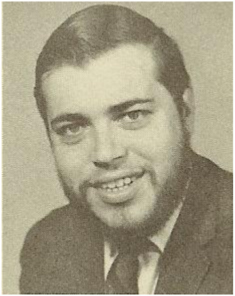


Sylvain Cappell - Opera Selecta

Markus Banagl

Universität Heidelberg

August 2018

Codim 1 Splitting Problems

- Y^{n+1} a connected, closed manifold (Poinc. cplx.), $n \geq 5$.
- $X \subset Y$ a connected, closed codim. 1 submanifold with trivial normal bundle.
- Assume $H = \pi_1(X) \to \pi_1(Y) = G$ injective.
- W^{n+1} a (smooth, PL, or top) manifold with h.e.

$$f : W \longrightarrow Y.$$

- **Def.** f is *splittable along* X if $f \simeq f'$, $f' \pitchfork X$, such that

$$f'| : M = f'^{-1}(X) \longrightarrow X$$

  is a homotopy equivalence.
- **S-Splitting Problem:** When is $f : W \to Y$ is splittable along X?
- **H-Splitting Problem:** When is W h-cobordant to W' such that the induced h.e. $f' : W' \to Y$ is splittable along X?

The Universal Cover

- 2 cases: $Y - X$ has two components, or one component.
- Will only discuss here the case of 2 components:

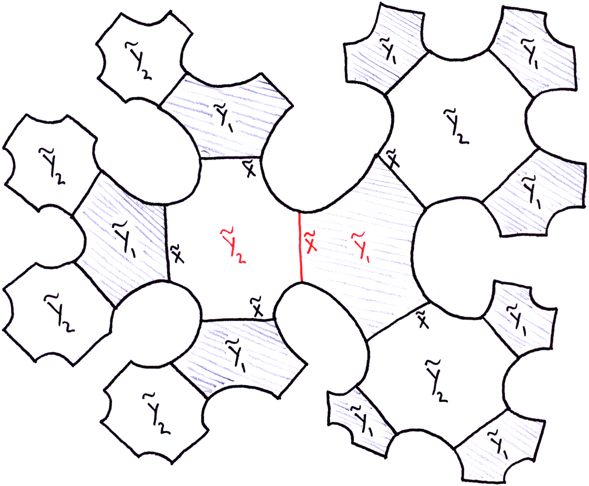
$$Y = Y_1 \cup_X Y_2, \; G = G_1 *_H G_2, \; G_i = \pi_1(Y_i), \; W = W_1 \cup_M W_2.$$

- Description of universal cover:

$$\widetilde{Y} = \bigcup_{\alpha \in [G, G_1]} \widetilde{Y}_1 g(\alpha) \cup_{\cup_{\alpha \in [G,H]} \widetilde{X} g(\alpha)} \bigcup_{\alpha \in [G, G_2]} \widetilde{Y}_2 g(\alpha),$$

$$\partial \widetilde{Y}_i = \bigcup_{\alpha \in [G_i, H]} \widetilde{X} g(\alpha),$$

cosets α, representatives $g(\alpha) \in \alpha$.


$\tilde{Y}_2$
$\tilde{Y}_1$
$\tilde{X}$

- Preferred (i.e. basepoint preserving) lifts

$$\widetilde{X} \subset \widetilde{Y}_i \subset \widetilde{Y}.$$

- $\widetilde{Y} - \widetilde{X}$ has 2 components with closures

$$Y_R \supset \widetilde{Y}_1, \ Y_L \supset \widetilde{Y}_2.$$

- $\widetilde{Y} = Y_L \cup_{\widetilde{X}} Y_R$.
- Cover $p : \widehat{Y} \to Y$, $p_* \pi_1(\widehat{Y}) = \mathrm{Im}(H \to G)$.
- Preferred (i.e. basepoint preserving) lifts

