Higgs bundles, real forms and the Hitchin fibration

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Contents

Introducción

Los fibrados de Higgs fueron introducidos por Hitchin en [44]. Desde ese momento, se han estudiado exhaustivamente, verificándose su importancia en áreas tan diversas como las representaciones de grupos de superficie, teoría de Teichmüller, teorías gauge, geometría hyperkähleriana, sistemas integrables, dualidad de Langlands y and mirror symmetry.

En esta tesis se estudia una generalización de la aplicación de aplicaci'on de Hitchin para grupos de Lie reales reductivos. Antes de entrar en los detalles, incluimos una breve revisión breve de los resultados conocidos hasta ahora.

Sea $G^{\mathbb{C}}$ un grupo de Lie reductivo complejo $G^{\mathbb{C}}$. Dada X una curva proyectiva compleja de género $g \geq 2$, considérese $K \to X$ el fibrado canónico sobre X. Un $G^{\mathbb{C}}$*-fibrado de Higgs* sobre X es un par (E, ϕ) con E un $G^{\mathbb{C}}$-fibrado principal holomorfo y $\phi \in H^0(X, E(\mathfrak{g}^{\mathbb{C}}) \otimes K)$ una sección del fibrado adjunto associado $E(\mathfrak{g}^{\mathbb{C}})$ con coeficientes en el fibrado canónico. La sección ϕ es conocida como *campo de Higgs*. Cuando $G^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, se puede ver a E como un fibrado vectorial holomorfo y $\phi : E \to E \otimes K$ un endomorfismo del mismo, salvo la tensorización por el canónico.

Hitchin [44] estudió de dos modos distintos el espacio de móduli de fibrados de Higgs vectoriales de rango dos. Por un lado, definió una noción de estabilidad para este tipo de fibrados de Higgs que generaliza la noción de Mumford de estabilidad para fibrados vectoriales. Por otro lado, consideró un móduli obtenido por técnicas se teorías gauge. Éste es el móduli de soluciones a un conjunto de ecuaciones diferenciales ahora conocidas como *ecuadiones de Hitchin*, que generalizan la condición de platitud para conexiones unitarias.

Hitchin demostró que un fibrado de Higgs estable da lugar a una solución a las ecuaciones y viceversa. Este resultado generaliza un teorema de Narasimhan y Seshadri [59] sobre fibrados vectoriales estables y conexiones unitarias planas.Los resultados de Hitchin fueron generalizacios por Simpson [72] a dimensión superior.

En su forma más general las ecuaciones de Hitchin para un grupo de Lie reductivo complejo arbitrario G^C and a un fibrado de Higgs (E, ϕ) se expresan como sigue:

$$F_h - [\phi, \tau_h \phi] = \alpha\omega. \quad (1)$$

Aquí, h es una reducción $\mathbb{C}^\infty$ del grupo de estructura de E a un subgrupo conexo maximal $U \leq G^{\mathbb{C}}$, ω es una forma de volumen en X, F_h es la 2-forma curvatura para la conexión A compatible con la reducción, h, y la estructura holomorfa de E, τ_h is the involución que define $U \leq G^{\mathbb{C}}$ y $\alpha \in \mathfrak{z}(\mathfrak{u})$ está determinado por la topología del fibrado E (donde $\mathfrak{z}(\mathfrak{u})$ es el centro del álgebra de Lie $\mathfrak{u}$ de U). Asimismo, existen nociones de (poli,semis)estabilidad para $G^{\mathbb{C}}$-fibrados de Higgs que generalizan las dadas por Ramanathan [63, 64] para fibrados principales. La correspondencia de Hitchin–Kobayashi dice que un fibrado de Higgs (E, ϕ) es polistable si y sólo si existe una solución a (1). Esto induce un homeomorfismo entre el móduli gauge $\mathcal{M}^\alpha_{gauge}(G^{\mathbb{C}})$ y el móduli de fibrados de Higgs poliestables $\mathcal{M}^\alpha(G^{\mathbb{C}})$.

Hitchin observó que en el caso de fibrados vectoriales,una solución (A, ϕ) a la secuaciones produce una conexión plana $A + \phi + \phi^*$. En general, la elección de una reducción C^∞ del grupo de estructura de E a $\mathrm{U}(n)$ permite descomponer cualquier conexión $B = A + \phi + \phi^*$ para alguna conexión A compatible con la métrica. Un teorema de Donaldson [26] (rango dos) y Corlette [21] (para dimensiones y grupos complejos reductivos arbitrarios) demuestra que una conexión plana reductiva produce una solución a las ecuaciones de Hitchin y viceversa. Este teorema, junto con la anterior correspondencia de Hitchin–Kobayashi (para $\alpha = 0$) es la base de la teoría de Hodge no abeliana, la cual realciona el espacio de móduli de Betti $\mathcal{R}(G^{\mathbb{C}})$ de representaciones reductibas del grupo fundamental de X en $G^{\mathbb{C}}$, el móduli de DeRham de conexiiones planas reductivas sobre el $G^{\mathbb{C}}$-fibrado, y el mmóduli de Dolbeault de fibrados de Higgs polistables (ver Goldman [36] para más detalles sobre el móduli de representaciones).

La construcción algebraica del móduli llega a cargo de Nitsure [61], para $\mathrm{GL}(n, \mathbb{C})$-fibrados de Higgs sobre una curva. Considera campos de Higgs tuisteados por un fibrado de línea holomorfo $L \to X$ en vex de sólo el tuisteo canónico. Simpson lo generaliza a dimensión superior [73, 74].

Como ya se ha señalado, en esta tesis se estudian fibrados de Higgs más generales. para grupos de Lie reductivos reales en lugar de sólo complejos. Sea $H \leq G$ un subgrupo compacto maximal subgrupo, y sea $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ una desocmposición de Cartan, con $\mathfrak{h}$, $\mathfrak{g}$ las álgebras de Lie de H, G respectivamente. Fijemos un fibrado de línea holomorfo $L \to X$. Un *par de Higgs L-tuistead* es un par (E, ϕ) donce $E \to X$ es un $H^{\mathbb{C}}$-fibrado principal holomorfo sobre X y $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$. Aquí, $H^{\mathbb{C}}$ y $\mathfrak{m}^{\mathbb{C}}$

son las complexificaciones de H y $\mathfrak{m}$ respectivamente, $E(\mathfrak{m}^{\mathbb{C}})$ es el fibrado asociado de fibra $\mathfrak{m}^{\mathbb{C}}$ via isotropía $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$. Así, se recupera la noción de $G^{\mathbb{C}}$-fibrado de Higgs tomando $(G^{\mathbb{C}})_{\mathbb{R}}$-pares de Higgs K-tuisteados, donde $(G^{\mathbb{C}})_{\mathbb{R}}$ es el grupo real subyacente al grupo complejo $G^{\mathbb{C}}$, visto como una forma real de $G^{\mathbb{C}} \times G^{\mathbb{C}}$.

