An Introduction to Contact Cuts

Tom Stalljohann

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Recapitulation: The Poincaré return map

Def.: (global surface of section)

- M closed 3-manifold, X (smooth) vector field on M
- $\Sigma \subset M$ embedded compact surface satisfying:
  i. Each component of $\partial\Sigma$ is a periodic orbit of X
  ii. Int(Σ) is transverse to X
  iii. The orbit of X through any point in $M \backslash \partial\Sigma$ intersects Int(Σ) in forward and backward time

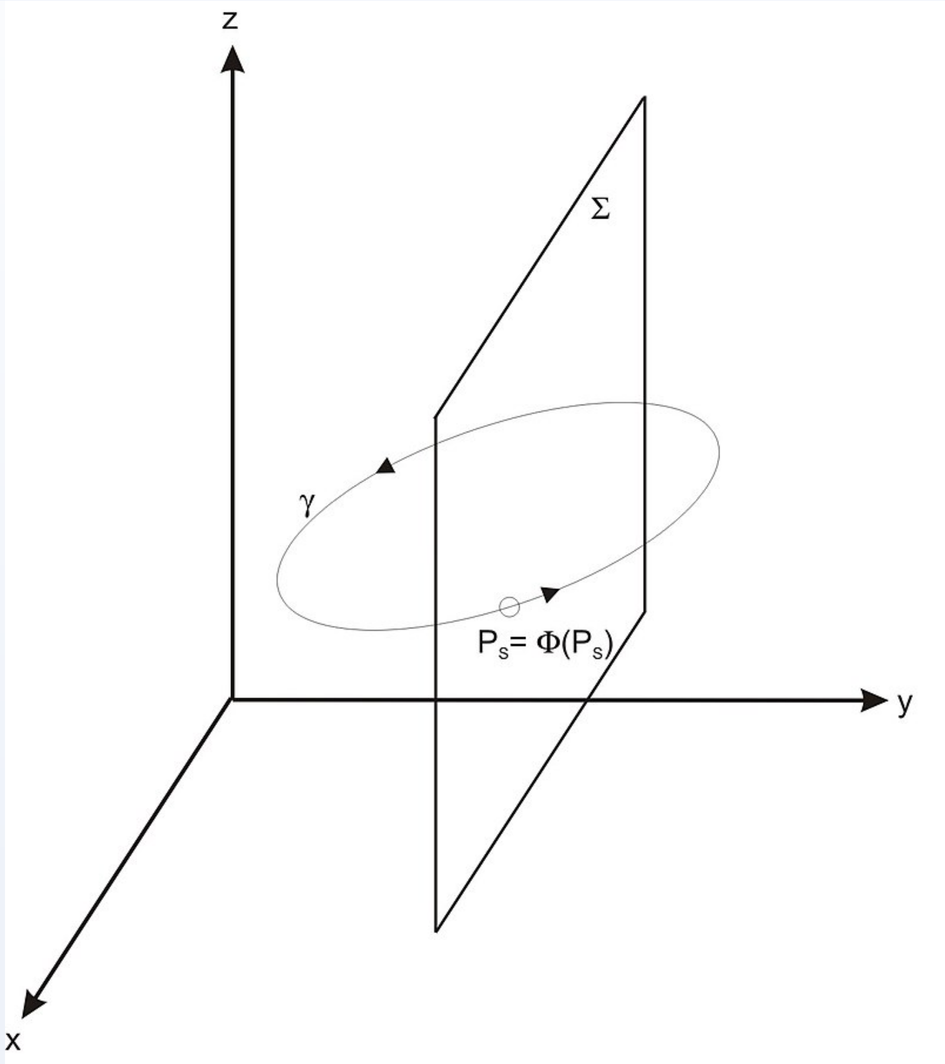
Then Σ is called a *global surface of section*.

Def: (Poincaré return map)
Let ϕ_X^t be the flow of X. The Poincaré return map of X is defined as

$$\psi\colon \mathrm{Int}(\Sigma) \longrightarrow \mathrm{Int}(\Sigma)$$
$$p \longmapsto q = \phi_X^t(p) \text{ with minimal } t \text{ so that } \phi_X^t(p) \in \mathrm{Int}(\Sigma)$$

Def.:

An area-preserving diffeomorphism $\psi\colon \Sigma \longrightarrow \Sigma$ *embeds into a Reeb flow on* M if ψ is the Poincaré return map for some Reeb vector field on M.



