theory is necessary to build a truncated bundle with the right properties. We require the link bundle to be glued from trivial bundles by means of cellular homeomorphisms. Generalized Poincaré duality is shown for pseudomanifolds with such a link bundle.

Dold fibrations may be useful in other contexts as well. They enjoy two key properties: firstly, they are preserved under fiberwise homotopy equivalence. This greatly lowers the difficulty of showing that a bundle is a Dold fibration. Secondly, under some mild restrictions, the mapping cylinder of a fiberwise homotopy equivalence between Dold fibrations is again a Dold fibration. The latter is from [3] and an overview of Dold fibrations and other weak fibrations is given in [4, Chapter IV, Section 1].

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Pseudodifferential calculus for multiply fibred cusps

Daniel Grieser

(joint work with Eugenie Hunsicker)

We construct a pseudo differential calculus that allows to construct parametrices for Dirac type and Laplace operators on non-compact Riemannian manifolds whose geometry at infinity is conformal to a multiply fibred cusp. Such manifolds arise naturally as $\mathbb{Q}$-rank one locally symmetric spaces and as the spaces studied by

Tian and Yau [5] in their proof of the non-compact Calabi conjecture, as well as in various physical moduli space contexts as described in [2]. This work is part of a broader project to study manifolds with corners which have fibration structures at boundary faces that glue in an appropriate way. These include all locally symmetric spaces, and conjecturally include moduli spaces such as the k-monopole moduli spaces (ongoing work of Singer and Melrose), good resolutions of projective varieties (ongoing work of Grieser and Melrose), and spaces arising in variations of mixed Hodge structures (ongoing work of M. Kerr).

We define a notion of split ellipticity for operators on such spaces and prove that certain Dirac and Laplace type operators (for example, the signature operator and Hodge Laplacian) satisfy this condition under certain integrability assumptions on the metrics at infinity. We then prove that split elliptic operators are Fredholm between suitable spaces, allowing to define an index, and that the asymptotics of elements in their kernel can be given, which is useful for Hodge theory.

To give more detail, we describe the setup in the case of two fibrations, for ease of notation. By a *manifold with doubly fibred boundary* we mean a compact manifold with boundary, M, together with two fibrations $\partial M \overset{\phi_2}{\to} B_1 \overset{\phi_1}{\to} B_0$ of compact manifolds. M is a compactification of the non-compact spaces mentioned above. We denote the fibres of ϕ_2 by F_2 and those of ϕ_1 by F_1. Also, denote the fibres of $\phi_1 \circ \phi_2 : \partial M \to B_0$ by F. Thus, the boundary is fibred by 'big' fibres F, which in turn are fibred by 'small' fibres F_2. We will also assume that a trivialization of a neighborhood U of the boundary is given, $U \equiv \partial M \times [0, \varepsilon)$. This fixes a boundary defining function near ∂M which we denote by x. The geometric structure on these manifolds that we consider is most easily described by a set of vector fields on M, whose elements are characterized by orders of tangency to the boundary and to the various fibres in the boundary: Given $\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2$ we consider vector fields on M which are tangent to order a_2 to the fibres F_2, to order $a_1 + a_2$ to the fibres F and to order $1 + a_1 + a_2$ to the boundary. More precisely, we consider

$$
{}^{\mathbf{a}}\mathcal{V}(M) := \left\{ V \in \Gamma(TM) \;\middle|\; \begin{array}{l} (\phi_2)_* V = O(x^{a_2}), \\ (\phi_1 \circ \phi_2)_* V = O(x^{a_1+a_2}), \\ x_* V = O(x^{1+a_1+a_2}) \end{array} \right\}.
$$

With respect to local coordinates adapted to the fibrations, i.e. the coordinate x, coordinates $y = (y_j)_j$ on B_0, supplementing coordinates z on B_1 (i.e. 'coordinates on F_1') and supplementing coordinates w on ∂M (i.e. 'coordinates on F_2') this set of vector fields is spanned over $C^\infty(M)$ by

$$
x^{1+a_1+a_2}\partial_x,\; x^{a_1+a_2}\partial_y,\; x^{a_2}\partial_z,\; \partial_w.
$$

and forms the set of sections of a rescaled tangent bundle ${}^{\mathbf{a}}TM$. Associated with ${}^{\mathbf{a}}\mathcal{V}(M)$ are $\mathbf{a}$-metrics, which are Riemannian metrics on the interior of M which, when considered as sections of $S^2({}^{\mathbf{a}}T^*M)$, are positive definite and smooth up to the boundary. Also, compositions of elements of ${}^{\mathbf{a}}\mathcal{V}(M)$ and smooth functions yield $\mathbf{a}$-differential operators, whose principal symbols are naturally homogeneous functions on ${}^{\mathbf{a}}T^*M$. Ellipticity of an operator is defined as invertibility of its principal symbol, and Dirac type and Laplace operators for **a**-metrics are elliptic. All of these constructions are straightforward generalizations of the so-called ϕ-calculus [3], which corresponds to one fibration and $a = 1$, and analogous to constructions carried out in other geometric contexts, see [1] for references.

Unlike in the case of closed manifolds ellipticity does not guarantee Fredholmness between the natural Sobolev spaces, so if one aims at index theory more work is required. In addition to the principal symbol of an operator P, one needs to consider a second symbol, the normal family, which is a family of operators $N_P(\mu, p)$ parametrized by $p \in B_1$ and $\mu \in (T_p B_1) \times \mathbb{R}_+$, acting on the fibre F_{2p} of ϕ_2 over p. For example, for the Hodge Laplacian Δ we have

$$N_\Delta(\mu, p) = |\mu|^2 + \Delta_p$$

where Δ_p is the Hodge Laplacian on F_{2p}. As in the ϕ-calculus, we call an operator *fully elliptic* if it is elliptic and its normal family is invertible. A 'small' pseudodifferential calculus which contains parametrices (up to compact remainders) of fully elliptic **a**-operators was constructed in [1], and it was shown there that these operators are Fredholm between the natural Sobolev spaces.

However, many operators of interest are not fully elliptic, for example the operator $D = d + d^*$ and the Hodge Laplacian Δ in case of non-trivial fibres F_2: The kernel of $N_\Delta(0, p)$ consists of the space of harmonic forms on the fibre F_{2p}. Due to the topological nature of this space and to other special features of these operators, they nevertheless satisfy a condition which we call *split ellipticity.* Essentially, an **a**-operator P of order m is called split elliptic if it is elliptic and the dimension of $\mathcal{H}_p = \ker N_P(0, p)$ is independent of p (thus yielding a vector bundle over B_1); if $N_P(\mu, p)$ leaves $\mathcal{H}_p$ and a complement $\mathcal{H}_p^\perp$ invariant for all μ, p; and if, when writing P as 2×2 matrix with respect to the decomposition $\mathcal{H} \oplus \mathcal{H}^\perp$, the $\mathcal{H} \to \mathcal{H}$ part is $x^{a_2 m}$ times an operator of similar type (but with respect to the single fibration $B_1 \times [0, \varepsilon) \to B_0 \times [0, \varepsilon)$ and acting on sections of $\mathcal{H}$), the $\mathcal{H}^\perp \to \mathcal{H}^\perp$ part is fully elliptic and the off-diagonal terms are small in a suitable sense. This is a recursive definition, and the definition for a single fibration recurs to the case of no fibration, where the corresponding operators are b- (or totally characteristic) operators. Our main theorems are then:

Theorem 1. *Assume a manifold with k-fold fibred boundary and* **a***-metric is given, and that the* **a***-metric induces a Riemannian submersion on the boundary.*

The operator $D = d + d^*$ is split elliptic if $k = 1$, and also for $k > 1$ under certain integrability assumptions on the boundary metric. The Hodge Laplacian is split elliptic (for any $k \geq 1$) under certain integrability assumptions on the boundary metric.

The precise assumptions involve the mean curvature of certain fibres in the tower of the fibration at the boundary, and also the integrability of certain partial horizontal distributions. These conditions are closely related to those introduced in [4] for the Laplacian and a simply fibred cusp.

Theorem 2. *Let P be a split elliptic operator on a manifold with multiply fibred boundary. Then P is Fredholm between suitable split Sobolev spaces. Also, elements in* $\ker P$ *have complete asymptotic expansions at ∂M.*

The split Sobolev spaces are combinations of Sobolev spaces of different types. Rather than give the general definition, we restrict here to the case of one fibration $\phi : \partial M \to B$ and $a = 1$ and integer orders. For $m \in \mathbb{N}_0$ let $H^m_\phi(M)$ be the space of those L^2-functions (or forms) on M so that applying m ϕ-vector fields still yields an L^2-function. Let $\mathcal{H}$ be the bundle over B defined by the operator. Then $H^m_{\mathrm{split},\mathcal{H}}(M)$ is the space of functions which are in $H^m_{\mathrm{loc}}(M)$ over the interior, whose $\mathcal{H}^\perp$ component (defined near the boundary) is in $H^m_\phi(M)$ and whose $\mathcal{H}$-component is in $x^{-m}H^m_b(B\times[0,\varepsilon),\mathcal{H})$. A special and inconvenient feature of these Sobolev spaces is that they do not form a scale (i.e. spaces with larger m are not contained in spaces with smaller m). But this is unavoidable, since considering simple product type situations shows that precisely these spaces occur as natural domains for split elliptic operators.

These results generalize results in [6] and in [2].

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Stratifolds and equivariant Poincaré duality

Matthias Kreck

Let G be a compact Lie group and X a G-space. Then equivariant (co)homology groups are defined as the ordinary (co)homology groups of the Borel construction $X\times_G EG$. If X is a closed oriented manifold (on which G acts orientation preserving), then for G trivial one of the most fundamental theorems is Poincaré duality. For G non-trivial this is in general false, in a way as false as possible. It is natural to ask whether for equivariant homology $H^G_n(X)$ there is a new equivariant cohomology theory, which one might call dual theory $DH^n_G(X)$, which is Poincré dual to $H^G_n(X)$ and similarly for a new theory $DH^G_n(X)$, which is Poincaré dual to $H^n_G(X)$. In a much more general context an answer to this problem was given for arbitratry cohomology theories by Greenlees and May [G-M] in terms of equivariant spectra. In this talk I reported about a more geometric answer, where the cocycles are certain equivariant stratified spaces together with an equivariant map to X.

The background of this is my interpretation of ordinary (co)homology in terms of bordism groups of stratifolds as defined in [K]. I explained this definition which might be useful in other contexts. The next step was the corresponding interpretation of equivariant homology $H_n(X)$ as equivariant stratifold homology groups $SH^G_{n+\dim G}(X)$, where the latter is the bordism group of closed oriented stratifolds with free G action together with an equivariant map to X. Following the principles of [K] one gets a Poincaré dual cohomology theory for manifolds X as bordism groups of non-compact oriented free G-stratifolds together with a proper equivariant map to X.

It is natural to ask for a more classical interpretation of equivariant stratifold cohomology $SH^n_G(X)$ in terms of chain complexes. It is not clear whether this can be done for arbitrary compact Lie groups G but for finite groups this was done by Haggai Tene in his thesis [T].

What are the relations between this new theory $SH^n_G(X)$ and $H^n_G(X)$? Also this was answered by Tene. He constructs a natural transformation from $SH^n_G(X)$ to $H^n_G(X)$ and a third cohomology theory which he calls equivariant Tate cohomology since it is a generalization of ordinary Tata cohomology and an exact sequence relating these three theories. In joint work with Tene we give a simple interpretation of this natural transformation using infinite dimensional Hilbert stratifolds.

Finally I explained how the definition of Tate cohomology groups in terms of stratifolds can be used to give a geometric interpretation of the ordinary cup product in negative degrees. This is also contained in Tene's thesis.

It is unclear what the relation between these new cohomology theories and those constructed by Greenlees and May are. One should expect that they are isomorphic.

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The Signature Package on Witt Space

Eric Leichtnam

(joint work with Pierre Albin, Paolo Piazza, Rafe Mazzeo)

This talk describes a joint paper "The Signature Package on Witt Spaces", by Pierre Albin, Eric Leichtnam, Paolo Piazza and Rafe Mazzeo that will appear at the Annales Scientifiques de l'Ecole Normale Supérieure.

We give a parametrix construction for the signature operator on any compact, oriented, stratified pseudomanifold X which satisfies the Witt condition. This construction is inductive over the 'depth' of the singularity. It is then used to show that the signature operator is essentially self-adjoint and has discrete spectrum of finite multiplicity, so that its index – the analytic signature of X – is well-defined. This provides an alternate approach to some well-known results due to Cheeger. We then show how to couple this construction to a $C^*_r\Gamma$ Mischenko bundle associated to any Galois covering of X with covering group Γ. The appropriate analogues of these same results are then proved, and it follows that we may define an analytic signature index class as an element of the K-theory of $C^*_r\Gamma$. We establish in this setting and for this class the full range of conclusions, which sometimes goes by the name of the signature package. In particular, we prove a purely topological theorem, asserting the stratified homotopy invariance of the higher signatures of X, defined through the homology L-class of X, whenever the rational assembly map $K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*_r\Gamma) \otimes \mathbb{Q}$ is injective.

Pseudodifferential operators on manifolds with fibred corners and Poincaré duality on statified spaces

Jean-Marie Lescure

(joint work with Claire Debord, Frédéric Rochon)

We present a pseudodifferential calculus adapted to stratified spaces. Pseudodifferential calculi are useful to study linear elliptic equations, index theory and many examples of them have been introduced over the years for certain type of singular spaces ([8, 16, 15, 7, 6, 11, 14, 5], and many others). Our work [4] is an attempt to give a unified treatment of this question in the case of (Thom-Mather) stratified spaces. Our approach is based on two observations.

Firstly, stratified spaces can be desingularized into manifolds with corners carrying fibrations on boundary hypersurfaces, called manifolds with fibred corners. This leads to a one-to-one correspondance between stratified spaces and manifolds with fibred corners [1] which can be roughly described as follows. If X is a stratified space of depth k, its unfolding $2X$ is obtained by doubling the stratified space with boundary obtained from X by removing the minimal strata. Repeating this operation k-times, we get a smooth manifold 2^kX containing 2^k copies of the regular part $X^\bullet$ of X separated by k smooth hypersurfaces which are provided with natural fibrations. The completion in 2^kX of any copy of $X^\bullet$ gives rise to the same manifold with fibred corners. Conversely, each manifold with fibred corners comes with a natural equivalence relation whose quotient space is a stratified space.

