RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 07.11.2014

Exercise sheet 3

Coverings and spaces of non-positive curvature

To hand in until Friday, November 14, 2014, 14:00

Exercise 1. (*30 points*)
The aim of this exercise is to prove the following proposition:

Proposition. Let B and C be manifolds and B be connected. Let $p : C \longrightarrow B$ be a local homeomorphism with the property of lifting paths[1]. Then p is a covering map.

Let B, C be as in the proposition and $p : C \longrightarrow B$ a local homeomorphism with the property of lifting paths. Show the following statements.

(a) Squares can be lifted. That is, if $\varphi : [0,1] \times [0,1] \longrightarrow B$ continuous with $b := \varphi(0,0)$ and $c \in p^{-1}(b)$ is given, then there is a unique $\widetilde{\varphi} : [0,1] \times [0,1] \longrightarrow C$ such that $\widetilde{\varphi}(0,0) = c$ and $p \circ \widetilde{\varphi} = \varphi$.

(b) Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow B$ be two paths of B with $\gamma_1(0) = \gamma_2(0) =: b$ and $\gamma_1(1) = \gamma_2(1) =: q$. Choose $c \in C$ such that $p(c) = b$. Let γ_1 and γ_2 be homotopic. Then also their lifts $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ with origin c are homotopic.

(c) Let X be a simply connected manifold and $f : X \longrightarrow B$ continuous. Then f can be lifted to some $\widetilde{f} : X \longrightarrow C$.

(d) Let C be path connected and B simply connected. Then p is a homeomorphism.

(e) Prove the proposition above.

Exercise 2. (*15 points*)
Let B and C be connected manifolds and $p : C \longrightarrow B$ a proper local homeomorphism. Show that p is a finite covering.
Hint: Use the proposition of exercise 1. To show that p has the path lifting property, show that the subset of $[0,1]$, where the lift is defined, is $[0,1]$.

Exercise 3. (*15 points*)
The aim of this exercise is to compute the sectional curvature of $\mathbb{M}^n_k$ for $k \neq 0$.

(a) Show that a geodesic through the point x with speed $v \in T_x\mathbb{M}^n_k = \operatorname{span}(x)^\perp$ is given by
$k > 0: \quad \gamma(t) = x\cos(t\sqrt{k}\|v\|) + \frac{v}{\sqrt{k}\|v\|}\sin(t\sqrt{k}\|v\|)$
$k < 0: \quad \gamma(t) = x\cosh(t\sqrt{-k}\|v\|) + \frac{v}{\sqrt{-k}\|v\|}\sinh(t\sqrt{-k}\|v\|)$.

(b) For $u, w \in T_x\mathbb{M}^n_k$ with $\|u\| = \|w\| = 1$ and $g(u,w) = 0$, a 1-parameter family of geodesics is given by $\gamma_{v(s)}(t) = \exp_x(t\,v(s))$ with $v(s) = u\cos(s) + w\sin(s)$. Use Jacobi fields to compute the curvature of $\mathbb{M}^n_k$ for both $k > 0$ and $k < 0$.

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[1]**Definition** A continuous map $p : C \longrightarrow B$ is said to have the property of lifting paths when every path of B may be lifted. That is for every path $\gamma : [0,1] \longrightarrow B$ and for all $c \in p^{-1}(\gamma(0))$ there is a path $\widetilde{\gamma} : [0,1] \longrightarrow C$ with $\widetilde{\gamma}(0) = c$ and $p \circ \widetilde{\gamma} = \gamma$.