Source:
https://de.wikipedia.org/wiki/Poincar%C3%A9-Abbildung#/media/Datei:Poincareschnitt.jpg

Main Theorem

Assumption:
Write (r, θ) for polar coordinates on D^2.
Let $H = (H_s)_{s \in \mathbb{R}/2\pi\mathbb{Z}}$ be a smooth family of Hamiltonian functions (i.e. functions) on the 2-disc D^2 and assume there is a neighbourhood of the boundary ∂D^2 in D^2 on which H only depends on r, not on θ or the 'time-parameter' s.

Def.: (Hamiltonian vector field)

$\lambda := r^2 \, d\theta = 2x \, dy$

$\omega := d\lambda = 2r \, dr \wedge d\theta$

Then (D^2, ω) is a symplectic manifold.

For $s \in \mathbb{R}/2\pi\mathbb{Z} = S^1$ define the time dependent vector field $X = (X_s)$ on D^2 via

$$\iota_{X_s}\omega = \omega(X_s, \cdot) = dH_s$$

X is called the *Hamiltonian vector field* of H_s.

Main Theorem:

Let H be as in our assumption and X the associated Hamiltonian vector field.

$\psi := \phi_X^{2\pi}$, where ϕ_X denotes the flow of the time dependent vector field X.

Then ψ embeds into a Reeb flow on S^3.

First steps

Notice:

- X_s is a multiple of the angular vector field $\partial\theta$ near the boundary

  $\partial(S^1 \times D^2) = S^1 \times \partial D^2$ of $V \coloneqq S^1 \times D^2$

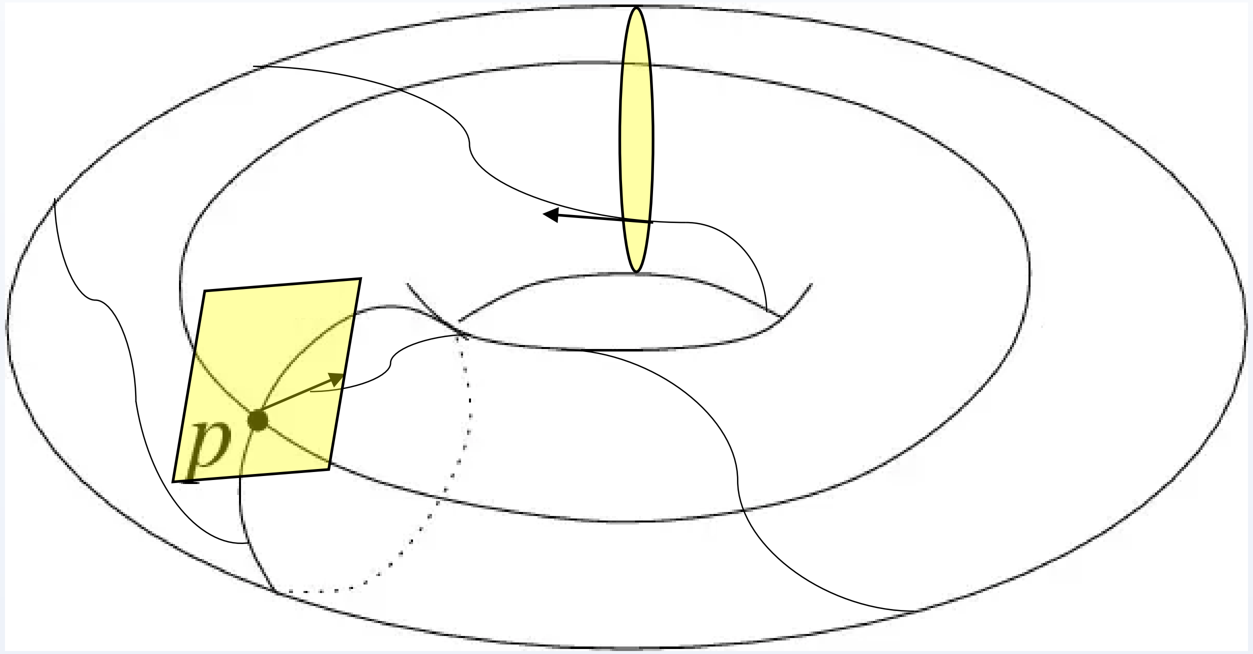
- We can add any constant to H_s without changing X_s, so we may assume:

  $H_s|_{\partial D^2} = h \in \mathbb{N}$ (applying our assumption)

- The canonical transformation of X into a autonomous vector field R on $S^1 \times D^2$ is $R \coloneqq \partial_s + X_s$

