Root systems and spherical Coxeter groups

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1 Root systems

Definition. [1] A subset $\Phi \subseteq V$ of a finite dimensional $\mathbb{R}$-vector space V is called an *(abstract) root system*, if:

(R1) Φ is finite, $0 \notin \Phi$ and $\langle \Phi \rangle_{\mathbb{R}} = V$,

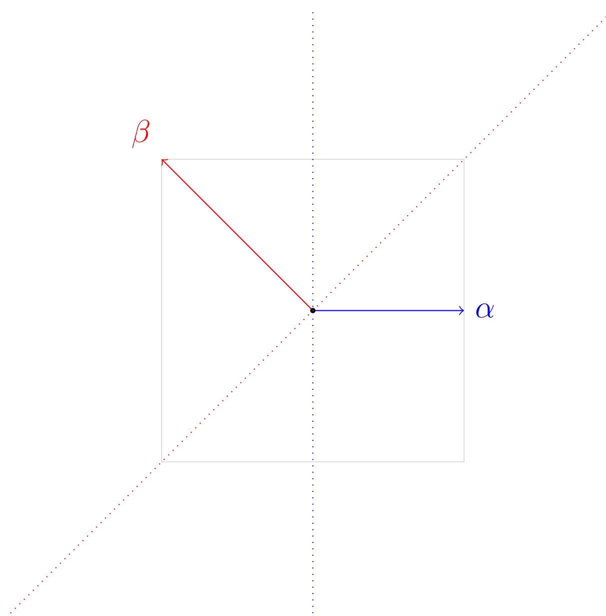
(R2) $\forall \alpha \in \Phi$: if $\lambda\alpha \in \Phi$ for $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$,

(R3) $\forall \alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{GL}(V)$ along α stabilising Φ,

(R4) (*crystallographic condition* or *integrality*) for all $\alpha, \beta \in \Phi$: $s_\alpha.\beta - \beta$ is an integral multiple of α

The group $W = W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called a *spherical Coxeter group* or *Weyl group* of Φ.

Running example. Consider the roots $\alpha, \beta \in \mathbb{R}^2$:



Then $\Phi = \{ \ \alpha, \ \ \beta$

2 Positive systems and bases

Assume the vector space V is equipped with a total ordering '$>$' compatible with addition and scalar multiplication by positive real numbers.

Definition. The *positive system* of Φ is the subset $\Phi^+ \subseteq \Phi$ consisting of all positive roots in Φ.

A subset $\Delta \subseteq \Phi$ is called a *base* of Φ if

- it is a vector space basis of V and
- any $\beta \in \Phi$ is a linear combination $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ with either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$.

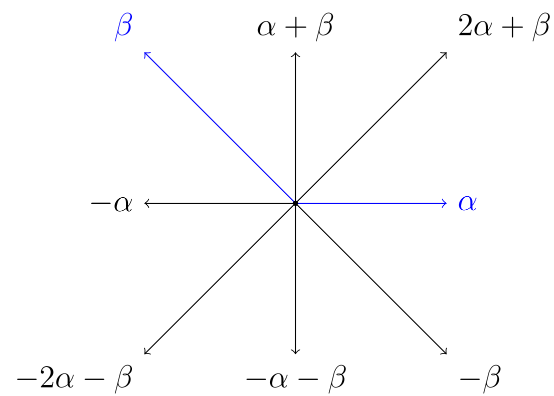
The roots in Δ are called *simple roots*.

Properties. *For any root system Φ:*

- *Every positive system contains a unique base.*
  Any base is contained in a unique positive system.
- *Any two bases of Φ are conjugate under $W(\Phi)$.*
- $W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi\rangle = \langle s_\alpha \mid \alpha \in \Delta\rangle$.

Running example. .

Consider the root system Φ from above:



Then $\Phi^+ = \{$

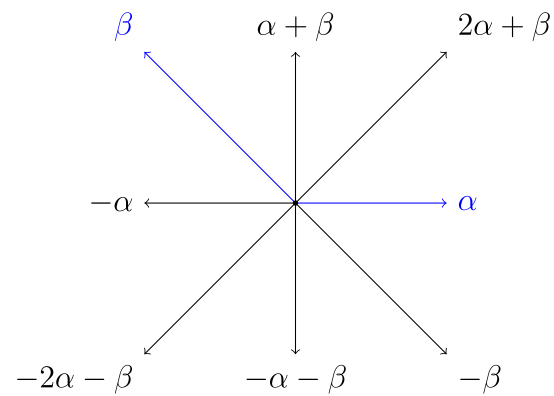
and $\Delta = \{$

3 Height and the highest root

Definition. Let Δ be a base of Φ.
The *height* of a root $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Phi$ is $\mathrm{ht}(\beta) := \sum_{\alpha \in \Delta} c_\alpha$.