The second observation is that blow-up technics [9] can be used on manifolds with fibred corners. Indeed, if X is a manifold with fibred corners, natural geometric considerations lead us to focus on the Lie algebra $\mathcal{V}_{\mathsf{S}}(X)$ of vectors fields tangent to the boundary which are moreover tangent to the fibers of the fibrations at the boundary hypersurfaces and also vanish at the second order in a normal direction to the boundary hypersurfaces. This Lie algebra is a straight generalization of that of [8] and already gives an algebra $\mathrm{Diff}^*_{\mathsf{S}}(X)$ of differential operators. Following [9], we define a suitable blow-up double space X^2_π. It is a smooth manifold with corners whose boundary hypersurfaces are labelled by the boundary hypersurfaces of X and sorted in two types: *front faces* and *boundary faces*. The manifold X^2_π is also provided with a *blow-down* map $\beta_\pi : X^2_\pi \to X^2$ and a lifted *diagonal* Δ_π. Then, one shows that elements of $\mathrm{Diff}^*_{\mathsf{S}}(X)$ can be intepreted as distibutions on X^2_π conormal to the lifted diagonal Δ_π. Then, we just define the space $\Psi^*_{\mathsf{S}}(X)$ of S-pseudodifferential operators as the space of distributions on X^2_π which are conormal with respect to Δ_π and which fulfill a suitable vanishing condition at any boundary faces of X^2_π. Conversely, we can interpret these distributions as linear operators mapping $C^\infty(X)$ into itself. Also, The Lie algebra $\mathcal{V}_{\mathsf{S}}(X)$ is a finitely generated projective module over $C^\infty(X)$ so there is a smooth vector bundle ${}^\pi TX \to X$ whose space of sections is precisely $\mathcal{V}_{\mathsf{S}}(X)$. In the groupoid language, ${}^\pi TX \to X$ is a Lie algebroid with anchor map given by a natural map $\iota_\pi : {}^\pi TX \to TX$, which is actually an isomorphism over $\overset{\circ}{X} = X \setminus \partial X$. The bundle ${}^\pi TX$, which is also isomorphic to TX but not in a canonical way, allows us to define appropriate metrics and densities on $\overset{\circ}{X}$. These geometric data yield natural Sobolev spaces $H^*_{\mathsf{S}}(X)$ and the maps

$$P : x^l H^p_{\mathsf{S}}(X) \to x^l H^{p-m}_{\mathsf{S}}(X),$$

are continuous for all $P \in \Psi^m_{\mathsf{S}}(X)$ and all l, p. Various symbol maps can be defined. First of all, there is the principal symbol map:

$$\Psi^m_{\mathsf{S}}(X) \xrightarrow{\sigma_m} S^{[m]}({}^\pi T^* X)$$

which induces the short exact sequence:

$$0 \longrightarrow \Psi^{m-1}_{\mathsf{S}}(X) \longrightarrow \Psi^m_{\mathsf{S}}(X) \longrightarrow S^{[m]}({}^\pi T^* X) \longrightarrow 0.$$

Next, by restricting the distributional kernels of S-operators to the front faces ff_i associated to the boundary hypersurfaces H_i of X, we get *conormal symbol maps*:

$$\sigma_{\partial_i} : \Psi^m_{\mathsf{S}}(X) \to \Psi^m_{\mathrm{ff}_i}(H_i), \tag{1}$$

for all i, where $\Psi^m_{\mathrm{ff}_i}(H_i)$ denotes the space of distibutions on ff_i conormal to $\mathrm{ff}_i \cap \Delta_\pi$ and satisfying suitable vanishing conditions. This gives the exact sequence:

$$0 \longrightarrow x_i \Psi^m_{\mathsf{S}}(X) \longrightarrow \Psi^m_{\mathsf{S}}(X) \longrightarrow \Psi^m_{\mathrm{ff}_i}(H_i) \longrightarrow 0 \tag{2}$$

where x_i denotes the intially chosen defining function of H_i. Next we prove the composition theorem for S-operators, following a method inspired from [6] (that is, by using computations in local charts and a inductive argument on the dimension and the depth of the manifold with fibred corners): if $A \in \Psi^m_{\mathsf{S}}(X)$ and $B \in \Psi^n_{\mathsf{S}}(X)$, then $A \circ B \in \Psi^{m+n}_{\mathsf{S}}(X)$ and $\sigma_{\partial_i}(A \circ B) = \sigma_{\partial_i}(A) \circ \sigma_{\partial_i}(B)$ for all i. At this point, we are able to discuss compactness, ellipticity and Fredholmness for the S-calculus. The conclusions are rather natural and exaclty in the spirit of calculi developped with blow-up technics:

(1) 0-orders operators are compact (on $H^0_{\mathsf{S}}(X) = L^2(X)$) if and only if their principal and conormal symbols all vanish.
(2) An operator $P \in \Psi^*_{\mathsf{S}}(X)$ is said to be *fully elliptic* when its principal and conormal symbols are all invertible. If $P \in \Psi^m_{\mathsf{S}}(X)$ is a fully elliptic operator then $P : H^p_{\mathsf{S}}(X) \to H^{p-m}_{\mathsf{S}}(X)$ is Fredholm for all m and its kernel is contained into $C^\infty(X)$.

While the micro-local analysis is performed on manifolds with corners, considerations in K-theory related with the S-calculus give us information about the original stratified space. For that purpose, we introduce a *semi-classical* S-calculus associated with the blow-up space $X^2_{\pi-\mathrm{sl}}$ obtained by blowing-up $\Delta_\pi \times \{0\}$ into $X^2_\pi \times [0,1]_\epsilon$. Once the boundary faces of $X^2_{\pi-\mathrm{sl}}$ have been removed, we get an amenable Lie groupoid $\mathcal{G}_{\pi-\mathrm{sl}}$ which can be thought as a generalization of the tangent groupoid construction of A. Connes [2]. The slice at $\epsilon = 1$ of this groupoid is a Lie groupoid $\mathcal{G}_\pi$ whose space of compactly supported pseudodifferential operators is contained in $\Psi^*_{\mathsf{S}}(X)$. The slice at $\epsilon = 0$ of $\mathcal{G}_{\pi-\mathrm{sl}}$ is the Lie groupoid ${}^\pi TX$, which is also the Lie algebroid of $\mathcal{G}_\pi$ and eventually $\mathcal{G}_{\pi-\mathrm{sl}}$ is the adiabatic deformation [13] of $\mathcal{G}_\pi$. Finally, the subgroupoid $T^{\mathsf{FC}}X$ of $\mathcal{G}_{\pi-\mathrm{sl}}$ given by $\epsilon < 1$ and $x\epsilon = 0$ (where $x = \prod x_i$) is a continuous family amenable groupoid which plays the role of a tangent space of ${}^{\mathsf{S}}X$ (ie, of the stratified space corresponding to X) in K-theory. Then we show:

(1) The C^*-algebra $C^*(T^{\mathsf{FC}}X)$ is Poincaré dual to $C({}^{\mathsf{S}}X)$ (see also [3]).
(2) Any fully elliptic operator P on X naturally provides a K-homology class $[P]$ on ${}^{\mathsf{S}}X$ while its noncommutative symbol (that is, the collection of its principal and conormal symbols) gives a K-theory class $[\sigma(P)]$ of $C^*(T^{\mathsf{FC}}X)$.
(3) Conversely, all elements of $K^0(C({}^{\mathsf{S}}X))$ and $K_0(C^*(T^{\mathsf{FC}}X))$ can be represented in this way.
(4) The Poincaré duality isomorphism $K^0(C({}^{\mathsf{S}}X)) \to K_0(C^*(T^{\mathsf{FC}}X))$ induced by (1) is precisely the map sending $[P]$ to $[\sigma(P)]$.

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Witten's holonomy theorem for manifolds with corners

Paul Loya
(joint work with Sergiu Moroianu, Jinsung Park)

We study the determinant line bundle for a family of Dirac operators on compact manifolds with corners and we describe the Quillen metric and Bismut-Freed connection in this context, then we compute the holonomy of the connection using the eta invariant. In other words, we extend the so-called "global anomaly formula", or Witten's holonomy theorem, to manifolds with corners. In more detail, Witten studied the limiting behavior of the eta invariant of a Dirac operator on the total space (assumed compact) of a fibration over $\mathbb{S}^1$, when the metric in the base $\mathbb{S}^1$ is blown-up in a certain manner, a process known as taking the adiabatic limit. He argued that the limit equals (minus the logarithm of) the holonomy of the determinant line bundle. This conjecture was subsequently proved independently by Bismut and Freed [1] and Cheeger [2]. In this talk, we allow the fibers to be compact manifolds with corners of arbitrary codimension endowed with cylindrical end metrics and we allow the base to be any interval in $\mathbb{S}^1$. Thus, we shall consider more generally parallel transport in the determinant line bundle. We also consider the adiabatic limit of the eta function instead of the eta invariant, as in [3] for the closed case. Assuming the fiber operators to be invertible, we compute the adiabatic limit of the natural eta function in this setting and relate it to the integral of the Bismut-Freed meromorphic family of one-forms and new correction terms coming from the boundaries of the fibers and the ends of the interval. Setting the eta function variable equal to zero, we derive a parallel transport formula in the manifolds with corners context. The "local anomaly formula" for the corners case is work in progress; the case when the fibers are manifolds with boundary was handled in [4].

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A comparison theorem between two complexes on a singular space

Ursula Ludwig

The aim of this talk is to present the de Rham type comparison result achieved in [9] between an analytic complex and a geometric complex on a spaces X with isolated cone-like singularities. The analytic complex is a subcomplex of the complex of L^2-forms on X, the geometric complex consists of cells which are allowed for the middle lower perversity in the sense of intersection homology as defined by Goresky and MacPherson in [6]. Both complexes are produced using Morse theory on the singular space.

The result generalises the comparison between the so called Morse-Thom-Smale complex and the Witten complex on a smooth compact manifold. Let us recall the smooth result first: Let (M, g) be a smooth compact Riemannian manifold and let $f : M \to \mathbb{R}$ be a smooth Morse function on M.

(a) The geometric complex (Morse-Thom-Smale complex): If the pair (f, g) satisfies the Morse-Smale transversality condition one can define the so called Morse-Thom-Smale complex (C^u_*, ∂_*). The complex (C^u_*, ∂_*) is generated by the critical points of the Morse function. The boundary operator ∂_* is given by counting trajectories of the gradient vector field between critical points of index difference 1. Note that as a corollary of the existence of the Morse-Thom-Smale complex and the well known fact that it computes the singular homology of M one gets the Morse inequalities. The results on the Morse-Thom-Smale complex were known already by Thom and Smale (for a proof see [11]).

(b) The analytic complex (Witten complex): Inspired by ideas from quantum field theory Witten in [12] proposed a new proof of the Morse inequalities. Its departure point is the deformation of the de Rham complex $(\Omega^*(M), d)$ of smooth forms into a complex $(\Omega^*(M), d_t)$, where $d_t = e^{-tf} d e^{tf}$. Hereby $t \in \mathbb{R}^+$ is the deformation parameter, and one is interested in the limit $t \to \infty$. Using the fact

that the two complexes $(\Omega^*(M), d)$ and $(\Omega^*(M), d_t)$ are isomorphic as well as the Hodge theory for the deformed complex one has that

$$\ker \Delta_t \simeq H^*(\Omega^*(M), d_t) \simeq H^*(\Omega^*(M), d) =: H^*_{dR}(M). \tag{1}$$

Here $H_{dR}(M)$ denotes the de Rham cohomology of the manifold M, which, by de Rham's theorem, is isomorphic to the singular cohomology of M. The operator $\Delta_t = (d_t + \delta_t)^2$ is called the Witten Laplacian; δ_t denotes the adjoint of d_t with respect to the L^2-metric on forms. The main result in this first part of Witten's program is the spectral gap theorem for the Witten Laplacian, stating that for t large enough and some appropriate constants $C_1, C_2, C_3 > 0$ one has that $\mathrm{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$. Moreover the number of the small eigenvalues of the Witten Laplacian (counted with multiplicities) equals the number of critical points of the Morse function. In view of (1) the finite dimensional subcomplex $(\mathcal{S}_t, d_t)$ of $(\Omega^*(M), d_t)$, generated by the eigenforms of the Witten Laplacian to small eigenvalues, still computes the de Rham cohomology of M. The Morse inequalities follow as a corollary from the spectral gap theorem.

(c) Comparison between the two complexes: The second step of Witten's program aims at comparing the complex $(\mathcal{S}_t, d_t)$ constructed in (b) with the Morse-Thom-Smale complex: Witten suggested that in some appropriate sense the complex $(\mathcal{S}_t, d_t)$ should "converge" to the complex $\mathrm{Hom}((C^u_*, \partial_*), \mathbb{R})$ as $t \to \infty$.

The rigorous proofs for the Witten deformation and the comparison result (c) for smooth manifolds and smooth Morse functions were given by Helffer and Sjöstrand in [8] using semi-classical analysis. In [2] Bismut and Zhang gave a second proof using a result by Laudenbach (see Appendix of [1]) on the structure of the boundary of the unstable cells of critical points of the Morse function. In [1] and [2] the Witten deformation was used to prove comparison results between analytic and Reidemeister torsion.

The Witten deformation on singular spaces: Let (X, g) be a Riemannian manifold with cone-like singularities. The singular set of X will be denoted by Σ. An important invariant for singular spaces is the intersection homology as defined by Goresky and MacPherson in [6]. It has an analytic description in terms of the L^2-cohomology of X as defined by Cheeger (see [4]): The de Rham complex $(\Omega^*_0(X \setminus \Sigma), d)$ of smooth forms, compactly supported outside the singular set Σ admits a maximal extension into a Hilbert complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$ in the space of L^2-integrable forms. Here $\langle\ ,\ \rangle$ denotes the the L^2-metric, we use the language of Hilbert complexes as introduced in [3]. The L^2-cohomology of X is defined as the cohomology of the Hilbert complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$,

$$H^*_{(2)}(X) := H^*((\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)).$$

Integration yields an isomorphism (see [5])

$$H^*_{(2)}(X) \simeq \mathrm{Hom}(IH^{\underline{m}}_*(X), \mathbb{R}),$$

where by $IH^{\underline{m}}_*(X)$ we denote the intersection homology of X with lower middle perversity $\underline{m}$. In [9] the Witten deformation has been generalised to the singular situation by deforming the complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$ using a *radial Morse function* f :

$X \to \mathbb{R}$. The Witten Laplacian $\Delta_t := (d_{t,\max} + \delta_{t,\min})^2$ is the Laplacian associated to the deformed Hilbert complex $(\mathcal{C}_t, d_{t,\max}, \langle\ ,\ \rangle)$. The L^2-Hodge theorem for this situation gives an isomorphism

$$\ker \Delta_t \simeq H^*((\mathcal{C}, d_{t,\max}, \langle\ ,\ \rangle)) \simeq H^*_{(2)}(X).$$

In [9] we show that the spectral gap theorem holds also in this singular context: For a radial Morse function f the restriction $f_{|X\setminus\Sigma}$ is a Morse function in the smooth sense and we denote by $c_i(f_{|X\setminus\Sigma})$ the number of critical points of $f_{|X\setminus\Sigma}$ of index i. For a singular point $p \in \Sigma$, we denote by L_p the link of X at p and by cL_p the cone over L_p.

Theorem 1. (1) *Let X be a Riemannian manifold with cone-like singularities and let $f : X \to \mathbb{R}$ be a radial Morse function on X. Then there exist constants $C_1, C_2, C_3 > 0$ and $t_0 > 0$ depending on X and f such that for any $t > t_0$,*

$$\operatorname{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset.$$

(2) *Let us denote by $(\mathcal{S}_t, d_t, \langle\ ,\ \rangle)$ the subcomplex of $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ generated by all eigenforms of the Witten Laplacian Δ_t to eigenvalues in $[0,1]$. Then, for $t \geq t_0$,*

$$\dim \mathcal{S}_t^i = c_i(f_{|X\setminus\Sigma}) + \sum_{p\in\Sigma} m_p^i =: c_i(f),$$

where the contribution of the singular point p to $c_i(f)$ is given by $m_p^i := \dim\left(\operatorname{Hom}(IH_i^{\underline{m}}(cL_p, L_p), \mathbb{R})\right)$.

The Witten deformation on singular spaces has been studied by the author already in previous articles (see [10]). The Morse functions considered there were called admissible Morse functions and were inspired from the Stratified Morse theory of Goresky and MacPherson in [6]. In the context treated in [9] we are however able to prove stronger results.

The geometric complex for the singular space: Let us shortly outline the strategy in [9] of the construction of the geometric complex (C_*^u, ∂_*) for the singular space X: First we decompose X into the unstable cells of critical points of the radial Morse function f. As one sees immediately, this decomposition does not have enough cells. Therefore we refine it and get a chain complex (T_*, ∂_*). Now, the subcomplex $(T_*^{all}, \partial_*) \subset (T_*, \partial_*)$ of allowed chains (in the sense of intersection homology) is too big. One gets the complex (C_*^u, ∂_*) as a subcomplex of (T_*^{all}, ∂_*). To pick this subcomplex in the right way, smooth Morse theory on the link is used.

The comparison result for the singular space: Finally in [9] we establish a comparison theorem between the Witten complex $(\mathcal{C}_t, d_{t,\max}, \langle\ ,\ \rangle)$ and the geometric complex (C_*^u, ∂_*), which generalises Theorem 6.11 and Theorem 6.12 in [2] to the singular setting.