  Then: $\psi = \phi_X^{2\pi} = \phi_R^{2\pi}$

Visualization of the flow of R on ∂V:



Source (modified version):
https://www.google.de/imgres?imgurl=https%3A%2F%2Fqph.fs.quoracdn.net%2Fmain-qimg-79d8ddf944f86ebcc57aedc4926780ad.webp&imgrefurl=https%3A%2F%2Fwww.quora.com%2FWhat-is-an-intuitive-explanation-of-a-fundamental-group&tbnid=E8n6APQkOwsn5M&vet=10CHsQMyidAWoXChMI2Pv547PZ6gIVAAAAAB0AAAAAEAM..i&docid=9_GM5ieaZ_Y6eM&w=600&h=303&q=2%20torus&ved=0CHsQMyidAWoXChMI2Pv547PZ6gIVAAAAAB0AAAAAEAM

Topological cuts

Setting:

- smooth action $S^1 \times M \longrightarrow M, (\lambda, m) \mapsto \lambda * m$ on manifold M
- $f: M \longrightarrow \mathbb{R}$ smooth S^1-invariant function with regular value $a \in \mathbb{R}$
- S^1-action on $f^{-1}(a)$ is free

Define the equivalence relation ~ on $f^{-1}([a, \infty))$ through: For

$m \neq n: m \sim n \quad \Longleftrightarrow \quad m, n \in f^{-1}(a)$ and

m, n are in the same S^1-orbit

$$M_{[a,\infty)} \coloneqq f^{-1}([a, \infty))/\sim$$

- We have the natural S^1-action $\lambda * (m, z) \coloneqq (\lambda * m, \lambda^{-1} z)$ on $M \times \mathbb{C}$

- $\Psi: M \times \mathbb{C} \longrightarrow \mathbb{R}, (m, z) \mapsto f(m) - |z|^2$ is S^1-invariant and a is a regular value of Ψ

- S^1 acts freely on $\Psi^{-1}(a)$

- $\sigma: f^{-1}\big([a, \infty)\big) \longrightarrow \Psi^{-1}(a), m \mapsto (m, \sqrt{f(m) - a})$ descends to a homeomorphism $\overline{\sigma}: M_{[a,\infty)} \xrightarrow{\sim} \Psi^{-1}(a)/S^1$ and hence $M_{[a,\infty)}$ carries a smooth structure

- $f^{-1}((a, \infty))$ is open and dense in $M_{[a,\infty)}$ and $M_{[a,\infty)} \setminus f^{-1}((a, \infty))$ is diffeomorphic to $f^{-1}(a)/S^1$.

Contact cuts

Setting:

- Contact manifold (N, α) with strict contact S^1-action generated by vector field Y (i.e. $(\phi_Y^t)^* \alpha = \alpha$)
- Define the *momentum map* $\mu_N : N \longrightarrow \mathbb{R}$, $\mu_N \coloneqq \alpha(Y)$

By Cartan, we have:

(1) $$d\mu_N = \mathcal{L}_Y \alpha - \iota_Y d\alpha = -\iota_Y d\alpha$$

Consequences of (1):

- Y is tangent to $\mu_N^{-1}(0)$
- $\mu_N^{-1}(0)$ regular $\Leftrightarrow$ $Y \neq 0$ along $\mu_N^{-1}(0)$
- the S^1-action restricts to $\mu_N^{-1}(0)$ and is locally free

Furthermore we assume:
S^1-action is free on $\mu_N^{-1}(0)$.

- By Quotient manifold theorem we have: $\mu_N^{-1}(0)/S^1$ smoooth manifold
- There is a unique contact form $\hat{\alpha}$ on $\mu_N^{-1}(0)/S^1$ with $\pi_N^*\hat{\alpha} = \alpha|_{T\mu_N^{-1}(0)}$
  with $\pi_N\colon \mu_N^{-1}(0) \longrightarrow \mu_N^{-1}(0)/S^1$ the projection

Now consider the contact manifold $(N \times \mathbb{C}, \alpha + xdy - ydx)$ with S^1-action generated by the vector field $Y - \left(x\partial_y - y\partial_x\right)$

Notice that this action is compatible with the action on $N \times \mathbb{C}$ defined in the 'Topological cut' section since

$$\phi^t_{Y-(x\partial_y - y\partial_x)}(p, x_0 + iy_0) = (\phi^t_Y(p), (x_0 + iy_0)(\cos(-t) + i\sin(-t)))$$

