Magnetic Geodesics on the Two-Sphere

A Twist Condition for Strong Magnetic Flows

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July 13, 2020

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- The particle q is subject to the Lorentz force $F_{\mathbb{R}^3}(v) = v \times B$, where v is the velocity of q.
- **Goal:** Want to find conditions for the existence of infinitely many periodic orbits.

Framework

- The Euclidean metric on $\mathbb{R}^3$ induces a metric g on M. Define a two-form $\sigma \in \Omega(M)$ by $\sigma := i_M^* \sigma_{\mathbb{R}^3}$, where $\sigma_{\mathbb{R}^3} = \iota_B vol_{\mathbb{R}^3}$ and $i_M : M \to \mathbb{R}^3$ is the inclusion. $\sigma_{\mathbb{R}^3}$ is closed because $div(B) = 0$ and thus σ is closed as well.

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- Let $\flat : TM \to T^*M$, $v \mapsto g_q(v, .)$. Switching to Hamiltonian formulation one gets the symplectic manifold $(\omega_\sigma := d\lambda - \pi^*\sigma, TM)$, where $\lambda := \flat\lambda^*$ and λ^* is the tautological one-form on T^*M.

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- The magnetic flow is generated by the Hamiltonian vector field X_E^σ associated to the energy function $E(q, v) = \frac{1}{2} g_q(v, v)$ via $dE = \iota_{X_E^\sigma} \omega_\sigma$.

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- The magnetic flow is generated by the Hamiltonian vector field X_E^σ on $(\omega_\sigma = d\lambda - \pi^*\sigma, TM)$ associated to E.

Remark

Compare this to the better known geodesic case $\nabla_v v = 0$, where the geodesic flow is generated by the Hamiltonian vector field on $(\omega := d\lambda, TM)$ associated to the same energy function E.

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- Because $dE(X_E^\sigma) = \omega_\sigma(X_E^\sigma, X_E^\sigma) = 0$, The level sets Σ_m are invariant under the magnetic flow. Therefore the restrictions of the Hamiltonian vector field $X^\sigma_{E\,|\Sigma_m}$ to the separate level sets can be studied.
- How are $X^\sigma_{E\,|\Sigma_m}$ for different m related?

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- The rescaling $(q, v) \mapsto (q, \frac{v}{m})$ sends $X^\sigma_{E\,|\Sigma_m}$ to $mX^{\frac{\sigma}{m}}_{E\,|\Sigma_1} = (mX + fV)_{|\Sigma_1}$.

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- Then $X^m \to fV$ and $\omega_m \to -\pi^*\sigma$ as $m \to 0$.

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- Since $\iota_{R_m}\omega'_m = \iota_{R_m} d\lambda_m = 0$, the Reeb vector field R_m lies in the one-dimensional kernel distribution of ω'_m and therefore we can study the Reeb flow ϕ_m to understand the magnetic flow.

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- It holds that $R_m \to V$ as $m \to 0$.

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- **Recall our task:** Want to find conditions for the existence of infinitely many periodic orbits of the magnetic flow.
- **Idea:** Find an annulus that is a global surface of section (SOS) for the Reeb flow and for which the first return map is twist (For twist maps the existence of infinitely many period orbits has been proven).

Approach to the problem

Global surface of section

Let ϕ be a flow on Σ without rest points and N a compact surface. A global surface of section for ϕ is an embedding $S : N \to \Sigma$ that has the following properties:

- $S(\mathring{N})$ is transverse to the flow ϕ and $S(\partial N)$ is the support of a finite collection of periodic orbits of ϕ.

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- For each $z \in \Sigma \setminus S(\partial N)$, there are $t_- < 0 < t_+$ such that $\phi_{t_-}(z)$, $\phi_{t_+}(z)$ lie in $S(\mathring{N})$.

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- And the first return map $P : S(\mathring{N}) \to S(\mathring{N})$, $P(z) := \phi_{\tau(z)}$.