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L^2approximation and homological growth

Wolfgang Lück

Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $[G : G_i]$ is finite for all $i \in I$ and $\bigcap_{i\in I} G_i = \{1\}$. Let K be a field. We denote by mg the minimal number of generators, by $\rho^{\mathbb{Z}}$ the integral torsion, by $b_n^{(2)}$ the p-th L^2-Betti number, and by $\rho^{(2)}$ the L^2-torsion. The symbol $\mathcal{N}(G)$ stands for the von Neumann algebra of the group G.

The starting point of this talk is the following result (see [7]).

Theorem *Lück (1994)* Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a G-covering. Then

$$b^{(2)}(\overline{X}; \mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\overline{X}; \mathbb{Q})}{[G : G_i]};$$

The analogous result for signatures and η-invariants has been proved by Lück-Schick [9].

Meanwhile the question has occurred whether a result like this is true also for characteristic p. Then the theory of von Neumann algebras is not available anymore. A partial result is given by Linnell-Lück-Sauer [6]

Theorem *Linnell-Lück-Sauer (2010)* Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a G-covering. Suppose that G is torsionfree and elementary amenable. Then one assign to a $\mathbb{F}_p$-module M its Ore dimension $\dim_{\mathrm{Ore}}$ and one has

$$\dim_{\mathrm{Ore}}\bigl(H_n(\overline{X};\mathbb{F}_p)\bigr) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\overline{X};\mathbb{F}_p)}{[G:G_i]};$$

Bergeron-Lück-Sauer are currently working on a non-amenable situation given by congruence subgroups.

The case $n=1$ is of special interest for group theory. For instance the following question is open

Question For which groups G, does the limit $\lim_{i\in I}\frac{b_1(G_i;K)}{[G:G_i]}$ exist for all systems $(G_i)_{i\in I}$ with $\bigcap_{i\in I} G_i = \{1\}$ and fields K and is independent of the choice of $(G_i)_{i\in I}$ and K?

Abért-Nikolov [1, Theorem 3] have shown for a finitely presented residually finite group G which contains a normal infinite amenable subgroup that the answer to the questions above is yes.

The questions above is related to questions of Gaboriau (see [3, 4, 5]), whether every essentially free measure preserving Borel action of a group has the same cost, and whether the difference of the cost and the first L^2-Betti number of a measurable equivalence relation is always equal to 1.

The answer to the questions above is negative in general if we drop the condition that the system $\{G_i \mid i\in I\}$ has non-trivial intersection, as an example by Lück shows.

The following two conjectures are motivated by [2, Conjecture 1.3] and [8, Conjecture 11.3 on page 418 and Question 13.52 on page 478].

Conjecture (*Approximation Conjecture for L^2-torsion*)
Let X be a finite connected CW-complex and let $\overline{X}\to X$ be a G-covering.

(1) If the G-CW-structure on $\overline{X}$ and for each $i\in I$ the CW-structure on $G_i\backslash\overline{X}$ come from a given CW-structure on X, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty}\frac{\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))}{[G:G_i]};$$

(2) If X is a closed Riemannian manifold and we equip $G_i\backslash\overline{X}$ and $\overline{X}$ with the induced Riemannian metrics, one can replace the torsion in the equality appearing in (1) by the analytic versions;

(3) If $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n\geq 0$, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty}\frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}.$$

Conjecture (*Homological growth and L^2-torsion for aspherical closed manifolds*) Let M be an aspherical closed manifold of dimension d and fundamental group $G=\pi_1(M)$. Then

(1) For any natural number n with $2n \neq d$ we have

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = 0.$$

If $d = 2n$ is even, we get

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = (-1)^n \cdot \chi(M) \geq 0;$$

(2) For any natural number n with $2n+1 \neq d$ we have

$$\lim_{i\in I} \frac{\ln\left(\left|\mathrm{tors}\big(H_n(G_i\backslash M)\big)\right|\right)}{[G:G_i]} = 0.$$

If $d = 2n+1$, we have

$$\lim_{i\in I} \frac{\ln\left(\left|\mathrm{tors}\big(H_p(G_i\backslash M)\big)\right|\right)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}\big(M;\mathcal{N}(G)\big) \geq 0.$$

Some evidence for the two conjectures above comes from the following result:

Theorem *(Lück)* Let M be an aspherical closed manifold with fundamental group $G = \pi_1(X)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$

$$\begin{aligned}
\lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\mathrm{mg}\big(H_n(G_i\backslash M)\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\ln\left(\left|\mathrm{tors}\big(H_n(G_i\backslash M)\big)\right|\right)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{(2)}\big(G_i\backslash\overline{X};\mathcal{N}(\{1\})\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{\mathbb{Z}}\big(G_i\backslash\overline{X}\big)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0;\\
\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0.
\end{aligned}$$

In particular the two conjectures above are true.

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The symmetric signature of Witt spaces

James McClure
(joint work with Greg Friedman)

For a compact oriented m-manifold M (and more generally for a Poincaré duality space) the symmetric signature $\sigma^*(M)$ is an element of the symmetric L-group $L^m(\pi_1(M))$. The symmetric signature was introduced by Miščenko in [6] as a tool for studying the Novikov conjecture, and since then it has become an important part of surgery theory.

The basic ingredient in the construction of $\sigma^*(M)$ is Poincaré duality on the universal cover. Another situation where Poincaré duality occurs is the middle perversity intersection homology of a certain class of pseudomanifolds, the *Witt spaces* ([7]), so it is natural to ask whether there is a symmetric signature for Witt spaces.

There are several constructions of the symmetric signature for Witt spaces in the literature. Cappell, Shaneson, and Weinberger [3] give a brief description of a construction which uses controlled topology; further information is given in [8, pages 209–210], but the complete account has not been published. Banagl [2, Section 4] uses the Ph. D. thesis of Thorsten Eppelmann [4] to construct an L-homology fundamental class for a Witt space and then defines the symmetric signature to be the image of this class under the assembly map. However, there are gaps in Eppelmann's work (Banagl, Laures and McClure are currently working on a corrected version of [4], using the work of Friedman and McClure reported on in this talk). An analytic construction of the symmetric signature (as an element of the L-theory of the $C*$ algebra of the fundamental group, for smoothly stratified Witt spaces) has been given by Albin, Leichtnam, Mazzeo, and Piazza [1].

We give a new approach which has several useful features. It is similar in spirit to that of Miščenko (and thus answers a question in [1]). It works for spaces more general than Witt spaces; the only requirement is that the map from lower middle to upper middle perversity intersection homology is an isomorphism. The actual construction uses only the diagonal map of the pseudomanifold and the cross product on intersection chains, and the supporting results use only the Künneth theorem of [5] and standard facts about intersection chains. We give a simple proof of stratified homotopy invariance; this is proved by a rather intricate analytic argument in [1] and it is not known how to prove it using the approach of [2]. We also give a simple proof of the product formula; to prove this using the approach of [2] one would need to show that Eppelmann's map $MIP \to L^\bullet$ is a map of ring spectra up to homotopy.

An argument due to Weinberger (see [1, Proof of Proposition 11.1]) shows that any two definitions of the symmetric signature for Witt spaces must agree rationally if (1) they are bordism invariant and (2) they agree with Miščenko's definition for smooth manifolds. Thus all of the known constructions of the symmetric signature agree rationally; it would be interesting to know whether they agree over the integers.

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Stratified spaces, Lie manifolds, and the well-posedness of Laplace's equation

Victor Nistor

(joint work with Bernd Ammann, Constantin Bacuta, Anna Mazzucato, Ludmil Zikatanov)

We discuss the role of stratified spaces in the statement and proof of a well-posedness (or continuous bijection) of the Laplacian on suitable weighted Sobolev spaces.

We consider the Poisson problem with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

defined on a bounded domain $\Omega \subset \mathbb{R}^d$, where Δ is the *analyst's* Laplacian $\Delta = \sum_{i=1}^d \partial_i^2$. When $\partial\Omega$ is smooth, it is well known that this Poisson problem has a unique solution $u \in H^{m+1}(\Omega)$ for any $f \in H^{m-1}(\Omega)$ and $g \in H^{m+1/2}(\partial\Omega)$ [6]. Moreover, u depends continuously on f and g. This result is the *classical well-posedness of the Poisson problem* on smooth domains.

On the other hand, when Ω is not smooth, it is also known [5, 8, 9, 10] that there exists a constant $s = s_\Omega$, such that $u \in H^s(\Omega)$ for any $s < s_\Omega$, but $u \notin H^{s_\Omega}(\Omega)$ in general, even if f and g are smooth functions. For instance, if Ω is a polygon (in two dimensions), then $s_\Omega = \pi/\alpha_{MAX}$, where α_{MAX} is the largest angle.

For polyhedra, we have $s_\Omega < \infty$, and this phenomenon is called *loss of regularity* and is responsible for the loss of accuracy in certain approximation methods for the solutions of Equation (1). It is therefore desirable to establish an alternative well-posedness result on polyhedra. In my presentation, I will argue that one way to obtain a convenient well-posedness result for the Poisson problem on a polyhedron Ω is to use the stratified space geometry of Ω. This leads, by successive conformal changes of the metric, to a metric for which the smooth part of $\overline{\Omega}$ is a smooth manifold with boundary whose double is complete. The resulting Sobolev spaces (defined by the new metric) will lead to spaces on which the Poisson problem is well-posed [4]. Let us now describe these results in more detail.

A *stratified curvilinear polyhedral domain* Ω is an open subset of a Riemannian manifold (M, g) of dimension d together with a stratification of $\overline{\Omega} = \Omega^{(d)} \supset \Omega^{(d-1)} \supset \ldots \supset \Omega^{(1)} \supset \Omega^{(0)}$. We then define stratified curvilinear polyhedral domains by induction as follows. For $d = 0$, Ω is just a finite set of points. For $d = 1$, Ω is a finite set of intervals. For simplicity, we shall consider only domains Ω that coincide with the interior of their closure $\overline{\Omega}$. The stratum S_0 for $d = 1$ will contain all the boundary points of the intervals, but may contain also other points. For $d > 1$, we require our domain Ω to satisfy the following conditions: for every point $p \in \partial\Omega$, there exist a neighborhood V_p in M such that if $p \in \Omega^{(l)} \setminus \Omega^{(l-1)}$, $l = 1, \ldots, n-1$, there is a stratified curvilinear polyhedral domain $\omega_p \subset S^{n-l-1}$, $\overline{\omega_p} \neq S^{n-l-1}$, and a diffeomorphism $\phi_p : V_p \to B^{n-l} \times B^l$ such that $\phi_p(p) = 0$ and

$$\phi_p(\Omega \cap V_p) = \{rx',\, 0 < r < 1,\, x' \in \omega_p\} \times B^l, \tag{2}$$

inducing a homeomorphism $\overline{\Omega} \cap V_p \to \{rx',\, 0 \le r < 1,\, x' \in \overline{\omega_p}\} \times B^l$ of stratified spaces that is a diffeomorphism on each stratum.

We then introduce the *desingularization* of Ω, denoted $\Sigma(\Omega)$, by gluing in a natural way all the sets $[0, 1) \times \overline{\omega_p} \times B^l$ as in Equation (2). The resulting set is a Lie manifold with boundary, in the sense of [1]. See also [2, 3, 4].

Let $r_0(x) \ge 0$ be the distance from x to the set $\Omega^{(0)}$ if x is close to that set. We assume that r_0 is smooth outside $\Omega^{(0)}$ and satisfying $r_0 \le 1$, $r_0(x) > 0$ for $x \notin \Omega^{(0)}$. We replace then the metric $g =: g_0$ with $g_1 := r_0^{-2} g$. We repeat this construction for the other strata, in the increasing order of dimension. Thus r_k is the smoothed distance to $\Omega^{(k)}$ in the metric g_k, and we let $g_{k+1} := r_k^{-2} g_k$, $k \le d-2$. Then g_{d-1} is a compatible metric on the desingularization $\Sigma(\Omega)$ [2, 4] and we can use the results on Sobolev spaces from those papers. Let $\rho := r_0 r_1 \ldots r_{d-2}$. The Lie algebra of vector fields on $\Sigma(\Omega)$ is $\mathcal{V} = \mathcal{C}^\infty(\Sigma(\Omega))\rho\Gamma(\overline{\Omega}, TM)$. Thus a basis of $\mathcal{V}$ over $\mathcal{C}^\infty(\Sigma(\Omega))$ is given by $\{\rho\partial_i\}$. The resulting Sobolev spaces are

$$\mathcal{K}^m_a(\Omega) := \{u, \rho^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq m\} = \rho^{a-n/2}H^m(\Omega, g_{d-1}),$$

where the space $H^m(\Omega, h)$ is the Sobolev space associated to the metric h. (Let $r_\Omega(x)$ denote the distance from x to $\Omega^{(d-2)}$. We have that r_Ω/ρ and ρ/r_Ω are both bounded, so in the above definition of Sobolev spaces we can replace ρ with r_Ω.)

The stratification plays a role also in the formulation of the problem, especially if one uses *mixed* boundary conditions. Indeed, the stratification of $\overline{\Omega}$ does not have to be determined by the geometry. One then chooses a set of *open* faces on which to require Neumann boundary conditions (that is, the faces on which to specify the normal derivative at the boundary). Let $\partial_N\Omega$ be the union of these faces and let $\partial_D\Omega = \partial\Omega \smallsetminus \partial_N\Omega$. The set $\partial_D\Omega$ is called the *Dirichlet part*of the boundary. The set $\partial_N\Omega$ is called the *Neumann part* of the boundary and are endowed with the named boundary conditions.

The fact that the Sobolev spaces $\mathcal{K}^m_a$ are associated to a Lie manifold guarantees that Laplacian Δ satisfies elliptic regularity in the scale of spaces $\mathcal{K}^m_a(\Omega)$. To this end, one also needs to establish that $\rho^2\Delta - \Delta_{g_{d-1}}$ is a lower order differential operator generated by $\mathcal{V}$ and $\mathcal{C}^\infty(\Sigma(\Omega))$. Assume that there are no adjacent faces that are endowed with Neumann boundary conditions, then one has a Hardy-Poincaré inequality since there exists $C > 0$ such that every point of x is at a distance $\leq C$ to the *Dirichlet part* of the boundary of $\Sigma(\Omega)$. The classical proof of well-posedness for the Poisson problem (based on the Lax-Milgram lemma) and a perturbation argument in a then give the following result [4]

Theorem. *Assume that there are no adjacent faces with Neumann boundary conditions. Then there exists $\eta_\Omega > 0$ such that the Laplacian*

$$\Delta : \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u = 0 \text{ on } \partial_D\Omega\} \cap \{\partial_\nu u = 0 \text{ on } \partial_N\Omega_D\} \to \mathcal{K}^{m-1}_{a-1}(\Omega)$$

is a continuous bijection (with bounded inverse) for any $|a| < \eta_\Omega$ and any $m \in \mathbb{Z}_+$.

In two dimensions (and more generally for domains with conical points), the above well-posedness result was established in [9]. The treatment of the case when adjacent faces are endowed with Neumann boundary conditions requires more work. In 2D it is based on a relative index theorem [7], but in higer dimensions it is still open.

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Quasi-isometry topological invariants for finitely presented groups

Daniele Ettore Otera
(joint work with Valentin Poénaru)

In the talk given at the Workshop, I reported some of my recent results on the geometry and the topology at infinity of finitely generated groups. I described several tameness conditions at infinity for groups, and I presented some new classes of QSF groups. Some of these results were obtained also with L. Funar (Grenoble).

