RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 21.11.2014

Exercise sheet 5

Curvature of submanifolds II and the Theorema Egregium

To hand in until Friday, November 28, 2014, 12:00

Exercise 1. (20 points)
Let (M, g) be an m-dimensional Riemannian manifold and $N \subset M$ a submanifold of dimension n with induced Riemannian metric, also denoted by g. For each $p \in N$ we can write the tangent space pointwise as $T_pM = T_pN \oplus T_pN^\perp$ where $T_pN^\perp = \{\xi \in T_pM | g_p(\xi, \eta) = 0 \ \forall \eta \in T_pN\}$ denotes the normal space. Similarly with the normal bundle $TN^\perp$ we obtain the decomosition $TM = TN \oplus TN^\perp$. That is for all $X \in TM$ there is a unique way to write it as $X = X^\top + X^\perp$ where $X^\top$ denotes the component tangent to N. Let ∇^M be the Levi-Civita connection on M. Show that the Levi-Civita connection ∇^N on (N, g) is given by

$$\nabla^N_X Y = (\nabla^M_{\overline{X}} \overline{Y})^\top$$

for all $X, Y \in \Gamma(TN)$ with local extensions $\overline{X}, \overline{Y} \in \Gamma(TM)$.

Exercise 2. (20 points)
Let the notations be as in Exercise 1, $N \subset M$ a hypersurface. On the last exercise sheet we introduced the first and second fundamental form for a surface in $\mathbb{R}^3$. On this sheet we will see a more general definition of the second fundamental. The *vector valued second fundamental form* on N is defined as $II(X, Y) = (\nabla^M_{\overline{X}} \overline{Y})^\perp$ for all $X, Y \in \Gamma(TN)$ with local extensions $\overline{X}, \overline{Y}$ respectively. Let ν be a local normal unit field on N which is unique up to a sign. Then the *real valued second fundamental form* l of N is defined[1] as $l_p(u, v) = -g(\nabla^M_u \nu, v)$ for $u, v \in T_pN$.

(a) Show that $II_p(u, v) = l_p(u, v)\nu_p$ for all $u, v \in T_pN$.

(b) Show that N is totally geodesic if and only if the second fundamental form at every point of N vanishes.

Exercise 3. (20 points)
Let the notations be as in Exercise 1 and 2 and $N \subset M$ a Riemannian hypersurface. Let R^M, κ^M (resp. R^N, κ^N) be the curvature tensor and the sectional curvature of M (resp. N). For a 2 dimensional subspace of TN generated by X and Y the Gaussian curvature of N is defined as

$$K(X, Y) = \frac{l(X, X)l(Y, Y) - l(X, Y)^2}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

(a) Show that for $X, Y, Z, W \in \Gamma(TN)$ and the notations above we have

$$R^N(X, Y, Z, W) = R^M(X, Y, Z, W) - l(X, Z)l(Y, W) + l(X, W)l(Y, Z)$$

and

$$\kappa^N(X, Y) = \kappa^M(X, Y) + K(X, Y).$$

Hint: Show that $g(\nabla^M_X \nabla^M_Y Z, W) = g(\nabla^N_X \nabla^N_Y Z, W) - l(Y, Z)l(X, W)$.

(b) What does this mean for the sectional and the Gaussian curvature of a surface in $\mathbb{R}^3$?

Remark The equations shown in (a) are called the Gauss equations. The result in (b) is also known as the Theorema Egregium of Gauss.

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[1] l is well defined up to the sign of ν.