Horofunction Compactification of $\mathbb{R}^n$ with Polyhedral Norms

Anna-Sofie Schilling (Heidelberg)

Introduction

I work on the **horofunction compactification of symmetric spaces** of non-compact type. Instead of compactifying a symmetric space X directly, it is often easier to **compactify the flats** lying in X and then use a group action on them. Flats are totally geodesic immersions of Euclidean space into X, therefore I will present here the **horofunction compactification of a finite-dimensional normed space** with polyhedral norm.

Theoretical Background

Unit and Dual Unit Balls

To every norm $\|\cdot\|$ there is a unit ball $B_{\|\cdot\|}$ associated to it:

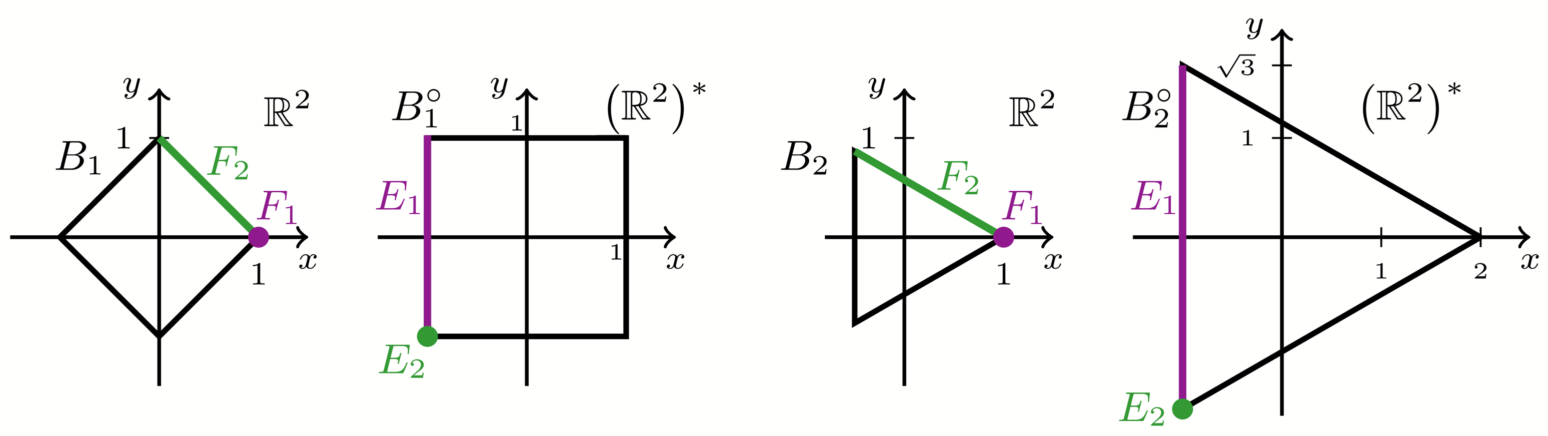
$$B_{\|\cdot\|} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

Denote by $\langle\cdot|\cdot\rangle$ the dual pairing of $\mathbb{R}^n$ and its dual space $(\mathbb{R}^n)^*$.
Then the **dual unit ball** B° of B is defined as

$$B^\circ = \{y \in (\mathbb{R}^n)^* \mid \langle y|x\rangle \geq -1 \; \forall x \in B\}.$$

If B is polyhedral, then so is B°. In this case, to every face F of B there is exactly one face $E = F^\circ$ of B°, called the **dual face of** F, satisfying

$$\dim(F) + \dim(F^\circ) = n - 1.$$



$\|(x,y)\|_{B_1} = |x| + |y|$ $\qquad$ $\|(x,y)\|_{B_2} = \begin{cases} 2|x|, & \text{if } x \leq -\frac{|y|}{2} \\ x + \frac{3}{2}|y|, & \text{else} \end{cases}$

Examples of unit balls and their duals. The colors indicate dual faces.

Horofunction Compactification

General setting

Let (X,d) be a nice[1] metric space allowing the metric to be non-symmetric (i.e. $d(x,y) \neq d(y,x)$ possible). The basic construction is to embed X via $\psi: z \mapsto \psi_z$ into the space $\widetilde{C}(X)$ of continuous real valued functions vanishing at a basepoint p_0 in the following way:

$$\psi_z(x) = d(x,z) - d(p_0,z).$$

The closure of the image $\overline{\psi(X)}$ is compact and called the **horofunction compactification of** X. We identify the space X with its image in $\widetilde{C}(X)$ and call the elements in the boundary $\partial_{hor}(X)$ **horofunctions**.

