Quiver Representation Theory

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Motivation

- Quiver representations are a natural generalization of more classical contents of representation theory.
- For example, any representation of a finite group G is particularly a representation of a quiver Q_G associated with G.
- The moduli space $\mathcal{M}$ of certain quiver representations forms a non-singular complex variety which has many interesting geometric applications.
- For example, $\mathcal{M}$ is closely related to flag manifolds and anti-self-dual connections on four dimensional manifolds.

Quivers

Definition

A *quiver* Q is a finite and directed multigraph, that means

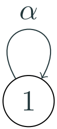
$$Q = (V, E, s, t)$$

consisting of finite sets V and E and maps $s, t \colon E \to V$.

- Elements of V are called *vertices*.
- Elements of E are called *edges*.
- For any edge $\alpha \in E$, the vertex $s(\alpha)$ is called the *source* of α and $t(\alpha)$ is called the *target* of α.

Examples of Quivers

- The *loop quiver* is given by $V = \{1\}$ and $E = \{\alpha\}$ (note that s and t are uniquely determined).



- The *Kronecker quiver* is given by $V = \{1, 2\}$, $E = \{\alpha\}$, $s(\alpha) = 1$, and $t(\alpha) = 2$.

- Given a finite group $G = \{g_1, \ldots, g_n\}$, we associate the *n-loop quiver* Q_G given by $V = \{1\}$ and $E = G$ (note that s and t are uniquely determined).

Quiver Representations

Throughout the talk, fix an algebraically closed field k.

Definition

Let Q be a quiver. A *representation* M of Q is a collection of k-vector spaces

$$\{ M_i \mid i \in V \}$$

and a collection of k-linear maps

$$\left\{ f_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)} \,\middle|\, \alpha \in E \right\}.$$

For example, a representation M of the loop quiver is given by a k-vector space X and a k-linear endomorphism $\varphi \colon X \to X$.

Quiver Representations vs. Group Representations

Let G be a finite group and let $Q_G = \left(\{1\}, G, s, t\right)$ be its associated quiver.

- A representation of G is given by a k-vector space X and a group homomorphism $G \to \mathrm{Aut}_k(X)$.
- A quiver representation M of Q_G is given by a k-vector space X and k-linear maps $\varphi_g \colon X \to X$ for all $g \in G$. Equivalently, one could give a map

$$G \to \mathrm{End}_k(X), \; g \mapsto \varphi_g.$$

Thus, any representation of G is particularly a representation of Q_G.

Category of Representations

For the remainder of the talk, fix a quiver $Q = (V, E, s, t)$. There is a natural notion of morphism $M \to N$ between two representations M and N of Q. This yields the category

$$\mathbf{Rep}\,(Q)\,.$$

Theorem

$\mathbf{Rep}\,(Q)$ *is an abelian category.*

Definition

The *quiver algebra* kQ is the unital and associative k-algebra with generator set $\{e_i \mid i \in V\} \cup E$ satisfying the relations

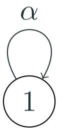
$$e_i^2 = e_i, \quad e_i e_j = 0, \quad e_{t(\alpha)}\alpha = \alpha = \alpha e_{s(\alpha)}$$

for all $i \in V$, $j \in V \setminus \{i\}$, and $\alpha \in E$.

- One should think of e_i as a path of length 0 and of α as a path of length 1. Then, all paths in Q define an element in kQ.
- The unit in kQ is given by $1 = \sum_{i \in V} e_i$.

Example: Quiver Algebra of Loop Quiver

Let L denote the loop quiver.



Since $s(\alpha) = t(\alpha)$, we obtain pairwise distinct elements $\alpha^n \in kL$ for all $n \in \mathbb{N}$. With $\alpha^0 = e_1 = 1$, one easily sees that

$$kL = k[\alpha].$$

Modules over Quiver Algebra vs. Quiver Representations

Let kQ-**Mod** denote the category of left kQ-modules.

Theorem

There is an equivalence of categories

$$\mathbf{Rep}\,(Q) \cong kQ\text{-}\,\mathbf{Mod}\,.$$

This means that we can use the structure theory of kQ-modules to understand representations of Q.

Terminology

Fix a kQ-module M.

Definition

- M is called *simple* if $M \neq 0$ and the only kQ-submodules of M are 0 and M.
- M is called *semisimple* if $M \neq 0$ and M can be written as a direct sum of simple kQ-modules.
- M is called *indecomposable* if $M \neq 0$ and for any direct sum decomposition $M = N_1 \oplus N_2$ there holds $N_1 = 0$ or $N_2 = 0$.
- M is called *projective* if there exists another kQ-module N such that $M \oplus N$ is a free A-module.