Este tipo de objetos llevan estudiándose muchos años. Los fibrados Higgs para grupos reales ya aparecen en Hitchin [44, 45]. Por otro lado, tuisteos por fibrados de línea arbitrarios se consideran en [61, 54, 12, 25]. Nociones de estabilidad y poliestabilidad pueden ser definidas en general, y dependen de un elemento $\alpha \in \mathfrak{z}(\mathfrak{h})$ (donde $\mathfrak{z}(\mathfrak{h})$ es el centro de $\mathfrak{h}$). El correspondiente móduli será denotado $\mathcal{M}_L^{\alpha}(G)$. Estos han sido estudiados en distintos contextos por Bradlow, García-Prada, Gothen, Mundet i Riera ([13], [15], [31], [16], [32]). Cuando $\alpha = 0$, $L = K$ y $(G^{\mathbb{C}})_{\mathbb{R}}$ es el grupo real subyacente a alguún grupo complejo, tenemos un isomorfismo $\mathcal{M}(G^{\mathbb{C}}) \cong \mathcal{M}_K^0((G^{\mathbb{C}})_{\mathbb{R}})$.

El parámetro α aparece de modo natural en las ecuaciones de Hitchin. Los mismos autores demuestran una correspondencia de Hitchin–Kobayashi para G-pares de Higgs [16, 32]. Siguiendo [32], donde se trata la teoría general para G-pares de Higgs L-tuisteados, tenemos que un par de Higgs (E, ϕ) es α-poliestable si y sólo si dada una métrica hermítica h_L sobre L, existe una reducción C^{∞} h del grupo de estructura a H satisfaciendo:

$$F_h - [\phi, \tau_h \phi] \otimes \omega = \alpha\omega. \tag{2}$$

Aquí, todos los elementos se definen como en (1), salvo que en este caso es necesarion fijar h_L para definir τ_h, y el corchete de Lie tiene que ser tensorizado con una forma de volumen. Al contrario de lo que ocurre en el caso complejo, para formas reales de tipo Hermítico, α podría pertenecer a $\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g})^{\perp}$. Sólo la proyección de α a $\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{z}(\mathfrak{g})$ está determinada por la topología del fibrado principal. Nótese asimismo que α puede tomar valores distintos de cero sólo en el caso de forma real Hermítica y grupos complejos no semisimples (ya que de otro modo, $\mathfrak{z}(\mathfrak{h}) = 0$).

Si $L = K$, $\alpha = 0$, tenemos un homeomorfismo entre $\mathcal{M}_K^0(G)$ y un espacio de módulo de representaciones reductivas $\rho : \pi_1(X) \to G$, $\mathcal{R}(G)$. En [45], Hitchin demuestra que existe una componente conexa de la imagen de $\mathcal{M}(G_{split})$ en $\mathcal{M}(G^{\mathbb{C}})$.

Los tuisteos por fibrados que no son el canónico aparecen de modo natural en el estudio del espacion de móduli de G-fibrados de Higgs (K-tuisteados). En efecto, cuando G es de tipo hermítico, (es decir, G/H es un espacio simétrico hermítico), a un par de Higgs se le puede asociar un invariante topológico llamado invariante de Toledo. Si G/H es de tipo tubo (es decir, biholomorfo a un tubo sobre un cono simétrico), es espacio de móduli de G-fibrados de Higgs con Toledo maximales isomorfo al espacio de móduli de H^*-pares de Higgs tuisteados por K^2, donde H^* is el dual no-compacto de H. Bradlow, Gothen, García-Prada and Mundet-i-Riera ([15, 13, 14, 31]) prueban esta correspondencia de Cayley usando teoría de clasificación de grupos de Lie reales reductivos, y de modo general por por Rubio-Núñez and Biquard–García-Prada–Rubio-Núñez [65, 8].

Existe un morfismo canónico del espacio de móduli de G-pares de Higgs a un espacio afín $\mathcal{A}_L(G)$. Esta aplicación juega un papel fundamental en la presente tesis. Para definirla, consiérese el campo de Higgs como una aplicación $H^{\mathbb{C}}$-equivariante $E \to \mathfrak{m}^{\mathbb{C}} \otimes L$. Dada $\mathfrak{a}$ una subaálgebra abeliana maximal de $\mathfrak{m}$, considérese $W(\mathfrak{a}) := N_H(\mathfrak{a})/C_H(\mathfrak{a})$, where $N_H(\mathfrak{a})$ is the normaliser of $\mathfrak{a}^{\mathbb{C}}$ in $H^{\mathbb{C}}$ and $C_H(\mathfrak{a})$ its centraliser. Este es un grupo de Weyl. Además, por el Teorema de Chevalley, $\mathfrak{m}^{\mathbb{C}}//H^{\mathbb{C}} \cong \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$, donde el doble cociente indica el cociente GIT afín. Ahora podemos evaluar $\mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$ sobre el campo de Higgs, obteniendo una aplicación del móduli de pares al espacio afín $\mathcal{A}_L(G) = H^0(X, \mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a}))$, denoted

$$h_{G,L} : \mathcal{M}_L^{\alpha}(G) \to \mathcal{A}_L(G).$$

Esta aplicación se llama la *aplicación de Hitchin*, por Nigel Hitchin, quien la introdujo en [41] para $G^{\mathbb{C}}$-fibrados de Higgs. Con esto, se recupera la aplicación de Hitchin original mediante:

$$h_{G^{\mathbb{C}},K} : \mathcal{M}(G^{\mathbb{C}}) \to \mathcal{A}_K(G^{\mathbb{C}})$$

.

Los objetivos fundamentales de esta tesis son la construcción de una sección a la aplicación de Hitchin para grupos reales arbitrarios y el estudio de sus fibras. Para grupos de Lie complejos reductivosy $L = K$, ambos aspectos fueron desarrollados por varioa autores ([41, 45, 25, 70, 27, 28]).

Hitchin construyó en [45] unaa sección s a $h_{G^{\mathbb{C}},K}$ para un grupo complejo simple $G^{\mathbb{C}}$ usando los resultados de Kostant [50]. La imagen de la sección es una componente conexa del espacio de móduli para G_{split}, que en el caso de $\mathrm{SL}(2,\mathbb{C})$ coincide con es espacio de Teichmüller. En general, este componente de Hitchin-Teichmüller corresponde, en términos de $\mathcal{R}(G^{\mathbb{C}})$, a representaciones en G_{split} que se deforman a representaciones que factorizan a través de una representación irreducible $SL(2,\mathbb{R}) \to G_{split}$.

En cuanto a la estructura global de la fibración para grupos de Lie complejos reductivos, Hitchin [41] probó que $h_{\mathbb{C}}$ es propio y confiere a $\mathcal{M}(G^{\mathbb{C}})$ la estructura de un sistema integrable sobre la base. Lo que es más, para grupo simples de tipos A, B, C y D, a cada punto genérico de la base $a \in \mathcal{A}_K(G^{\mathbb{C}})$, le asició la llamada curva espectral $\overline{X}_a \subseteq K$ y demostró que los $G^{\mathbb{C}}$-fibrados de Higgs sobre a se corresponden con una subconjunto abuerto de una variedad de Prym de $Jac(\overline{X}_a)$. En particular, las fibras son isomorfas a variedades abelianas, y en esos casos el sistema es completamente algebraicamente integrable. Simpson [74] diúna descripción espectral para $G^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ en dimensión superior. Generalizaciones a otros grupos en contextos distintos fueron llevadas a cabo por Katzarkov and Pantev [47] for $G = G_2$. En ese trabajo se define un recubrimiento distinto, (que juega el papel de $\overline{X}_a$), más adelante llamado recubrimiento cameral en honor a Donagi [24, 23]. Beilinson–Kazhdan [7], Kanev [46] y Scognamillo [70] introdujeron nociones similares. Para grupos complejos reductivos aribitrarios, Donagi–Gaitsgory, Faltings, y Scognamillo ([25, 70, 69, 71, 27]), describieron la fibración en términos de recubrimientos camerales. En este trabajo, usaremos las técnicas desarrolladas por Donagi y Gaitsgory, as'i como la formulación de Ngô's [60]. Donagi–Gaitsgory provaron que la aplicación de Hitchin induce una estructura de gerbo en el móduli stack de $G^{\mathbb{C}}$-pares. Lo que es más, demustran que la banda es un haz de toros sobre la base $\mathcal{D}_W$. En particular, las fibras son categorías de torsores abelianos. Esta abelianización generaliza la de Hitchin, ya que para grupos de tipos A, B, C, D el recubrimiento cameral es un recubrimiento de Galois sobre la curva espectral, y los datos especrrales y camerales son equivalentes, Véase [24, 23, 25].

