Asymptotic geometry of the Higgs moduli space

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Motivation

Why solutions of

$$\mathbf{F}_A^{\perp} + [\Phi, \Phi^{\dagger}] = 0 = \overline{\partial}_A \Phi? \quad (1)$$

For mathematicians:

Moduli spaces of Higgs bundles have an interesting geometry

Results can be translated into topology, Riemann surfaces, and harmonic analysis

For physicists:

Yang-Mills theory $d_A \star \mathbf{F}_A^{\perp} = 0$

Magnetic monopoles $D_A \star \mathbf{F}_A^{\perp} + [\Phi, D_A \Phi^{\dagger}] = 0 = D_A \star D_A \Phi$

Differential geometry on bundles

Bundle geometry I

Consider holomorphic, hermitian bundle $\mathbb{C}^r \to E \to M$.

Problem: Metric $h_{\alpha\overline{\beta}}$ on $E_x \neq$ metric $g_{m\overline{n}}$ on T_xM.

Define local frame $\{e_\alpha\}_{\alpha=1}^r$ of sections: $e_\alpha = e_\alpha(x)$.

Under transition maps $e_\alpha \to e'_\alpha = (T_x)^\alpha_\gamma e_\alpha$, the fibre metric transforms as $h \to h' = ThT^*$

We have the Kähler $(1,1)$-form $\omega = h_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$

For E, there is a unique bundle Dolbeault operator $\overline{\partial}^E$ with $\overline{\partial}^E(\eta s) = \eta\overline{\partial}^E s + (\overline{\partial}\eta)s$ for complex differential forms η on M and sections s of E. Locally, $\overline{\partial}^E s = \overline{\partial} s$.

Get unique connection $\nabla^h = \nabla + \overline{\partial}^E$ with $dh(s,t) = h(\nabla^h s, t) + h(s, \nabla^h t)$, the **Chern connection**

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Bundle geometry II

Consider $M = \Sigma$ Riemann surface. We want to compute the Chern classes of the bundle, hence find curvature $\mathbf{F}_\nabla$ of Chern connection

$\rightarrow$ It is $\mathbf{F}_\nabla = \overline{\partial}(h^{-1}\partial h)$ by Cartan formalism.

Then the **Chern classes** of the bundle are given by

$$\det\left(\frac{\mathrm{i}t\mathbf{F}}{2\pi} + \mathbf{1}\right) =: \sum_{k=0}^{r_E} c_k(E)t^k. \tag{2}$$

Two important invariants for vector bundle:

Rank $r_E = \dim E_x$,

Degree $d_E = \int_M c_1(E) = \frac{\mathrm{i}}{2\pi}\int_M \operatorname{tr}\mathbf{F}_\nabla$

We also define the **slope** $\mu_E = \frac{d_E}{r_E}$

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Higgs bundles

Stability of Higgs bundles

Definition

A **Higgs bundle** is a tupel $(\overline{\partial}^E, \Phi)$ where $(E, h) \to \Sigma$ is a hermitian vector bundle over a Riemann surface Σ with metric h and holomorphic structure $\overline{\partial}^E$, and $\Phi = \Phi^\alpha_{\beta m} dz^m$ is a $(1,0)$-form with $\overline{\partial}^E \Phi = 0$.
A **Φ-invariant sub-bundle** $F < E$ is such that $\Phi(F) < F \otimes K_\Sigma$.

Definition

A vector bundle is **stable**, if for any subbundle $F < E$, $\mu_F < \mu_E$.
A Higgs bundle is **stable**, if for all Φ-invariant subbundle $F < E$ holds $\mu_F < \mu_E$. Direct sums of stable Higgs bundles of the same degree are **polystable**.

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Remarks and examples

The moduli space of polystable Higgs bundles by a gauge action $\mathrm{GL}(E)$ is a noncompact smooth complex manifold $\mathcal{M}^D_{\mathrm{GL}}(r_E, d_E)$. Its dimension depends on the genus g of Σ: $\dim_{\mathbb{C}} \mathcal{M}^D_{\mathrm{GL}}(r_E, d_E) = 2 + r^2(2g - 2)$ (Hitchin 1987).

Example (Higgs bundles)

$(\overline{\partial}^E, 0)$ is a stable Higgs bundle if E is stable as a vector bundle.

