Quiver Varieties

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Chapter 1

Introduction

As a motivation, let us consider a compact Kähler manifold M and $\pi = \pi_1(M)$ its fundamental group. If G denotes a complex reductive Lie group, then the character variety

$$\mathrm{Hom}\,(\pi, G)/G$$

exhibits various interesting structures such as the structure of a symplectic space, or even the structure of a hyper-Kähler space. As a matter of fact for now, quiver varieties present the same structure as character varieties is a slightly richer manner. Therefore, anyone interested in the study of character varieties should be curious about quiver varieties.

One way to introduce quiver varieties consists in starting from groups representation. Let $G \to \mathrm{End}\,(V)$ be a representation of some group G in a vector space V. For $g \in G$, write $A_g \colon V \to V$ for the associated linear transformation. The group G can be drawn as a graph with a single vertex (representing G) and an arrow for every element of the group. The same cartoon can be drawn to illustrate the representation $A_\bullet \colon G \to \mathrm{End}\,(V)$.

What makes the study of groups with linear representation very hard is the non-linearity arising with the constraints

$$A_{g^{-1}} = (A_g)^{-1}, \quad A_{gh} = A_g \circ A_h, \quad \forall g, h \in G.$$

If the constraints are dropped, the resulting object is a graph (on a single vertex) where each arrow is freely chosen to be any linear transformation of the vector space. Such an object, is called the representation of a quiver. This survey is an introduction to quivers and the study of their representations.

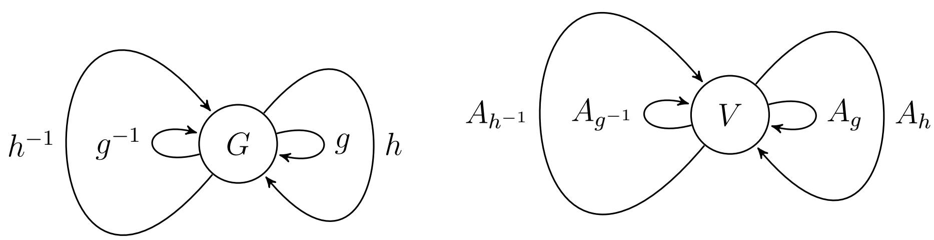


Figure 1.1: Visualization of the linear representation $A_\bullet \colon G \to \mathrm{End}\,(V)$ as a graph. Observe that the graph does not transcribe the inherent group laws.

Chapter 2

Quiver varieties and representations

We start with the definition of a quiver.

Definition 1 (Quiver)**.** A *quiver* Q is a finite directed graph (Q_0, Q_1, s, t) where Q_0 is the set of vertices, Q_1 is the set of edges and $s, t\colon Q_1 \to Q_0$ are the two functions indicating the source and the target of each edge.

Note that multiple edges between the same pair of vertices is allowed in the definition of a quiver. Consider the following first example of a quiver.

Example 2. Let the quiver Q be given by $Q_0 = \{0, 1\}$, $Q_1 = \{\alpha, \beta, \gamma\}$ and s, t being defined as follows: $s(\alpha) = 1$, $t(\alpha) = 2$ and $s(\beta) = t(\beta) = s(\gamma) = t(\gamma) = 2$. The quiver Q can be drawn as:

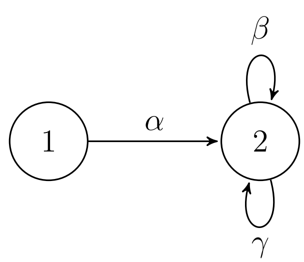


Figure 2.1: Illustration of the graph corresponding to the quiver Q.

Another source of examples of quivers are finite groups that give rise to quivers via the graph introduced in Figure 1.1.

The next crucial concept is the definition of a representation of quiver. Let k denote a field.

Definition 3 (Representation of a quiver)**.** A *k-representation of a quiver Q* is a collection of k-vector spaces $\{V_i\}_{i \in Q_0}$ together with a collection of k-linear maps $\{f_\alpha\colon V_{s(\alpha)} \to V_{t(\alpha)}\}_{\alpha \in Q_1}$.

It was mentioned in introduction that representation of quivers should be thought of group representations where constraints corresponding to the underlying group structures have been dropped. In particular, we emphasize that no conditions of commutativity have been assumed in the definition of a representation of a quiver.

As with any new mathematical object, a notion of morphism shall be defined.

Definition 4 (Morphism of representation)**.** A *morphism between two representations* $M = (\{V_i\}, \{f_\alpha\})$ and $N = (\{W_i\}, \{g_\alpha\})$ of the same quiver $Q = (Q_0, Q_1, s, t)$ is a collection $\{u_i\}_{i \in Q_0}$ of k-linear maps such that the following diagram commutes for every α in Q_1.

$$\begin{tikzcd} V_{s(\alpha)} \arrow[r, "f_\alpha"] \arrow[d, "u_{s(\alpha)}"] & V_{t(\alpha)} \arrow[d, "u_{t(\alpha)}"] \\ W_{s(\alpha)} \arrow[r, "g_\alpha"] & W_{t(\alpha)} \end{tikzcd}$$

Such a representation is denoted $u \colon M \to N$.

Immediate properties of morphisms of quiver representations can be derived from the definition. Let $M = (\{V_i\}, \{f_\alpha\}), N, P$ be representations of a quiver $Q = (Q_0, Q_1, s, t)$.