La aplicación de Hithin para curvas elípticas fue estudiada por Franco-Gómez [28]. Observó que las fibras no siempre son abelianaa. En su trabajo, la naturaleza no abeliana de las fibras de Hitchin aparece al considerar $G = \mathrm{U}^*(2n)$, y otros grupos reales. Schaposnik considera curvas de género superior [66, 67]. Utiliza las técnicas espectrales de Hitchin, así como una involución en el espacio de móduli de pares de Higgs complejos para describir la fibración para formas reales split, $\mathrm{U}(p, p)$ y $\mathrm{Sp}(2p, 2p)$. Obsrevó que en el caso de $\mathrm{Sp}(2p, 2p)$, las fibras son no abelianas. En trabajo conjunto con Hitchin [43], extienden esto a grupos lineales definidos sobre los cuaterniones.

Ahora pasamos a explicar los resultados principales de la tesis:

En el Capítulo 1, estabIedemos las bases de teoría de Lie siguiendo a Knapp [48] and la teoría de Kostant y Rallis' sobre órbitas y representaciones de pares simétricos [51]. Definimos un *grupo de Lie reductivo* como una tupla (G, H, θ, B) donde G es un grupo de Lie real con álgebra de Lie reductiva, $H \leq G$ un compacto maximal, θ una involucción de Cartan sobre $\mathfrak{g}$ que induce $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ y B una forma bilineal en $\mathfrak{g}$ Ad G invariant, definida positiva en $\mathfrak{h}$ y negativa en $\mathfrak{m}$. Asimismo definimos la noción de *grupo de Lie fuertemente reductivo*, lo cual coincide con la reductividad de Knapp.

Nos ha parecido más a decuado separar ambas nociones ya que la reductividad fuerte es una condición excesiva para la mayoría de las aplicaciones.

El primer resultado del Capítulo 1 es una clasificación de las álgebras de Lie reductivas. Este resultado es clásico en el caso semisimple ([40, 39, 62]), pero no conocemos ninguna referencia que trate el caso de centro no trivial.

Proposition (1.1.14)**.** *Dada un álgebra de Lie compleja reductiva $\mathfrak{g}^{\mathbb{C}}$, existe una correspondencia 1 a1 entre classes de conjugación por automorfismos internos de formas reales y clases de conjugación por automorfismos internos de automorfismos lineales $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.*

En la Sección 1.2 consideramos grupos de Lie reductivos. Dado un grupo de este tipo (G, H, θ, B) con complexificación $(G^{\mathbb{C}}, H^{\mathbb{C}}, \theta, B_{\mathbb{C}})$, definimos el subgrupo split maximal $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B}) \leq (G, H, \theta, B)$, es decir, un subgrupo que es split dentro de su propia complexificación. Ests noción ya era conocida para Humphreys y Borel–Tits [11] para grupos algebraicos de matrices. Tenemos:

Proposition (1.2.27)**.** *Si (G, H, θ, B) es reductivo, también lo es $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B})$.*

Este resultado nos será útil en las secciónes 1.2 y 1.3 para extender los resulados de [51] a grupos no necesariamente de tipo adjunto.

El resultado principal de la Sección 1.4 es la clasificación geoméetrica de los centralizadores de elementos regulares de $\mathfrak{m}^{\mathbb{C}}$. Sean $\mathfrak{m}_{reg}$ el subconjunto de elementos regulares de $\mathfrak{m}^{\mathbb{C}}$, es decir, elementos con centralizadores en $H^{\mathbb{C}}$ de dimensión mínima, $Ab^a(\mathfrak{m}^{\mathbb{C}}) \subset Gr(a, \mathfrak{m}^{\mathbb{C}})$ la subvariedad de álgebras abelianas de $\mathfrak{m}^{\mathbb{C}}$ de dimensión a igual a la dimensión del cetralizador regular, y $\mathfrak{c}_{\mathfrak{h}}(x)$ el centralizador en $\mathfrak{h}^{\mathbb{C}}$ del punto $x \in \mathfrak{m}^{\mathbb{C}}$. Finalmente, consideramos la variedad de incidencia μ_{reg} de $\mathfrak{m}_{reg} \times Ab^a(\mathfrak{m}^{\mathbb{C}})$.

Proposition (1.4.6)**.** *La aplicación*

$$\psi : \mathfrak{m}_{reg} \to Ab^a(\mathfrak{m}^{\mathbb{C}}), \ x \mapsto \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x)$$

es lisa con imagen lisa, y su grafo es μ_{reg}.

Nuestra prueba del resultado anterior se basa en técnicas desarrolladas por Donagi y Gaitsgory [25], aunque más tarde encontramos un resultado similar de Le Barbier-Grünewald [52] ya publicado.

El Capítulo 2 contiene material preliminar sobre el espacio de móduli space de G-pares de Higgs L-tuisteados α-poliestables. Es aquí que todos los ingredientes en al definición de reductividad de Knapp se vuelven necesarios para definir el espacio de móduli. Esta es la idea en [32] que retomamos aquí. La mayoría de los resultados de esta sección son conocidos, salvo excepciones que no hemos encontrado en la literatura (Proposiciones 2.3.18, 2.3.7 y 2.4.3). Sea $C_{H^{\mathbb{C}}}(\alpha)$ el centralizador de $\alpha \in \mathfrak{g}^{\mathbb{C}}$ en $H^{\mathbb{C}}$, y sea $C_H(\alpha)$ el centralizador en H.

Proposition (2.3.7). *Sea (E, ϕ) un G-fibrado de Higgs α-poliestable. Sea $M_{(E,\phi)}$ el espacio de móduli de soluciones $h \in \Omega^0(X, E(H^{\mathbb{C}}/H)$ a la ecuación $F_h - [\phi, \tau_h\phi]\omega = \alpha\omega$. Tómese una solución h, y sea E_h la correspondiente reducción de E a un H-fibrado principal. Entonces $M_{(E,\phi)} \cong \mathrm{Aut}(E, \phi) \cap C_{H^{\mathbb{C}}}(\alpha)/\mathrm{Aut}(E_h, \phi) \cap C_H(\alpha)$.*

Una consecuencia de esto es que las soluciones irreducibles no son necesariamente únicas, al contrario de lo que ocurre en el caso complejo.

