RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 16.01.2015

EXERCISE SHEET 10

A Third Model of the Hyperbolic Plane

To hand in by Friday, January 23, 2015, 12:00

On this sheet we will use the notations and resultes from the last exercise sheet (sheet 9).

Exercise 1. (Upper Half-Plane Model) (20 points)
Consider the Cayley transform

$$\begin{aligned} c : \mathbb{C} &\longrightarrow \mathbb{C}; \\ z &\longmapsto \frac{z-i}{z+i} \end{aligned}$$

which is a Möbius map and can be extended to a map $c : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$.

(a) Show that the inverse of the Cayley transform is given by $c^{-1} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}; \quad z \longmapsto i\frac{1+z}{1-z}$.

(b) Show that c^{-1} map $\mathbb{D}^2$ one-to-one onto $H^2 := c^{-1}(\mathbb{D}^2) = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$. This gives a third model for the hyperbolic space, called the upper half-plane-model.

(c) Show that the goedesics in this model are the straight vertical lines or intersections of circles in $\mathbb{R}^2$ orthogonal to the boundary $\{x_1 = 0\}$ (as a subspace of $\mathbb{R}^2$).
Hint: You can use that a Möbius transformation is a conformal map.

(d) $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm Id$ acts on H^2 by Möbius transformations. Show that every element of $PSL(2, \mathbb{R})$ induces an isometry on H^2.

Exercise 2. (20 points)
The distance between two points x, y in $\mathbb{D}^2$ is given by

$$d(x,y) = \operatorname{arcosh}\left(1 + 2\frac{\|x-y\|^2}{(1-\|x\|^2)(1-\|y\|^2)}\right)$$

whereas for $u, v \in H^2$ it is given by

$$d(u,v) = \operatorname{arcosh}\left(1 + \frac{\|u-v\|^2}{2u_2v_2}\right).$$

The boundary of $\mathbb{D}^2$ (as a set) is the set $\{x \in \mathbb{R}^2 \mid \|x\| = 1\}$, and the one of H^2 is the set $\{x \in \mathbb{R}^2 \mid x_2 = 0\}$. By Exercise 1 you already know that the Cayley transform sends the boundary of H^2 homeomorphically to the boundary of $\mathbb{D}^2$, so these two definitions coincide. Choose one of the two models and show that two geodesics parametrized by arclength converge to the same boundary point if and only if their distance is bounded, that is, for two geodesics γ_1, γ_2 parametrized by arclength there is an $M \geq 0$ such that $d(\gamma_1(t), \gamma_2(t)) < M$ for all $t \geq 0$.
Remark: The aim of this exercise is to see that the boundary of $\mathbb{H}^2$ depends only on the hyperbolic metric and can be defined by equivalence classes of geodesics without referring to a particular model.

Exercise 3. (20 points)
In this exercise we want to classify the orientation preserving isometries of $\mathbb{H}^2$. In the upper half-plane model H^2 they are actually all given by the elements of $PSL(2,\mathbb{R})$, see also Exercise 1 (c).

(a) Show that, for $A \in PSL(2,\mathbb{R})$, $\operatorname{tr}(A) > 2$ if and only if the action of A has two fixed points on the boundary of $\mathbb{H}^2$. Draw a rough picture of how this action looks like. Such isometries are called hyperbolic.

(b) Show that, for $A \in PSL(2,\mathbb{R})$, $\operatorname{tr}(A) = 2$ if and only if the action of A has one fixed point on the boundary. Draw a rough picture of how this action looks like. Such isometries are called parabolic.

(c) Show that, for $A \in PSL(2,\mathbb{R})$, $\operatorname{tr}(A) < 2$ if and only if the action of A has one fixed point in the interior. Draw a rough picture of how this action looks like. Such isometries are called elliptic.