RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 12.12.2014

Exercise sheet 8

Miscellaneous about Curvature

To hand in by Friday, December 19, 2014, 12:00

Exercise 1. (20 points)
Let (M, g) be a complete connected Riemannian manifold, $p \in M$ a point and $D > 0$ a real number. Assume that for all $q \in M$ and for all tangent vectors $v, w \in T_qM$ we have for the sectional curvature: $K(v, w) > 0$. Assume additionally that if $d(p, q) \geq D$ then

$$K(v, w) \geq \frac{\pi^2}{d(p, q)^2}$$

for all $v, w \in T_qM$. Show that M is compact.

Exercise 2. (20 points)
Let $\mathbb{C}^{1,n}$ be the vector space $\mathbb{C}^{n+1}$ endowed with the following Hermitian form h of signature $(1, n)$:

$$h(x, y) = -x_0\overline{y_0} + \sum_{j=1}^{n} x_j\overline{y_j}.$$

Denote by $U(1, n)$ the group of matrices preserving the form h and by $\pi : \mathbb{C}^{1,n} \setminus \{0\} \longrightarrow \mathbb{CP}^n$ the usual projection into the projective space. Then the complex hyperbolic n-space $\mathbb{CH}^n$ is defined as:

$$\mathbb{CH}^n = \{x \in \mathbb{CP}^n \mid x = \pi(v) \text{ with } h(v, v) < 0\}.$$

(a) Show that if $h(v, v) < 0$, then $d\pi_v$ identifies the tangent space at $x = \pi(v)$ with the orthogonal space at v with respect ot h.

(b) Define a Riemannian metric on $\mathbb{CH}^n$ by using the real part of the restriction of h to these orthogonal subspaces.

(c) Show that every element of $U(1, n)$ induces an isometry of this metric.
Remark: Note that $\mathrm{Isom}(\mathbb{CH}^n)$ contains more isometries than these, there is also one induced by complex conjugation on $\mathbb{C}^{n+1}$.

(d) Show that $\mathbb{CH}^n$ is homogeneous.

(e) Show that $\mathbb{CH}^n$ has pinched curvature.

Exercise 3. (20 points)
Let $M(n,\mathbb{R})$ be the algebra of $n \times n$-matrices over $\mathbb{R}$ and $S(n,\mathbb{R})$ the subspace of symmetric matrices. Denote by $P(n,\mathbb{R}) \subseteq S(n,\mathbb{R})$ the open cone of symmetric and positive-definite matrices.

(a) Show that $P(n,\mathbb{R})$ is a Riemannian manifold where the scalar product on the tangent space $T_pP(n,\mathbb{R})$ is given by

$$(X,Y)_p = \mathrm{tr}(p^{-1}Xp^{-1}Y)$$

where tr denotes the trace of a matrix.

$GL(n,\mathbb{R})$ acts on $M(n,\mathbb{R})$ by $g.A = gAg^t$ (where g^t is the transpose of g).

(b) Show that this action leaves $S(n,\mathbb{R})$ and $P(n,\mathbb{R})$ invariant and is transitive on $P(n,\mathbb{R})$.

(c) Show that the action is by Riemannian isometries.

(d) Determine the Stabilizer of the identity matrix.