En el Capítulo 3 damos una construcción de una secció a h_L que generaliza la dada por Hitchin [45]. Usamos la teoría de Kostant–Rallis [51], quienes probaron, entre otras cosas, la existencioa de una sección del morphismo de Chevalley $\mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}//H^{\mathbb{C}}$. Como ya hemos mencionado, consideramos espacios de móduli para valores de α arbitraios y tuisteo arbitrarios, lo que hace necesario modificar los argumentos de Hitchin, dado que se desconoce si existe una interpreción de estos espacios de móduli en términos de representaciones del grupo fundamental. Tomamos un punto de vista intrínseco construyendo la sección directamente en $\mathcal{M}(G)$ en lugar de su imagen en $\mathcal{M}(G^{\mathbb{C}})$. En ocasiones,la sección factoriza a través de $\mathcal{M}_L^{\alpha}(\widetilde{G})$. en particular, se recupera una versión intrínseca de la sección de Hitchin en el case de formas split. Lo que es más, damos una interpretación de las componentes de Hitchin en el mduli de la forma split empleando una extensión $H_\theta^{\mathbb{C}}$ de $H^{\mathbb{C}}$, orginalmente definida por Kostant–Rallis [51] en el caso adjunto, y por García-Prada–Ramaman [33] en general.

El modo más sencillo de proceder a la construcción de la sección de Hitchin en esta generalidad es el modo intrínsco, en lugar del involutiorio dado por Hitchin.Hay varias razones para esto, la más importante de las cuales es que uno de los módulis $\mathcal{M}_L^{\alpha}(\widetilde{G})$, $\mathcal{M}_L^{\alpha}(G)$ puede ser vacío mientras que el otro no cuando $\alpha \in \mathfrak{z}(\widetilde{\mathfrak{g}}) \cap \mathfrak{z}(\mathfrak{g})^{\perp}$ o $\alpha \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{z}(\widetilde{\mathfrak{g}})^{\perp}$, lo que hace insuficiente considerar sólo el caso split.

Para enunciar el resultado principal del Capítulo 3, fijamos un morfismo θ-equivariante e irreducible $\rho : \mathrm{SL}(2, \mathbb{C}) \to G^{\mathbb{C}}$ de manera que $\mathrm{SL}(2, \mathbb{R})$ cae en G. Asumimos las condiciones siguientes en el parámetro y el grado del fibrado de línea L:

$$\deg L := d_L > 0, \qquad i\alpha \leq d_L/2. \tag{3}$$

Denotamos con un superíndice ^{smooth} el locus de puntos lisos del espacio de móduli.

Theorem (3.2.10). *Under the hypothesis (3), assuming $d\rho(\alpha) \in i\mathfrak{z}(\mathfrak{h})$, there exists a section s of the map*

$$h_L : \mathcal{M}_L^{d\rho(\alpha)}(G) \to \mathcal{A}_L(G)$$

for G a reductive Lie group. This section takes values in the smooth locus of the moduli space.

Moreover, the section factors through $\mathcal{M}_L^{d\rho(\alpha)}(\widetilde{G})^{smooth}$, where $\widetilde{G}$ is the connected maximal split subgroup of G, if and only if $d\rho(\alpha) \in i\mathfrak{z}(\widetilde{\mathfrak{h}})$.

Theorem (3.2.10). *Si se da (3) y además $d\rho(\alpha) \in i\mathfrak{z}(\mathfrak{h})$, entonces existe una a sección s de la aplicación*

$$h_L : \mathcal{M}_L^{d\rho(\alpha)}(G) \to \mathcal{A}_L(G)$$

Para cualquier grupo fuertemente reductivo G. Esta sección toma valires en el locus liso.

Además, la sección factoriza a través de $\mathcal{M}_L^{d\rho(\alpha)}(\widetilde{G})^{smooth}$, donde $\widetilde{G}$ es el subgrupo split maximal conexo de G, si y sólo si $d\rho(\alpha) \in i\mathfrak{z}(\widetilde{\mathfrak{h}})$, caso este último en que determina una componente connexa de $\mathcal{M}_L^{d\rho(\alpha)}(\widetilde{G})$.

Esto demuestra en particular que h_L es sobreyectiva.

Demostramos asimismo que, para grupos simples, $\alpha = 0$ and $L = K$, la imagen de la sección de Hitchin–Kostant–Rallis es una componente conexa de $\mathcal{M}_L^{\alpha}(G)$ si y solamente si G es split.

Proposition (3.2.13). *Sea $G \leq G^{\mathbb{C}}$ una forma real de un grupo de Lie simple. Entonces, la sección de HKR consiste en una componente conexa del moduli de G-fibrados de Higgs (tuisteados por K) si y sólo si G es la forma split.*

De la prueba de la anterios proposición se deduce que para grupos de tipo Hermítico no-tubo, la imagen está contenida en una componente con Toledo no maximal. En la Sección 3.3 calculamos la componente que contiene la sección para los grupos $\mathrm{SU}(2,1)$ and $\mathrm{SU}(p,p)$.

Proposition (3.4.5). *1. La sección de Hitchin–Kostant–Rallis spara $\mathcal{M}_L^{\beta}(SU(2,1))$, $\beta \in \mathfrak{z}(\mathfrak{su}(2)) = i\mathbb{R}$ existse si y solamente si $i\beta \leq 0$.*

En ese caso, su expresión exacta es

$$\begin{array}{rcc} s : H^0(X, L^2) & \to & \mathcal{M}_L^0 SU(2,1) \\ \omega & \mapsto & \left[L \oplus L^{-1} \oplus \mathcal{O}, \begin{pmatrix} 0 & 0 & \omega \\ 0 & 0 & 1 \\ 1 & \omega & 0 \end{pmatrix} \right], \end{array}$$

y su tipo topológico es $\tau = 0$.

2. La imagen de la sección de HKR está contenida en el locus estrictamente estable.

3. La sección de HKR section factoriza a través de $\mathcal{M}_L^\alpha(SO(2,1))$ para todo $i\alpha \leq 0$.

Proposition (3.4.10)**.** *Existe una sección de la aplicación de Hitchin*

$$h_L : \mathcal{M}_L^\alpha(SU(p,p)) \to \mathcal{A}_L(SU(p,p)) = \bigoplus_{i=0}^{p-1} H^0(X, L^{2i})$$

donde $\alpha \in \mathfrak{z}(\mathfrak{u}(1)) \cong i\mathbb{R}$ si y solamente si $i\alpha \leq 0$. En esta situcación l sección factoriza a través de $\mathcal{M}_L^\alpha(Sp(2p,\mathbb{R}))$ y

$$s : \oplus_{i=0}^{p-1} H^0(X, L^{2i}) \to \mathcal{M}_L^\alpha(G)$$

envía $\oplus\omega_i$ a

$$\sum_{\substack{2\not|i,\ 2|j \\ j\geq i-1}} \omega_{\frac{j-i+1}{2}} E_{ip+j} + \delta \cdot \sum_{\substack{2\not|i,\ 2\not|j \\ p-2\geq j+i\geq 2}} \omega_{\frac{p-(i+j)+2}{2}} E_{ip+j} + \sum_{\substack{2|i,\ 2\not|j \\ i\geq j-1}} \omega_{\frac{i-j+1}{2}} E_{ip+j} +$$

$$\sum_{\substack{2|i,\ 2\not|j \\ j\geq i-1}} \omega_{\frac{1+j-i}{2}} E_{p+i,j} + \delta \sum_{\substack{2|i,\ 2|j \\ p-2\geq j+i\geq 2}} \omega_{\frac{p-(j+i)+2}{2}} E_{p+i,j} + \sum_{\substack{2\not|i\ 2|j \\ p\leq j\geq i-1}} \omega_{\frac{1+j-i}{2}} E_{p+i,j}$$

donde $\delta = 1$ si p es par, y cero en otro caso. El invariante de Toledo de la imagen es $\tau = p(g-1)$, y la sección está contenida en el locus estrictamente estable.