0.1. **(n)-Connectivity at infinity and its growth.** Among the main asymptotic topological properties one finds the *(n)-connectivity at infinity* (i.e. spheres 'very close' to infinity bound balls which are 'near' infinity), and for groups, the number of ends is already an essential characteristic (e.g. J. Stallings proved that if a finitely generated group has infinitely many ends, then it is a non-trivial amalgam (or an HNN-extension) over finite groups). For one-ended groups, or for groups that are simply connected at infinity (sci), there is a very natural question, that we firstly studied in [5]. The idea being to measure the 'minimal way' two points (resp. a loop) near infinity can be connected (resp. filled). More precisely:

Definition 1: Let X be a one-ended metric space and e be a base-point. Consider the function $f : r \to V_0(r) = \inf(R)$ with the property that any two points which sit outside the ball of radius R centered at e, can be joined by a path outside $B(e, r)$. The *end-depth* of X is then the (growth of the) function V_0.

Definition 2: Let X be a simply connected non-compact metric space which is simply connected at infinity. The rate of vanishing of π_1^∞ (or the *sci growth*), denoted $V_1(r)$, is the infimal $N(r)$ with the property that any loop which sits outside the ball $B(N(r))$ of radius $N(r)$ bounds a 2-disk outside the ball $B(r)$.

After considering the equivalence relation: f and g are *rough equivalent* if there exist constants c_i, C_j (with $c_1, c_2 > 0$) such that $c_1 f(c_2 R) + c_3 \leq g(R) \leq C_1 f(C_2 R) + C_3$, we looked at the growth-type of V_0 and V_1, and we proved:

Theorem 1: [Funar-Otera, [2, 4]]

- The sci growth and the end-depth are well-defined quasi-isometry invariants for finitely presented groups.
- The function V_1 is linear for many co-compact lattices in Lie groups (semi-simple, nilpotent and several classes of solvable Lie groups); and also for some non-uniform lattices in higher rank Lie groups (those of $\mathbb{Q}$-rank 1).
- All geometric 3-manifold groups have a trivial V_1.
- Simply connected at infinity Coxeter and Artin groups have linear V_1.
- The end-depth is linear for ALL finitely presented groups.
- The amalgamated free product of two sci groups over a one-ended group is sci with a linear sci growth.

In particular all one-ended groups have the same type of connectedness at infinity, while we expect the same not to be true for the sci growth.

0.2. **Other tameness conditions: qsf and '*easy*' groups.** The simple connectivity at infinity have been used for characterizing Euclidean spaces as being the contractible manifolds that are simply connected at infinity (Siebenmann, Stallings), and it has been conjectured for a long time that contractible universal coverings of compact 3-manifolds were homeomorphic to $\mathbb{R}^3$ (i.e. simply connected at infinity). V. Poénaru, in [8, 9], gave a partial solution to the 3-dimensional covering conjecture, 'approximating' such universal cover by compact and simply-connected 3-manifolds. Then S. Brick in [1] adapted this idea for arbitrary finitely presented groups defining the *quasi-simple filtration* (abbreviated QSF) as follows:

Definition 3: A complex X is QSF if for any compact sub-complex $C \subset X$ there exists a simply connected compact complex K and a PL-map $f : K \to X$ so that $C \subset f(K)$ and $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a PL-homeomorphism.

This notion should be compared with the following two ones. The first is the *geometric simple connectivity*, a topological condition well-known in differential topology: a smooth manifold is said to be GSC if it admits Morse functions f without critical points of index $\lambda = 1$. There is also a more combinatorial version, in terms of handlebody decompositions, the condition being then that each 1-handle should be in cancelling position with some 2-handle. This definition makes also sense for cell-complexes and hence for groups (for more see [3, 5, 7, 8]).

The second one is an extension of this concept in the realm of polyhedra:

Definition 4: A polyhedron P is *weakly geometrically simply connected* (WGSC) if any compact subspace of it is contained in a simply connected sub-polyhedron.

Theorem 2: [Funar-Otera, [3]] A finitely presented group Γ is QSF iff there is a smooth compact manifold M such that $\pi_1 M = \Gamma$ and $\widetilde{M}$ is GSC or WGSC.

The GSC condition was heavily used by Poénaru since long ago. One of his methods was to "represent" in a special way several low-dimensional objects (homotopy 3-spheres, universal covering spaces of smooth closed 3-manifolds [8, 9, 10]), and then groups [6, 7], where such an *(inverse)-representation* is defined as follows:

Definition 5: A GSC-*representation* for a finitely presented group Γ is a non-degenerate simplicial map $f : X^2 \longrightarrow \widetilde{M}^3(\Gamma)$, with the following features:

(1) X^2 is a GSC (i.e. geometrically simply connected) 2-complex.

(2) The $\widetilde{M}^3(\Gamma)$ is the universal cover of a compact (necessarily singular) 3-manifold $M^3(\Gamma)$ associated to (a presentation of) the group Γ (see [6]).
(3) $\Psi(f) = \Phi(f)$ (see [8] for the definition of these equivalence relations); this condition means that f is "realizable via a sequence of folding maps".
(4) The map f is "essentially surjective", in the sense that one can get $\widetilde{M}^3(\Gamma)$ from $\overline{fX^2} \subset \widetilde{M}^3(\Gamma)$ by adding 2- and 3-cells (possibly infinitely many).

Poénaru's first representation-result (the "collapsible pseudo-spine representation theorem") says that given a homotopy 3-sphere Σ^3, there exist a collapsible finite 2-complex K^2 and a non-degenerate simplicial map $f\colon K^2 \to \Sigma^3$ for which the complement of $f(K^2)$ is a collection of open 3-cells, and for which one can pass from K^2 to $f(K^2)$ by a sequence of elementary "zipping moves" which push a singular point along the line of double points of f in certain ways. Recently Poénaru and Tanasi in [10] gave an extension of these ideas to the case of a simply-connected open 3-manifold V^3. In such a general case, at the source of the representation, the set of double points is, generally speaking, no longer closed. However, if $V^3 = \widetilde{M}^3$ is the universal cover of a closed 3-manifold, then they construct an X^2 with a free $\pi_1(M^3)$-action and having the equivariance property $f(gx) = gf(x)$, $g \in \pi_1 M^3$.

Definition 6: A representation is called *easy* if the sets Imf (i.e. fX^2) $\subset \widetilde{M}^3(\Gamma)$ and $M_2(f)$ (i.e. the set of double points of f) $\subset X^2$ are closed subsets.

Theorem 3: [Otera-Poénaru, [6]] Groups admitting an easy GSC-representation are QSF. In such a case we call the group *easily-representable* (or just *easy group*).

Comments: First of all V. Poénaru has developed a program for possibly showing that all finitely presented group are QSF. In this spirit, the latter theorem can be seen as a model for the full program and serve as a good introduction for it (but observe that in general Poénaru works with representations which are not easy, and so things become much more complicated and of difficult manipulation). Secondly, one may try to construct easy-representations for known geometric classes of groups for which one already knows that they are QSF (e.g. hyperbolic groups, almost-convex groups, combable groups, Tucker groups). It is worthy to note that, as for the QSF, there is no example of a group which does not admit an easy GSC-representation. So it is an intriguing question to ask whether the QSF property is equivalent, for groups, to the condition to be easily-representable (we think so!).

We end by saying that if the QSF property were really valid for all (finitely presented) groups, then it will be the very first example on a non-trivial geometric condition for groups valid for all of them, and this will 'stem' Gromov's philosophy saying that such a property should have a trivial proof (by its thesis that "every property of all groups is either false or trivial").

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Geometry of orbit spaces of proper Lie groupoids

Markus J. Pflaum

(joint work with Hessel Posthuma and Xiang Tang)

Orbit spaces of proper Lie group actions and their stratification theory are well studied in the mathematical literature. However, much less is known about the quotient spaces of general proper Lie groupoids. In the paper [4], we study the stratification theory and metric structure of such orbit spaces.

Let G be a proper Lie groupoid. Denote by G_1 its arrow set and by G_0 its set of objects. Let $X := \mathsf{G}_0/\mathsf{G}$ be the orbit space, and $\pi : \mathsf{G}_0 \to X$ the projection of the groupoid G. Choose a point $x \in \mathsf{G}_0$. By the Linearization Theorem of Zung [6] and Weinstein [5] it follows that there is an open neighborhood $U \subset \mathsf{G}_0$ of x such that the groupoid $\mathsf{G}_{|U}$ is isomorphic to an action groupoid $\mathsf{G}_x \ltimes V$, where G_x is the isotropy group of x, and V is an open G_x-invariant open neighborhood of the origin in the normal space $N_x := T_x\mathsf{G}_0/T_x\mathcal{O}$, where $\mathcal{O}$ denotes the orbit in G_0 through x. Moreover, one can choose U in such a way that $\mathsf{G}_{|U}$ and $\mathsf{G}_{|\pi^{-1}\pi(U)}$ are Morita equivalent.

It has been proved in [4] that under a linearization $\varphi : \mathsf{G}_{|U} \to \mathsf{G}_x \ltimes V$ the set germ $\left[\pi\big(\varphi^{-1}(V^{\mathsf{G}_x})\big)\right]_{\mathcal{O}}$ does not depend on the particular point x in $\mathcal{O}$, but only on the orbit $\mathcal{O}$. Denote this set germ by $\mathcal{S}_{\mathcal{O}}$. The following result then holds true.

Theorem 1. *The map $X \ni \mathcal{O} \to \mathcal{S}_{\mathcal{O}}$ defines on the quotient space X of a proper Lie groupoid G a stratification in the sense of Mather* [1]*. Moreover, X carries in a natural way the structure of a differentiable space (cf.* [2, 3]*) compatible with the stratification. The stratification of X satisfies Whitney's condition B and has a system of smooth control data.*

Since every stratified space with a smooth system of control data is triangulable, one immediatley obtains:

Corollary. *The orbit space X of a proper Lie groupoid G is triangulable.*

There does not exist a meaningful notion of an invariant riemannian metric on a general proper Lie groupoid, since a proper Lie groupoid does not act in general on its tangent bundle, but only on the normal bundles to its orbits. We therefore introduce the following notion.

Definition. Given a riemannian metric η on G_0, we shall say that it is *adapted*, if, restricted to each orbit $\mathcal{O}$, the induced metric on the normal bundle $N_{\mathcal{O}}$ is invariant under the canonical action of G.

It has been proved in [4] that every proper Lie groupoid carries a complete adapted riemannian metric. Using this and a slice theorem for proper Lie groupoids generalizing the Linearization Theorem by Zung and Weinstein, we could show the following result on the metric structure of orbit spaces of proper Lie groupoids.

Theorem 2. *Let G be a proper Lie groupoid such that the orbit space X is connected. Let η be an adapted riemannian metric on G. Then there exists a metric $\overline{d}$ on the orbit space X such that the following properties hold true:*

(1) *The metric $\overline{d}$ is uniquely determined by the property that for each orbit $\mathcal{O}$ in G_0 and every point q of an appropriate metric tubular neighborhood of $\mathcal{O}$ the relation*
$$\overline{d}(\mathcal{O},\mathcal{O}_q) = d(q,\mathcal{O})$$
holds true, where $\mathcal{O}_q$ is the orbit through q.

(2) *The canonical projection $\pi : \mathsf{G}_0 \to X$ onto the orbit space is a submetry, i.e. every ball $B_r(p)$ in (G_0,η) with respect to the geodesic distance on the riemannian manifold (G_0,η) is mapped under π onto the ball $B_r(\mathcal{O}_p)$ in X.*

(3) *$(X,\overline{d})$ is a length space, i.e. the geodesic distance with respect to $\overline{d}$ coincides with $\overline{d}$. Moreover, the topology induced on X by $\overline{d}$ coincides with quotient topology with respect to π.*

(4) *In case (G_0,η) is a complete riemannian manifold, $(X,\overline{d})$ is even a complete locally compact length space, and every bounded closed ball is compact.*

(5) *In case (G_0,η) has curvature bounded from below, $(X,\overline{d})$ is an Alexandrov space globally of dimension $\leq \dim \mathsf{G}_0$.*

Finally, in [4], we proved a de Rham theorem for orbit spaces of proper Lie groupoids.

Theorem 3. *The singular cohomology $H^\bullet_{\mathrm{sing}}(X,\mathbb{R})$ of the orbit space X of a proper Lie groupoid G coincides naturally with the basic cohomology $H^\bullet_{\mathrm{basic}}(\mathsf{G},\mathbb{R})$ of G, i.e. the cohomology of the complex of basic differential forms on G_0. Moreover, if X is compact, the basic cohomology $H^\bullet_{\mathrm{basic}}(\mathsf{G},\mathbb{R})$ is finite dimensional.*

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Localized index theory on Lie groupoids and the van Est map

Hessel Posthuma

(joint work with M. Pflaum, X. Tang)

Lie groupoids are natural generalisations of manifolds, Lie groups, actions of Lie groups on manifolds and foliations. As such, they are models for singular spaces and there are several connections to the theory of stratified spaces. Here we are concerned with the index theory of longitudinally elliptic operator on such Lie groupoids, generalizing the Atiyah–Singer families index theorem.

Let G be a Lie groupoid over the unit space M, and we denote the source and target map by $s, t : \mathsf{G} \to M$. The composition $g_1 g_2$ of two elements $g_1, g_2 \in \mathsf{G}$ is defined only if $t(g_1) = s(g_2)$. A longitudinal pseudodifferential operator [3, 4] on G is a family of pseudodifferential operators on the t-fibers $t^{-1}(x)$, $x \in M$ that is smooth in x and invariant under the action of G. With the right conditions on the support of such pseudodifferential operators, they form an algebra denoted by $\Psi^\infty(\mathsf{G})$. For us, two facts of this pseudodifferential calculus are important:

i) The universal enveloping algebra $\mathcal{U}(A)$, where A is the Lie algebroid associated to G embeds into $\Psi^\infty(\mathsf{G})$ as families of invariant differential operators on the t-fibers.

ii) The ideal of smoothing operators $\Psi^{-\infty}(\mathsf{G}) \subset \Psi^\infty(\mathsf{G})$ is isomorphic to the convolution algebra $\mathcal{A}$ of G. Recall that the convolution algebra is given by $\mathcal{A} = \Gamma_c^\infty(\mathsf{G}; s^* \bigwedge^{\mathrm{top}} A^*)$ equipped with the product
$$(a_1 * a_2)(g) := \int_{h \in \mathsf{G}_{t(g)}} a_1(gh^{-1}) a_2(h),$$
where G_x is the submanifold of all arrows $g \in \mathsf{G}$ having target $x \in M$.

All this is easily extended to the case of operators acting on sections on a vector bundle pulled back from M. Therefore, for $E \to M$ a vector bundle, we write $\mathcal{U}(A; E)$ for the tensor product $\mathcal{U}(A) \otimes \mathrm{End}(E)$. We say an element $D \in \mathcal{U}(A; E)$ is *elliptic* if it defines an elliptic differential operator D_x on $t^{-1}(x)$ for each $x \in M$.

Standard arguments using this pseudodifferential calculus construct an *index class*
$$[\mathrm{Ind}(D)] \in K_0(\mathcal{A})$$
of such an elliptic operator. Unfortunately, the K-theory of $\mathcal{A}$ is still very poorly understood in general, with the exception of so-called *foliation groupoids* where the Connes–Skandalis index theorem gives a topological construction of the index class

above out of the symbol of D. To go beyond the case of foliations, we therefore apply the Chern-Connes character to cyclic homology and study the class

$$\mathrm{Ch}([\mathrm{Ind}(D)]) \in HC_\bullet(\mathcal{A}).$$

In general, the cyclic homology of $\mathcal{A}$ is (again) not understood beyond the foliation case where there is a complete computation due to Brylinski–Nistor and Crainic.

To circumvent this lack of understanding, we shall construct certain cyclic *co*homology classes, and compute the pairing with the homology class above. The result is an index theorem valid for all Lie groupoids, not just the foliation ones, cf. [5].

