Blaschke conjecture and Hopf rigidity

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Outline

History and formulation of the Blaschke conjecture

Setting: (M, g) connected, complete Riemannian manifold.

Definition

We define the **unit tangent bundle** to be the subset $SM \subset TM$ given by:

$$SM = \{(p, v) \in TM \,|\, g_p(v, v) = 1\}.$$

Definition

Let $v \in SM$. We define $con(v) \in (0, \infty]$ to be the first positive time t such that $\gamma_v(0)$ is conjugate to $\gamma_v(t)$ along γ_v. If no such time exists we set $con(v) = \infty$. For $p \in M$ we define the **first conjugate locus of p**

$$Con(p) := \{\gamma_v(con(v)) \,|\, v \in S_pM\}.$$

Definition

(M, g) is called **wiedersehen manifold** if for all $p \in M$, $Con(p)$ consists of one single point. (M, g) is called **wiedersehen surface** if in addition $dim M = 2$.

Example

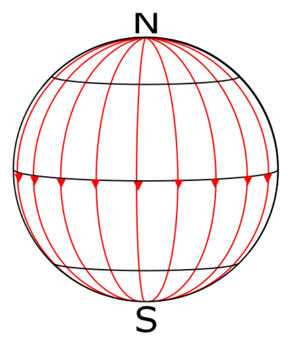
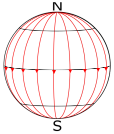


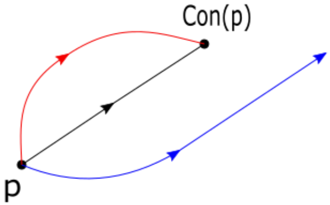
Figure: The red lines are the geodesics starting at the north pole N and meeting at the south pole S.

Why the name "wiedersehen"?

What does happen: There exists a time $a > 0$ such that any two unit speed geodesics starting at a common point p will meet again after time a at the conjugate point of p.



What doesn't happen:



Blaschke conjecture

1921: Blaschke conjectures that up to isometry the only wiedersehen surface in $\mathbb{R}^3$ is the round sphere.

Example

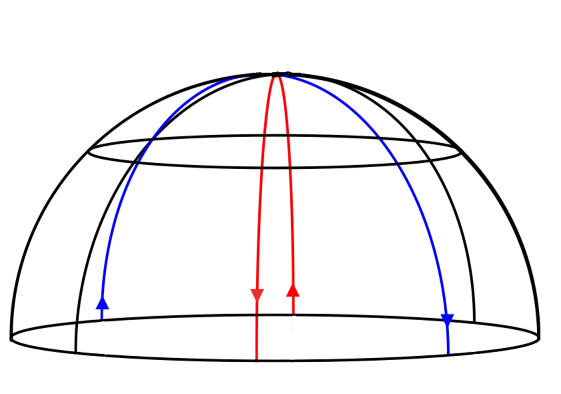


Figure: $\mathbb{RP}^2$ with the canonical metric. The red and blue lines are geodesics starting at the north pole N. After time π they meet there again.

Theorem

(L. W. Green, 1963) Every wiedersehen surface has constant positive Gaussian cuvature.

Remark

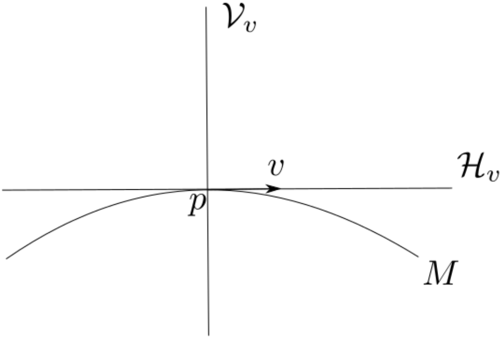
Every wiedersehen surface is thus isometric to the sphere or $\mathbb{RP}^2$ *with (a positive multiple of) the canonical Riemannian metrics.*

Splitting of the double tangent bundle

The double tangent bundle $TTM \to TM$. There are horizontal and vertical subbundles $\mathcal{H}$ and $\mathcal{V}$ of $TTM \to TM$ with:

1. $TTM = \mathcal{H} \bigoplus \mathcal{V}$.
2. For each $v \in TM$, $\mathcal{H}_v \cong T_{\pi(v)}M, \mathcal{V}_v \cong T_{\pi(v)}M$.

