Rational homotopy theory for singular spaces

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Forms of Poincaré duality

- Poincaré duality is a cornerstone of modern algebraic topology
- setting: closed oriented n-manifolds M^n
- Cap product with the **fundamental class** $[M^n] \in H_n(M^n;\mathbb{Z})$:

$$D\colon H^i(M^n;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M^n;\mathbb{Z}), \quad D(a) = a \cap [M^n].$$

- Non-degenerate bilinear form (**intersection pairing**):

$$H^i(M^n;\mathbb{Q}) \otimes H^{n-i}(M^n;\mathbb{Q}) \to \mathbb{Q}, \quad a \otimes b \mapsto \langle a \cup b, [M^n] \rangle.$$

- Symmetry of **Betti numbers** $b_i(M) = \operatorname{rank}_{\mathbb{Z}} H_i(M^n;\mathbb{Z})$:

$$b_i(M) = b_{n-i}(M^n).$$

. . . les nombres de Betti également distants des extrêmes sont égaux. *(H. Poincaré, 1895)*

Geometric perspectives on the intersection pairing

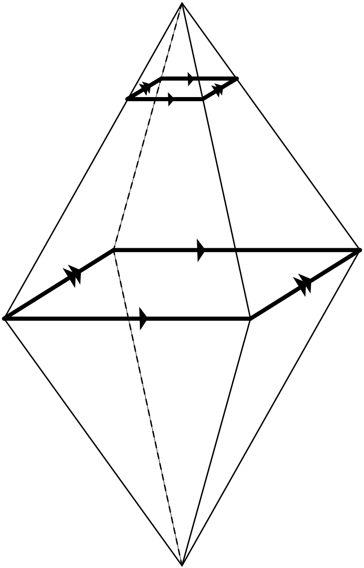
- If M^n is smooth, then, using smooth differential forms $\Omega^*(M)$, **de Rham's theorem** provides an $\mathbb{R}$-algebra isomorphism $H^*_{dR}(M) \cong H^*(M;\mathbb{R})$, and the intersection pairing reads

$$H^i_{dR}(M) \otimes H^{n-i}_{dR}(M) \to \mathbb{R}, \quad [\omega] \otimes [\eta] \mapsto \int_M \omega \wedge \eta.$$

- If M^n is not equipped with a smooth structure, then rational homotopy theory provides a similar perspective via Sullivan's **piecewise linear polynomial differential forms** $A_{PL}(M)$.
- There is a natural $\mathbb{Q}$-algebra isomorphism $H^*(A_{PL}(M)) \cong H^*(M;\mathbb{Q})$, and the intersection pairing reads

$$H^i(A_{PL}(M)) \otimes H^{n-i}(A_{PL}(M)) \to \mathbb{Q}, \quad [x] \otimes [y] \mapsto \int_M x \cdot y.$$

Beyond manifolds: the suspension of a 2-torus



Poincaré's own example:
$X^3 = \text{suspension}(S^1 \times S^1)$

the two cone points have no Euclidean neighborhood in X

$H_1(X) = \widetilde{H}_0(S^1 \times S^1) = 0$
$H_2(X) = H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$
$\Rightarrow \boldsymbol{b_1(X) \neq b_2(X)}$

the filtration
$X^3 \supset X^0 = \{\text{two cone points}\}$
gives X the structure of a **stratified pseudomanifold**

Formal definition of stratified pseudomanifolds

- idea: collect equisingular points in strata
- A **0-dimensional stratified pseudomanifold** is a countable set with the discrete topology.
- For $n > 0$, an ***n*-dimensional stratified pseudomanifold** is a paracompact Hausdorff space X equipped with a filtration

$$X = X_n \supseteq X_{n-1} = X_{n-2} \supseteq X_{n-3} \supseteq \cdots \supseteq X_1 \supseteq X_0 \supseteq X_{-1} = \emptyset$$

  by closed subsets such that the "pure strata" $X_j \setminus X_{j-1}$ are j-dimensional manifolds, the "top stratum" $X_{reg} = X_n \setminus X_{n-2}$ is dense in X, and **local normal triviality** holds as follows.
- Every $x \in X_j \setminus X_{j-1}$ has an open neighborhood $U \subset X$ (stratified by $U_k = U \cap X_k$) which is **stratification preserving** homeomorphic to $\mathbb{R}^j \times C(L)$, where
  - L is a compact stratified pseudomanifold of dimension $n - j - 1$,
  - $C(Z) = Z \times [0,1)/(Z \times \{0\})$ is the open cone on a space Z, and
  - $(\mathbb{R}^j \times C(L))_k = \mathbb{R}^j \times C(L_{k-j-1})$.