Polyhedral normed spaces

For a finite-dimensional normed space with polyhedral norm $\|\cdot\|_B$, Walsh [1] determines all horofunctions explicitly. Using his results we obtain [2] the following **characterization of horofunctions:**

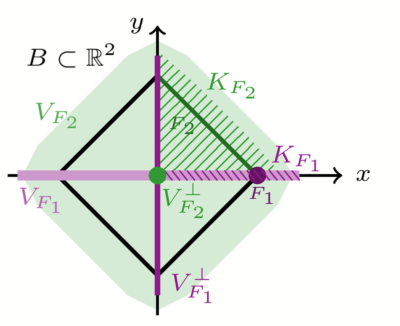
$$\partial_{hor}(X) = \{h_{E,p} \mid E \subset B^\circ \text{ is a proper face and } p \in \mathbb{R}^{\dim(E)}\},$$

where the functions $h_{E,p}: X \longrightarrow \mathbb{R}$ can be calculated explicitely.
In other words, to each face $E \subset B^\circ$ and a point $p \in \mathbb{R}^{\dim(E)}$ there is exactly one horofunction associated to it.

[1] "nice" here means that X is geodesic, d is symmetric with respect to convergence and that the symmetrized distance $d_{sym}(x,y) = d(x,y) + d(y,x)$ is proper.

Convergence of Sequences

To reveal more structure of the compactification, we examine the behavior of sequences at infinity. From now on let $X = \mathbb{R}^n$ and let $F \subset B$ be a face and $E = F^\circ \subset B^\circ$ be its dual face. First we fix some notation:



K_F Cone over face F

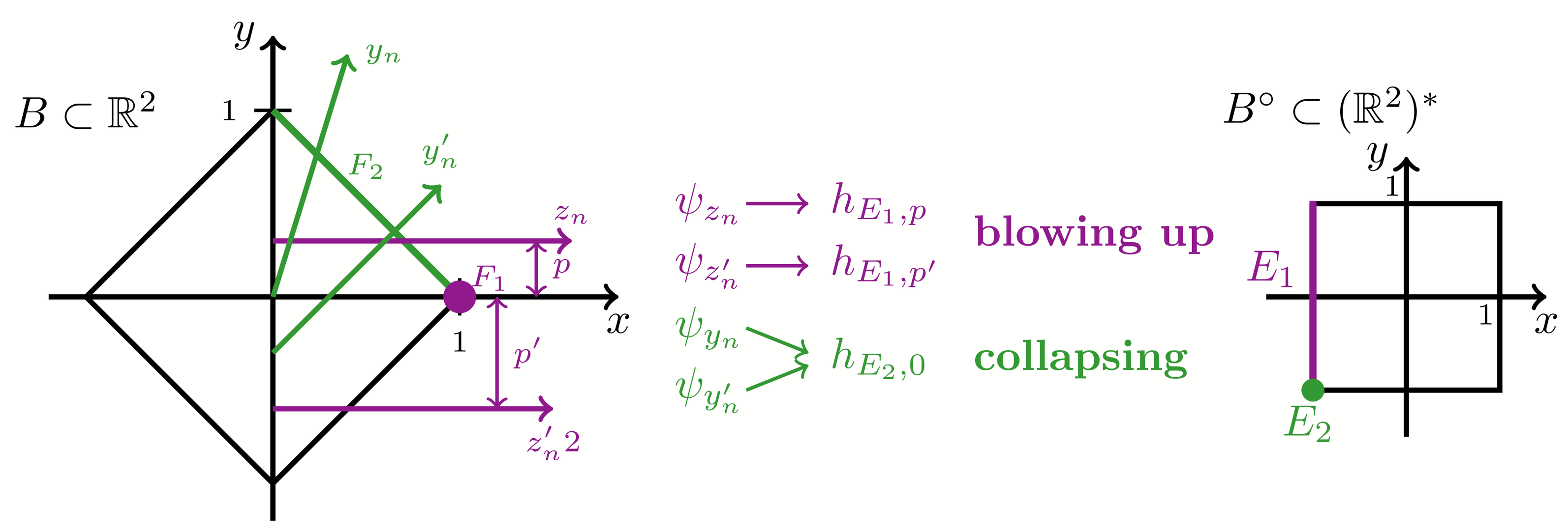
V_F Subspace over face F

$V_F^\perp$ Orthogonal complement of V_F $(\dim(V_F^\perp) = \dim(F^\circ))$

Then an unbounded **sequence** $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ (i.e. sequence $(\psi_{z_n}) \subset \widetilde{C}(X)$) **converges to a horofunction** $h_{E,p} \in \partial_{hor}(\mathbb{R}^n)$ if and only if the following conditions are satisfied:

- $\mathrm{proj}_{V_F}(z_n) \in K_F$ for all $n \gg 0$ (projected sequence lies in cone K_F),
- $d(\partial_{rel}K_F, \mathrm{proj}_{V_F}(z_n)) \to \infty$ (infinite distance to the relative boundary of K_F),
- $\|\mathrm{proj}_{V_F^\perp}(z_n) - p\|_B \to 0$ (orthogonal part of sequence converges to p)

Roughly speaking, a sequence in the direction of a face F of B converges to a horofunction associated to the dual face $E = F^\circ \subset B^\circ$ and $p \in \mathbb{R}^{\dim(E)}$.



Blowing up and collapsing behavior of converging sequences.

We observe the following behavior:

- sequences in a **singular direction** (i.e. $\dim(F) < (n-1)$) converge to different horofunctions $h_{E,p}, h_{E,p'}$ associated to the same face E but with different parameters p, p'. We call this behavior **blowing up**.
- sequences in a **regular direction** (i.e. $\dim(F) = (n-1)$) all converge to the same horofunction $h_{E,0}$. We call this behavior **collapsing**.