$$X \subset \widehat{Y}_i \subset \widehat{Y}.$$

- Quotients

$$Y_l = Y_L/H, \ Y_r = Y_R/H.$$

- Low degrees $0, 1$: Make M, W_1, W_2 connected and

$$\pi_1(M) \to \pi_1(X),\ \pi_1(W_i) \to \pi_1(Y_i)$$

  isomorphisms.
- Then above description of universal cover applies to $\widetilde{W}$. Lift f to $\widetilde{W} \to \widetilde{Y}$, $\widehat{W} \to \widehat{Y}$, get preimages W_L, W_R, W_l, W_r.
- f h.e. $\Rightarrow f| : M \to X$ deg 1.
- $$K_j(M) := \ker(H^t_j(M; \mathbb{Z}H) \twoheadrightarrow H^t_j(X; \mathbb{Z}H)).$$
- Suppose inductively $K_i(M) = 0$ for $i < j$ and j is below the middle, $j < (n-1)/2$.

- f h.e. $\Rightarrow K_*(\widehat{W}) = K_*(W) = 0$.
- So Mayer-Vietoris $\Rightarrow$ incl. induces iso. of $\mathbb{Z}\pi_1 M$-modules

$$K_j(M) \xrightarrow{\cong} K_j(W_l) \oplus K_j(W_r).$$

- **Key device**:

$$P := \ker(K_j(M) \to K_j(W_r)),\ \ Q := \ker(K_j(M) \to K_j(W_l)).$$

- Then

$$K_j(M) = P \oplus Q,\ \ Q \otimes_{\mathbb{Z}H} \mathbb{Z}G_1 \xrightarrow{\cong} K_j(W_1).$$

- How do nilpotence phenomena appear?

Paradigm:
If a class in $K_j(M)$ vanishes on the right of $\widetilde{M}$ in s steps (i.e. crossing boundary components of translated $\widetilde{W_i}$ $s-1$ times), then, by a homology in $\widetilde{W_1}$, the intersection of a cobounding disc ("tentacle") with $\partial\widetilde{W_1} - \widetilde{M}$ vanishes (after translation) *on the left* of $\widetilde{M}$ in $s-1$ steps.

In more detail: Let $\alpha \in P$.

- Then $\alpha = \partial D$ for a tentacle (disc) $D \subset W_R$.
- A geometric description of

$$\rho_1 : P \hookrightarrow K_j(M) \to K_j(W_1) \cong Q \otimes_{\mathbb{Z}H} \mathbb{Z}G_1$$

is given by

$$\rho_1(\alpha) := D \cap (\partial\widetilde{W_1} - \widetilde{M}).$$

- Similarly,

$$\rho_2 : Q \hookrightarrow K_j(M) \to K_j(W_2) \cong P \otimes_{\mathbb{Z}H} \mathbb{Z}G_2,$$

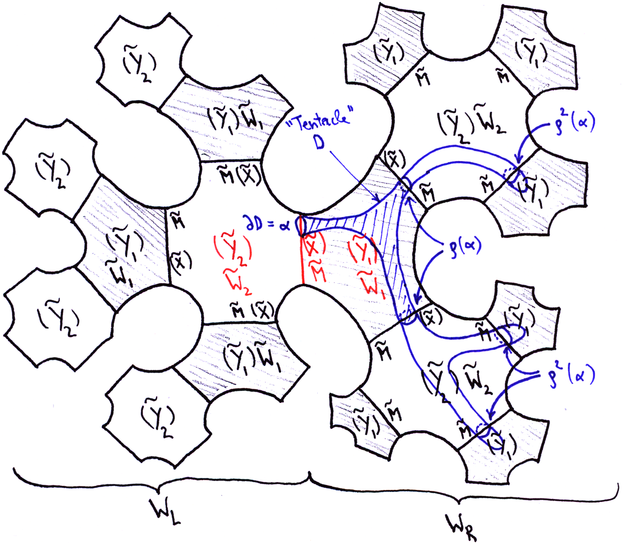
  and by extension

$$\rho : (P \oplus Q) \otimes_{\mathbb{Z}H} \mathbb{Z}G \xrightarrow{\rho_1 + \rho_2} (Q \oplus P) \otimes_{\mathbb{Z}H} \mathbb{Z}G.$$

- Disc D is compact $\Rightarrow$ D intersects only finitely many copies of $\widetilde{W}_1, \widetilde{W}_2$ in $\widetilde{W}$. So

$$\rho^s(\alpha) = 0$$

  for sufficiently large s.
- There is a finite maximal s that works for all α. Thus ρ is nilpotent.