Generalized forms of Poincaré duality

The frequent appearance of stratified pseudomanifolds as ...

- simplicial complexes,
- orbit spaces of group actions,
- real or complex algebraic varieties,
- orbifolds and conifolds in physics,
- as well as certain compactifications of noncompact spaces, e.g. various moduli spaces in number theory,

... motivated the search for generalized forms of Poincaré duality:

- intersection homology (M. Goresky, R.D. MacPherson, 1980)
- L^2 cohomology (J. Cheeger, $\sim$1980)
- **intersection spaces** (M. Banagl, 2009)

Intersection homology

- setting: n-dimensional stratified pseudomanifold X^n
- around 1980, Goresky and MacPherson defined intersection homology groups $IH_*^{\overline{p}}(X)$ depending on a **perversity function** $\overline{p}\colon \{2,3,4,\dots\} \to \{0,1,2,\dots\}$ with certain growth conditions
- geometric intuition: for PL spaces, $IH_*^{\overline{p}}(X)$ is the homology of the complex of $\overline{p}$-allowable chains, i.e., PL chains that deviate from being transverse to the singular strata of X in a way controlled by the parameter $\overline{p}$

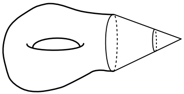
Theorem (Generalized Poincaré duality; Goresky-MacPherson)

Let $\overline{p}$ and $\overline{q}$ be **complementary** *perversities, i.e., $\overline{p}+\overline{q}=(0,1,2,3,4,\dots)$. If X^n is compact and oriented, then there is a non-degenerate bilinear form*

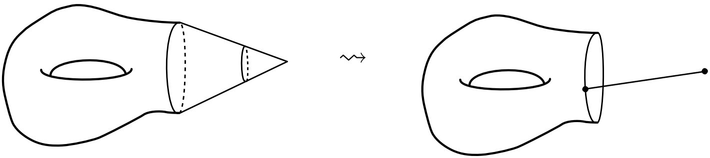
$$IH_i^{\overline{p}}(X;\mathbb{Q}) \otimes IH_{n-i}^{\overline{q}}(X;\mathbb{Q}) \to \mathbb{Q}.$$

Intersection space construction: a toy example

- setting: X^n is an n-dimensional stratified pseudomanifold with one isolated singularity obtained by coning off the boundary of a connected compact oriented n-manifold M^n



- fix a **truncation degree** $k > 0$
- truncate the homology $H_*(\partial M; \mathbb{Z})$ in degrees $\geq k$ by choosing a **spatial homology truncation** $h\colon (\partial M)_{<k} \to \partial M$
- **Banagl, 2009**: $X^n \rightsquigarrow \boldsymbol{I^k X = M \cup_{\partial M} \mathrm{cone}(h)}$



- Generalized Poincaré duality: $\widetilde{H}_i(I^k X; \mathbb{Q}) \cong \widetilde{H}^{n-i}(I^{n-k} X; \mathbb{Q})$

Conception of intersection spaces

- setting: n-dimensional stratified pseudomanifold X^n
- the **intersection spaces** of X should be a family $I^{\overline{p}}X$ of homotopy theoretic desingularizations of X in the sense that for complementary perversities $\overline{p}$ and $\overline{q}$,

$$\boldsymbol{HI^{\overline{p}}_*(X)} := \widetilde{H}_*(I^{\overline{p}}X;\mathbb{Q}) \text{ and } \boldsymbol{HI^*_{\overline{q}}(X)} := \widetilde{H}^*(I^{\overline{q}}X;\mathbb{Q})$$

  satisfy generalized Poincaré duality $HI^{\overline{p}}_i(X) \cong HI^{n-i}_{\overline{q}}(X)$
- to form $I^{\overline{p}}X$, we are only allowed to modify the stratified space X near the singular strata: the links of X are replaced iteratively by so-called spatial homology truncations, where the truncation degrees k are controlled by the parameter $\overline{p}$
- direct benefits compared to intersection homology: have perversity internal cup product on $\widetilde{H}^*(I^{\overline{p}}X)$; can immediately apply other generalized homology theories $E_*(-)$ to $I^{\overline{p}}X$
- $HI^{\overline{p}}_*(X)$ is in general not isomorphic to intersection homology, but they are related by mirror symmetry on Calabi-Yau 3-folds