Nontrivial: Let $E = K_\Sigma^{1/2} \oplus K_\Sigma^{-1/2}$, and let $q = q(z)dz \otimes dz$ be a holomorphic quadratic differential on Σ. A Higgs field is given by

$$\Phi = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \tag{3}$$

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Kobayashi-Hitchin correspondence

Definition

A **Hitchin pair** (A, Φ) of a connection $D_A = d + A$ and an $\mathrm{End}(E)$-valued $(1,0)$-form Φ fulfills

$$\overline{\partial}_A \Phi = 0 \tag{4}$$

$$\mathbf{F}_A + [\Phi, \Phi^\dagger] = -2\pi \mathrm{i} \mu_E \mathbf{1}_E \omega. \tag{5}$$

We may write $\mathbf{F}_A^\perp := \mathbf{F}_A + 2\pi\mathrm{i}\mu_E \mathbf{1}_E \omega$ and thus $\mathbf{F}_A^\perp + [\Phi, \Phi^\dagger] = 0$.

Theorem (Kobayashi-Hitchin, proven by Uhlenbeck-Yau 1986)

There exists an isomorphism between the moduli spaces of irreducible Hitchin pairs and of polystable Higgs bundles, given by $(A, \Phi) \mapsto (\overline{\partial}_A, \Phi)$. *Also,* $\mathbb{D} = \overline{\partial}_A + \partial_A^h + \Phi + \Phi^\dagger$ *gives a projectively flat connection.*

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Asymptotic decoupling

The decoupling problem

If (A, Φ) solves the equation $\overline{\partial}_A \Phi = 0 = \mathbf{F}_A^{\perp} + t^2[\Phi, \Phi^{\dagger}]$, then $(A, t \cdot \Phi)$ is a Hitchin pair.

For $t \to \infty$ we (heuristically) get the **decoupled selfduality equations**:

$$\mathbf{F}_A^{\perp} = 0 = [\Phi, \Phi^{\dagger}]. \tag{6}$$

Problem: Solutions (A_j, Φ_j) may not converge to solutions of the decoupled equations!

However, we can at least get local decoupling of solutions under certain conditions for Φ and E.

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Mochizuki's theorem

Theorem (Asymptotic decoupling on discs, Mochizuki 2016)

Let $\{U_\alpha\}_{\alpha \in S}$ be a finite covering of Σ and $\Delta(R)$ an open disc of radius R, such that $(E, \Phi)|_{U_\alpha} = \bigoplus_\alpha (E_\alpha, \Phi_\alpha)$ for all $P \in \Delta(R)$ is a decomposition of E_x into eigenspaces of Φ, where $\Phi_\alpha = f_\alpha dz$. If:

d minimum distance of points in S fulfills $d \geq 1$,

λ eigenvalues of f_α have distance $\leq \frac{d}{100}$ from α,

We have $M, C > 0$ st. $|\lambda| < M$ and $Cd \geq M$ on $\Delta(R)$,

($r_{E_\alpha} = 1$ and $d_E = 0$),

then we find constants K and ϵ such that on a smaller disc $\Delta(R_2)$

$$|\mathbf{F}_A^\perp|_{g,h} = |[\Phi, \Phi^\dagger]|_{g,h} \leq Ke^{-\epsilon d}. \tag{7}$$

Case $r_E = 2$ *quantitatively* by Swoboda, Mazzeo, Weiß and Witt in 2015

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Let $\{U_\alpha\}_{\alpha \in S}$ be a finite covering of Σ and $\Delta(R)$ an open disc of radius R, such that $(E, \Phi)|_{U_\alpha} = \bigoplus_\alpha (E_\alpha, \Phi_\alpha)$ for all $P \in \Delta(R)$ is a decomposition of E_x into eigenspaces of Φ, where $\Phi_\alpha = f_\alpha dz$. If:

d minimum distance of points in S fulfills $d \geq 1$,

λ eigenvalues of f_α have distance $\leq \frac{d}{100}$ from α,

We have $M, C > 0$ st. $|\lambda| < M$ and $Cd \geq M$ on $\Delta(R)$,

($r_{E_\alpha} = 1$ and $d_E = 0$),

then we find constants K and ϵ such that on a smaller disc $\Delta(R_2)$

$$|\mathbf{F}_A^\perp|_{g,h} = |[\Phi, \Phi^\dagger]|_{g,h} \leq K e^{-\epsilon d}. \tag{7}$$

Case $r_E = 2$ *quantitatively* by Swoboda, Mazzeo, Weiß and Witt in 2015

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Asymptotic decoupling

Proof of Mochizuki's theorem I

For the decomposition $(E, \Phi)|_{U_\alpha} = \bigoplus_\alpha (E_\alpha, \Phi_\alpha)$, we obtain two projections onto the E_α

- π_α induced by the eigendecomposition of $\Phi = fdz$ at P, and
- $\pi'_\alpha = (\pi'_\alpha)^\dagger$ obtained by orthogonalisation.