Approach to the problem

Twist maps

Let $h : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be a diffeomorphism. We say that h is twist if $h(x+1,\theta) = h(x,\theta) + (1,0)$ and it holds the following properties:

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- h preserves the orientation and volume,
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- there exists a $c \in \mathbb{R}$, such that for every $x \in \mathbb{R}$, $h_0(x,0) < x + c < h_0(x,1)$.

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- there exists a $c \in \mathbb{R}$, such that for every $x \in \mathbb{R}$, $h_0(x,0) < x + c < h_0(x,1)$.
- We retrieve a map $\bar{h} : S^1 \times [0,1] \to S^1 \times [0,1]$ by quotienting.

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- Consider polar coordinates $(\varphi, \theta) \in (0, \pi) \times S^1$. For $m = 0$ we have a surface of section $J : (0, \pi) \times S^1 \to \Sigma$.

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In our case

- Recall that $R_m \to V$ as $m \to 0$.
- Consider polar coordinates $(\varphi, \theta) \in (0, \pi) \times S^1$. For $m = 0$ we have a surface of section $J : (0, \pi) \times S^1 \to \Sigma$.
- J extends to $[0, \pi] \times S^1$.
- The first time return map is then simply the identity.

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- Can we find a surface of section for $m > 0$ that gives us a twist map?
- Let's assume that f has non-degenerate min, max points $p_{\pm}$ at the South and North Pole respectively.
- If $\pi^{-1}(p_{\pm})$ were still periodic orbits for Reeb flows of λ_m for m small enough. Then the SOS for λ_0 would still be one for λ_m. Additionally, if and we can find some nice local expression for λ_m close to $p_{\pm}$, then we can check the behavior of the Reeb flow close to the North and South Pole in coordinates and see whether the return map is twist.

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- We later want to study the Reeb vector fields near $p_{\pm}$.
- Choose local coordinates in a neighborhood of $p_{\pm}$ such that $\lambda_0 = d\theta - r^2 d\phi$, where θ parametrizes the fibers.
- For λ_m there is no such local expression, but there is a diffeomorphism $\psi_1 : \Sigma \to \Sigma$ such that $\psi_1^* \lambda_m = e_m^q \lambda_0$, where $q_m : \Sigma \to \mathbb{R}$ admits the Taylor expansion at $m = 0$, $q_m = \frac{m^2}{2f} + o(m^2)$.

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- There is a function $S_m : \Sigma \to \mathbb{R}$ whose critical points are the support of periodic orbits.
- The construction is done by sending $(q, v) \in \Sigma$ to two-periodic loops in Σ which are then evaluated by some action functional.

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- Write $S_m = 2\pi + m^2 \bar{S}_m$.
- For $m = 0$ the critical points for $\bar{S}_m$ are $S_{p_\pm} S^2$.
- For m small enough, we can still find critical points close to the fibers since f has a non-degenerate critical point at $p_\pm$.

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- Need to find a diffeomorphism $\psi_2 : \Sigma \to \Sigma$ so that $\psi_2(\pi^{-1}(p_\pm)) = \gamma_\pm$.
- Exists, but I didn't understand it in detail yet.

Putting everything together

- The form $\psi_2^* \psi_1^* \lambda_m$ shares the same periodic orbit with λ_0 at $p_\pm$. Therefore, the SOS for λ_0 is one for $\psi_2^* \psi_1^* \lambda_m$ as well.

Putting everything together

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- It can also be shown that $\psi_2^*\psi_1^*\lambda_m = \psi_2^* e^q \lambda_0 = \frac{\lambda_0}{1-\frac{m^2}{f}} + o(m^2)$, for m small enough.

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- It can also be shown that $\psi_2^*\psi_1^*\lambda_m = \psi_2^* e^q \lambda_0 = \frac{\lambda_0}{1-\frac{m^2}{f}} + o(m^2)$, for m small enough.
- Continue to study the Reeb flow locally and we might discover that the first time return map is twist. . ..