Magnetic Systems in Symplectic Geometry

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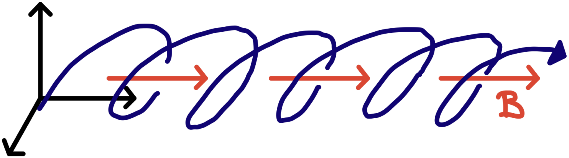
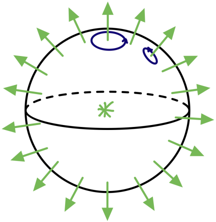
Magnetism in Classical Mechanics



Newtons law: $\dot{x} = v$ and $\dot{v} = v \times B = \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix} \cdot v$

$$\Leftrightarrow \qquad \mathrm{d}\left(\tfrac{1}{2}|v|^2\right) = (\dot{x}, \dot{v}) \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix}$$



$$\Leftrightarrow \qquad \mathrm{d}E = \iota_{\dot{\gamma}}\omega_\mu; \quad \omega_\mu = \mathrm{d}v \wedge \mathrm{d}x + \pi^*\mu$$

Standard Cotangent Bundle

- Configuration space: M smooth manifold
- Phase space: T^*M cotangent bundle
- Canonic 1-form: $\alpha_{(x,p)} := p \circ \mathrm{d}\pi_{(x,p)} \in T^*_{(x,p)}(T^*M)$
- Canonic symplectic structure: $\mathrm{d}\alpha \in \Omega^2(T^*M)$
- Hamiltons equations: $\dot{\gamma} = X_H; \quad \iota_{X_H}(\mathrm{d}\alpha) = \mathrm{d}H$

Standard Tangent Bundle

For a Riemannian manifold (M, g) there is also a canonic symplectic structure on the tangent bundle TM:

$$\omega_0 = g^* \mathrm{d}\alpha$$

Proposition

Take the kinetic Hamiltonian $E(x, v) = \frac{1}{2} g_x(v, v)$*, then the Hamiltonian flow is the geodesic flow.*

Magnetic Systems

Take $\mu \in \Omega^2(M)$ closed, then

$$\omega_\mu := \omega_0 - \pi^* \mu$$

is a symplectic form on TM and the triple (M, g, μ) is called **magnetic system**.

We can define a bundle map $F : TM \to TM$ via

$$g_x(F_x(v), \cdot) = \mu_x(v, \cdot).$$

It is called **Lorentz force**.

Magnetic Flow

The flow of the Hamiltonian vector field X_E determined by $\mathrm{d}E = \omega_\mu(X_E, \cdot)$ is called magnetic flow.

Proposition

The Hamiltonian vector field for a magnetic system is given by

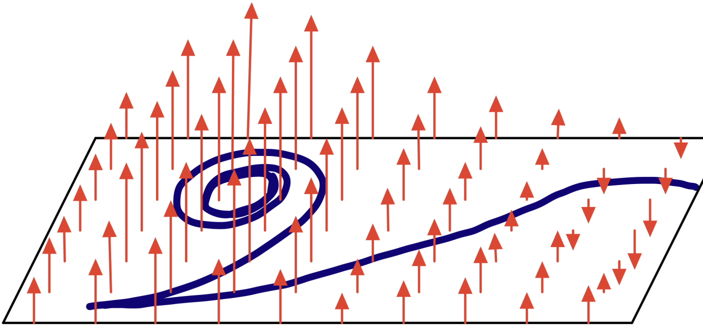
$$(X_E)_{(x,v)} = \mathcal{L}^{\mathcal{H}}_{(x,v)}(v) + \mathcal{L}^{\mathcal{V}}_{(x,v)}(F_x(v)).$$

$\Rightarrow$ A Hamiltonian trajectory $\gamma(t) = (x(t), v(t))$ satisfies

$$\dot{x} = v \quad \text{and} \quad \mathrm{D}_t v = F_x(v).$$

Magnetic vs. Geodesic Flow

- Denote $S_m := \{(x,v) | E(x,v) = m\}$, it is invariant under magnetic (and geodesic) flow.
- $\mathrm{d}\varphi_a^{-1}\left(X_E^0\right)\big|_{S_{am}} = a\left(X_E^0\right)\big|_{S_m} \Rightarrow$ dynamics are the same up to reparametrization
- $\mathrm{d}\varphi_a^{-1}\left(X_E^\mu\right)\big|_{S_{am}} = a\left(X_E^{\frac{\mu}{a}}\right)\big|_{S_m} \Rightarrow$ dynamics change with scaling