"Tentacle"
D
$\partial D = \alpha$
$\rho(\alpha)$
$\rho^2(\alpha)$
$(\tilde{Y}_1)\tilde{W}_1$
$(\tilde{Y}_2)\tilde{W}_2$
$(\tilde{Y}_1)$
$(\tilde{Y}_2)$
$\tilde{M}(\tilde{X})$
$\tilde{M}$
$(\tilde{X})$
W_L
W_R

- Applying iterates of ρ to $P, Q \rightsquigarrow$ finite filtration by f.g. $\mathbb{Z}H$ modules
$$P = P_0 \supset P_1 \supset \cdots \supset P_r = 0,$$
$$Q = Q_0 \supset Q_1 \supset \cdots \supset Q_r = 0,$$
with
$$\rho_1(P_i) \subset Q_{i+1} \otimes_{\mathbb{Z}H} \widetilde{\mathbb{Z}G_1},\ \rho_2(Q_i) \subset P_{i+1} \otimes_{\mathbb{Z}H} \widetilde{\mathbb{Z}G_2},$$
where $\widetilde{\mathbb{Z}G_i} \subset \mathbb{Z}G_i$ is the $\mathbb{Z}H$ submodule generated additively by $G_i - H$; $\mathbb{Z}G_i \cong \mathbb{Z}H \oplus \widetilde{\mathbb{Z}G_i}$.
- Take $s =$ largest index such that $P_s \oplus Q_s \neq 0$, say $P_s \neq 0$.
- Represent lifts to $\ker(\pi_{j+1}(W_1, M) \to \pi_{j+1}(Y_1, X))$ of generators z_i of P_s by embeddings $(D^{j+1}, S^j) \to (W_1, M)$.

- As j is below middle, can perform handle exchanges on these: $f \simeq f'$, $f'^{-1}(Y_2) = W_2 \cup \mathrm{Im}(\mathrm{emb})$, $f'^{-1}(M) = M'$ obtained from M by surgery on $z_i : S^j \times D^{n-j} \to M$.
- Then

$$K_j(M') \cong K_j(M)/\{z_i\} = P/\{z_i\} \oplus Q = P' \oplus Q',$$

- The new filtration

$$P' = P'_0 \supset P'_1 \supset \cdots \supset P'_t = 0,$$

$$Q' = Q'_0 \supset Q'_1 \supset \cdots \supset Q'_t = 0$$

has $P'_s = P_s/\{z_i\} = 0$, so inductively get

$$K_j(M') = 0.$$

- Assume $n = 2k$ even.
- Need to analyze the middle dimension k. By previous argument, can now assume $K_i(M) = 0$ for $i < k$.
- Argument of Wall $\Rightarrow$ $K_k(M)$ stably free. So

$$[P] = -[Q] \in \widetilde{K}_0(H) \text{ (reduced projective class group of } \mathbb{Z}H).$$

- As $K_k(W_i)$ is stably free,

$$[P] = [K_k(W_2)] = 0 \text{ in } \widetilde{K}_0(G_2).$$

(Sim. $[Q] = 0 \in \widetilde{K}_0(G_1)$.) So

$$[P] \in \ker(\widetilde{K}_0(H) \longrightarrow \widetilde{K}_0(G_1) \oplus \widetilde{K}_0(G_2)).$$