First of all, we construct the line bundle $L = \bigwedge^{\mathrm{top}} T^*M \otimes \bigwedge^{\mathrm{top}} A$ of "transversal densities". It was first noticed by Evens–Lu–Weinstein that the groupoid G naturally acts on this line bundle and therefore we can consider its *differentiable groupoid cohomology* $H^\bullet_{\mathrm{diff}}(\mathsf{G}; L)$. This is a straightforward generalization of differentiable group cohomology with values in a representation of a Lie group, viz. the case when M is a point. This cohomology is the domain of a canonical map

$$\chi : \bigoplus_{i \geq 0} H^{\bullet+2i}_{\mathrm{diff}}(\mathsf{G}; L) \to HC^\bullet(\mathcal{A}).$$

This map can be thought of as the characteristic map associated to an action of a Hopf algebroid on $\mathcal{A}$. It enables us to pair elements in $K_0(\mathcal{A})$, such as the index class, with differentiable groupoid cohomology classes.

Second, we construct the index class in $K_0(\mathcal{A})$ is such a way that it is represented by idempotents in $\mathcal{A}$ with support arbitrarily close to the unit. In fact, one can construct a "localized K-theory" $K_0^{\mathrm{loc}}(\mathcal{A})$ build from idempotents with exactly this property, equipped with a canonical forgetful map $K_0^{\mathrm{loc}}(\mathcal{A}) \to K_0(\mathcal{A})$. The remark above then boils down to the statement that there is a natural refinement

$$[\mathrm{Ind}(D)]_{\mathrm{loc}} \in K_0^{\mathrm{loc}}(\mathcal{A})$$

of the index class. The crucial feature of the localized K-theory is that it naturally pairs with *Lie algebroid cohomology*:

$$\langle\ ,\ \rangle : K_0^{\mathrm{loc}}(\mathcal{A}) \times H^{\mathrm{ev}}_{\mathrm{Lie}}(A; L) \to \mathbb{C}.$$

Similar to differentiable groupoid cohomology, Lie algebroid cohomology generalizes the cohomology theory of Lie algebras and the representation of A on L is just the infinitesimal part of the representation of G. As for Lie groups, there is a natural "van Est" map for Lie groupoids

$$E : H^\bullet_{\mathrm{diff}}(\mathsf{G}; L) \to H^\bullet_{\mathrm{Lie}}(A; L).$$

The first result relates the global pairing with the localized one via this van Est map:

Theorem 1. *Let G be a Lie groupoid, $E \to M$ a vector bundle over the unit space, and $D \in \mathcal{U}(A, E)$ an elliptic element. Then, for $\alpha \in H^{2k}_{\mathrm{diff}}(\mathsf{G}; L)$,*

$$\langle \chi(\alpha), \mathrm{Ch}([\mathrm{Ind}(D)]) \rangle = \langle E(\alpha), [\mathrm{Ind}(D)]_{\mathrm{loc}} \rangle\,.$$

This reduces the computation of the index to a local computation near the unit space. We perform this computation using the fact that the pseudodifferential calculus on G is a quantization of the Lie–Poisson structure on A^*, and reduce it to the *algebraic index theorem* for this Poisson manifold. The final result is given as follows:

Theorem 2. *Let $A \to M$ be an integrable Lie algebroid, E a vector bundle over M and $D \in \mathcal{U}(A, E)$ an elliptic element. For $c \in H^{2k}_{\mathrm{Lie}}(A; L)$ we have*

$$\langle c, [\mathrm{Ind}(D)]_{\mathrm{loc}} \rangle = \frac{1}{(2\pi\sqrt{-1})^k} \int_{A^*} \pi^* c \wedge \hat{A}(\pi^! A) \wedge \rho^*_{\pi^! A} \mathrm{ch}(\sigma(D)).$$

Here, the right hand side is a topological expression using the usual characteristic classes, only now given in Lie algebroid cohomology rather than de Rham cohomology. The notation $\pi^! A$ denotes the pull-back (in the category of Lie algebroids) of A along the projection $\pi : A^* \to M$. This is a Lie algebroid over A^* with anchor map $\rho^*_{\pi^! A} : \pi^! A \to TA^*$, which has the same Lie algebroid cohomology as A.

Together, these two theorems give a complete understanding of the pairing between the index class and Lie groupoid cohomology classes for any Lie groupoid. Possibly, the localized index Theorem 2 is much more powerful and has more applications. We can consider some special cases to get some more insight:

i) The pair groupoid $M \times M$ of any manifold is proper, and there is therefore only one nonzero differentiable groupoid cohomology class which lives in degree zero. In this case, we find the Atiyah–Singer index theorem for elliptic operators on M. On the other hand, the associated Lie algebroid is simply TM and its Lie algebroid cohomology is given by $H^\bullet_{dR}(M)$. With this, Theorem 2 recovers Connes–Moscovici's localized index theorem [2]. The *covering index theorem* of Connes–Moscovici is a very natural statement in the present framework about two Lie groupoid that induce the same Lie algebroid.

ii) For a foliation $\mathcal{F} \subset TM$, we can apply this theory to the holonomy groupoid $\mathsf{G}_{\mathcal{F}}$ of $\mathcal{F}$. In this case we find Connes' index theorem [1, §III.7.γ] for the pairing between the index class and elements in $H^\bullet(B\mathsf{G}_{\mathcal{F}})$, but only for those classes that come from differentiable groupoid cohomology of the holonomy groupoid. This restriction is the price we have to pay for being able to extend the index theorem from foliation groupoids to arbitrary Lie groupoids.

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Witt groups of perverse sheaves

Jörg Schürmann
(joint work with Jon Woolf)

1. Witt groups of Abelian categories. First we recall some classical results and notions. Let $(A, \oplus)$ be an additive category with duality $D : A^{op} \to A$, e.g. with $\chi : id \xrightarrow{\sim} D^2$. An (anti)symmetric selfdual object $b \in obj(A)$ is a morphism $\beta : b \to Db$ such that $\beta = \pm D\beta \circ \chi$. For β an isomorphism it is called nondegenerate, with the obvious notion of isomorphism of selfdual objects called isometry. Assume now that A is Abelian. The orthogonal complement of a subobject $i : a \to b$ is given by $a^\perp := kern(Di \circ \beta)$, with a isotropic (resp. Lagrangian) in case $Di \circ \beta i = 0$, i.e. it factorizes as $a \to a^\perp$ (resp. as an isomorphism $a \xrightarrow{\sim} a^\perp$). b is anisotropic (resp. metabolic) if it contains no isotropic subobject $\neq 0$ (resp. it contains a Lagrangian subobject). The Witt group of A is defined as

$$W^\pm(A) := \{\text{isom. classes of nondeg. (anti)symmetric selfdual obj.}\}/\sim\,,$$

with $b \sim b' := b \oplus m \simeq b' \oplus m'$ for some metabolic m, m'. For an isotropic subobject $i : a \to b$ one gets by isotropic reduction $a^\perp/a \sim b$, so that for b Noetherian there is a maximal isotropic subobject a with $a^\perp/a$ anisotropic. Since A has the exact duality D, the following are equivalent:

$$A \text{ Noetherian} \Leftrightarrow A \text{ Artinian} \Leftrightarrow \text{any object is of finite length}\,,$$

so that in this case any object has a finite decomposition series with simple factors.

Theorem 1. *Let A be an Abelian and Noetherian category with exact duality D.*
1) Any nondegenerate selfdual object is isometric to an anisotropic one, which is a finite orthogonal sum of simple nondegenerate selfdual objects.
2) $W^\pm(A) = \bigoplus_{[s \simeq Ds]} W^\pm(\langle s \rangle)$ as a direct sum over isometry classes of simple selfdual objects s, with $\langle s \rangle$ the full subcategory generated by self-extensions of s.

An example is given by $A = Rep_G(k)$ the category of finite dimensional representations of a group G over a field k, with $DV = V^\vee$ the dual representation. For $G = \pi_1(X)$ the fundamental group of a nice connected topological space, this corresponds to the category $Loc(X)$ of local systems on X.

2. Witt groups of triangulated categories. Here we do not give the general abstract definitions, but illustrate them by some important related results.
a) Cappell-Shaneson [3] studied in some steps mapping theorems for L-classes for a stratified submersion $f : Y \to X$ of compact Whitney stratified spaces with even dimensional strata and Y oriented by studying the total direct image $F = f_* IC_Y \in D^b_c(X)$ of the intersection (co)homology complex of Y (for some

field coefficients k), which is selfdual with respect to Verdier duality D on the corresponding constructible derived category:

(1) They introduced a cobordism relation $\sim$ for such selfdual constructible sheaf complexes F, and showed
(2) $F \sim {}^p h^0(F)$, with ${}^p h^0$ the cohomology functor of the middle perversity t-structure on $D^b_c(X)$.
(3) a decomposition formula ${}^p h^0(F) \sim \bigoplus_S IC_S(L_S)$ into twisted intersection complexes, with L_S selfdual local systems on the strata S.
(4) an explicit formula $L_S = j_S^* i_S^{!*} F$ for $j_S : S \to \bar{S}$ the open inclusion of the stratum into its closure, and $i_S : \bar{S} \to X$ the closed inclusion, with

$$i_S^{!*} F := im \left({}^p h^0(i_S^! F) \to {}^p h^0(i_S^* F) \right) .$$

Here the image is taken in the Abelian heart $Perv(X) \subset D^b_c(X)$ of perverse sheaves, and as observed in [4], the last explicit formula is only true under some additional assumptions, but not in general. Youssin [7] generalized the steps (1-3) to abstract triangulated categories with a selfdual t-structure, by introducing a corresponding cobordism group $\Omega^{\pm}(X) := \Omega^{\pm}(D^b_c(X))$ of selfdual objects, with

$$\Omega^{\pm}(D^b_c(X)) \xrightarrow[\sim]{{}^p h^0} W^{\pm}(Perv(X)) .$$

b) Balmer [1] introduced more general (4-periodic) Witt groups W_i ($i \in \mathbb{Z}$) for triangulated categories with duality, together with an important localization sequence (under some mild assumptions, e.g. these categories are also $\mathbb{Z}[1/2]$-linear). Using this, Woolf [6] showed for a compact PL-space X, that Balmer's Witt groups

$$W_i(X; k) := W_i \left(D^b_{pl-c}(X) \right)$$

for the PL-constructible derived category (for some field coefficients k of $char(k) \neq 2$) form a generalized homology theory in this PL-context. Woolf also explained the relation to Siegel's bordism theory of PL-Witt spaces [5] and the symmetric L-homology (with the colimit taken by crossing with $[P^2(\mathbb{C})] \in \Omega_4^{Witt}(pt)$):

$$KO_*(X)[1/2] \xleftarrow[Siegel]{Sullivan} colim\, \Omega_*^{Witt}(X) \xrightarrow{\sim} W_*(X; \mathbb{Q}) \xleftarrow{\sim} L_*^{sym}(X; \mathbb{Q}) .$$

Similar results can be shown for $k = \mathbb{R}$ instead of $\mathbb{Q}$ if one uses Banagl's PL-version [2] of Minatta's signature homology instead of Siegel's Witt bordism.

3. Witt groups of perverse sheaves. In this last section with explain some new results from [4]. These hold more generally in the context of triangulated categories with a selfdual t-structure. But for simplicity, we only state the results in the geometric context of perverse sheaves on stratified spaces fitting with the theme of the workshop.

Theorem 2. *Let X be a locally cone-like topologically stratified space with even dimensional strata, and consider the Balmer Witt groups $W_i(X) := W_i\left(D^b_c(X)\right)$ of the corresponding constructible derived category with respect to Verdier duality* *(for some field coefficients k of $char(k) \neq 2$). Then $W_i(X) = 0$ for i odd, and if we write $+$ or $-$ for $i \equiv 0$ or 2 mod. 4, then*

$$W_{\pm}\left(D^b_c(X)\right) \xrightarrow[\sim]{{}^p h^0} W^{\pm}\left(Perv(X)\right) \xleftarrow[\sim]{Youssin} \Omega^{\pm}\left(D^b_c(X)\right) .$$

If X is compact and the stratification can be refined to a triangulation (e.g. it is a Whitney stratification), then we get canonical group homomorphisms (for $i \equiv 0$ or 2 mod. 4):

$$W^{\pm}\left(Perv(X)\right) \simeq W_{\pm}\left(D^b_c(X)\right) \to W_i\left(D^b_{pl-c}(X)\right) ,$$

so that the following theorem implies similar results in $L^{sym}_*(X;\mathbb{Q}), KO_*(X)[1/2]$ as well as for L-classes.

Assume X has only finitely many strata (e.g. X is compact), so that $Perv(X)$ is Noetherian with simple objects given by twisted intersection complexes $IC_S(L_S)$ for L_S a simple local system on a stratum S. Consider the "gluing context" of stratified subspaces (i.e. union of strata)

$$Y \xrightarrow[closed]{i} X \xleftarrow[open]{j} U := X \backslash Y .$$

Theorem 3. *1) Assume $F \in Perv(X)$ is selfdual and nondegenerate. Then*

$$[F] = \left[i_* i^{!*} F\right] + \left[j_{!*} j^* F\right] \in W^{\pm}\left(Perv(X)\right) .$$

2) $W^{\pm}(Perv(X)) \simeq W^{\pm}(Perv(Y)) \oplus W^{\pm}(Perv(U))$. Here the inclusion resp. projection

$$W^{\pm}\left(Perv(Y)\right) \xrightarrow{i_*} W^{\pm}\left(Perv(X)\right) \xrightarrow{j^*} W^{\pm}\left(Perv(U)\right)$$

is induced from the exact extension resp. restriction functors i_*, j^*, which commute with Verdier duality. The other inclusion resp. projection

$$W^{\pm}\left(Perv(U)\right) \xrightarrow{j_{!*}} W^{\pm}\left(Perv(X)\right) \xrightarrow{i^{!*}} W^{\pm}\left(Perv(Y)\right)$$

is induced from the intermediate extension resp. restriction functors $j_{!*}, i^{!*}$ applied to an anisotropic representative of the corresponding Witt class.

Warning: For F not anisotropic, the decompositions in 1) and 2) above need not be related! Inductively one gets from 2) above (with $\epsilon(S) := (-1)^{dim(S)/2}$):

$$W^{\pm}\left(Perv(X)\right) \simeq \bigoplus\nolimits_S W^{\pm\cdot\epsilon(S)}\left(Loc(S)\right) .$$

For the more complicated explicit decomposition formulae corresponding to 1) above we refer to [4].

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Motivic integration and invariants of singular points of complex hypersurfaces: Applications to the quasi-ordinary case

Manuel González Villa
(joint work with N. Budur, P. González-Pérez)

This talk aims to illustrate the usefulness of the space of *arcs* (or *n-jets*) for the study of the singularities of algebraic varieties.

This idea was proposed in the late sixties by Nash, who conjectured a bijection between the essential divisors of the singularity and the irreducible components of the space of arcs centered at the singularity. Ishii and Kollar have stablished the *Nash conjecture* for toric singularities and found a counterexample in dimension 4. More recently Fernández Bobadilla and Pe have settled the conjecture for normal surfaces.

The interest in the arc space of algebraic varieties reflourished after the introduction of *motivic integration* by Kontsevich in 1995. He introduced a notion of *motivic measure* on a certain class of subsets of arcs. This measure takes values on the Grothendieck ring of algebraic varieties, which is a universal additive invariant. With help of a *transformation rule* for motivic integrals under birational maps, Kontsevich showed that *birational equivalent Calabi-Yau manifolds have the same Hodge numbers.*

The theory of motivic integration has been further developed by Batyrev, Denef and Loeser and has found a wealth of applications to birational geometry and singularity theory. See [3] for a survey on arc spaces, motivic integration and its applications.

Finally, we announce some applications to the quasi-ordinary case.

1. Invariants of singularities of hypersurfaces

Let $f : (\mathbb{C}^{d+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a singular point at $0 \in f^{-1}(0)$. For $0 < \delta << \epsilon < 1$ small enough, the map $f : f^{-1}(\mathbf{D}^*_\delta) \cap \mathbb{B}(0, \epsilon) \to \mathbf{D}^*_\delta$ is a C^∞-locally trivial fibration. Call $F_f := f^{-1}(t) \cap \mathbb{B}(0, \epsilon)$ the *Milnor fibre* of f at 0.