Krull-Schmidt Decomposition

Assume that $M \neq 0$ and that $\dim_k(M) < \infty$.

Theorem (Krull-Schmidt)

There exist pairwise non-isomorphic indecomposable kQ-modules $M_1, \ldots, M_r$ and positive integers $m_1, \ldots, m_r$ such that $M \cong \bigoplus_{i=1}^r M_i^{m_i}$. This decomposition is unique up to isomorphism of the modules M_i and permutation of the index i.

Let $M = \bigoplus_{i=1}^r M_i^{m_i}$ denote the Krull-Schmidt decomposition of M. Then, there is an isomorphism

$$\mathrm{End}_{kQ}(M) \cong I \oplus \prod_{i=1}^{r} \mathrm{Mat}_{m_i}(k)$$

of k-vector spaces for some two-sided nilpotent ideal $I \subseteq \mathrm{End}_{kQ}(M)$.

Decompositions of the Quiver Algebra

- For all vertices $i \in V$, we define the left ideal $P(i) = kQe_i$ of kQ. The Krull-Schmidt decomposition of kQ is given by

$$kQ = \bigoplus_{i \in V} P(i) \in kQ\text{-}\mathbf{Mod}.$$

In particular, all $P(i)$ are projective and indecomposable.

- Define $kQ_{\geq 1} \subseteq kQ$ as the ideal generated by E. Then, we obtain

$$kQ \cong \mathrm{End}_{kQ}(kQ) \cong kQ_{\geq 1} \oplus \prod_{i \in V} k \in k\text{-}\mathbf{Mod}.$$

Simple and Projective Indecomposable Representations

Theorem

All projective and indecomposable kQ-modules are isomorphic to $P(i)$ for some vertex $i \in V$.

Assume now that Q does not contain any cycles (equivalently, $\dim_k(kQ) < \infty$), and let $S(i) = P(i)/kQ_{\geq 1}P(i)$.

Theorem

$S(i)$ is a simple kQ-module and all simple kQ-modules are isomorphic to $S(i)$ for some vertex $i \in V$.

Moral: We can classify simple representations of Q as well as projective indecomposable representations of Q.

The Corresponding Representations (Part 1)

Fix a vertex $i \in V$. Then, the representation

$$P(i) = \left(\left\{ P(i)_j \right\}, \{f_\alpha\} \right),$$

corresponding to the module $P(i)$, is given by

$$P(i)_j = \begin{cases} k, & \text{if there exists a path } i \to j \text{ in } Q \\ 0, & \text{else} \end{cases}$$

and

$$f_\alpha = \begin{cases} \mathrm{id}_k, & \text{if } P(i)_{s(\alpha)} = P(i)_{t(\alpha)} = k \\ 0, & \text{else.} \end{cases}$$

The Corresponding Representations (Part 2)

By construction, it follows that the representation

$$S(i) = \left(\left\{ S(i)_j \right\}, \{g_\alpha\} \right),$$

corresponding to the module $S(i)$, is given by

$$S(i)_j = \begin{cases} k, & \text{if } i = j \\ 0, & \text{else} \end{cases}$$

and

$$g_\alpha = 0.$$

Example 1: The Kronecker Quiver

Consider the Kronecker quiver.

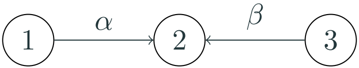
There holds $P(2) = S(2)$, since there are no paths with source $P(2)$ which have length ≥ 1. Thus, we obtain the representations

$$S(1) : k \to 0,$$
$$S(2) = P(2) : 0 \to k,$$
$$P(1) : k \xrightarrow{\text{id}} k.$$

One can show that in this case these are the only indecomposable representations.

Example 2

Consider the following quiver.



Then, the representations $P(i)$ and $S(i)$ are given by

$$S(1) : k \to 0 \leftarrow 0, \quad S(2) = P(2) : 0 \to k \leftarrow 0,$$

$$S(3) : 0 \to 0 \leftarrow k, \quad P(1) : k \xrightarrow{\text{id}} k \leftarrow 0,$$

$$P(3) : 0 \to k \xleftarrow{\text{id}} k.$$

In this case, there is another indecomposable representation, which is not isomorphic to any $P(i)$ or $S(i)$, given by

$$k \xrightarrow{\text{id}} k \xleftarrow{\text{id}} k.$$

The Dimension Vector

For the remainder, let $r = \operatorname{card}(V)$ be the number of vertices.

Definition

Let $M = \left(\{M_i\}, \{f_\alpha\}\right)$ be a representation of Q.