- $\lambda :=$ nonsingular intersection form on $K_k(M)$.
- Standard piping argument (Wall, Zeeman) $\Rightarrow$ finite sets of elements of P, Q can be represented by *disjoint* (embedded, framed) spheres. So

$$\lambda|_P = 0, \ \lambda|_Q = 0, \ \text{adj}(\lambda) : P \cong Q^*.$$

Whitehead group:

$$
\begin{array}{c}
\mathrm{Wh}(G_1 *_H G_2) \\
\downarrow \\
\mathrm{Wh}(G_1 *_H G_2)/\operatorname{Im}(\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2)) \\
\Vert\wr \qquad \text{Waldhausen} \\
\ker(\widetilde{K}_0(H) \longrightarrow \widetilde{K}_0(G_1) \oplus \widetilde{K}_0(G_2)) \oplus (\text{nilpotent maps}) \\
\downarrow \text{proj} \\
\ker(\widetilde{K}_0(H) \longrightarrow \widetilde{K}_0(G_1) \oplus \widetilde{K}_0(G_2)).
\end{array}
$$

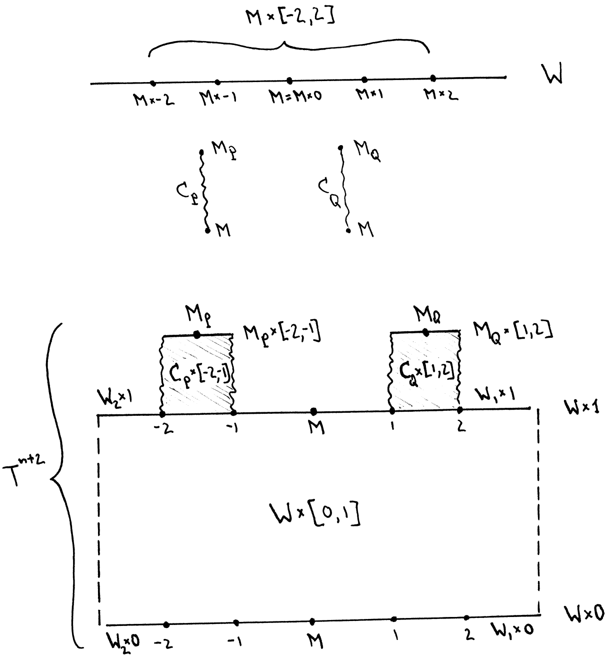
$$\Phi : \mathrm{Wh}(G_1 *_H G_2) \longrightarrow \ker(\widetilde{K}_0(H) \longrightarrow \widetilde{K}_0(G_1) \oplus \widetilde{K}_0(G_2))$$

h.e. f has Whitehead torsion

$$\tau(f) \in \mathrm{Wh}(G).$$

Fact: $\Phi(\tau(f)) = [P]$.

- Now suppose that
$$\Phi(\tau(f)) = 0.$$
Thus $[P] = 0$.
- By trivial ambient surgeries (to stabilize P), may assume P (and $Q \cong P^*$) free $\mathbb{Z}H$ modules, basis $\{a_i\}$ for P, dual basis $\{b_i\}$ for Q.
- Represent by disjoint (embedded, framed) spheres in M,
$$a_i \cap b_j = \varnothing (i \neq j),\ a_i \cap b_i = \mathrm{pt}\,.$$
- Surgery on $\{a_i\} \rightsquigarrow$ normal cobordism C_P from M to $M_P \simeq X$.
- Similarly, surgery on $\{b_i\} \rightsquigarrow$ normal cobordism C_Q from M to $M_Q \simeq X$.


M × [-2,2]
W
M×-2
M×-1
M=M×0
M×1
M×2
M_P
C_P
M
M_Q
C_Q
M
M_P
M_P×[-2,-1]
C_P×[-2,-1]
M_Q
M_Q×[1,2]
C_Q×[1,2]
W_2×1
W_1×1
W×1
-2
-1
M
1
2
T^{n+2}
W×[0,1]
W×0
W_2×0
-2
-1
M
1
2
W_1×0

- T is a cobordism from $W = W \times 0$ to W_{mod},

$$W_{\text{mod}} : (W_2 \cup_M C_P) \cup_{M_P} (C_P \cup_M C_Q) \cup_{M_Q} (C_Q \cup_M W_1).$$