We define $\rho_\alpha = \pi_\alpha - \pi'_\alpha$ to be the **skewedness** of the decomposition in direction α.

Lemma

If $|f_P|_h \leq G_1 d + G_2$, then $d \cdot \delta \cdot |\rho_\alpha|_h \leq |[f_{h,P}{}^\dagger, \pi_\alpha]|_h$ for $\delta = \delta(G_i, r_E)$.

Also, $|\rho_\alpha|_h \leq |\pi_\alpha|_h \leq B$ for some constant $B(G_i, r_E)$.

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Asymptotic decoupling

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Lemma

Let now s be a section of $\mathrm{End}(E)$ with $\overline{\partial} s = 0 = [\Phi, s]$. Then,

$$-\partial\overline{\partial}\ln|s|_h^2 \leq -\frac{|[f_h{}^\dagger, s]|_h^2}{|s|_h^2}. \tag{8}$$

Idea:

$$-|s|_h^2\partial\overline{\partial}\ln|s|_h^2 \leq -|h(s, \overline{\partial}\partial s - \partial\overline{\partial} s)| = -|h(s, R(\overline{\partial}, \partial)s)|$$
$$= -|h(s, [\Phi, \Phi^\dagger](\overline{\partial}, \partial)s)| \leq -|h([f_h{}^\dagger, s], [f_h{}^\dagger, s])|$$

Asymptotic decoupling

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Asymptotic decoupling

Proof of Mochizuki's theorem III

Lemma

For any $R_1 \in (0, R)$ *and* $R_2 \in (0, R_1)$ *there are constants* C_1, C_2, C_{11} *dependent on* R, R_1, r_E*:*
$|f_P|_h \le C_1 M + C_2$ *on* $\Delta(R_1)$ *and* $|\rho_\alpha|_h \le C_{11} e^{-\epsilon_0 d}$ *on* $\Delta(R_2)$*.*

Proofs are very similar:
First use (8) to find a differential inequality:

$$-\partial\overline{\partial}\ln|f_P|_h^2 \le -\frac{C_3^2}{4}|f_P|_h^2, \qquad -\partial\overline{\partial}\ln\left(\frac{|\pi_\alpha|_h^2}{r_\alpha}\right) \le -\epsilon_1 d^2 \ln\left(\frac{|\pi_\alpha|_h^2}{r_\alpha}\right)$$

Then find solutions to equal case:

$$|f_P|_h^2 = \frac{B}{(R^2 - z\bar{z})^2}, \qquad \ln\left(\frac{|\pi_\alpha|_h^2}{r_\alpha}\right) = e^{-\epsilon_2 z\bar{z}d}$$

The following subsets of $\Delta(R_1)$ have minima higher than the boundary:

$$\left\{P \middle| |f_P|_h^2 > B(R^2 - |P|^2)^{-2}\right\}, \quad \left\{P \middle| \ln(|\pi_\alpha|_h^2 r_\alpha^{-1}) > Ce^{(|P|^2 - R_1^2)\epsilon d}\right\}$$

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Asymptotic decoupling

Proof of Mochizuki's theorem IV

By $|f_{|P}|_h \leq C_1 M + C_2$ and $|\rho_\alpha|_h \leq C_{11} e^{-\epsilon_0 d}$, we arrive at

$$|[f, \pi_\alpha^\dagger]|_h = |[f^\dagger, \pi_\alpha]|_h \leq C_{20} e^{-\epsilon_{20} d} \text{ on } \Delta(R_2). \tag{9}$$

Because $\mathbf{F}_A^\perp = [\Phi, \Phi^\dagger]$, it is now enough that $[\Phi, \Phi^\dagger]$ decays exponentially with d. This is indeed the case.