We are interested in invariants of the Milnor fibre such as the Betti numbers $b_i(F_f) := \dim_{\mathbb{C}} H^i(F_f, \mathbb{C})$ and Euler characteristic $\chi(F_f) := \Sigma_{i>0}(-1)^i b_i(F_f)$.

Going once along the border of the disc $\mathbf{D}_\delta$ induces the *monodromy*, a diffeomorphism of the Milnor fiber $F_f \to F_f$ (defined up to isotopy). The monodromy operator $M_f : H^\bullet(F_f, \mathbb{C}) \to H^\bullet(F_f, \mathbb{C})$ is the induced map in cohomology. The map M^i_f is quasi-unipotent, hence the eigenvalues of M^i_f are roots of unity.

The cohomology group $H^i_c(F_f, \mathbb{Q})$ carries a mixed Hodge structure (MHS) which is compatible with the monodromy operator M_f (Steenbrink, Saito, Navarro-Aznar). The Hodge-Steenbrink spectrum $hsp(f,0)$ of f at 0 is a finer invariant which describes the interplay between the eigenvalues of the monodromy and the mixed Hodge structure of the cohomology of F_f.

2. Motivic zeta functions

The motivic viewpoint on the study of invariants of singularities has been developed by Denef and Loeser.

We denote by $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ the *Grothendieck ring* of algebraic varieties equipped with a good action of the group μ_n (of n-th roots of unit) for some n. We denote by $\mathbb{L} \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ the class of the affine line.

The space $\mathcal{L}_n(\mathbb{C}^{d+1})_0$ of n-jets of $\mathbb{C}^{d+1}$ at 0 consists of $\varphi = (\varphi_1(t), ..., \varphi_{d+1}(t)) \in (\mathbb{C}[[t]]/(t^{n+1}))^{d+1}$ such that $\varphi_i(0) = 0$. The set $\mathcal{X}_{n,1}$ given by

$$\mathcal{X}_{n,1} = \{\varphi \in \mathcal{L}_n(\mathbb{C}^{d+1})_0 \mid f \circ \varphi = t^n + a_{n+1}t^{n+1} + \cdots\} \tag{1}$$

is a constructible subset of $\mathcal{L}_n(\mathbb{C}^{d+1})_0$, equipped with a good μ_n-action given by $(\lambda, \varphi) \mapsto \varphi(\lambda t)$. Hence the class $[\mathcal{X}_{n,1}]$ is a well defined element of $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$.

The *motivic zeta function* of the germ f is

$$Z(f,T) := \sum [\mathcal{X}_{n,1}]\mathbb{L}^{-n(d+1)}T^n. \tag{2}$$

Denef and Loeser proved that $Z(f,T)$ is a rational function of the form

$$Z(f,T) = \sum_J A_J \prod_{i \in J} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1 - \mathbb{L}^{-\nu_i}T^{N_i}} \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}](T), \tag{3}$$

where the data $\nu_i, N_i \in \mathbb{Z}_{\geq 0}$ and $A_J \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ are expressed in terms of an embedded resolution of $\{f = 0\}$.

There are interesting connections between zeta functions and monodromy.

For example, the *Monodromy Conjecture* predicts that relation between the *poles* of the *naive* motivic zeta function and the eigenvalues of the monodromy. Here naive means a weaker condition on the definition of (1). It should be remarked that not all the pairs (ν_i, N_i) appearing in (3) give poles of the zeta function. There are in general many cancellations between the candidates and the determination of the poles is a difficult problem. This conjecture was originally proposed by Igusa in the framework of p-adic zeta functions.

Another connection comes from the coefficients A_J in (3). There exists a notion of limit (heuristically $\mathbb{L}^{-\nu_i}T^{N_i}(1 - \mathbb{L}^{-\nu_i}T^{N_i})^{-1}$ tends to -1 as T goes to ∞) for expressions of the form (3) which allows to define *motivic Milnor fibre* of f at $0 \in \mathbb{C}^{d+1}$ as

$$\mathcal{S}_{f,0} := -\lim_{T\to\infty} Z(f,T) \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}).$$

According to Denef and Loeser's viewpoint $\mathcal{S}_{f,0}$ is the *motivic incarnation* of the classic Milnor fibre (equipped with $M_{f,0}$): They proved that the Milnor fibre F_f and the motivic Milnor fibre $\mathcal{S}_{f,0}$ define the same class in the Grothendieck ring $K_0(\mathrm{HS}^{\mathrm{mon}})$ of Hodge structure with a quasi-unipotent endomorphism. In particular, $\mathcal{S}_{f,0}$ determines the Hodge-Steenbrink spectrum of f at 0.

3. Applications to Quasi-ordinary hypersurfaces

A germ of analytic function $f : (\mathbb{C}^{d+1}, 0) \to (\mathbb{C}, 0)$ is *quasi-ordinary* if it can be defined in suitable local coordinates $(x_1, ..., x_d, y)$ by a Weierstrass polynomial $f \in \mathbb{C}\{x_1, ..., x_d\}[y]$ such that the discriminant $\Delta_y f$ is of the form $x_1^{\delta_1} \cdots x_d^{\delta_d} \epsilon$, with ϵ a unit in $\mathbb{C}\{x_1, ..., x_d\}$. The germ $(V, 0)$ defined by $\{f = 0\}$ at $0 \in \mathbb{C}^{d+1}$ is a quasi-ordinary hypersurface.

Quasi-ordinary singularities arise classically in the study of parametrizations and resolutions of complex surface singularities (Jung, Walker, Zariski). Quasi-ordinary singularities include plane curve singularities ($d = 1$) but are in general non-isolated. The Whitney Umbrella ($f = y^2 - x_1^2 x_2$) is a well known example.

Jung-Abhyankar Theorem claims that quasi-ordinary hypersurfaces are parametrized by a special type of fractional power series $\xi = \Sigma c_\lambda \mathbf{x}^\lambda \in \mathbb{C}\{x_1^{1/n}, ..., x_d^{1/n}\}$. A quasi-ordinary branch ξ has a finite and totally ordered set of *characteristic exponents* $\{\lambda_1 < \cdots < \lambda_g\} \subset \frac{1}{n}\mathbb{Z}_{\geq 0}^{d+1}$, which generalize the Newton-Puiseux exponents of plane branches.

The characteristic monomials determine most of the geometry and the topology $(V, 0)$. In particular, Lipman and Gau proved that the characteristic monomials of $(V, 0)$ are equivalent to its embedded topological type of $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$.

Together with González-Pérez we prove in [4] that

> *the motivic zeta function $Z(f, T)$, the motivic Milnor fibre $S_{f,0}$ and the spectrum $hsp(f, 0)$ of an irreducible quasi-ordinary hypersurface singularity are determined by its embedded topological type.*

These results generalize the work of Guibert [5] for plane curve singularities and improve the results of [1] which is devoted to the study of naive zeta functions and the Monodromy Conjecture in the quasi-ordinary case.

As an application of the previous results we consider in collaboration with Budur and González-Pérez the *log canonical threshold*, which is an important invariant of the singularity defined in terms of an embedded resolution as the minimun of the ratios ν_i/N_i in (3). We prove in [2] that

> *the log canonical threshold of an irreducible quasi-ordinary hypersurface singularity is determined by its embedded topological type.*

We give explicit formulas for all the above mentioned invariants in terms of the characteristic exponents.

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An Introduction to Intersection Cohomology and Perverse Sheaves

Jon Woolf

We survey intersection cohomology and perverse sheaves from an algebraic perspective. For simplicity we work with a complex projective variety X, with singular set Σ, embedded in a non-singular projective variety M (which can be taken to be $\mathbb{CP}^m$). We consider sheaves of $\mathbb{C}$-vector spaces in the classical (rather than Zariski) topology on X, and let $D_c(X)$ denote the algebraically constructible derived category of such. All functors will be derived, so we write f_* etc not Rf_*. A 'local system', or locally constant sheaf, on a stratum S will be considered to lie in degree $-\dim S$. Poincaré–Verdier duality $D : D_c(X)^{\mathrm{op}} \to D_c(X)$ is an equivalence whose square is (isomorphic to) the identity. One advantage of the above shifting convention for local systems is that the dual of a local systems (with this shift) on a closed stratum is once again a local system.

Many of the results in this survey, as well as generalisations to other settings, can be found in [Dim04, Sch03, KS90, GM88, dCM09].

When X is non-singular and $\mathcal{L} \cong D\mathcal{L}$ is a self-dual local system

$$H^i(X;\mathcal{L}) = H^i(p_*\mathcal{L}) \cong H^i(p_*D\mathcal{L}) \cong H^i(Dp_*\mathcal{L}) \cong DH^{-i}(p_*\mathcal{L}) \cong DH^{-i}(X;\mathcal{L})$$

where $p : X \to \mathrm{pt}$ is the map to a point. However, when X is singular $D\mathcal{L}$ is not in general a local system so the above Poincaré duality breaks down. The idea of Goresky and MacPherson's intersection cohomology theory [GM80, GM83a] is to modify the coefficients $\mathcal{L}$ so that they become self-dual even on singular X.

Perverse sheaves and intermediate extensions. A self-dual local system $\mathcal{L}$ on a stratum $\jmath_S : S \hookrightarrow X$ has two (dual) extensions, connected by a natural morphism $\jmath_{S!}\mathcal{L} \to \jmath_{S*}\mathcal{L}$. An algebraically natural way to proceed would be to take the image of this morphism, which should then be a self-dual extension. But this we cannot do, since images are not well-defined in triangulated categories such as $D_c(X)$. However,

Theorem 1 ([BBD82])**.** *There is a self-dual t-structure on $D_c(X)$. The heart, the category* $\mathrm{Perv}(X)$ *of perverse sheaves, is a full abelian subcategory preserved by duality, and there is a cohomological functor ${}^pH^0 : D_c(X) \to$* $\mathrm{Perv}(X)$ *left-inverse to the inclusion.*

The intermediate extension $\jmath_{S!*}\mathcal{L}$ is the image of the morphism obtained by applying ${}^pH^0$ to the morphism between the two dual extensions. Thus we have maps ${}^pH^0(\jmath_{S!}\mathcal{L}) \twoheadrightarrow \jmath_{S!*}\mathcal{L} \hookrightarrow {}^pH^0(\jmath_{S*}\mathcal{L})$. The intermediate extension exists for any $\mathcal{L}$, and is self-dual whenever $\mathcal{L}$ is. By definition the intersection cohomology complex $\mathrm{IC}_{\overline{S}}(\mathcal{L}) = \jmath_{S!*}\mathcal{L}$.

But what are these perverse sheaves? For a Whitney stratification $\mathbb{S}$ of X by complex varieties we say $\mathcal{E}$ is perverse $\iff$ $\mathcal{E}$ is $\mathbb{S}$-constructible and $H^i(\jmath_S^!\mathcal{E})_x = 0$ for $i < -\dim S$ and $H^i(\jmath_S^*\mathcal{E})_x =$ for $i > -\dim S$, for all x in each S. For example, a local system on a closed stratum S is perverse and, by construction intermediate extensions are perverse. If $\mathcal{E}$ is perverse for one stratification then it is perverse for any stratification for which it is constructible. Let $\mathrm{Perv}(X) = \mathrm{colim}_{\mathbb{S}}\, \mathrm{Perv}_{\mathbb{S}}(X)$.

Whilst the above definition is brief, it is not very illuminating. More intuitively $\mathrm{Perv}_{\mathbb{S}}(X)$ is obtained by glueing together the categories of local systems (with our shift) on the strata. This intuition can be made precise by using the technique of glueing t-structures. Since each of these categories of local systems is preserved by duality, so are the perverse sheaves.

It is traditional to remark that perverse sheaves are neither sheaves nor perverse. But they do have nice algebraic properties. For instance perverse sheaves form a stack (i.e. a sheaf of categories, so that both objects and morphisms can be understood locally on X), $\mathrm{Perv}(X)$ has finite length (i.e. each perverse sheaf has a finite composition series with simple factors), and the simple perverse sheaves are the $\jmath_{S!*}\mathcal{L}$ for S and $\mathcal{L}$ irreducible. A much deeper property is

Theorem 2 ([BBD82, Sai88, Sai90, dCM05])**.** *The pushforward under a proper map of a simple perverse sheaf 'of geometric origin' (see* [BBD82] *for the definition) is a direct sum of shifted simple perverse sheaves of geometric origin.*

This algebraic result has many important consequences. For instance, it implies that $H^*(\widetilde{X}) \cong IH^*(X) \oplus A^*$ for any resolution $\widetilde{X} \to X$, and combining it with Hodge theory yields the Hard Lefschetz Theorem for $IH^*(X)$.

Perverse sheaves and Morse theory. A different approach to perverse sheaves is provided by stratified Morse theory. Fix a stratification $\mathbb{S}$ of $X \subset M$. Say $x \in S$ is critical for smooth $f : M \to \mathbb{R}$ if it is critical for $f|_S$. Then f is Morse if

- the critical values are distinct,
- each critical point in S is non-degenerate for $f|_S$,
- $d_x f$ is non-degenerate at each critical point x.

(This last condition means that the derivative does not annihilate any limit of tangent spaces to an adjacent stratum.) The main new ingredient in *stratified* Morse theory is the normal Morse data: given $\mathcal{E} \in D_c(X)$ and critical $x \in S$ this is defined by $\mathrm{NMD}\,(\mathcal{E}, f, x) = R\Gamma_{\{f \geq f(x)\}}(\mathcal{E}|_{N\cap X})_x$ where N is a complex analytic normal slice to S in M. It turns out that this depends only on $\mathcal{E}$ and the stratum $S \ni x$, so we write $\mathrm{NMD}\,(\mathcal{E}, S)$.

Example 3. *When X is a curve and $\mathcal{E} = IC_X(\mathbb{C})$ one has $NMD\,(\mathcal{E}, x) = \mathbb{C}^{m_x - b_x}$, where m_x is the multiplicity and b_x the number of analytic branches, at singular x and $\mathbb{C}$ at non-singular x. Note that the 'Morse group' may not be one-dimensional, e.g. for a higher order cusp, and also that it may vanish, e.g. for an ordinary double point.*

We say $\mathcal{E}$ is pure if, for each stratum S, the normal Morse data $\mathrm{NMD}(\mathcal{E}, S)$ is concentrated in degree $-\dim S$. If $x \in S$ is critical for Morse f and $\mathcal{E}$ is pure then

$$H^i(X_{\leq fx-\epsilon}, X_{\leq fx+\epsilon}; \mathcal{E}) \cong \begin{cases} \mathrm{NMD}(\mathcal{E}, S) & i = \lambda - \dim S \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda =$ index at x of $f|_S$. So for pure $\mathcal{E}$, critical points in S 'contribute' only in degrees from $-\dim S$ to $\dim S$. Hence $H^i(X; \mathcal{E}) = 0$ for $|i| > \dim X$.

There is a beautiful Morse-theoretic characterisation of the perverse sheaves, which says that they are exactly the pure objects of $D_c(X)$. [KS90]

If $S \subset \mathbb{C}^n$ then it is a classical fact that any Morse critical point for a distance function $f|_S$ has index $\leq \dim S$. Hence for an affine morphism $\jmath : U \hookrightarrow X$ we have $H^i(U; \mathcal{E}|_U) = 0$ for $i > 0$ whenever $\mathcal{E}$ is perverse. Many important vanishing theorems follow; here are two examples, the first obtained by applying this globally and the second by applying it locally.

