RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 19.12.2014

Exercise sheet 9

Möbius Transformations and the Poincaré Disc Model

To hand in by Friday, January 16, 2015, 12:00

Exercise 1. (15 points and 5 extra christmas bonus points)
Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. $GL(2,\mathbb{C})$ acts on $\hat{\mathbb{C}}$ in the following way:
For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$ and $z \in \hat{\mathbb{C}}$ we have

$$A.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

with the convention that if $cz + d = 0$, then $A.z = \infty$, and that $A.\infty = \frac{a}{c}$ if $c \neq 0$ and ∞ otherwise. Consider the map $\pi : \mathbb{C}^2 \setminus \{0\} \longrightarrow \hat{\mathbb{C}}$ that sends $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\frac{x}{y}$ if $y \neq 0$, and to ∞ otherwise.

(a) Prove that for every $A \in GL(2,\mathbb{C})$ and $v \in \mathbb{C}^2 \setminus \{0\}$, $\pi(A \cdot v) = A.\pi(v)$.

(b) Find the set of all elements of $GL(2,\mathbb{C})$ that fix every point of $\hat{\mathbb{C}}$, and prove that is is a normal subgroup. The quotient by this subgroup is denoted $PGL(2,\mathbb{C})$ and called the group of Möbius transformations.

(c) Prove that this action is transitive, that is for every two elements $z_1, z_2 \in \hat{\mathbb{C}}$ there is an $A \in GL(2,\mathbb{C})$ such that $A.z_1 = z_2$.

(d) Prove that v is an eigenvector for the matrix A if and only if $\pi(v)$ is in the fixed set of A.

(e) Give a classification of the elements of $GL(2,\mathbb{C})$ up to conjugation.

(f) Describe the fixed set of every element of $GL(2,\mathbb{C})$.

Exercise 2. (15 points and 5 extra christmas bonus points)
A generalised circle in $\hat{\mathbb{C}}$ is either a circle in $\mathbb{C}$ or a straight line union with $\{\infty\}$. Every generalised circle disconnects $\hat{\mathbb{C}}$ in two parts, these parts are called discs.

(a) Prove the following identity: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{bc-ad}{c^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$.

(b) Prove that the image of a generalised circle for a matrix A is again a generalised circle.

(c) Prove that for every two discs D_1, D_2, there exists a matrix A such that $A(D_1) = D_2$.

Exercise 3. (15 points and 5 extra christmas bonus points)
The aim of this exercise is to prove a generalisation of the Schwarz Lemma:
Let $D = \{z \mid |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Let $f : D \longrightarrow D$ be a holomorphic map such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$.

The generalisation of this is the following theorem (Schwarz-Pick theorem):
Let $f : D \longrightarrow D$ be holomorphic. Then for all $z_1, z_2 \in D$ it holds

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|.$$

Prove the second statement.
Hint: Choose two suitable Möbius transformations mapping D to itself such that a composition of them with f fixes 0 and also maps D to itself. Use then the Schwarz Lemma.

Exercise 4 (Poincaré Disc Model)**.** (15 points and 5 extra christmas bonus points)
On $\mathbb{R}^3$ we consider the scalar product $\langle \cdot, \cdot \rangle$ where $\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{2} x_i y_i$. Define $I_2 := \{x \in \mathbb{R}^3 \mid \langle x, x \rangle = -1,\ x_0 > 0\}$ to be the upper fold of the hyperboloid. We call the pair $(I_2, \langle \cdot, \cdot \rangle|_{T_x I_2})$ the hyperboloid model of the hyperbolic space $\mathbb{H}^2$. From the lecture you already know a lot about this model, for example the metric, the geodesics and the isometries. On this exercise sheet we want to introduce another model for the hyperbolic plane, namely the Poincaré disc model. Consider the map $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $(x_0, x_1, x_2) \mapsto \frac{1}{1+x_0}(x_1, x_2)$. It describes the stereographic projection onto the plane $x_0 = 0$ with respect to the point $p = (-1, 0, 0)$. Then the image of the hyperboloid is $\pi(I_2) =: \mathbb{D}^2$, the open disc of radius 1 in $\mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ as a subset of $\hat{\mathbb{C}}$.

(a) Show that $\pi|_{I_2}$ is a diffeomorphism.

(b) Show that the metric on $\mathbb{D}^2$, which is given as the pull-back by $\pi^{-1}|_{I_2}$, is

$$ds^2 = \frac{4}{(1 - x_1^2 - x_2^2)^2}(dx_1^2 + dx_2^2).$$

(c) Show that the distance of two points $x, y \in \mathbb{D}^2$ is given by

$$d(x, y) = \operatorname{arcosh}\left(1 + 2\frac{\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}\right).$$

(d) Show that the group of isometries is $PU(1, 1)$. Here $U(1, 1) = \{A \in GL(2, \mathbb{C}) \mid A^* J A = J\}$ where $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and A^* is the conjugate transposed of A and $PU(1, 1) = U(1, 1)/\{\lambda \operatorname{Id}\}$ with $\lambda \in \mathbb{C}$ and $\lambda\overline{\lambda} = 1$.

(e) Show that the goedesics in this model are the diameters or intersections of $\mathbb{D}^2$ with circles in $\mathbb{R}^2$ orthogonal to the boundary of $\mathbb{D}^2$ (as a subspace of $\mathbb{R}^2$).