The action on $N \times \mathbb{C}$ is also a strict contact S^1-action with momentum map

$$\mu(p, z) = \mu_N(p) - |z|^2$$

(with the notation from the 'Topological cut' section:

$M = N,\ f = \mu_N,\ \Psi = \mu)$

Using the results from above for an arbitrary contact manifold satisfying our assumptions and the section 'Topological cut', we get:

$(\mu^{-1}(0)/S^1, \bar{\alpha})$ is a contact form of dimension dim N where

$\pi\colon \mu^{-1}(0) \longrightarrow \mu^{-1}(0)/S^1$ and

$\pi^*\bar{\alpha} = (\alpha + xdy - ydx)|_{T\mu^{-1}(0)}$

Also

$$\mu_N^{-1}\big((0,\infty)\big) \hookrightarrow \mu^{-1}(0)/S^1$$
$$\mu_N^{-1}/S^1 \hookrightarrow \mu^{-1}(0)/S^1$$

are contact embeddings.

Contact cuts on the disc D^2 and related constructions

Lemma 1:

For H_s sufficiently large, the 1-form

$$\alpha \coloneqq H_s ds + \lambda$$

is a positive contact form on $S^1 \times D^2$.

The condition for α to be a positive contact form is given by

(2) $H_s + \lambda(X_s) > 0$ or equivalently

(2') $r\frac{\partial H_s}{\partial r} < 2H_s$

Proof:

(3) $\alpha \wedge d\alpha = (H_s ds + \lambda) \wedge (dH_s \wedge ds + \omega) = ds \wedge (H_s\omega + \lambda \wedge dH_s)$

(4) $\lambda \wedge dH_s = \lambda(X_s)\omega$

$\Longrightarrow \alpha \wedge d\alpha = (H_s + \lambda(X_s))ds \wedge \omega$

$\Longrightarrow$ (2)

$\lambda = \iota_{\frac{r}{2}\partial_r}\omega$

$\Longrightarrow \lambda(X_s) = -dH_s(\frac{r}{2}\partial_r)$

$\Longrightarrow$ (2')

$\square$

Now assume that the contact condition is fulfilled.

Lemma 2:
$R = \partial_s + X_s = fR_\alpha$ for some positive function f, where R_α denotes the Reeb vector field of α.

Proof:

$$\iota_R d\alpha = \iota_R(dH_s \wedge ds + \omega) = -dH_s + dH_s = 0$$

and

$$\alpha(R) = H_s + \lambda(X_s) \underset{(2)}{>} 0$$

$\square$

Lemma 3:

On a collar neighbourhood of $\partial V = \partial(S^1 \times D^2)$ in $S^1 \times D^2$ where $\mathrm{H} = (H_s)_s$ depends only on r, the S^1-action generated by $Y \coloneqq \partial_s - h\partial_\theta$ is a strict contact S^1-action with respect to α.

The momentum map is $\mu_V = \alpha(Y) = H_s - hr^2$ and since $H_s|_{\partial D^2} = h$ we have:
$\partial V \subset \mu_V^{-1}(0)$

$Y \neq 0 \quad \Longrightarrow \quad \partial V$ regular component

(2‘) on $\partial V \quad \Longrightarrow \quad d\mu_V(\partial_r) < 0$ on $\partial V \quad \Longrightarrow \quad \mu_V > 0$ on an interior neighbourhood of ∂V

Lemma 4:

The contact cut $(S^1 \times D^2)/\sim$ is contactomorphic to S^3 endowed with the standard contact structure.

Proof of the main theorem

- $D^2 \cong \{0\} \times D^2 \hookrightarrow S^1 \times D^2 \longrightarrow (S^1 \times D^2)/{\sim} \cong S^3$ is an embedding, smooth on Int(D^2)

- Since X_s is a multiple of the angular vector field $\partial\theta$ near the boundary it suffices to consider the flow of X_s on Int(V)$\cong (V/{\sim})\backslash(\partial V\ /\ {\sim})$

- On Int(V)=Int($S^1 \times D^2$) this follows from Lemma 2

- Under the above identification $S^1 = \partial D^2 \cong \mu_V^{-1}(0)/S^1$ and $\mu_V^{-1}(0)/S^1 \hookrightarrow S^3$is a contact embedding and therefore ∂D^2 is a periodic orbit of the Reeb vector field on S^3

References

- Peter Albers, Hansjörg Geiges, Kai Zehmisch.
  “Pseudorotations of the 2-disc and Reeb flows on the 3-sphere“.
  arXiv:1804.07129

- Eugene Lerman. “Contact cuts“.
  arXiv:math/0002041