Selected References

[1] Cormac Walsh, *The horofunction boundary of finite-dimensional normed spaces, Math. Proc. Cambridge Philos. Soc.*, 142(3):497–507, 2007.

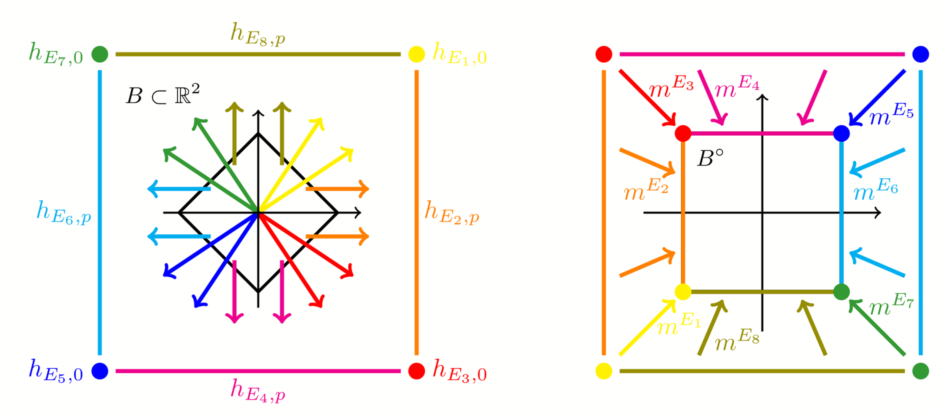
[2] Lizhen Ji, Anna-Sofie Schilling, *Polyhedral Horofunction Compactification as Polyhedral Ball, ArXiv e-prints* arXiv:1607.00564v2, Aug. 2016.

[3] T.Haettel, A.Schilling, C.Walsh, A.Wienhard, *Horofunction Compactifications of Symmetric Spaces, ArXiv e-prints* arXiv:1705.05026v2, Sept. 2018.

Homeomorphism between Compactification and B°

This behavior of convergence leads to a homeomorphism m between the compactification $\overline{\mathbb{R}^n}^{hor}$ and the dual unit ball B°.
For each face $E \subset B^\circ$ we construct a homeomorphism $m^E: \mathbb{R}^{\dim(E)} \to \mathrm{int}(E)$ to the interior of E, which is compatible with the convergence of sequences. Putting all these maps m^{E_i} together, we obtain a **homeomorphism** m between the horofunction compactification $\overline{\mathbb{R}^n}^{hor}$ and the dual unit ball B°:

$$\begin{aligned} m: \overline{\mathbb{R}^n}^{hor} &\longrightarrow B^\circ \\ x \in \mathbb{R}^n &\longmapsto m^{B^\circ}(x) \\ h_{E,p} \in \partial_{hor}\mathbb{R}^n &\longmapsto m^E(p) \end{aligned}$$

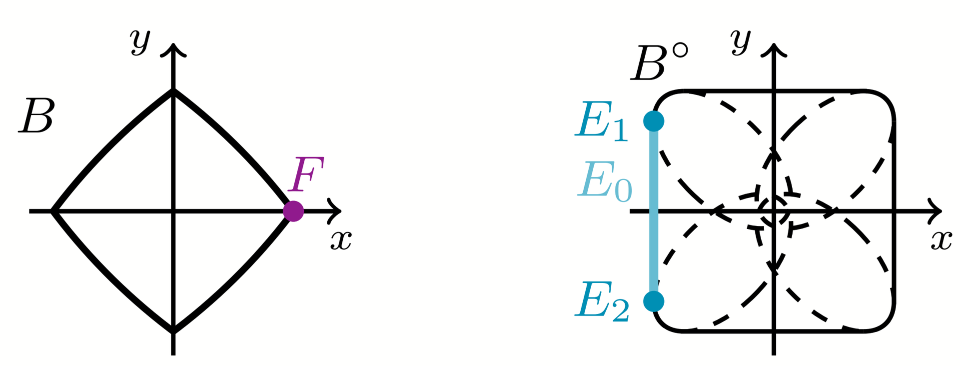


Connection between the horofunctions as limits of sequences (left) and the maps m^{E_i} (right).

Further Work

Based on the results for polyhedral norms, I am working on different projects:

- I try to generalize the results to **norms that are not polyhedral**, for example to a blown up L^1-norm. There is no 1-1 correspondence between the faces of B and those of B° anymore as shown in the picture. Therefore the behavior of sequences at infinity changes. Additionally we now have uncountably many faces of B and B°, which makes it difficult to define the maps m^C for the homeomorphism m.



There is more than one dual face to F if B and B° are not polyhedral.

- There are many well-known compactifications of symmetric spaces apart from the horofunction compactification. Some of them can also be determined by compactifying the flats, which gives us a nice way to compare compactifications. We have already shown [3] that any (generalized) Satake compactification can be realized as a special horofunction compactification. I want to continue in this direction and **compare** the horofunction compactification **with other known compactifications of** X.