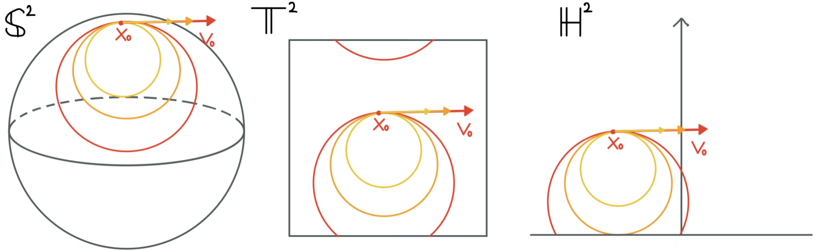
Magnetic Systems on Surfaces

- Σ: oriented smooth surface,
- g: Riemannian metric of constant curvature κ,
- σ: Riemannian area form induced by metric and orientation.

For any $\mu \in \Omega^2(\Sigma)$ there exists a unique function $f : \Sigma \to \mathbb{R}$ such that $\mu = f \cdot \sigma$. We consider f to be constantly $s \in \mathbb{R}$.

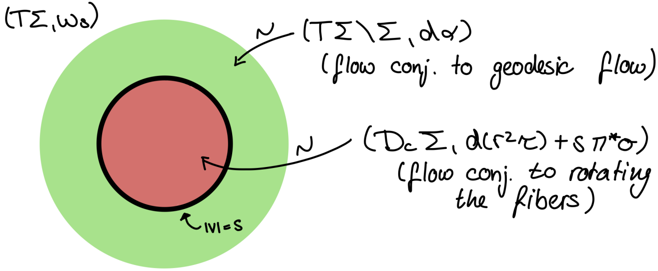
$\Rightarrow$ $F_x(v) = s\iota_x v$

$\Rightarrow$ $x(t)$ is a curve with constant geodesic curvature $\frac{s}{|v_0|}$



Magnetic Flow on Hyperbolic Surfaces

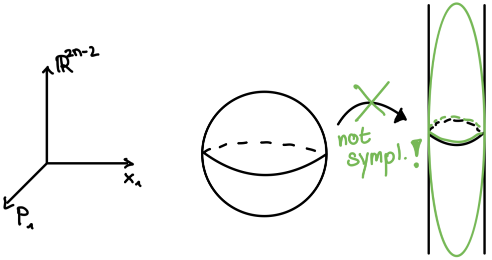
- $PSL(2,\mathbb{R}) \tilde{\to} S\mathbb{H}; \quad A \mapsto (A(i), \mathrm{d}A_i(i_i))$
- $X_E \equiv \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + s \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \Rightarrow \det(X_E) = \frac{1}{4}(s^2 - 1)$
- $s > 1$ conjugate to rotating the fibers, $s = 1$ conjugate to horocycle flow, $s < 1$ conjugate to geodesic flow



Hofer-Zehnder Capacity

- ω symplectic $\Rightarrow \omega^n$ is a volume form $\Rightarrow$ volume is a symplectic invariant

Gromov's Non-Sqeezing Theorem



Symplectic capacities are symplectic invariants that measure the 'size' of a symplectic manifold.

The Hofer-Zehnder capacity does this in terms of the possible Hamiltonian dynamics on the symplectic manifold.

Hermitian Symmetric Spaces

- (M, J, μ) a Kähler manifold, study $(TM, \omega_{s\mu})$

$\Rightarrow$ $F_x(v) = sJ_x(v)$

$\Rightarrow$ $(X_E)_{(x,v)} = \mathcal{L}^{\mathcal{H}}_{(x,v)}(v) + s\mathcal{L}^{\mathcal{V}}_{(x,v)}(J_x(v))$

- Suppose $Q \subset M$ complex totally geodesic submanifold and $(x, v) \subset TQ$

$\Rightarrow$ $(X_E)_{(x,v} \subset T(TQ) \subset T(TM)$

$\Rightarrow$ magnetic flow preserves TQ

Theorem (Polydisc/ Polysphere Theorem)

Every element in the compact/ noncompact Hermitian symmetric space $M = (H/K)$ is in the K-orbit of a point in the polysphere/ polydisc.

Further Directions

- Hofer–Zehnder capacity for magnetic systems
- Magnetic and sub-Riemannian billiards
- Magnetic curvature
- Systolic inequalities for magnetic geodesics