- $f : W \to Y$ extends to normal map $F : T \to Y \times I$.
- Restriction $f_{\text{mod}} = F| : W_{\text{mod}} \to Y$.
- From construction,

$$M_P \longrightarrow X, \; M_Q \longrightarrow X, \; C_P \cup_M C_Q \longrightarrow X$$

  are all homotopy equivalences.
- Mayer-Vietoris $\Rightarrow$ f_{mod} h.e.
- From construction, $f_{\text{mod}} \simeq f'_{\text{mod}}$ with

$$f'^{-1}_{\text{mod}}(X) = M_P, \; M_P \xrightarrow{\simeq} X.$$

  So f_{mod} is **splittable along** X.
- Need to modify T further to get an h-cobordism.

$$K_i(T) = \begin{cases} (P \oplus Q) \otimes_{\mathbb{Z}H} \mathbb{Z}G, & i = k+1 \\ 0, & i \neq k+1. \end{cases}$$

- Intersection form λ_T on $K_{k+1}(T)$ can be described in terms of P, Q and ρ.
- **Def.** $H < G$ is called *square-root closed* if for all $g \in G$, $g^2 \in H$ implies $g \in H$.
- H $\sqrt{\ }$ closed in $G_1 *_H G_2$ iff H $\sqrt{\ }$ closed in both G_1 and G_2.
- Normal cobordism T has surgery obstruction

$$x = [((P \oplus Q) \otimes_{\mathbb{Z}H} \mathbb{Z}G, \lambda_T, \mu)] \in L^h_{n+2}(G),$$

λ_T given by above description in terms of ρ.

- If H is $\sqrt{\ }$ closed in G, then a purely algebraic argument shows that

$$x \in \mathrm{Im}(L^h_{n+2}(H) \longrightarrow L^h_{n+2}(G)).$$

- So there exists a normal cobordism T_1 on $W_{\mathrm{mod},1} \xrightarrow{\simeq} Y_1$ (rel $\partial W_{\mathrm{mod},1}$) with surgery obstruction

$$x_1 \in L^h_{n+2}(G_1),\ x_1 \mapsto -x.$$

- $T' := T \cup_{W_{\mathrm{mod},1}} T_1$.
- Then T' has vanishing surgery obstruction, and is a still a cobordism from W to a *split* h.e. manifold.
- Do surgery on T' to get an h-cobordism.

Splitting Theorem. (Cappell.)
If $\pi_1(X)$ is square-root closed in $\pi_1(Y^{2k+1})$ and $f : W \xrightarrow{\simeq} Y$ has $\Phi(\tau(f)) = 0$, then W is h-cobordant to a homotopy equivalence which is splittable along X.

- n odd, and the s-splitting problem, can also be treated.
- Led to computations of Wall L-groups, instances of the Novikov conjecture on higher signatures.
- The above P/Q-decomposition/filtration can be further systematized by the UNil obstruction groups.
- Cappell showed $(\mathbb{Z}/_2)^\infty \subset \mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}/_2 - e], \mathbb{Z}[\mathbb{Z}/_2 - e])$. So $\exists$ closed smooth $M^{4k+1} \simeq \mathbb{R}P^{4k+1} \# \mathbb{R}P^{4k+1}$ which is not a nontrivial connected sum.
- Special cases: theorems of Browder (simply conn.), Wall ($H = G_1$), R. Lee ($H = 0$, G has no 2-tor), Farrell (fibration problem),...

Homology Surgery (Cappell-Shaneson.)

