Brauer Groups and cohomological Brauer Groups

Oberseminar der Arbeitsgruppen Arithmetische Geometrie,
Arithmetische Homotopietheorie SS 2015
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Topic

Similarly to central simple algebras, an Azumaya-algebra on a scheme X is defined as an $\mathcal{O}_X$-algebra that étale locally is isomorphic to the endomorphism sheaf of a vector bundle. There is an analogue notion of Brauer-equivalence and the set of isomorphism classes of Azumaya-algebras over X modulo Brauer-equivalence forms an abelian group $\mathrm{Br}(X)$ under the $\otimes$-product, the Brauer-group of X. The Brauer group plays an important role in arithmetic geometry, e.g. via the Brauer-Manin-obstruction for a Hasse-principle for rational points over number fields or the Tate-Lichtenbaum duality of p-adic curves.

Let X be a quasi-compact, separated and quasi-projective scheme. A descent-theoretical interpretation of Azumaya algebras leads to a monomorphism $\mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_\mathrm{m})$. The goal of the seminar is to prove that via this monomorphism, $\mathrm{Br}(X)$ is isomorphic to the cohomological Brauer-group $\mathrm{Br}'(X)$, the torsion part of $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_\mathrm{m})$. The affine case was proven by Gabber in his PhD thesis, followed by an unpublished proof of the quasi-projective case. There is an independent proof by de Jong, which we will follow in the seminar.

$\mathrm{Br} = \mathrm{Br}'$ allows us to apply powerful cohomological machinery to study $\mathrm{Br}(X)$ – a group defined in purely geometric terms. On the other hand, $\mathrm{Br} = \mathrm{Br}'$ gives explicit geometrical interpretations to abstractly defined cohomology classes. This latter view is exactly the leading idea of de Jong's proof: Using the machinery of stacks and gerbes, we will learn how to interpret cohomology classes in $\mathrm{H}^2(E, \mathbb{G}_\mathrm{m})$ as isomorphism classes of $\mathbb{G}_\mathrm{m}$-gerbes, where E denotes either the étale site $X_{\text{ét}}$ or the fppf site X_{fl}. These objects carry a rich geometric structure, allowing us to translate the problem of finding a Brauer-class belonging to a given torsion class of $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_\mathrm{m})$ to the geometric problem of constructing a certain locally free sheaf on the gerbe.

Time and Place

Thursday, 11 a.m. – 1 p.m. , INF 288, HS 4; first meeting 16.04.2013

Date Talks

16.04. ***Talk 0:*** **Introduction.**

23.04 ***Talk 1:*** **Torsors and $\mathbf{H}^1$.**

For $\mathcal{G}$ a sheaf of groups in $\tilde{E}$, define $\mathcal{G}$-torsors in $\tilde{E}$. Let $\pi^1(E, \mathcal{G})$ be the set of isomorphism classes of $\mathcal{G}$-torsors in $\tilde{E}$. Discuss functoriality in the site (for $f\colon X_{\text{fl}} \to X_{\text{ét}}$ the map $f^*\colon \pi^1(X_{\text{ét}}, \mathcal{G}) \to \pi^1(X_{\text{fl}}, f^*\mathcal{G})$) and in the coefficients (via the contracted product $\mathcal{T} \wedge^{\mathcal{G}} \mathcal{H}$ for a morphism $\mathcal{G} \to \mathcal{H}$ of sheaves of groups). Explain $\check{\mathrm{H}}^1(E, \mathcal{G})$ and show the analogue of [Mil, III §4 Prop 4.6]: $\pi^1(E, \mathcal{G}) \cong \check{\mathrm{H}}^1(E, \mathcal{G})$ $(=: \mathrm{H}^1(E, \mathcal{G}))$. Main references are [Mil, III §4] and [Gir, III §1].

30.04. ***Talk 2:*** **Long exact sequence of a central extension.**

Assume that X is either quasi-projective or work with hypercoverings. For a central extension of sheaves of groups $\mathbf{1} \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to \mathbf{1}$ in $\tilde{E}$ (in particular, $\mathcal{G}'$ is a sheaf of abelian groups), construct the long exact sequence

$$\begin{array}{l} \mathbf{1} \longrightarrow \mathrm{H}^0(E,\mathcal{G}') \longrightarrow \mathrm{H}^0(E,\mathcal{G}) \longrightarrow \mathrm{H}^0(E,\mathcal{G}'') \\ \hookrightarrow \mathrm{H}^1(E,\mathcal{G}') \longrightarrow \mathrm{H}^1(E,\mathcal{G}) \longrightarrow \mathrm{H}^1(E,\mathcal{G}'') \longrightarrow \mathrm{H}^2(E,\mathcal{G}')\,. \end{array}$$

Main references is [Gir, III §3 and IV §3.5, especially Cor. 3.5.4]. See also the 3$^{\mathrm{rd}}$-step in the proof of [Mil, IV Thm. 2.5].