El Capítulo 4 está dedicado al estudio de la fibración de Hitchin para un grupo algebraico real conexo G. Seguimos la estrategia de Donagi y Gaitsgory', combinada con la teoría de Kostant y Rallis [51]. La generalidad de este método nos permite tratar todas las fibras (regulares) de manera simultánea, así como considerar esquemas de dimensión arbitraria (para los últimos, sin embargo, no se impone ninguna condición de integrabilidad)..

Ocurre, como en el caso complejo, que la aplicación de Hitchin es un gerbe, pero sólo es abeliano para formas quasi-split. Esto es consistente con Hitchin–Schaposnik para formas split, $\mathrm{U}(p,p)$ y grupos clásicos definidos sobre los cuaterniones [43], y provee una lista completa de grupos para los que la fibración es abeliana. Las algebras simples quasi split son: las formas split, $\mathfrak{su}(p,p)$, $\mathfrak{su}(p,p+1)$, $\mathfrak{so}(p,p+2)$ y $\mathfrak{e}II$.

Theorem (4.3.13)**.** *Sea $(G,H,\theta,B) < (G^{\mathbb{C}},U,\tau,B)$ una forma real de un grupo algebraico complejo reductivo conexo. Sea X un esquema complejo, y fíjese un fibrado de línea $L \to X$. Entonces:*

1. *El stack de G-fibrados de Higgs regulares tuisteados por L sobre X es un gerbo sobre* $\mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})$ *que es abeliano si y sólo si la forma G es quasi-split.*

2. *Si la forma real G es quasi-split, el gerbo es bandeado por el haz de grupos* $J^L_{\mathfrak{m}} \to \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})$, *cuyas secciones sobre* $s : U \to \mathcal{A}_L(G)$ *son* $J^L_{\mathfrak{m}}(U) = Hom_{H^{\mathbb{C}}}(L \times_s \mathfrak{m}_{reg}, C_{\mathfrak{m}}|_{L\times_s \mathfrak{m}_{reg}})$. *Nótese que en la última expresiónthat interpretamos L como un* $\mathbb{C}^\times$*-fibrado principal, y s como un morfismo* $\mathbb{C}^\times$*-equivariante* $L \to \mathfrak{a}_{reg}/W(\mathfrak{a}^{\mathbb{C}})$.

3. *Si X es una curva, y* $L \to X$ *un fibrado de línea de grado par,* $\left[\mathfrak{m}_{reg} \otimes L/H^{\mathbb{C}}\right] \cong BJ^L_{\mathfrak{m}}$.

Obsérvese que la geometría del par simétrico es crucial en la estructura del gerbo. Un ejemplo de esto es la naturaleza no abeliana del gerbo para formas no-quasi-split.

En la situación abeliana (quasi-split), definimos un haz de toros sobr ela base de Hitchin cuyo stack clasificador actúa simple y transitivamente sobre las fibras, lo cual es también común con el caso complejo.

Tómese el W-recubrimiento $\mathfrak{d}^{\mathbb{C}} \to \mathfrak{d}^{\mathbb{C}}//W$, donde $\mathfrak{d}^{\mathbb{C}} \subset \mathrm{lg}^{\mathbb{C}}$ is una subálgebra de Cartan con grupo de Weyl W. Dada una sección $a : X \to \mathfrak{d}^{\mathbb{C}} \otimes L//W$, definimos su recubrimiento cameral asociado como $\widehat{X}_a := X \times_a \mathfrak{d}^{\mathbb{C}} \otimes L$ (véase [24, 23]). Seal $\mathfrak{r}_L = \mathrm{Im}(\mathfrak{m}_{reg} \to \mathfrak{d}^{\mathbb{C}} \otimes L/W)$.

Para cada raíz simple, definimos $D^X_\alpha = \{\widehat{x} \in \widehat{X} \ : \ w_\alpha(\widehat{x}) = \widehat{x}\}$. Sobre $\mathfrak{r}_L$ consideramos el haz

$$\mathcal{D}^L_{\widetilde{W}}(U) = \left\{ f : \widehat{U} \to D^{\mathbb{C}} \ : \ \begin{array}{c} w \circ f \circ w \equiv f, \\ \breve{\alpha}(f(\widehat{x})) = 1 \text{ for all } x \in D^U_\alpha \end{array} \right\}.$$

Aquí $\widetilde{W} = \mathbb{Z}_2 \ltimes W$.

El resultado principal de esta sección es el siguiente corolario al Theorema 4.4.9.

Corollary (4.4.13)**.** *La imagen del gerbo* $\mathcal{H}iggs_L(G)^{reg}$ *en* $\mathcal{H}iggs_L(G^{\mathbb{C}})^{reg}$ *es un gerbo sobre* $\mathfrak{r}_L$ *bandeado por* $\mathcal{D}^L_{\widetilde{W}}$.

Par terminar, reintepretamos las fibras en términos de categorías de torsores sobre el recubrimiento cameral. Dada $a : X \to \mathfrak{d}^{\mathbb{C}} \otimes L//W$, si a toma valores en $\mathfrak{r}_L$, entonces $\widehat{X}$ es una union de trasladadas de un subrecurbimiento $\widehat{X}_0$, imagen de un recubrimiento cameral real.

Dado el fibrado de línea $\mathcal{O}(D^X_\alpha)$ sobre $\widehat{X}$, tómese el fibrado principal asociado por la coraíz $\breve{\alpha}$: $\mathcal{R}_\alpha := \breve{\alpha}\mathcal{O}(\mathcal{D}_\alpha)$. Del mismo modo, denifimos $\mathcal{R}_{nilp} = \otimes_{\alpha\in\Delta^+}\mathcal{R}_\alpha$. Considérese una terna $(P, \gamma, \underline{\beta})$, donde $P \to \widehat{X}$ es un $D^{\mathbb{C}}$-fibrado principal, γ un conjunto de isomorfismos, $\gamma_\alpha : P \cong w_\alpha^* P \times_\alpha D^{\mathbb{C}} \otimes \mathcal{R}_\alpha$ y $\underline{\beta}$ un conjunto de ismorfismos $\beta_n : \alpha(P)|_{D_\alpha} \cong \mathcal{O}(D_\alpha)|_{D_\alpha}$. Estos datos han de satisfacer las condiciones obvias de compatibilidad (see [25]). Sea $\mathcal{C}am_a$ la categoría cuyos objectos son los triples triples $(P, \underline{\gamma}, \underline{\beta})$ where $P \to \widehat{X}_a$ definidos como antes.