- $M^n \subset W^{n+k}$ be a PL embedding of manifolds.
- For $k \geq 3$, Zeeman unknotting $\Rightarrow M \subset W$ locally flat, and $\pi_1(W - M) \to \pi_1(W)$ is an iso.
- For $k = 2$, not nec. loc. flat and $\pi_1(W - M) \to \pi_1(W)$ is surjective, but rarely an iso.
- Given a group π and epimorphism $\mathbb{Z}\pi \to \Lambda$, study manifolds V (such as $V = W - M$) with $\pi_1 V = \pi$ and given homology type over Λ (e.g. $\Lambda = \mathbb{Z}\pi_1 W$).
- CS define reduced Grothendieck groups $\Gamma_n(\mathbb{Z}\pi \to \Lambda)$.
- $\Gamma_n(\mathrm{id}_\Lambda) = L_n(\Lambda)$ (Wall group).
- $\Gamma_{\mathrm{odd}}(\mathbb{Z}\pi \to \Lambda) \subset L_{\mathrm{odd}}(\Lambda)$.
- $\Gamma_{\mathrm{even}}(\mathbb{Z}\pi \to \Lambda)$ usually much larger than $L_{\mathrm{even}}(\Lambda)$.

Homology Surgery: Obstruction

- A deg. 1 normal map (f, b)

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f} & X^n \end{array}$$

  with $\pi_1 X = \pi$ has an obstruction

$$\sigma(f, b) \in \Gamma_n(\mathbb{Z}\pi \to \Lambda).$$

- **Thm.** (Cappell, Shaneson) $\sigma(f, b) = 0 \Leftrightarrow (f, b)$ is normally cobordant to a (simple) homology equivalence over Λ.

Codimension 2 PL embeddings

- M^n, W^{n+2} oriented closed PL manifolds, $n \geq 3$.
- **Def.** A *Poincaré embedding* (PE) of M in W is a triple (ξ, C, h), where
  - ξ is a 2-plane bundle over M,
  - C is a CW complex such that $S(\xi) \subset C$,
  - $$\begin{array}{ccc} W & \xrightarrow[h]{\text{s.h.e.}} & D(\xi) \cup_{S(\xi)} C \\ & \underset{\text{deg } 1}{\searrow} & \downarrow \\ & & D(\xi)/S(\xi) = T(\xi) \end{array}$$
- Think of C as candidate homotopy type for a complement.
- Rem.: In general, only a spherical fiber space is given. But in codim 2, G_2/O_2 is contractible, so can specify a vector bundle.
- **Question**: Can the PE be realized by a PL embedding $M \hookrightarrow W$? Idea: Do not attempt $W - M \simeq C$, more natural: homology equivalence.

Normal Invariant of Poincaré embedding.

The PE provides a *normal invariant* $\eta \in [M, G/PL]$ as follows:

- Make $h \pitchfork$ to 0-section of ξ in $D\xi \cup C$.
- $N := h^{-1}(\text{0-section}) \subset W$,
  $\nu :=$ normal bundle of N in W.
- Get deg 1 normal map

$$\begin{array}{ccc} D\nu & \xrightarrow{h|_{D\nu}} & D\xi \\ \downarrow & & \downarrow \\ N & \xrightarrow{h|} & M, \end{array}$$

has normal invariant as usual (Sullivan).

Thickenings

- Let A^n be a closed PL manifold.
- **Def.** A *codim 2 thickening* of A is a PL embedding $f : A \hookrightarrow R$, R a compact $(n+2)$-dimensional PL manifold, $f^{-1}(\partial R) = \varnothing$, R is a regular neighborhood of $f(A)$ in itself.
- A *concordance of codim 2 thickenings* is a codim. 2 thickening of $A \times [0,1]$.
- $\mathcal{H}(A) :=$ concordance classes of codim. 2 thickenings.
- Stratified Transversality (D. Stone): cont. $f : A \to B \rightsquigarrow f^* : \mathcal{H}(B) \to \mathcal{H}(A)$.
- $\mathcal{H}$ is a homotopy functor.
- Brown $\Rightarrow \mathcal{H}$ represented by a space BRN_2 (oriented: $BSRN_2$).
- $\pi_*(BSRN_2) = \{\text{cod. 2 thickenings of spheres}\}/_{\text{conc}}$ $= \{\text{1 isol. sing.}\}/_{\text{conc}} =$ knot cob.
- Application of homology surgery: Homotopy type of $BSRN_2$.

