RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 14.11.2014

Exercise sheet 4

Curvature of submanifolds

To hand in until Friday, November 21, 2014, 12:00

Exercise 1. (20 points)
Let (M, g) be a Riemannian manifold and P, Q curvature-like tensors such that $K_P(v, w) = K_Q(v, w)$ for all $v, w \in T_pM$, where $K_P(v, w) = \frac{P(v,w,w,v)}{\|v\|^2\|w\|^2 - g(v,w)^2}$.
Show that $P = Q$.
Hint: Compute the second derivative

$$\frac{\partial^2}{\partial\alpha\partial\beta}\left(P(X + \alpha Z, Y + \beta W, Y + \beta W, X + \alpha Z) - P(X + \alpha W, Y + \beta Z, Y + \beta Z, X + \alpha W)\right)$$

for $\alpha = \beta = 0$.

Exercise 2. (40 points)
Let M be an oriented parameterised surface in $\mathbb{R}^3$ and $q = (q_1, q_2, q_3) \in M$ such that there is an open neighbourhood $U \subset M$ of p and a C^2-function $f : W \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that (q_1, q_2) is a critical point[1] of f and U is the graph of f. Then the *Gaussian curvature* $K(q)$ of M at q is the determinant of the Hessian matrix of f at q. As this matrix is symmetric, we can diagonalise it to obtain two directions which are called the *principal directions*. The second derivative restricted to these directions are the eigenvalues and are called the *principal curvatures*. The Gaussian curvature then is the product of these derivatives.
Let $\langle\cdot,\cdot\rangle$ be the standard scalar product in $\mathbb{R}^3$ and $h : V \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ a regular local parameterisation. Let (x_1, x_2) be the corresponding local coordinates of $M = h(V)$. The *first fundamental form* b with $b_p : T_pV \times T_pV \longrightarrow \mathbb{R}$ is defined as $b_p(v, w) = h^*\langle v, w\rangle = \langle dh(v), dh(w)\rangle$. Let ν be a normal field to M of norm 1, which is unique up to a sign. The *second fundamental form* l with $l_p : T_pV \times T_pV \longrightarrow \mathbb{R}$ is given by $l_p(v, w) = -\langle \nabla_{dh(v)}\nu, dh(w)\rangle$.

(a) Show that in local coordinates we have $b_{ij} := b\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial h}{\partial x_i}, \frac{\partial h}{\partial x_j}\right\rangle$ and $l_{ij} := l\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \nu, \frac{\partial^2 h}{\partial x_i \partial x_j}\right\rangle$ and therefore, with the appropiate choice of the sign of ν,

$$l_{ij} = \det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}, \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}\right) \cdot \sqrt{\det(b_{kl})}^{-1}.$$

(b) Show that the Gaussian curvature can be expressed by $K(p) = \frac{\det(l_{ij})}{\det(b_{ij})}(p)$.

(c) Determine the Gaussian curvature of a parabola rotated around the z-axis.

(d) Let M be the surface parameterized by $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ (x, y)^t \mapsto (x, y, xy)^t$. Compute the Gaussian curvature of M at each point and compute also the principal curvature at the origin and draw a sketch of it.

(e) Let M be the surface parameterized by $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ (x, y)^t \mapsto (x+y, x-y, x^2)^t$. Compute the Gaussian curvature of M at each point and compute also the principal curvature at the origin and draw a sketch of it.

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[1] It turns out that by a suitable choice of f every point of M can be made to a critical point of f.