Theorem (4.4.30)**.** *Dado un* $\mathfrak{r}_L$ *esquema* $a : X \to \mathfrak{r}_L$*, sea* $\mathcal{H}iggs_L(G)_a$ *la fibra* $[\chi]_L^{-1}(a)$ *en* $\mathcal{H}iggs_L(G)^{reg}$*. Entonces* $\mathcal{H}iggs_L(G)_a$ *y* $\mathcal{C}am_a$ *son categorías equivalentes.*

Para grupos de rango real uno, la prueba del Teorema 4.4.17 se reduce a la del caso split. Esto permite relacionar la fibración de Hitchin de la split maximal con la de G. En particular, podemos relacionar invarantes topológicosde moduli de G-fibrados de Higgs y $\widetilde{G}$-fibrados de Higgs, tal y como hacemos para el $\mathrm{SU}(2,1)$ (ver Capítulo 5).

En la Sección 4.5, analizamos el case de formas reales que no son quasi split. A nivel de fibrados de Higgs abstractos, damos una caracterización de un substack denso abierto con fibras isomorfas a categorías de torsores no abelianos sobre el cameral. Más precisamente, el gerbo se reduce al gerbo trivial sobre el recubrimiento cameral universal $\mathfrak{a}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a})$. Esto, junto con la naturaleza no abeliana del gerbo constituye la mayor diferencia con el caso abeliano. Las fibras no pueden ser descritas como categorías de torsores sobre la base, pero sí sobre el recubrimiento cameral.

Lo anterior se extiende al móduli de pares tuisteados.

Proposition (4.5.4)**.** *Sea* $a : X \to \mathfrak{a} \otimes L//W(\mathfrak{a})$ *una* $\mathfrak{a} \otimes L//W(\mathfrak{a})$*-curva projectiva. Sea* $X_{reg} \subseteq X$ *el subconjunto densto de puntos en la preimnagen por* a *de* $\mathfrak{a}_{reg} \otimes L//W(\mathfrak{a})$*. Sea* $\widetilde{X}_{reg} = X_{reg} \times_a \mathfrak{a} \otimes L$*. Entonces, el conjunto de clases de isomorfismo de* G*-fibrados de Higgs tuisteados por* L *sobre* X_{reg} *es isomorfo a* $H^1(\widetilde{X}_{reg}, C_H(\mathfrak{a}))$*.*

Terminamos la sección discutiendo las dificultades consistentes en extender la descripcción a la ramicación del cameral.

En la sección 4.6, se exponen cuestiones por resolver relacionadas con el capítulo: la descripción intrínseca de la fibración, la relación entre (semi,poli)estabilidad de los datos camerales y (semi,poli)estabilidad del par de Higgs y extensión de la descripción de las fibras no abelianas a la ramificación.

En el Capítulo 5 calculamos los datos camerales de los $\mathrm{SU}(2,1)$-fibrados de Higgs como aplicación del Theorem 4.4.17. Comenzamos por estudiar la noción de regularidad (Proposición 5.3.1 y Lema 5.3.6), de los que se deduce que los puntos regulares son lisos en sus respectivas fibras. Además, vemos que los punos poliestables no son sino $\mathrm{SL}(2,\mathbb{R})$-fibrados de Higgs (Proposición 5.3.3).

En cuando a la descripción de las fibras:

Theorem (5.4.2)**.** *Sea* $\omega \in H^0(X, K^2)$*, y sea* $\mathcal{F}_\omega := \kappa\left(\mathcal{M}(SU(2,1))^{reg}\right) \cap h_{\mathbb{C}}^{-1}(\omega)$*. La fibra* $\mathcal{F}_\omega$ *es un esquema de grupos sobre* $Pic^{-(g-1)<d<(g-1)}(X)$*, cuya fibra es isomorfa a* $(\mathbb{C}^\times)^{4g-5}$*. Aquí* $Pic^{-(g-1)<d<(g-1)}(X)$ *denota la unión* $\bigsqcup_{-(g-1)<d<2(g-1)} Pic^d(X)$*. En particular, la componente conexa de la fibra es una variedad abeliana con operación dada por multiplicación en* $(\mathbb{C}^\times)^{4g-5}$ *y producto tensorial en la base* $Pic^0(X)$*.*

Asimismo, comparamos este resultado con el método espectral desarrollado por Hitchin Hitchin [41] y Schaposnik [66, 67].

Acabaremos esta sección introductoria explicando algunos problemas interesantes relacionados con este trabajo que planeamos abordar en un futuro próximo.

Caso no abeliano Como se explicó en la Sección 4.5, la manera de atacar este caso sobre la ramificación requiere en primer lugar estudiar si el gerbo o su pullback al cameral cover son bandeados. Una vez esto hecho, se puede proceder a la caracterización de la banda Esto daría la descripción más completa posible, en paralelo con el caso abeliano.

Espacios de móduli Una pregunta interesante es cómo la estabilidad de los fibrados de Higgs se refleja en los datos camerales corresponcientes. Simpson ([74]) aborda esta cuestión para fibrados vectoriales sobre variedades de dimensión mayor que 1. Una noción de estabilidad de datos camerales tendría sentido incluso en el caso abeliano, dado que dichos fibrados principales van acompañados de datos extra.

Geometría simpléctica del sistema de Hitchin La fibración de Hitchin para un grupo de Lie complejo reductivo $G^{\mathbb{C}}$ es un sistema completamente integrable. Para grupos y tuisteos generales, se ha observado que el espacio de móduli admite una estructura simpléctica natural. La mejor manera de entenderlo es en términos de móduli gauge. (cf. [16]).

El estudio de la geometría simpléctica del sistema de Hitchin para formas reales se relaciona con los siguientes puntos:

- Integrabilidad del sistema y programa Langlands geométrico: Existen amplias referencias sobre este aspecto para grupos complejos y tuisteos arbitrarios ([41, 47, 71, 54, 12]). La fibración de Hitchin es estable por dualidad de Langlands, de manera que una sitema integrable le corresponde un dual.

  El caso de grupos reales ha sido tratado por Hitchin [42] y Baraglia–Schaposnik [3, 4]. Encuentran una dualidad entre el sistema de Hitchin del grupo G y el de un grupo complejo.

- Los stacks de De Rham y Betti. El punto anterior está directamente relacionado a la definición de un stack de De Rham para G-pares de Higgs. Esto aportaría una nueva perspectiva desde la que estudiar los pares de Higgs, así como constituiría un puente hacia la definición del stack de Betti de representaciones y uan posible teoría de Hodge no abeliana en nuestro contexto.

  Existe trabajo por García-Prada–Ramanan [34], e esta dirección, que se basan en Simpson [] para estudiar los aspectos Tannakianos de los G-pares de Higgs.

α-moduli Cuanto G es de tipo hermítico, existen fenómenos interesantes asociados a la existendia de una familia continua de espacios de móduli. Por un lado, fenómenos de wall-crossing phenomena se dan en los puntos críticos del parámetro. Por otro lado, el 0-moduli puede verse como un caso límite. Cuando la norma del parámetro es suficientemente grande, la fibración de Hitchin trivializa sobre la base de Hitchin (común para todos los valores del parámetro). Esto es potencialmente una gran fuente de información para estudiar el límite en 0.

