RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II

Heidelberg, 05.12.2014

Exercise sheet 7

Lengths in a Riemannian Manifold and Triangles in Model Spaces

To hand in by Friday, December 12, 2014, 12:00

Exercise 1. (15 points)
For a metric space (X, d) the length of a smooth curve $\gamma : [0,1] \longrightarrow X$ was defined by

$$L_d[\gamma] = \sup_{n \in \mathbb{N}} \sup_{0=t_0<\ldots<t_n=1} \sum_{i=1}^{n} d\left(\gamma(t_{i-1}), \gamma(t_i)\right).$$

On a Riemannian manifold (M, g) we additionally can define the (Riemannian) length $L_g[\gamma]$ of such a curve γ by the metric g:

$$L_g[\gamma] = \int_{\gamma(0)}^{\gamma(1)} \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}dt.$$

Show that on a Riemannian manifold these lengths coincide, that is, $L_d[\gamma] = L_g[\gamma]$ for some smooth curve $\gamma : [0,1] \longrightarrow M$.

Hint: One inequality is clear. For the other one, show for $t \in (0,1)$ and $\delta > 0$ the inequality

$$\frac{1}{\delta} d(\gamma(t), \gamma(t+\delta)) \leq \frac{1}{\delta} L_d[\gamma|_{[t,t+\delta]}] \leq \frac{1}{\delta} \int_t^{t+\delta} \sqrt{g\left(\dot\gamma(s), \dot\gamma(s)\right)}ds.$$

Then show by using the exponential map that both sides converge to $\sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}$ as $\delta \longrightarrow 0$ and finally use the fundamental theorem of calculus.

Exercise 2. (15 points)
Show that every geodesic triangle in $\mathbb{M}^n_\kappa$ lies in a 2-dimensional totally geodesic submanifold isometric to $\mathbb{M}^2_\kappa$.

Exercise 3. (15 points)
Let Δ, $\Delta' \subset \mathbb{M}^2_\kappa$ be two comparable triangles without antipodale vertices. Show that there is an isometry $\in \mathrm{Isom}(\mathbb{M}^2_\kappa)$ that sends Δ to Δ'.

Exercise 4. (15points)
Let $\kappa \in \mathbb{R}$ be a real number and consider three non-negative real numbers $a, b, c, \in \mathbb{R}_{\geq 0}$ with $a + b + c \leq 2D_\kappa$ and $a \leq b + c,\ b \leq a + c$ and $c \leq a + b$. Show that there is a triangle in $\mathbb{M}^2_\kappa$ with side-lengths equal to a, b and c.