- M is called *finite dimensional* if $\dim_k(M_i) < \infty$ holds for all vertices i.
- If M is finite dimensional, we call

  $$\mathbf{dim}_k(M) = \left(\dim_k(M_i)\right)_{i \in V} \in \mathbb{N}_0^r$$

  the *dimension vector* of M.
- We say that Q is of *finite orbit type* if for all given $\mathbf{n} \in \mathbb{N}_0^r$, there are only finitely many isomorphism classes of representations $M \in \mathbf{Rep}(Q)$ with $\mathbf{dim}_k(M) = \mathbf{n}$.

Underlying Undirected Multigraph

- By $|Q| = (V, E)$, we denote the underlying undirected multigraph of Q.
- For example, let Q denote the Kronecker graph. Then, $|Q|$ is given by the following graph.

Goal: We want to classify quivers of finite orbit type by means of their underlying undirected multigraphs.

Simply Laced Dynkin Diagrams (Part 1)

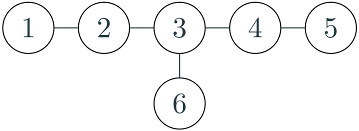
The following graphs are called *simply laced Dynkin diagrams*.

- A_r for some positive integer r:

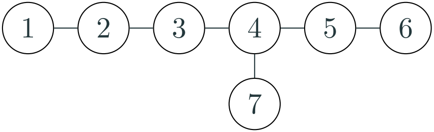
- D_r for some integer $r \geq 4$:

Simply Laced Dynkin Diagrams (Part 2)

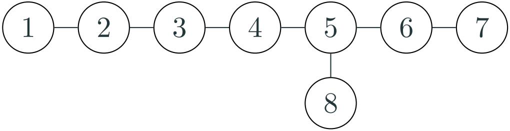
- E_6:



- E_7:



- E_8:



Gabriel's Theorem

Theorem (Gabriel)

The quiver Q is of finite orbit type if and only if $|Q|$ is a simply laced Dynkin diagram.

Euler form

Definition

The $\mathbb{R}$-bilinear form $\langle \cdot, \cdot \rangle_Q : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$, given by

$$\langle \mathbf{m}, \mathbf{n} \rangle_Q = \sum_{i \in V} m_i n_i - \sum_{\alpha \in E} m_{s(\alpha)} n_{t(\alpha)}$$

for all $\mathbf{m} = (m_i)_{i \in V}, \mathbf{n} = (n_i)_{i \in V} \in \mathbb{R}^r$, is called *Euler form* of Q.

- The Euler form depends on the direction of the edges $\alpha \in E$.
- In particular, the Euler form is non-symmetric.

Tits Form

Definition

The quadratic form $q_Q \colon \mathbb{R}^r \to \mathbb{R}$ associated to the Euler form, that means

$$q_Q(\mathbf{n}) = \langle \mathbf{n}, \mathbf{n} \rangle_Q = \sum_{i \in V} n_i^2 - \sum_{\alpha \in E} n_{s(\alpha)} n_{t(\alpha)}$$

for all $\mathbf{n} = (n_i)_{i \in V} \in \mathbb{R}^r$, is called *Tits form* of Q.

- The Tits form only depends on $|Q|$.
- $|Q|$ is a simply laced Dynkin diagram if and only if q_Q is positive definite.

Outlook (Part 1)

- Fix $k = \mathbb{C}$.
- For a given multigraph G, consider the double quiver $Q = (V, E, s, t)$.
- For example

- For families $M = \left\{ M_i \mid i \in V \right\}$ and $N = \left\{ N_i \mid i \in V \right\}$ of hermitian vector spaces, we consider

$$R\left(Q, M, N\right) = \bigoplus_{\alpha \in E} \operatorname{Hom}\left(M_{s(\alpha)}, M_{t(\alpha)}\right) \oplus \bigoplus_{i \in V} \operatorname{Hom}\left(M_i, N_i\right) \oplus \bigoplus_{i \in V} \operatorname{Hom}\left(N_i, M_i\right)$$

Outlook (Part 2)

- Elements of $R(Q, M, N)$ look as follows.

$$
\begin{array}{ccc}
M_1 & \overset{x}{\underset{y}{\rightleftarrows}} & M_2 \\
\psi_1 \Big\uparrow \Big\downarrow \varphi_1 & & \psi_2 \Big\uparrow \Big\downarrow \varphi_2 \\
N_1 & & N_2
\end{array}
$$

- There is a natural action $\mathrm{GL}(M) \curvearrowright R(Q, M, N)$, where $\mathrm{GL}(M) = \prod_i \mathrm{GL}(M_i)$.
- There are two different ways to build quotients along this action. These quotients are called *quiver varieties*.
- (Twisted) GIT quotient yields that quiver varieties are non-singular complex symplectic varieties.
- Hyperkähler quotient yields that quiver varieties are hyperkähler manifolds.