La correspondencia de Cayley Dado un grupo de tipo hermítico no-tubo, la componente de $\mathcal{M}(G)$ correspondiente a fibrados de Higgs con Toledo maximal está en correspondencia con el móduli de H^*-pares de Higgs K^2-tuisteados. Esta correspondencia respeta la fibración, lo cual apunta a un cierta dualidad entre ambos sistemas.

Estudio intrínseco del stack. Properness de la aplicación de Hitchin Una cuestión sin resolver es si la aplicación de Hitchin para grupos reales es propia. A través del estudio intrínseco de la misma, se obtendría información de la fibración para el grupo real relativa a la del complejo, lo cual facilitaría en gran medida el estudio de la propiedad de la aplicación.

Una aplicación: componentes conexas del espacio de móduli Dada una forma real G quasi-split, el stack de G-pares de Higgs es un gerbo abeliano. Siguiendo la notación del Capítulo 4, sea $\mathcal{D}_{\widetilde{W}}$ la banda del gerbo. Por definición, el $B\mathcal{D}$ actúa simple y transitivamente sobre las fibras. Esto fue usado por Ngô [60] para calcular las componentes conexas de las fibras, información que puede ser integrada estudiando deformaciones a lo largo de la base.

Chapter 1

Lie theory

We start by establishing the basic Lie theoretic results that will be necessary in what follows.

1.1 Reductive Lie algebras and real forms

In this section, the superscript $^{\mathbb{C}}$ will denote complex Lie algebras, whereas the lack of notation is reserved for real Lie algebras; the complexification of a real Lie algebra $\mathfrak{l}$ will be denoted by $\mathfrak{l}^{\mathbb{C}}$. We make this notation extensive to any subspace of a real Lie algebra.

Definition 1.1.1. A reductive Lie algebra over a field k is a Lie algebra $\mathfrak{g}$ over k whose adjoint representation is completely reducible.

Simple and semisimple Lie algebras are reductive. It is well known that any reductive Lie algebra decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})$$

where $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie subalgebra (the semisimple part of $\mathfrak{g}$) and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$, thus an abelian subalgebra.

We will focus on Lie algebras over the real and complex numbers and the relation between them. As a first example, note that any complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with its underlying real structure ${\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}}$ is a real reductive Lie algebra. Conversely, given a real reductive Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex reductive Lie algebra.

We will denote real Lie algebras by a subscript 0.

1.1.1 Real forms of complex Lie algebras

Definition 1.1.2. Let $\mathfrak{g}^{\mathbb{C}}$ be a complex Lie algebra. A **real form** $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is the subalgebra of fixed points of an antilinear involution $\sigma \in \mathrm{Aut}_2({\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}})$.

Example 1.1.3. Any real Lie algebra $\mathfrak{g}$ is a real form of its complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Indeed

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}.$$

So that the involution $(X, Y) \mapsto (X, -Y)$ has $\mathfrak{g}$ as its set of fixed points.

Example 1.1.4. Given a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, one can obtain it as a real form of $\mathfrak{g}^{\mathbb{C}} \otimes \mathbb{C}$. Indeed, choose a maximal compact subalgebra $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ defined by the antilinear involution $\tau \in \mathrm{Aut}_{\mathbb{R}}({\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}})$ where ${\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}}$ denotes $\mathfrak{g}^{\mathbb{C}}$ with its underlying real structure. Choose an isomorphism $\mathfrak{g}^{\mathbb{C}} \otimes \mathbb{C} \cong \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$, and define on it the antilinear involution

$$\tau^{\mathbb{C}}(x, y) := (\tau(x), -\tau(y)).$$

Its subalgebra of fixed points is isomorphic to $\mathfrak{u} \oplus i\mathfrak{u}$, namely, it is isomorphic to ${\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}}$ as a real Lie algebra.

Remark 1.1.5. *There is another natural definition of a real form of a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ as a real subalgebra $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ (more precisely, a subalgebra $\mathfrak{g} \subset {\mathfrak{g}^{\mathbb{C}}}_{\mathbb{R}}$) such that the natural embedding $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{g}^{\mathbb{C}}$ is an isomorphism*

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}^{\mathbb{C}}.$$

It is an easy exercise to prove that both definitions are equivalent.

In order to classify real forms of a complex reductive Lie algebras $\mathfrak{g}^{\mathbb{C}}$, note that it is enough to classify real forms of complex semisimple Lie algebras and real forms of complex abelian Lie algebras separately. Indeed, any automorphism of a Lie algebra leaves both the semisimple part ${\mathfrak{g}^{\mathbb{C}}}_{ss}$ and the center $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ invariant, so that an antilinear involution $\sigma : \mathfrak{g}^{\mathbb{C}} = {\mathfrak{g}^{\mathbb{C}}}_{ss} \oplus \mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \to {\mathfrak{g}^{\mathbb{C}}}_{ss} \oplus \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ splits as $\sigma = \sigma_{ss} \oplus \sigma_a$ where $\sigma_{ss} : {\mathfrak{g}^{\mathbb{C}}}_{ss} \to {\mathfrak{g}^{\mathbb{C}}}_{ss}$ defines a real form of ${\mathfrak{g}^{\mathbb{C}}}_{ss}$ and $\sigma_a : \mathfrak{z}(\mathfrak{g}^{\mathbb{C}}) \to \mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$ defines a ral form of $\mathfrak{z}(\mathfrak{g}^{\mathbb{C}})$.

Definition 1.1.6. Two real forms $\mathfrak{g}$ and $\mathfrak{g}'$ of $\mathfrak{g}^{\mathbb{C}}$ (defined respectively by antilinear involutions $\sigma,\ \sigma' \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$) are said to be isomorphic if and only if there exists

$\phi \in \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ making the following diagram commute

$$\begin{array}{ccc} \mathfrak{g}^{\mathbb{C}} & \xrightarrow{\phi} & \mathfrak{g}^{\mathbb{C}} \\ \sigma\downarrow & & \downarrow\sigma' \\ \mathfrak{g}^{\mathbb{C}} & \xrightarrow[\phi]{} & \mathfrak{g}^{\mathbb{C}}. \end{array}$$

We will say they are inner isomorphic if furthermore ϕ can be chosen inside $\mathrm{Inn}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$.

What follows is probably known, but we have not found a reference where it is treated systematically, so we include it in here.

Cartan theory for semisimple Lie algebras Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra, and let σ be an antilinear involution on it defining a real form $\mathfrak{g} := \mathfrak{g}^{\mathbb{C}^{\sigma}}$. It is well know (see for example [62, 39]) that there exists a compact antilinear involution $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ (that is, an antilinear involution defining a compact real form $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$, namely, such that its adjoint group is compact) commuting with σ. Hence, the composition $\theta = \sigma\tau$ is a linear involution of $\mathfrak{g}^{\mathbb{C}}$, but linear this time. This gives the following.