$BSRN_2$

- A thickening $M \subset R$ has an Euler class χ. Get map

$$BSRN_2 \to BSO_2.$$

- Furthermore, thickenings $M \subset R$ have normal maps η. Get $\mathcal{H}_{\text{or}}(M) \to [M, G/PL]$ and so

$$BSRN_2 \to G/PL.$$

- So have

$$(\chi, \eta) : BSRN_2 \to BSO_2 \times G/PL.$$

- **Thm.** (Cappell, Shaneson) (χ, η) has a section

$$\varphi : BSO_2 \times G/PL \to BSRN_2.$$

(So get sufficiently many regular neighborhoods.)

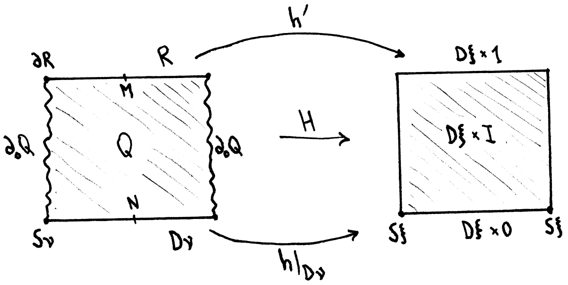
Construction of thickening for M

- From PE, have $\xi \in [M, BSO_2]$.
- Using φ, can construct $\alpha \in [M, BSRN_2]$ such that $\chi(\alpha) = \xi$ and $\eta(\alpha) = \eta$.
- Let $M \subset R$ be a representative of α.
- Note: Usually, cannot choose $M \subset R$ to have a uniform block bundle structure; will generally have high-dimensional singular sets (non-locally flat points). $M \subset R$ must accommodate fairly general PL L-classes.
- Main issue to be solved: How to put R into W?

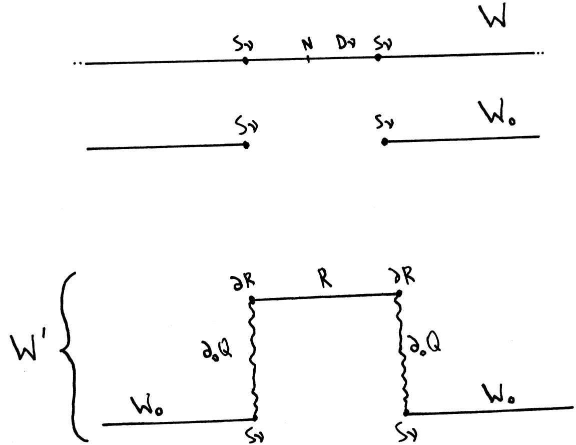
- Obstruction theory $\rightsquigarrow$ (simple) $\mathbb{Z}\pi_1 M$-homology equivalence $h' : (R, \partial R) \to (D\xi, S\xi)$.
- As $\eta(\alpha) = \eta$, there is a normal bordism

$$H : Q \to D\xi \times I$$

between $h' : R \to D\xi$ and $h| : D\nu \to D\xi$:



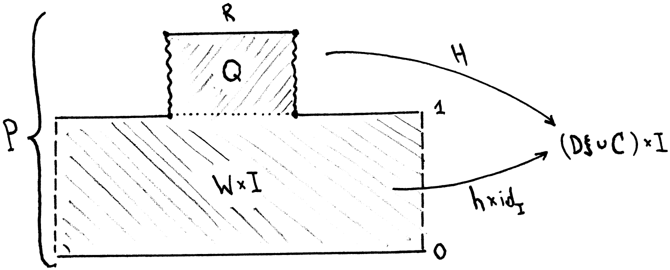
- $W_0 := \mathrm{cl}(W - D\nu)$.
- $W' := R \cup_{\partial R} \partial_0 Q \cup_{S\nu} W_0$.



- Glue Q to $W \times I$:

$$P := W \times I \cup_{D\nu \times 1} Q.$$

- Then $\gamma := (h \times \mathrm{id}) \cup H : P \to (D\xi \cup C) \times I$ is a normal cobordism from h to a map $W' \to D\xi \cup C$.