Theorem 4 ([GM83b]). *If H is a generic hyperplane in $\mathbb{CP}^m$ then $IH^i(X) \to IH^i(X \cap H)$ is an isomorphism for $i < -1$ and injective when $i = -1$.*

Theorem 5 ([BBD82]). *The extensions $\jmath_!$ and $\jmath_*$ preserve perverse sheaves. In particular if U is a stratum with local system $\mathcal{L}$ then $\jmath_!\mathcal{L}$ and $\jmath_*\mathcal{L}$ are perverse.*

Glueing and quiver descriptions. Let $h : X \to \mathbb{C}$ be regular and $X_t = h^{-1}(t)$. Then there are nearby cycles ${}^p\psi_h(\mathcal{E})$, with

$$H^i\left({}^p\psi_h(\mathcal{E})\right)_x \cong H^i(MF_x; \mathcal{E}),$$

where F_x is the (local) Milnor fibre at $x \in h^{-1}(0)$, and vanishing cycles ${}^p\varphi_h(\mathcal{E})$.

Theorem 6 ([GM83b, KS90, Mas09]). *The nearby and vanishing cycles functors, ${}^p\psi_h$ and ${}^p\varphi_h$, preserve perverse sheaves, and commute with duality.*

There is a *canonical* morphism $c : {}^p\psi_h(\mathcal{E}) \to {}^p\varphi_h(\mathcal{E})$, and also a *variation* morphism $v : {}^p\varphi_h(\mathcal{E}) \to {}^p\psi_h(\mathcal{E})$. These can be used to describe the respective monodromy operators $1 + vc$ and $1 + cv$ on the nearby and vanishing cycles.

The data in the unipotent (with respect to the monodromy) nearby and vanishing cycles, together with the canonical and variation morphisms between them are exactly what is required to glue together perverse sheaves on a divisor and its complement to obtain a perverse sheaf on X. More precisely

Theorem 7 ([Beĭ87]). *The categories* $\mathrm{Perv}(X)$ *and* $\mathrm{Glue}(X, h)$ *are equivalent via*

$$\mathcal{E} \mapsto \left(\mathcal{E}|_{X-X_0}, {}^p\varphi_h^{un}(\mathcal{E}), c, v\right).$$

Here $\mathrm{Glue}(X, h)$ *is the category with objects* $(\mathcal{E}, \mathcal{F}, c, v)$ *where* $\mathcal{E} \in \mathrm{Perv}(X - X_0)$ *and* $\mathcal{F} \in \mathrm{Perv}(X_0)$ *with* $\mathcal{F} \xrightarrow{v} {}^p\psi_h^{un}(\mathcal{E}) \xrightarrow{c} \mathcal{F}$ *where $\mu = 1 + vc$, and morphisms given by commuting diagrams.*

This glueing theorem is a key ingredient in obtaining linear algebra, or quiver, descriptions of perverse sheaves which allow one to actually compute. For example the category $\mathrm{Perv}_{\mathbb{S}}(\mathbb{CP}^n)$ of perverse sheaves on $\mathbb{CP}^n$, equipped with the standard stratification with strata $\mathbb{C}^i$ for $i = 0, 1, \ldots, n$ is equivalent to representations of the quiver

$$\begin{tikzcd} 0 \arrow[r, bend left, "p"] & 1 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & n \arrow[l, bend left, "q"] \end{tikzcd}$$

with $1 + qp$ invertible and all other length two paths zero.

In fact it turns out that every category of perverse sheaves, with respect to a fixed stratification, has a quiver description [GMV96]. However, the description is not unique and it is, in general, very hard to find a tractable one. Conversely, when one does, it is often important. For instance, work of Braden [Bra02], Khovanov [Kho00] and Stroppel [Str09] culminated in a diagrammatic description of perverse sheaves on the Grassmannian $Gr_m(\mathbb{C}^{2m})$ which has close connections with Khovanov's categorification of the Jones polynomial (as well as being of independent interest in representation theory).

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Structure Set of Multiaxial Manifolds

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(joint work with Sylvain Cappell, Shmuel Weinberger)

A manifold M with $U(n)$-action is *multiaxial*, if it is locally $U(n)$-homeomorphic to an open subset of the $U(n)$-representation $k\rho_n \oplus j\epsilon$, where ρ_n is $\mathbb{C}^n$ with the defining $U(n)$-action, and ϵ is $\mathbb{R}$ with trivial $U(n)$-action. The condition is equivalent to that the action is locally linear, and any isotropy group is conjugate to a unitary subgroup $U(i)$.

In [2], M. Davis gave classification of multiaxial $U(n)$-manifolds for the case $k \leq n$. In [3], M. Davis and W. C. Hsiang classified the concordance class of multiaxial $U(n)$-homotopy spheres, also for the case $k \leq n$. Here we study the structure set $S_{U(n)}(M)$, i.e., the homeomorphism classes of $U(n)$-manifolds isovariantly homotopy equivalent to M. Our results cover $k \leq n$ as well as $k \geq n$.

A multiaxial $U(n)$-manifold is naturally stratified

$$M = M_0 \supset M_{-1} \supset M_{-2} \supset \cdots \supset M_{-n}, \quad M_{-i} = U(n)M^{U(i)}.$$

The quotient $\bar{M} = M/U(n)$ is correspondingly stratified with $\bar{M}_{-i} = M_{-i}/U(n)$. The stratification has the following special properties.

1. (1) $\bar{M}_{-i} = M^{U(i)}/U(n-i)$, where $M^{U(i)}$ is a multiaxial $U(n-i)$-manifold modeled on $k\rho_{n-i} \oplus j\epsilon$.
2. (2) If $k \leq n$, then the whole M is fixed by $U(n-k)$, so that $\bar{M} = M/U(n) = \bar{M}_{n-k} = M^{U(n-k)}/U(k)$. Since $M^{U(n-k)}$ is a multiaxial $U(k)$-manifold modeled on $k\rho_k \oplus j\epsilon$, the study for the case $k \leq n$ is equivalent to the case $k = n$.
3. (3) If $k \geq n$, then the link of $\bar{M}_{-i-1}$ in $\bar{M}_{-i}$ is $\mathbb{C}P^{k-n+i}$.
4. (4) For any $i < j$, the pure strata of $\bar{M}_{-j}$ in $\bar{M}_{-i}$ are all connected and simply connected.

By the third property, half of the links between adjacent strata are $\mathbb{C}P^{\text{even}}$, which are closed oriented manifolds of signature 1. The product with such a manifold induces "periodicity isomorphism" on surgery theory [6, Chapter 9]. The periodicity can be extended to the transfer along orientable bundles with such a manifold as fibre [4, 5]. As a consequence of this and the stratified surgery theory of [7], the computation of the structure set of the quotient space of the multiaxial $U(n)$-manifold fits into the following in half the cases.

Lemma. Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a stratified space, satisfying the following properties.

1. (1) The link of X_{-1} in X is a closed oriented manifold of signature 1.
2. (2) The link bundle of $X_{-1} - X_{-2}$ in X is orientable.
3. (3) The pure strata of all links are connected and simply connected.

Then there is a natural homotopy equivalence of structure *spectra*

$$\mathbb{S}(X) = \mathbb{S}(X, \text{rel } X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Moreover, $\mathbb{S}(X, \text{rel } X_{-2})$ is the homotopy fibre of the assembly map

$$\mathbb{H}(X, X_{-1}; \mathbb{L}) \to \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

Here $\mathbb{S}, \mathbb{H}, \mathbb{L}$ are the spectra whose homotopy groups are the usual structure set S, the homology H, and the usual surgery obstruction group L. Note that for the assembly map in the lemma, if X were a manifold with boundary X_{-1}, then the classical surgery theory of [6] says that the homotopy fibre would compute the structure set of the manifold with boundary. Here for a pair of spaces (X, X_{-1}), the computation can still be algebraically carried out. To emphasize the algebraic nature of such computation, we write

$$\mathbb{S}(X, \text{rel } X_{-2}) = \mathbb{S}^{\text{alg}}(X, X_{-1}).$$

Correspondingly, for our multiaxial $U(n)$-manifold, if $k \geq n$ and $k - n$ is even, then

$$S(\bar{M}) = S^{\text{alg}}(\bar{M}, \bar{M}_{-1}) \oplus S(\bar{M}_{-2}).$$

The left side is the isovariant structure set $S_{U(n)}(M)$. By the first special property of $\bar{M}$, the second factor on the right is

$$S(\bar{M}_{-2}) = S(M^{U(2)}/U(n-2)) = S_{U(n-2)}(M^{U(2)}).$$

Since $k - (n-2)$ is still even, the decomposition can be further applied to the second factor. This leads to the following decomposition.

Theorem. Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$. If $k \geq n$ and $k - n$ is even, then we have natural decomposition

$$\begin{aligned} S_{U(n)}(M) &= \oplus_{i \geq 0} S_{U(n-2i)}(M^{U(2i)}, \text{rel } U(n-2i) M^{U(2i+2)}) \\ &= \oplus_{i \geq 0} S^{\text{alg}}(\bar{M}_{-2i}, \bar{M}_{-2i-1}). \end{aligned}$$

As remarked in the second special property for $\bar{M}$, the theorem actually also includes the case $k \leq n$.

If $k - n$ is odd, similar decomposition still holds if the multiaxial $U(n)$-manifold can fit into a larger multiaxial $U(n+1)$-manifold, for which $k - (n+1)$ becomes even.

Theorem. Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$, such that $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold modeled on $k\rho_{n+1} \oplus j\epsilon$. If $k \geq n$ and $k - n$ is odd, then we have natural decomposition

$$\begin{aligned} S_{U(n)}(M) = S_{U(n)}(M, &\text{rel } U(n) M^{U(1)}) \oplus \\ &\left(\oplus_{i \geq 0} S_{U(n-2i-1)}(M^{U(2i+1)}, \text{rel } U(n-2i-1) M^{U(2i+3)}) \right) \\ = S^{\text{alg}}&(\bar{M}) \oplus \left(\oplus_{i \geq 0} S^{\text{alg}}(\bar{M}_{-2i-1}, \bar{M}_{-2i-2}) \right). \end{aligned}$$

We remark that in both theorems, the natural restriction

$$S_{U(n)}(M) \to S_{U(n-i)}(M^{U(i)})$$

is (split) onto when $k - n + i$ is even. Geometrically, this means that, if F is a $U(n-i)$-manifold and $g\colon F \to M^{U(i)}$ is a $U(n-i)$-isovariant homotopy equivalence, then there is a $U(n)$-manifold N and a $U(n)$-isovariant homotopy equivalence $f\colon N \to M$, such that $F = N^{U(i)}$, and the restriction of f to N is $U(n-i)$-isovariantly homotopy equivalent to g. In other words, the $U(i)$-fixed points of M can be homotpically replaced. In [1], we showed that it is possible to homotopically replace the fixed points of the whole action group under certain general condition. The replacement here is not for the fixed points of the whole action group, and the kind of condition in [1] may not be satisfied. Therefore we get a new type of replacement.

For the unit sphere of the multiaxial representation $M = S(k\rho_n \oplus j\epsilon)$, we may compute the structure set explicitly. The main computation is

$$S^{\mathrm{alg}}(\bar{M}, \bar{M}_{-1}) = H_{2kn-n^2-1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}), \quad N = S(k\rho_n).$$

The homology may be computed by the spectral sequence

$$E_2^{p,q} = H_p(\bar{N}, \bar{N}_{-1}; \pi_q\mathbb{L}) = \begin{cases} H_p(\bar{N}, \bar{N}_{-1}; \mathbb{Z}), & \text{if } q = 0 \text{ mod} 4, \\ H_p(\bar{N}, \bar{N}_{-1}; \mathbb{Z}_2), & \text{if } q = 2 \text{ mod} 4, \\ 0, & \text{if } q \text{ is odd}. \end{cases}$$

Note that $\bar{N}$ consists of k-tuples of vectors in $\mathbb{R}^n$ with total unit length 1, and the quotient space $\bar{N}$ has Schubert cell structures similar to the complex Grassmannians. Then we get $S^{\mathrm{alg}}(\bar{M}, \bar{M}_{-1}) = \mathbb{Z}^{A_{k,n}} \oplus \mathbb{Z}_2^{B_{k,n}}$, where $A_{k,n}$ counts the number of cells in $\bar{N} - \bar{N}_{-1}$ of dimension 0(mod 4) and $B_{k,n}$ counts the number of cells of dimension 2(mod 4). This generalizes the classical computation of the structure set of $\mathbb{C}P^n$ in [6, Section 14C].

The computation of $S^{\mathrm{alg}}(\bar{M}, \bar{M}_{-1})$ can be applied to the other factors in the decomposition, simply by replacing $k\rho_n$ with $k\rho_{n-2i}$. Then for $k \geq n$ and $k - n$ even, we get

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{i\geq 0} A_{k,n-2i}} \oplus \mathbb{Z}_2^{\sum_{i\geq 0} B_{k,n-2i}}.$$

The formula is almost correct, except the last term in the decomposition may have the empty singular part. This is the case $j = 0$ and n is odd, for which we have one less copy of $\mathbb{Z}$.

Similarly, for $k \geq n$ and $k - n$ odd, we have

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{A_{k,n}+\sum_{i\geq 0} A_{k,n-2i-1}} \oplus \mathbb{Z}_2^{B_{k,n}+\sum_{i\geq 0} B_{k,n-2i-1}}.$$

Again the number of copies of $\mathbb{Z}$ or $\mathbb{Z}_2$ need to be slightly adjusted in some exceptional cases.

Finally, all the work can be done for multiaxial $Sp(n)$-manifolds in parallel way, and we get (with slight adjustment in exceptional cases).

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \begin{cases} \mathbb{Z}^{\sum_{i\geq 0}\binom{k-1}{n-2i}}, & k \geq n,\ k-n \text{ even},\\ \mathbb{Z}^{\binom{k-1}{n}+\sum_{i\geq 0}\binom{k-1}{n-2i-1}}, & k \geq n,\ k-n \text{ odd}. \end{cases}$$

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Genera and characteristic classes of singular varieties

Shoji Yokura

(joint work with Jean-Paul Brasselet and Jörg Schürmann)

1. Genera

The cardinality $c(F)$ of a finite set F, i.e., the number of elements of F, satisfies that

(1) $X \cong X'$ (set-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2) $c(X) = c(Y) + c(X \setminus Y)$ for a subset $Y \subset X$ (a *scissor formula*),
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

Let us consider a similar thing on the category $\mathcal{TOP}$ of topological spaces, by modifying the condition (1) and (2), with (3) and (4) not changed, by

(1)' $X \cong X'$ ($\mathcal{TOP}$-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2)' $c(X) = c(Y) + c(X \setminus Y)$ for a closed subset $Y \subset X$.

If such a topological cardinality exists, then $c(\mathbb{R}^1) = -1$, hence $c(\mathbb{R}^n) = (-1)^n$. Thus, for a finite CW-complex X, $c(X)$ is equal to the Euler–Poincaré characteristic $\chi(X)$. The existence of such a topological cardinality is *guaranteed by the ordinary homology theory*, more precisely $c(X) = \chi_c(X) := \sum (-1)^i \dim_{\mathbb{R}} H^i_c(X;\mathbb{R}) = \sum (-1)^i \dim_{\mathbb{R}} H^{BM}_i(X;\mathbb{R})$. Here $H^{BM}_*(X)$ is the Borel–Moore homology group of X.

Furthermore let us consider a similar thing on the category $\mathcal{VAR}$ of complex algebraic varieties, say "$\mathcal{VAR}$-cardinality", by modifying (1)' and (2)' by

(1)” $X \cong X'$ ($\mathcal{VAR}$-isomorphism) $\Longrightarrow c(X) = c(X')$,
(2)” $c(X) = c(Y) + c(X \setminus Y)$ for a closed subvariety $Y \subset X$,

If such a cardinality exists, by setting $y := -c(\mathbb{C}^1)$ (we cannot do the same trick as we do for the above $c(\mathbb{R}^1) = -1$) we have that $c(\mathbb{P}^n) = c(\mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \cdots \sqcup \mathbb{C}^n) = 1 - y + y^2 - \cdots + (-y)^n$. The existence of such an algebraic cardinality is *guaranteed by Deligne's theory of mixed Hodge structures.* Let u, v be two variables, then the Deligne–Hodge polynomial $\chi_{u,v}$ is defined by $\chi_{u,v}(X) = \sum(-1)^i(-1)^{p+q} \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q}(H^i_c(X;\mathbb{C}))u^p v^q$. In particular, $\chi_{u,v}(\mathbb{C}^1) = uv$. Hence, if $uv = -y$, then $\chi_{u,v}$ is such an algebraic cardinality. Let us consider $u = y, v = -1$.
Then we have $\chi_y(X) := \chi_{y,-1}(X) = \sum(-1)^i(-1)^q \dim_{\mathbb{C}} Gr^p_F(H^i_c(X;\mathbb{C}))y^p$. This is called χ_y-*genus* of X.