Hence $T_v TM \cong T_{\pi(v)}M \bigoplus T_{\pi(v)}M$.



Sasaki metric

- Splitting of TTM leads to natural metric g^S on TM.
- Pullback under $SM \to TM$ gives Sasaki metric on SM.

Theorem

Geodesic flow of M is volume preserving, both considered as a map $\Phi^t : TM \to TM$ and $\Phi^t : SM \to SM$.

Proof of the Blaschke conjecture

Theorem

Let (M, g) be a simply connected wiedersehen manifold. Then:

1. *M is diffeomorphic to S^m.*
2. *$injM = diamM = a$.*
3. *For all $p \in M$ and $v \in S_pM$ we have $\gamma_v(a) = Con(p)$.*
4. *All unit speed geodesics in M are periodic with (least) period $2a$.*
5. *Con is an involutive isometry with d(p,Con(p))=a.*

Theorem

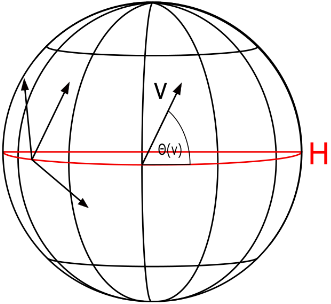
Let (M, g) be a closed Riemannian surface and let there be a time $a > 0$ such that along all unit speed geodesics no conjugate point appears before time a. Then

$$vol(M) \geq \frac{2a^2}{\pi}\chi(M)$$

and equality holds iff the Gaussian curvature is constant $K = \frac{\pi^2}{a^2}$.

Goal: Compute $vol(M)$ for simply connected wiedersehen surface M.
Easier: Compute $vol(SM)$.
Idea: Take closed geodesic $H \subset M$, set $SM_H = \{(p, v) \in SM \mid p \in H\}$ and consider

$$F : [0, a) \times SM_H \to SM, \qquad F(t, v) := \Phi^t(v).$$

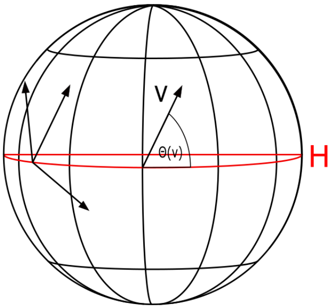


Theorem

(Santalo's formula)Let H be a hypersurface in (M, g). With F defined as above:

$$|F|_*(t, v) = sin(\theta(v)),$$

where $\theta(v) \in [0, \pi/2]$ is the angle between $T_{\pi(v)}H$ and v.



Proof.

(Blaschke conjecture)

$$\begin{aligned} vol(SM) &= \int_{[0,a)\times SM_H} sin\theta(v) vol_{\mathbb{R}\times SM_H} \\ &= \int_{[0,a)} vol_{\mathbb{R}} \int_{SM_H} sin\theta(v) vol_{SM_H} \\ &= a \int_H \int_{S_pM} sin\theta(v) vol_{S_pM}(v) vol_H(p) \\ &= a \int_0^{2\pi} |sin\theta| d\theta \int_H vol_H = a\cdot 4\cdot 2a = 8\cdot a^2. \end{aligned}$$

Hence $vol(M) = \frac{vol(SM)}{2\pi} = \frac{4a^2}{\pi}$. □

Generalisation of Blaschke's conjecture

How about higher dimensions?
Problem: Dimension 2 in Green's proof is crucial in applying

- Characterisation for constant sectional curvature
- Santalo's formula

Theorem

Let g be a Riemannian metric on S^m with $inj(S^m, g) = diam(S^m, g)$. Then (S^m, g) has constant positive sectional curvature.

Corollary

Every wiedersehen manifold has constant positive sectional curvature.

Further Generalisation: Classify Riemannian manifolds (M, g) with $inj(M, g) = diam(M, g)$.

Closed surfaces without conjugate points

Theorem

Let (M, g) be a closed Riemannian surface. If on M no conjugate points exist, then

$$\int_M K\mu_M \leq 0$$

and equality holds if and only if the Gaussian curvature K is identically zero.