Construction of complement for n odd

- Assume n odd.
- Then $\Gamma_{n+2}(\mathbb{Z}\pi_1 C \to \mathbb{Z}\pi_1 W) \to L_{n+2}(\pi_1 W)$ is injective.
- Implies for the homology surgery obstruction $\sigma(\gamma) = 0$.
- So by the above homology surgery obstruction theorem, γ is normally cobordant (rel $W \times 0 \cup R$) to a simple $\mathbb{Z}\pi_1 W$-homology equivalence

  $$f : (B, W\times 0, R, V) \to ((D\xi\cup C)\times I, (D\xi\cup C)\times 0, D\xi\times 1, C\times 1).$$

- The manifold V is the sought complement!
- Seifert-van Kampen $\Rightarrow$ f induces $\pi_1(V \cup_{\partial R} R) \cong \pi_1(D\xi \cup C)$.

Codim. 2 PL Embedding Theorem

- Then B is an s-cobordism.
- So by s-cob. thm. (using $n \geq 3$), get PL homeo.

$$\psi : (B, W \times 0, V \cup R) \cong (W \times I, W \times 0, W \times 1).$$

- The sought embedding is

$$M \subset R \xrightarrow{\psi|_R} W.$$

- Rem.: $f| : V \to C$ is in general *not* a homotopy equivalence.
- **Thm.** (Cappell, Shaneson) Let M^n, W^{n+2} be oriented closed PL manifolds, $n \geq 3$, n odd. Then any oriented Poincaré embedding of M in W can be realized by a PL embedding $M \hookrightarrow W$.
- Rem.: Also works for n even if $\pi_1 W = 0$. If n even and $\pi_1 W \neq 0$, CS construct "spineless" manifolds.

Cappell-Weinberger on Singular Spaces

- M a closed, smooth, simply connected manifold of even dimension $n \geq 5$.
- Browder-Novikov-Sullivan $\Rightarrow$

$$\begin{array}{ccc} S(M) \otimes \mathbb{Q} & \overset{L}{\hookrightarrow} & \bigoplus H^{4j}(M;\mathbb{Q}), \\ [h : N \simeq M] & \mapsto & (h^*)^{-1} L^*(TN) - L^*(TM), \end{array}$$

  is injective.
- In other words: M is determined, up to finite ambiguity, by *its homotopy type and its L-classes*.
- Extension to singular spaces?

Cappell-Weinberger

- Weinberger: TOP surgery fibration sequence for stratified spaces, topologically invariant characteristic classes. But C-W preceded the general theory.
- **Thm.** (Cappell-Weinberger) X an even dim. stratified pseudomanifold that has no strata of odd dimension. All strata S have dim ≥ 5, all strata and all links simply connected. Then:

$$S(X) \otimes \mathbb{Q} \hookrightarrow \bigoplus_{S \subset X} \bigoplus_{j} H_j(\overline{S}; \mathbb{Q}),$$

  where S ranges over the strata of X.
- Use intersection homology $IH_*^{50\%}(-)$ to define L-classes.

Some other directions:

- **Cappell-Weinberger-Yan**: classif. of TOP $U(n)$-actions on manifolds, all isotropy groups unitary subgroups ("multiaxial"); existence of closed aspherical manifolds with Center$(\pi_1) = \mathbb{Z}$, but not admitting nontrivial TOP S^1-actions; replacement problems for fixed sets;...
- **Cappell-Miller**: Extending flat vector bundles from part of ∂M to M^3 (compact 3-mfd); extension of analytic torsion to general flat bundles + extension of Cheeger-Müller thm. on top. invariance (Reidemeister-Franz).
- **Cappell-Lee-Miller**: Perturbative $SU(3)$-Casson invariant of integral homology 3-spheres.
- In algebraic geometry: M. Saito's theory of mixed Hodge modules $\rightsquigarrow$ intersection Hirzebruch characteristic classes IT_{y*}: **Cappell, Libgober, Maxim, Schürmann, Shaneson**.

Thank you.