Remark

Thus follows asymptotic decoupling because for rescaling Φ to $t \cdot \Phi$, $|K|$ in the theorem scales to $t^n |K|$ for some n, but d scales $t \cdot d$. Therefore $|[t \cdot \Phi, t \cdot \Phi^\dagger]|_{g,h} \leq t^n K e^{-\epsilon t d}$ goes to zero for $t \to \infty$.

Asymptotic decoupling

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Thanks for your attention!

Questions ?

Appendix
Important sources

Uhlenbeck, Yau 1986: On the existence of Hermitian Yang-Mills connections in stable vector bundles

Hitchin 1987: The self-duality equations on a Riemann surface

Mazzeo, Swoboda, Weiß, Witt 2015: Ends of the moduli space of Higgs bundles

Mochizuki 2016: Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces

Appendix

Fibre and vector bundles I

Definition

Let E, M and F topological spaces, and $p : E \twoheadrightarrow M$ a continuous surjection with $p^{-1}(x)$ homeomorphic to F for all $x \in M$.
Each point shall possess a trivialisation, that is a neighbourhood U with a homeomorphism $t : E_x = p^{-1}(U) \to F \times U$
The tupel (F, E, p, M) is then called a **fibre bundle**.
Suggestively write: $F \to E \xrightarrow{p} M$

Definition

A **section** of a fibre bundle is an inclusion $s : M \to E$, with $p \circ s = \mathrm{id}_M$.

Definition

A fibre bundle (F, E, M, p) is called **vector bundle** if F is a vector space.

Appendix

Fibre and vector bundles II

Definition

A **Fibre bundle morphism** between (F, E, p, M) and (F', E', p', M') is a pair (ψ, f) of continuous maps $\psi : E \to E'$ and $f : M \to M'$ such that $p' \circ \psi = f \circ p$.
A **vector bundle morphism** is a fibre bundle morphism between two vector bundles, for which $\psi| : E_x \to E'_{f(x)}$ is linear everywhere on M.

Definition

Between two trivialisations (U_i, t_i) and (U_j, t_j) there is a map $T_{ij} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F, (x, v) \to t_j \circ t_i^{-1}(x, v) = (x', v')$. This map can be viewed as a diffeomorphism $T_x : F \to F, v \to v'$. For vector bundles, T_x is always a matrix from $\mathrm{GL}_r(\mathbb{K})$. We call $T_x = T(x)$ the **transition map**.

Appendix

Fibre and vector bundles III

Examples of vector bundles:

- Trivial bundles: $\mathbb{K}^r \to \mathbb{K}^r \times M \to M$.
- Möbius bundle: $\mathbb{R} \to \mathrm{Mb} \to S^1$.
- Tangential and cotangential bundles: $E = TM$, $E = T^*M$.
- Canonical bundle with fibre $(K_M)_x = \det\left(T_x^*M\right) = \bigwedge_{i=1}^r T_x^*M$

Examples of other fibre bundles:

- Transition maps of vector bundles are **principal bundles** $\mathrm{GL}(F) \to P \to M$.
- The associated principal bundle of the Möbius bundle is $\mathbb{Z}_2 \to P \to S^1$.
- The **frame bundle** of a manifold $\mathrm{GL}_r(\mathbb{K}) \to \mathrm{GL}M \to M$ is associated to the tangential bundle.

Appendix

Complex differential forms

We want to extend the concepts of real exterior calculus to complex manifolds:

- Instead of n-forms $\omega = \omega_I dx^I$: (p, q)-**forms** $\omega = \omega_{IJ} dz^I \wedge d\bar{z}^J$
- Decompose $d = \partial + \overline{\partial}$ with $\partial f = \partial_i f dz^i$ and $\overline{\partial} f = \overline{\partial}_i f d\bar{z}^i$
- We get $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$
- Cauchy-Riemann equations: ω holomorphic iff $\overline{\partial}\omega = 0$
- Rules for computation: $\overline{\omega \wedge \eta} = \overline{\omega} \wedge \overline{\eta}$, $\overline{d\omega} = d\overline{\omega}$, $\overline{f^* \omega} = \bar{f}^* \overline{\omega}$
- ∂ and $\overline{\partial}$ obey product rules, $\overline{\partial\omega} = \overline{\partial}\overline{\omega}$, $\partial\overline{\partial} = -\overline{\partial}\partial$
- Cohomology group $H^{p,q} = \ker \overline{\partial}^{p,q} / \mathrm{im} \overline{\partial}^{p,q-1}$