2. χ_y-genus and motivic Hirzebruch class

Now let $\mathrm{Iso}(\mathcal{VAR})$ be the free abelian group generated by the isomorphism classes of varieties. Then the above χ_y can be considered as the homomorphism $\chi_y : \mathrm{Iso}(\mathcal{VAR}) \to \mathbb{Z}[y]$ defined by $\chi_y([X]) := \chi_y(X)$. Because of the condition (2)” we get

$$\chi_y : K_0(\mathcal{VAR}) := \frac{\mathrm{Iso}(\mathcal{VAR})}{\{[X] - [Y] - [X \setminus Y] \mid Y \subset X\}} \to \mathbb{Z}[y] \hookrightarrow \mathbb{Q}[y].$$

Here Y is a closed subset of X and $\{[X]-[Y]-[X\setminus Y] \mid Y \subset X\}$ is the free abelian group generated by the elements of the form $[X] - [Y] - [X \setminus Y]$. $K_0(\mathcal{VAR})$ is called the Grothendieck group (or ring) of complex algebraic varieties. $K_0(\mathcal{VAR})$ can be extended to a covariant (and also contravariant) functor $K_0(\mathcal{VAR}/-)$ by

$$K_0(\mathcal{VAR}/X) := \frac{\{[V \to X]\}}{\{[W \xrightarrow{h} X] - [Z \xrightarrow{h|_Z} X] - [W \setminus Z \xrightarrow{h|_{W\setminus Z}} X] \mid Z \subset W\}}.$$

Here Z is a closed subvariety of W. For a complex vector bundle E, the Hirzebruch class or the generalized Todd class of E is defined by

$$td_y(E) := \prod_{i=1}^{\mathrm{rank}\, E} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right),$$

where α_i is the Chern root of E, i.e., $c(E) = \prod_{i=1}^{\mathrm{rank}\, E}(1+\alpha_i)$. Note the Hirzebruch class unifies the three classes which are important in geometry and topology:

- $y = -1$: $td_{-1}(E) = \prod_{i=1}^{\mathrm{rank}\, E}(1+\alpha_i) = c(E)$, the total Chern class
- $y = 0$: $td_0(E) = \prod_{i=1}^{\mathrm{rank}\, E} \frac{\alpha_i}{1-e^{-\alpha_i}} = td(E)$, the total (original) Todd class,
- $y = 1$: $td_1(E) = \prod_{i=1}^{\mathrm{rank}\, E} \frac{\alpha_i}{\tanh \alpha_i} = L(E)$, the total Thom–Hirzebruch L-class.

Now we can formulate a Grothendieck–Riemann–Roch-type theorem for the χ_y-genus:

Theorem 1 ([3] (cf. [12], [16])). *Let the set-up be as above.*

(1) *There exists a unique natural transformation*

$$T_{y*} : K_0(\mathcal{VAR}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}[y]$$

such that for a smooth variety X $T_{y*}([X \xrightarrow{\mathrm{id}_X} X]) = td_y(TX) \cap [X]$. *Whether* X *is singular or non-singular,* $T_{y*}(X) := T_{y*}([X \xrightarrow{id_X} X])$ *is called the* motivic Hirzebruch class of X.

(2) *When* $X = pt$ *is a point,* $T_{y*} : K_0(\mathcal{VAR}/pt) = K_0(\mathcal{VAR}) \to \mathbb{Q}[y]$ *equals* χ_y.

3. A "unification" theorem

The above Hirzebruch class $T_{y*} : K_0(\mathcal{VAR}/-) \to H_*^{BM}(-) \otimes \mathbb{Q}[y]$ "unifies" the following three well-known characteristic classes of singular varieties:

(1) MacPherson's Chern class transformation [11]:$c_*^{Mac} : F(-) \to H_*(-) \otimes \mathbb{Q}$, which is the unique natural transformation from the constructible function functor $F(X)$ to the Borel–Moore homology such that $c_*(1\!\!1_X) = c(TX) \cap [X]$ for smooth X.

(2) Baum–Fulton–MacPherson's Todd class [1]: $td_*^{BFM} : G_0(X) \to H_*(X) \otimes \mathbb{Q}$, which is the unique natural transformation from the Grothendiekc group functor $G_0(X)$ of coherent sheaves such that $td_*^{BFM}(\mathcal{O}_X) = td(TX) \cap [X]$ for smooth X.

(3) Goresky– MacPherson's homology L-class [9], which is extended as a natural transformation by Cappell-Shaneson [6] (see also [3, 14, 15]): $L_*^{CS} : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$ defined on the cobordism group $\Omega(X)$ of selfdual constructible sheaf complexes, such that $L_*^{CS}([\mathbb{Q}_X[\dim X]]) = L(TX) \cap [X]$ for X smooth and compact.

Now the "unification" means the following:

Theorem 2 ([3] (cf. [12], [16])). *The following diagrams of natural transformations are commutative:*

$$\begin{tikzcd} & K_0(\mathcal{VAR}/X) \arrow[dl, "const"'] \arrow[dr, "T_{-1*}"] & \\ F(X) \arrow[rr, "c_*^{Mac}\otimes\mathbb{Q}"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{VAR}/X) \arrow[dl, "coh"'] \arrow[dr, "T_{0*}"] & \\ G_0(X) \arrow[rr, "td_*^{BFM}"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{VAR}/X) \arrow[dl, "sd"'] \arrow[dr, "T_{1*}"] & \\ \Omega(X) \arrow[rr, "L_*^{CS}"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

Here const : $K_0(\mathcal{VAR}/X) \to F(X)$ *is defined by* $const([V \xrightarrow{h} X]) := h_*1\!\!1_V$. *The other two comparison transformations are characterized by* $coh([V \xrightarrow{h} X]) = h_*([O_V])$ *and* $sd([V \xrightarrow{h} X]) = h_*([\mathbb{Q}_V[\dim V]])$ *for* V *smooth and* h *proper.*

For the details see [3, 12, 16], and for further and related works, e.g., see [4, 5] for genera, [7, 17] for the motivic Hirzebruch–Milnor class, [8] for the equivariant analogue of T_{y*}, [13] for a bivariant-theoretic analogue of T_{y*}, and [2] for fiberwise bordism groups.

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Problem Session

Question by James Davis. By a theorem of Thom, the image of the following map is a lattice of full rank.

$$\begin{aligned} \Pi_n : \Omega_{4n} &\to \mathbb{Z}^{\{\text{partitions of n}\}} \\ [M] &\mapsto (I \mapsto p_I(M)) \end{aligned}$$

It is injective up to torsion.

The question is to compute the image of Π_n through a range of n's and to develop techniques to compute the images of Π_n^{top} and Π_n^{PL}.

Example: Zhixu Su (2009) computed the image of Π_8 and concluded: there exists a 32-dimensional, simply-connected, smooth, closed manifold M with

$$H^*(M;\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 16, 32 \\ 0 & \text{otherwise} \end{cases}$$

It only has p_4 and p_8, so the image under Π_8 is

$$\begin{array}{lll} 4+4 & \mapsto & p_4^2(M) \\ 8 & \mapsto & p_8(M). \end{array}$$

Question by Greg Friedmann. Let F be a field. An F-Witt space is a piecewise linear stratified pseudomanifold such that if x is any point in a stratum of codimension $2k+1$ and L^{2k} is the link at x, then $I^{\bar{m}}H_k(L;F) = 0$, where $\bar{m}$ indicates the lower middle perversity and $I^{\bar{m}}H$ is the corresponding intersection homology group (with coefficients in F). The interest in Witt spaces is that this condition is sufficient to ensure $I^{\bar{m}}H_k(L;F) \cong I^{\bar{n}}H_k(L;F)$, where $\bar{n}$ is the upper middle perversity, which is complementary to $\bar{m}$. It follows that if X is a compact n-dimensional F-oriented F-Witt space, then there is a nonsingular Poincaré duality pairing

$$I^{\bar{m}}H_i(L;F) \otimes I^{\bar{m}}H_{n-i}(L;F) \to F.$$

In particular, when $n = 4j$, there is a nonsingular symmetric middle-dimensional pairing

$$I^{\bar{m}}H_{2j}(L;F) \otimes I^{\bar{m}}H_{2j}(L;F) \to F.$$

These spaces were first studied by Siegel [3] for $F = \mathbb{Q}$. Siegel showed that the middle-dimensional pairing provides a bordism-invariant element of the Witt group $W(\mathbb{Q})$ (hence the name "Witt space") and used this to compute the bordism groups of compact oriented $\mathbb{Q}$-Witt spaces, $\Omega_*^{\mathbb{Q}-\text{Witt}}$. It follows from this computation, and in particular Siegel's construction of generators for these groups, that every $\mathbb{Q}$-Witt space is bordant (via a $\mathbb{Q}$-Witt space) to a $\mathbb{Q}$-Witt space with at worst isolated singularities (in fact at most one isolated singularity).

Question 1: Is it possible to demonstrate this last fact (that every $\mathbb{Q}$-Witt space is $\mathbb{Q}$-Witt bordant to a $\mathbb{Q}$-Witt space with at worst isolated singularities) using a purely topological construction?

The motivation for this question arises from an error in [2], in which the author computes the bordism groups $\Omega_i^{F-\text{Witt}}$ for other fields F. The error occurs in the computation when $F = \mathbb{Z}_2$, the field with two elements, and $i \equiv 2 \mod 4$. For fields not of characteristic 2, one can use a version of the singular surgery of [3] to conclude that all $4j+2$-dimensional F-Witt spaces are boundaries and so

$\Omega_{4j+2}^{F-\mathrm{Witt}} = 0$. This surgery argument is predicated on the assumption that the self-intersection number of middle-dimensional intersection homology elements must be 0. When the characteristic of F is not 2, this follows from the anti-symmetry of the middle-dimensional pairing. When the characteristic is 2, one cannot draw this conclusion; overlooking this point was the source of the error. However, there is still a homomorphism $\Omega_{4j+2}^{F-\mathrm{Witt}} \to W(\mathbb{Z}_2) \cong \mathbb{Z}_2$, and one may still use singular surgery to show it is injective. Thus the unsettled question is whether or not there is a $\mathbb{Z}_2$-Witt space of dimension $4j+2$ whose middle-dimensional pairing represents the non-trivial element of $W(\mathbb{Z}_2)$. If one only asks that $\mathbb{Z}_2$-Witt spaces be $\mathbb{Z}_2$-orientable (which would be reasonable as such spaces would possess the desirable $\mathbb{Z}_2$-Poincaré duality), then the real projective spaces provide such elements. However, one might still ask about bordism groups of $\mathbb{Z}$-oriented $\mathbb{Z}_2$-Witt spaces, in which case the question is still open:

Question 2: Is there a $\mathbb{Z}$-oriented compact $\mathbb{Z}_2$-Witt space of dimension $4j+2$ whose middle-dimensional pairing represents the non-trivial element of $W(\mathbb{Z}_2)$?

An affirmative answer to Question 1 would provide a negative answer to Question 2 as follows: If X is a Witt space with at worst isolated singularities, then one can choose an open neighborhood N of the set of singularities such that $M = X - N$ is a compact oriented manifold-with-boundary (if there are no singularities, M is closed). Then the intersection pairing on $I^{\bar{m}}H_{2j+1}(X;\mathbb{Z}_2)$ can be identified with the intersection pairing on

$$\mathrm{im}(H_{2j+1}(M;\mathbb{Z}_2) \to H_{2j+1}(M,\partial M;\mathbb{Z}_2)).$$

It is known that such a pairing cannot represent the non-trivial element of $W(\mathbb{Z}_2)$ (by an argument using cohomology operations; see [1]). Hence if one could construct Witt bordisms "by hand" to Witt spaces with at worst isolated singularities, then the answer to Question 2 must be "no." It also follows from this argument that any possible example giving an affirmative answer to Question 2 must have singular set of dimension greater that 0.

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Questions by Paolo Piazza. Let X be a Witt space. By definition this is a stratified space X such that for all $x \in X$ for which the link L_x is even dimensional we have $\mathrm{IC}^{\bar{m}}_{\dim(L_x)/2}(L_x) = 0$. On a Witt space we can define the homology L-class $L_*(X) \in H_*(X)$. It turns out that the L-class is also equal to the Chern character of the signature class in analytic K-homology.

Question 1: under which hypothesis can we define, topologically, the homology $\hat{A}$-roof class $\hat{A}_*(X)$? A sufficient condition should be the existence of a metric with positive scalar curvature on the links. How would it depend on the choice

of that metric? Is it equal to the Chern character of a suitable K-homology class defined by the spin Dirac operator?
Question 2: Paolo also requests an update on the status of the relation between L^2-cohomology and intersection cohomology for projective varieties X with Fubini-Study-metric.

Daniel Grieser replies to the second question that there is ongoing work to obtain an analytically satisfactory answer, via two steps: First prove that every projective variety has a resolution which not only resolves the differentiable structure, but also the metric. The latter means that the metric pulled back to the resolution has a certain normal form. This normal form is defined in terms of towers of fibrations of the exceptional divisor and associated weight vectors. To D.G.'s knowledge some people in the resolution of singularities community are working on this conjecture. As a second step analyze the Hodge Laplacian on spaces with such metrics. Also here there is some work but no conclusive answers yet. This program generalizes the approach taken by Hsiang and Pati in the two-dimensional case.

Question by Julius Shaneson. Let M be a simply connected piecewise linear manifold and $M \simeq N$ a homotopy equivalence. Then there exists an embedding $M \hookrightarrow N \times D^2$, even though their Pontryagin might not agree. Notice that a PL-Manifold might not have a normal bundle. Take, for example,

Cone() $\subseteq \mathbb{R}^4$.

Can you stratify the embedding using only strata of even codimension?

Questions by Shmuel Weinberger. Question 1: Can you give a cycle theory for KO where the cycles are "reasonable geometric objects"

The fundamental obstacle is a result of Browder-Liulivicious-Peterson which shows that (in modern terminology) any cycle theory that is closed under crossing with smooth oriented manifolds, must be Eilenerg-MacLane at 2. Spin conditions could help.

I would be happy with something using almost complex structures as well.

Question 2: This question was adressed to Markus and involved the question of whether one can rewrite his Annals paper on "non-Witt spaces" to view the output as being, not a well defined self-dual complex of sheaves, but rather a well defined cobordism class of these. Then the inductive step would be a "transport" from a Lagrangian in one such realization to a Lagrangian of the other.

This would then imply the sequence of obstructions he defines is more canonical: if one set of choices is obstructed, then so would another.

The result about well definedness of signature and characteristic classes would then also have a more natural understanding. Markus seemed to believe that this was all correct, and (maybe I am now being optimistic) would mainly involve reorganizing the ideas in the original paper.

Matthias Kreck gives two answers to the first question. First answer: Use the theorem of Hopkins-Hovey:

$$\Omega_*^{\mathrm{Spin}}(X) \otimes KO_*(pt) \simeq KO_*(X).$$

This is a Conner-Floyd type description, but the proof is much more complicated.

Second answer (Kreck-Stolz): Consider Spin bordism of X and add a new relation, which identifies total spaces of HP^2-bundles with 0. Invert the Bott-manifold. The result is again $KO_*(X)$.

Conjectural answer: Consider Riemannian stratifolds with all strata spin and all links with positive scalar curvature. Conjecture: This gives a homology theory that is equivalent to $ko_n(X)$.

Reporter: Martin Ruderer