One can show that there exist a unique root in Φ with largest height. This root is called *highest root*.

Running example. Consider the root system Φ from above with $\Delta = \{\alpha, \beta\}$:



4 Decomposition of root systems

Definition. A non-empty root system Φ with base Δ is called *decomposable* if there exists a non-trivial partition $\Delta = \Delta_1 \sqcup \Delta_2$ such that $(\alpha_1, a_2) = 0$ for all $\alpha_1 \in \Delta_1$, $\alpha_2 \in \Delta_2$.

Φ is called *indecomposable* if no such decomposition exists.

Proposition. *Let Φ be a root system. There exists an (up to reordering) unique decomposition into a disjoint orthogonal union $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$ of indecomposable root systems Φ_i. Also, $W(\Phi) \cong W(\Phi_1) \times \cdots \times W(\Phi_r)$.*

Proposition. *Let Φ be a root system and $V = \langle \Phi \rangle_{\mathbb{R}}$. Then the following are equivalent:*

(a) Φ is indecomposable,

(b) $W(\Phi)$ acts irreducibly on V and

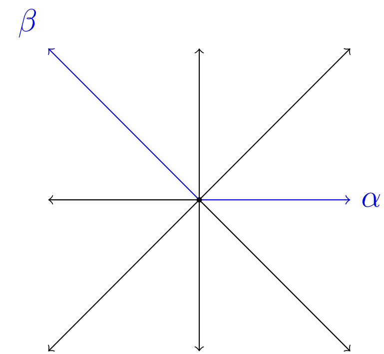
(c) $W(\Phi)$ is irreducible.

Corollary. *Let Φ be indecomposable. Then:*

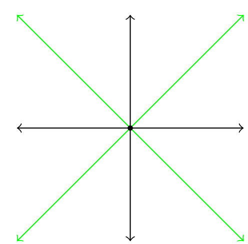
(a) There are at most two different root lengths in Φ.

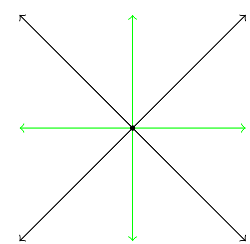
(b) All roots of the same length are conjugate under $W(\Phi)$.

Running example. The root system Φ from above with $\Delta = \{\alpha, \beta\}$ is indecomposable:



These green subsystems are decomposable:





5 Closed subsystems

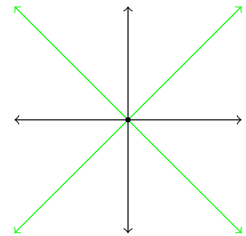
Definition. A subset $\Psi \subseteq \Phi$ is called *closed*, if for all $\alpha, \beta \in \Psi$:

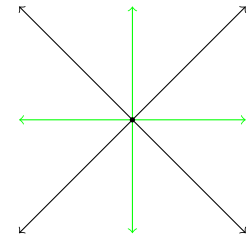
(C1) $s_\alpha.\beta \in \Psi$ and

(C2) if $\alpha + \beta \in \Phi$ also $\alpha + \beta \in \Psi$.

Closed subsets are indeed subsystems.

Running example. Consider the subsystems of Φ from above:





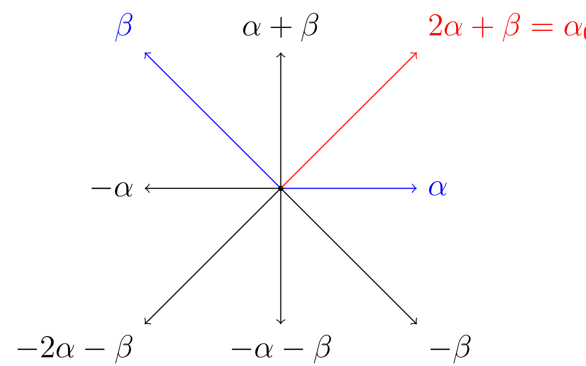
Theorem (Borel-de Siebenthal)**.** *Let Φ be an indecomposable root system with base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and highest root $\alpha_0 = \sum_{i=1}^{l} n_i \alpha_i$ with respect to Δ.*

Then the maximal closed subsystems of Φ up to conjugation by $W(\Phi)$ are those with bases

(1) $\Delta \setminus \{\alpha_i\}$ with $n_i = 1$, and

(2) $\Delta \setminus \{\alpha_i\} \cup \{-\alpha_0\}$ for $1 \leq i \leq l$ with n_i a prime.

Again, let $\Delta = \{\alpha, \beta\}$ and thus the highest root is $2\alpha + \beta$.



Thank you for your attention!

References

[1] G. Malle and D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2011.