Approaches to $HI_{\bar{p}}^{*}(X)$

- **linear algebra**

  C. Geske, Algebraic intersection spaces, J. Topol. Anal. (2020)

- **sheaf theory**

  M. Banagl, N. Budur, L. Maxim, Intersection Spaces, Perverse Sheaves and Type IIB String Theory, Adv. Theor. Math. Phys. (2014)

- **PL polynomial differential forms**

  D.J. Wrazidlo, On the rational homotopy type of intersection spaces, J. of Singularities (2020)

- **smooth differential forms**

  M. Banagl, Foliated stratified spaces and a de Rham complex describing intersection space cohomology, J. Differential Geometry (2016)

- **L^2 harmonic forms**

  M. Banagl, E. Hunsicker, Hodge theory for intersection space cohomology, Geometry & Topology (2019)

Approach by PL polynomial differential forms

- use Sullivan's contravariant functor A_{PL} from the category of topological spaces and continuous maps to the category of commutative cochain algebras and cochain algebra morphisms
- recall that the graded algebras $H^*(Z)$ and $H(A_{PL}(Z))$ are naturally isomorphic for any topological space Z

Theorem (for X with isolated singularities: W., 2020)

There is a differential ideal $\iota_{\overline{p}}\colon AI_{\overline{p}}(X) \hookrightarrow A_{PL}(X_{reg})$ such that ...

- *...the commutative cochain algebras $AI_{\overline{p}}(X) \oplus \mathbb{Q}$ and $A_{PL}(I^{\overline{p}}X)$ are* **weakly equivalent**, *that is, there exists a chain*

$$AI_{\overline{p}}(X) \oplus \mathbb{Q} \xleftarrow{\simeq} A_0^* \xrightarrow{\simeq} \dots \xleftarrow{\simeq} A_r^* \xrightarrow{\simeq} A_{PL}(I^{\overline{p}}X)$$

  of quasi-isomorphisms between commutative cochain algebras
- *...generalized Poincaré duality is realized by a* **nondegenerate intersection pairing** *of the form*

$$H^*(AI_{\overline{p}}(X)) \times H^{n-*}(AI_{\overline{q}}(X)) \to \mathbb{Q},\ ([x],[y]) \mapsto \int_{X_{reg}} \iota_{\overline{p}}(x) \cdot \iota_{\overline{q}}(y)$$

Construction of the differential ideal $AI_{\overline{p}}(X) \hookrightarrow A_{PL}(X_{reg})$

- 1 isolated singularity: $X^n = M \cup_{\partial M} \operatorname{cone}(\partial M)$, ∂M connected
- fix truncation degree $k = n - 1 - \overline{p}(n)$ (> 0)
- Choose **standard k-cotruncation**

$$\vartheta^D_{\geq k} \colon \tau^D_{\geq k} A_{PL}(\partial M) \to A_{PL}(\partial M),$$

that is, writing $C^* = A_{PL}(\partial M)$, we take $\vartheta^D_{\geq k}$ to be the inclusion $\vartheta^D_{\geq k} \colon \tau^D_{\geq k} C^* \to C^*$ of the k-cotruncation subcomplex

$$\tau^D_{\geq k} C^* : \cdots \to 0 \to D \xrightarrow{d|} C^{k+1} \xrightarrow{d} C^{k+2} \xrightarrow{d} \ldots$$

of C^* determined by the choice of a direct sum complement $C^k = D \oplus \operatorname{im}(d \colon C^{k-1} \to C^k)$.

- define $AI_{\overline{p}}(X)$ by fiber product

$$\begin{array}{ccc} \mathbf{AI_{\overline{p}}(X)} & \xrightarrow{\rho_{\overline{p}}} & \tau^D_{\geq k} A_{PL}(\partial M) \\ \downarrow{\scriptstyle \iota_{\overline{p}}} & & \downarrow{\scriptstyle \vartheta^D_{\geq k}} \\ A_{PL}(M) & \xrightarrow{\text{incl}^*} & A_{PL}(\partial M) \end{array}$$

Beyond the toy example

- setting: n-dimensional stratified pseudomanifold X^n
- assumption: X has link bundles that are **compatibly trivializable**
- Agustín, Fernández de Bobadilla: construction of **relative intersection space**, that is, a space pair $(Y^{\bar{p}}, Z^{\bar{p}})$ such that $I^{\bar{p}}X = Y^{\bar{p}}/Z^{\bar{p}}$
- How can the approach by PL polynomial differential forms be generalized to intersection space pairs?

Thank you for your attention!