- Given two morphisms $u \colon M \to N$ and $v \colon N \to P$, then there is a well-defined composed morphism $u \circ v \colon M \to P$ given by the compositions of the underlying linear maps: $(u \circ v)_i = u_i \circ v_i$ for every $i \in Q_0$.
- The composition of quiver representations morphisms is associative because composition of linear maps is associative.
- There is an obvious identity morphism $id \colon M \to M$.

These three properties define a category $\mathrm{Rep}\,(Q)$ whose objects are representations of the quiver Q. Observe furthermore that

- two morphisms $u \colon M \to N$ and $v \colon M \to N$ give rise to a sum morphism $u + v \colon M \to N$ defined by $(u + v)_i = u_i + v_i$ for every $i \in Q_0$.,
- and that given a morphism $u \colon M \to N$, then $\mathrm{Ker}\,(u)$ defines a new object of $\mathrm{Rep}\,(Q)$ whose associated vector spaces are the kernels $\{\mathrm{Ker}\,(u_i)\}_{i \in Q_0}$ and associated linear maps $\{f_\alpha \restriction_{\mathrm{Ker}\,(u_{s(\alpha)})}\}_{\alpha \in Q_1}$ are the restrictions to the kernels of the linear maps associated to the representations M.

Cokernel representation can be defined in the same way. Therefore, the category $\mathrm{Rep}\,(Q)$ is an abelian category. Note that the zero object in $\mathrm{Rep}\,(Q)$ is given by the representation of Q that associates the vector space $\{0\}$ to every vertex. Observe, that

Rep (Q), as an abelian category, is thus equivalent to a full subcategory of Left$R$-mod for some ring R.

From now on, we will be studying representations of quivers in which every vector space has finite dimension. Let $Q = (Q_0 = \{1, \ldots, |Q_0|\}, Q_1, s, t)$ be a quiver and $M = (\{V_i\}, \{f_\alpha\})$ be a representation of Q such that $\dim_k(V_i) < \infty$ for all $i \in Q_0$.

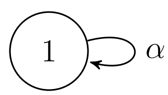
Definition 5 (Dimension vector)**.** The *dimension vector* of M is

$$\underline{\dim}\,(M) := (n_1, \ldots, n_{|Q_0|}), \quad n_i = \dim_k(V_i), \quad i \in Q_0.$$

For a choice of nonegative integers $n_1, \ldots, n_{|Q_0|}$, denote by $\mathrm{Rep}_{(n_1,\ldots,n_{|Q_0|})}(Q)$ the subcategory of Rep (Q) of representations with dimension vector $(n_1, \ldots, n_{|Q_0|})$.

One of the major goals in the study of quiver varieties is to classify the representations with a fixed dimension vector.

Example 6. Consider the following quiver Q with one vertex and one edge:



A representation M of Q is simply a pair (f, V) of a k-vector space V and a linear map $f\colon V \to V$. Further, an isomorphism between two representations (f, V) and (g, W) is simply a linear isomorphism $u\colon V \to W$ such that $g = ufu^{-1}$.

In particular, if $M_{n\times n}(k)$ denotes the $n \times n$-matrices with entries in k, then to every matrix $A \in M_{n\times n}(k)$ we can associate the representation (A, k^n) of Q. More precisely, if we introduce the category $\mathcal{M}_{n\times n}(k)$ induced by the action by conjugation of $GL_n(k)$ on $M_{n\times n}(k)$, i.e. the category whose objects are the matrices $M_{n\times n}(k)$ and morphisms between $A, B \in M_{n\times n}(k)$ are the elements $g \in GL_n(k)$ such that $B = gAg^{-1}$, then we get a functor

$$F\colon \mathcal{M}_{n\times n}(k) \to \mathrm{Rep}_n(Q).$$

This functor is faithful. Moreover, it is essentially surjective, as for any representation (f, V) of Q with $\dim_k(V) = n$, by choosing an isomorphism $V \cong K^n$ and conjugating f by the corresponding change of basis, one gets an isomorphic representation (A_f, k^n) in the image of F. Nevertheless, the functor F is not necessarily full as the following pair of morphisms needs not be invertible.

$$\begin{tikzcd} k^n \arrow[r, "A"] \arrow[d, "u"] & k^n \arrow[d, "u"] \\ k^n \arrow[r, "A"] & k^n \end{tikzcd}$$

Hence, F is not an equivalence of category. However, there is a bijection at the level of isomorphisms induced by F:

$$\mathrm{Isom}\,(\mathrm{Rep}_n(Q)) \leftrightarrow \mathrm{Isom}\,(\mathcal{M}_{n\times n}(Q)).$$

Observe that isomorphism classes in $\mathcal{M}_{n\times n}(Q)$ are nothing but conjugacy classes for the action of $GL_n(k)$ on $M_{n\times n}(k)$. Thus $\mathrm{Isom}\,(\mathcal{M}_{n\times n}(Q)) = M_{n\times n}(k)/GL_n(k)$. Note furthermore that if k is algebraically closed, then such classes are enumerated by Jordan canonical forms.

As a last remark, observe that the action of $GL_n(k)$ on $M_{n\times n}(k)$ is not proper and therefore the quotient $M_{n\times n}(k)/GL_n(k)$ is not Hausdorff and is not an algebraic variety in any reasonable way.

As a moral consequence of Example 6, we emphasize the omnipresence of geometric invariant theory in the study of quiver varieties.