07.05. ***Talk 3:*** **Br and Br′.**

Define Azumaya-algebras over local rings and present [Mil, IV §1 until Prop. 1.6] – this is the injective part of Azumayas Theorem. Go on with Azumaya-algebras and the Brauer group over schemes and present [Mil, IV §2 until Rem. 2.8]. In particular, explain how $\mathrm{H}^1(E,\mathrm{PGL}_n)$-classifies Azumaya-algebras of rank n^2, explain the monomorphism $\mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X_{\text{ét}},\mathbb{G}_{\mathrm{m}})$ and show that $\mathrm{Br}(X)$ is torsion. Define the cohomological Brauer group $\mathrm{Br}'(X) := \mathrm{H}^2(X_{\text{ét}},\mathbb{G}_{\mathrm{m}})_{\mathrm{tors}}$. If time permits: show $\mathrm{Br}'(X) = \mathrm{H}^2(X_{\text{ét}},\mathbb{G}_{\mathrm{m}})$ for regular X. Main reference is [Mil, IV], see also [Gro] and [Gir, V §4].

21.05. ***Talk 4:*** ***K*-theory of projective modules.**

Show resp. explain [Lie, Lem. 3.1.4.3 and Cor. 3.1.4.4.] and explain the relevant K-theory. Main reference for K-theory is [Bas, IX §4].

28.05. ***Talk 5:*** **Algebraic spaces and groupoids.**

Recall the notion of an algebraic space and introduce representable morphisms ([LMB, §1]). If S is an algebraic space, define the 2-category of S-groupoids and explain some of the examples in [LMB, §2]. A nice overview of the concept of categories fibred in groupoids can be found in [EHKV, §2] and you may also consult the Stacks Project.

11.06. ***Talk 6:*** **(Algebraic) Stacks.**

Define prestacks and stacks and briefly present some examples. Go on with representable stacks, representable morphisms and “morphisms with property P”. Eventually, give the definition of an algebraic stack and some examples. Discuss criteria for a stack to be algebraic. The main reference is [LMB, §3 and §4],

18.06. ***Talk 7:*** **Gerbes and $\mathbf{H^2}$.**

Define and explain gerbes and gerbes bound by a sheaf of (abelian) groups $\mathcal{A}$ over $X_{\text{ét}}$. Show that the isomorphism classes of gerbes bound by an abelian $\mathcal{A}$ are classified by $\mathrm{H}^2(X_{\text{ét}},\mathcal{A})$ (e.g. follow the instructions of (i) in the proof of [Mil, Thm. 2.5] using [Gir, IV §3.4]). See also [Lie, 2.2] and [EHKV, §3] for an overview.

25.06. ***Talk 8:*** **Sheaf theory on algebraic stacks.**

Introduce the lisse-étale site on an algebraic stack, characterize its associated topos ([LMB, §12]) and briefly mention its “non-functoriality” ([Ols]). Define module sheaves and quasi-coherent sheaves and present different ways of giving the data of a quasi-coherent sheaf ([LMB, §13]).

02.07. ***Talk 9:*** **Twisted sheaves.**

Define and explain twisted sheaves in the classical sense of “descent data + error term” (see [deJ, 2.3]) as well as in the sense of sheaves on $\mathbb{G}_{\mathrm{m}}$-gerbes and present [deJ, Sect. 2] resp. [Lie, Sect. 3] until the twisted interpretation of $\mathrm{Br} = \mathrm{Br}'$ of [Lie, Prop. 3.1.2.1]. Show that both interpretations of twisted sheaves are equivalent ([deJ, Lem. 2.10]).

09.07. ***Talk 10:*** **$\mathbf{Br = Br'}$, affine case.**

Explain the general problem and present the proof of the affine case of $\mathrm{Br} = \mathrm{Br}'$ following [Lie, Lem. 3.1.3.5, 3.1.3.6 and Sect. 3.1.4]. Main reference is [Lie, Sect. 3], see also [Hoo].

16.07. ***Talk 11:*** **$\mathbf{Br = Br'}$, quasi-projective case.**

Present the proof of the quasi-projective case of $\mathrm{Br} = \mathrm{Br}'$ following [deJ, Sect. 3].

23.07. ***Talk 12:*** **The Theorem of Rumely.**

The last talk reduced $\mathrm{Br} = \mathrm{Br}'$ to a diophantine problem. Solve this problem by proving the Theorem of Rumely ([M-B, Thm. 1.7]). Follow the proof in [M-B, Sect. 3] and explain the used results on the Picard group of [M-B, Sect. 2].

References

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