Corollary

Let g be a Riemannian metric without conjugate points on the two-dimensional torus T. Then (T, g) is flat.

Proof.

Idea: Use nonconjugacy to construct an integrable function $u : SM \to \mathbb{R}$ such that $u(\Phi^t(v))$ solves the Ricatti equation, i.e.

$$\frac{d}{dt}u(\Phi^t(v)) + u^2(\Phi^t(v)) + K(\pi \circ \Phi^t(v)) = 0.$$

Integrate with respect to t and over SM

$$\begin{aligned} -\int_{SM}\int_0^1 u^2(\Phi^t(v))dt &= \underbrace{\int_{SM} u(\Phi^1(v)) - u(v)}_{=0} + \int_{SM}\int_0^1 K(\pi \circ \Phi^t(v))dt \\ &= \int_0^1\int_{SM} K(\pi \circ \Phi^t(v))dt = \int_0^1\int_{SM} K(\pi(v))dt \\ &= 2\pi \int_M K. \end{aligned}$$

□

Summary

- Wiedersehen manifolds
  - Definition: $Con(p)$ is a singleton for all $p \in M$.
  - Why "wiedersehen"?
- Blaschke conjecture
  - Statement: wiedersehen surfaces have constant Gaussian curvature.
  - Green's proof: Volume inequality characterising constant curvature.

Left to do: show theorem that gives characterisation for constant sectional curvature. This follows from

Theorem

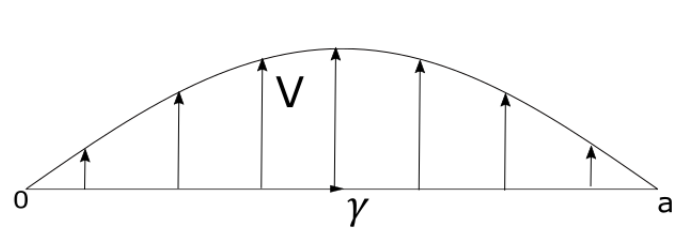
Let (M, g) be a closed Riemannian manifold and $a > 0$ such that for every $v \in SM, con(v) \geq a$. Then

$$\frac{a^2}{\pi^2} \int_M scal \mu_g \leq m(m-1) vol_g(M)$$

where equality holds if and only if M has constant sectional curvature $\frac{\pi^2}{a^2}$. Here $scal : M \to \mathbb{R}$ denotes the scalar curvature.

Proof.

$\gamma_v : [0, a] \to M$ unit-speed geodesic, E parallel normal vector field along γ_v, $V(t) = sin(\frac{\pi t}{a})E(t)$.



$$0 \leq \int_0^a g(\dot{V}, \dot{V}) - R(V, \dot{\gamma}_v, \dot{\gamma}_v, V)dt.$$

Summation of inequality for orthonormal frame along γ_v:

$$\int_0^a sin^2(\frac{\pi t}{a})Ric(\dot{\gamma}_v)dt \leq (m-1)\frac{\pi^2}{2a}.$$

Splitting of the double tangent bundle

$\pi : TM \to M$ tangent bundle, $V : (-\epsilon, \epsilon) \to TM$. There are two natural maps $C, \pi_* : TTM \to TM$:

1. $\pi_* \dot{V}(0) = \frac{d}{dt}_{|t=0} \pi \circ V(t)$, and
2. $C(\dot{V}(0)) = {}^{\pi \circ V}\nabla_{\partial_t} V(0)$.

Then $\mathcal{H} := kerC, \mathcal{V} := ker\pi_*$ are subbundles of $TTM \to TM$ with

1. $TTM = \mathcal{H} \bigoplus \mathcal{V}$.
2. For each $v \in TM, \pi_* : \mathcal{H}_v \to T_{\pi(v)}M$ and $C : \mathcal{V}_v \to T_{\pi(v)}M$ are isomorphims.

Sasaki metric

$$g_v^S(Z_1, Z_2) = g_{\pi(v)}(\pi_* Z_1, \pi_* Z_2) + g_{\pi(v)}(CZ_1, CZ_2),$$

where $Z_1, Z_2 \in T_v TM$.