Theorem 1.1.7. *Given a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ the following 1-1 correspondences hold:*

$$\left\{ \begin{array}{c} \textit{Linear involutions} \\ \textit{in } \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}) \end{array} \right\} / \mathrm{Inn}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}) \longleftrightarrow \left\{ \begin{array}{c} \textit{Antilinear involutions} \\ \textit{of } \mathfrak{g}^{\mathbb{C}} \end{array} \right\} / \mathrm{Inn}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}),$$

$$\left\{ \begin{array}{c} \textit{Linear involutions} \\ \textit{in } \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}) \end{array} \right\} / \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}) \longleftrightarrow \left\{ \begin{array}{c} \textit{Antilinear involutions} \\ \textit{of } \mathfrak{g}^{\mathbb{C}} \end{array} \right\} / \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}).$$

Cartan theory for abelian Lie algebras Abelian Lie algebras are simply vector spaces. So we can reduce the classification problem to the simplest case: $\mathfrak{g}^{\mathbb{C}} \cong \mathbb{C}$. Now, the following lemma shows that there is no possible Cartan theory for Lie algebras and we need to look at groups in order to obtain a coherent theory.

Lemma 1.1.8. *Real forms of $\mathbb{C}$ are in correspondence with real vectorial lines in $\mathbb{C} \cong \mathbb{R}^2$*

Proof. Any real form of $\mathbb{C}$ has real dimension 1, so that it is the real line generated by an element $X \in \mathbb{C}$, and since any such line is a real form, we have them all. □

We introduce the following definition:

Definition 1.1.9. Let $G^{\mathbb{C}}$ be a complex Lie group. A **real form** $G < G^{\mathbb{C}}$ is the subgroup of fixed points under an antiholomorphic involution $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$.

Now, not all real forms of a Lie algebra exponentiate to real forms of a given Lie group.

Lemma 1.1.10. *There exist only two isomorphism classes of real forms of $\mathbb{C}^\times$, understood as fixed point sets of antilinear involutions. These are: $U(1)$ and $\mathbb{R}^\times$.*

Proof. Note that the real forms of $\mathbb{C}$ exponentiate to $\mathrm{U}(1)$ (imaginary line), $\mathbb{R}_{>0}$ (real line), or spirals around the origin in all other cases, defined parametrically by points of the form $e^{r\sin\theta}e^{ir\cos\theta}$ for fixed θ determined by the line and varying $r \in \mathbb{R}$. Now, any antilinear involution of $\mathbb{C}$ has associated matrix $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$, corresponding to the reflection with respect to the line of angle ϕ. The latter exponentiates to an algebraic map if and only if $\phi = k\frac{\pi}{2}$ $k \in \mathbb{Z}$. □

Remark 1.1.11. *For reasons that will become clear in Section 1.2, we will only consider the real forms $i\mathbb{R}$ and $\mathbb{R}$ of $\mathbb{C}$, correponding to the groups $\mathbb{R}^\times$, $U(1)$. So from now on, when we refer to a real form of $\mathbb{C}$, we will mean one of the latter. On the other hand, as a real form $\mathfrak{g}$ of a higher dimensional abelian Lie algebra $\mathfrak{g}^{\mathbb{C}}$ we will only consider those satisfying the following condition: given a basis $e_i \in \mathfrak{g}^{\mathbb{C}}$ such that $\mathbb{C}\cdot e_i$ is σ-invariant, for σ the involution defining $\mathfrak{g}$, it must happen that $\mathbb{C}\cdot e_i \cap \mathfrak{g}$ be either $\mathbb{R}$ or $i\mathbb{R}$.*

Corollary 1.1.12. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex abelian Lie algebra. Then, there is a 1-1 correspondence between real forms $\mathfrak{g}$ and linear involutions $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.*

Proof. As in the semisimple case, the proof consists in, given a real form, compose the antilinear involution defining it with a compact involution commuting to it.

Now, a compact Lie subalgebra of an abelian complex Lie algebra is isomorphic to $i\mathbb{R}^{\dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}}}$. So all we need to prove is that the involution

$$\tau : (z_1, \dots, z_s, w_1, \dots, w_l) \mapsto (-\overline{z_1}, \dots, -\overline{z_s}, -\overline{w_1}, \dots, -\overline{w_l})$$

commutes with any other antilinear involution. The above involution is defined modulo a choice of a basis. So let σ be an antilinear involution on $\mathfrak{g}^{\mathbb{C}}$, or equivalently, fix a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Then, by choosing a basis e_i of $\mathfrak{g}^{\mathbb{C}}$ as in Remark 1.1.11, modulo permutations, we may assume the involution σ to take the form

$$\sigma : (z_1, \dots, z_s, w_1, \dots, w_l) \mapsto (\overline{z_1}, \dots, \overline{z_s}, -\overline{w_1}, \dots, -\overline{w_l}).$$

So clearly both τ and σ commute and their composition is a linear involution.

For the converse, note that any linear involution is determined by the dimension of its (-1) and $(+1)$ eigenspaces, so that the above argument together with example 1.1.10 tells us that all the linear involutions are of the form $\sigma\tau$. $\square$

Remark 1.1.13. *This result classifies real forms of an abelian Lie algebra "up to inner isomorphism". Unlike in the semisimple case, the result is not true anymore when considering outer isomorphism classes. Indeed, $i\mathbb{R}$ and $\mathbb{R}$ are outer isomorphic, as multiplication by i is a Lie algebra morphism in the abelian case.*

We may summarise the results of the previous two paragraphs as follows.

Proposition 1.1.14. *Given a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$, there is a 1-1 correspondence between inner conjugacy classes of real forms and inner conjugacy classes of linear automorphisms $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.*

Definition 1.1.15. An involution on a real reductive Lie algebra $\mathfrak{g}$ defining a maximal compact form is called a Cartan involution.

Now, given a reductive Lie algebra $\mathfrak{g}$, there are two ways of inducing a Cartan decomposition on it:

1. By fixing a Cartan involution θ.

2. By choosing a non degenerate Ad G invariant bilinear form B on $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a maximal compact subalgebra. Then B allows us to choose a direct summand $\mathfrak{m}$ of $\mathfrak{h}$ and establish $\theta|_{\mathfrak{h}} \equiv +1$, $\theta|_{\mathfrak{m}} \equiv -1$, which defines a Cartan involution.

In any of these two ways we obtain a decomposition of $\mathfrak{g}$ into $(+1)$ and (-1) θ-eigenspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

satisfying the relations

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

There is an action of $\mathfrak{h}$ on $\mathfrak{m}$ is induced by the adjoint action of $\mathfrak{g}$ on itself.

There are two important examples of real reductive Lie algebras: given a complex semisimple Lie algebra, we will consider its split real form and its compact subalgebra, defined in terms of their Cartan subalgebras. Let us first make the following definition precise.

Definition 1.1.16. A Cartan subalgebra of a real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is a subalgebra whose complexification is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, which in turn is a nilpotent self normalising subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Now, fixing a Cartan involution θ on $\mathfrak{g}$, let $\mathfrak{c}_0 \subset \mathfrak{g}$ be θ-stable Cartan subalgebra. We define **the real rank of** $\mathfrak{g}$ (also called **split** rank) to be the dimension $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \dim\ \mathfrak{c} \cap \mathfrak{m}$. We define a real form to be **compact** if and only if $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 0$. On the other hand, a real form is **split** if $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}\mathfrak{g}^{\mathbb{C}}$.

Remark 1.1.17. *Let $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c})$ denote the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{c} = \mathfrak{c}^{\mathbb{C}}$. Then we have the following characterisation: $\mathfrak{g}$ is compact (respectively, split) if and only if for all Cartan subalgebras $\mathfrak{c}_0 \subset \mathfrak{g}$ and all $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$, $X \in \mathfrak{c}_0$ we have $\alpha(X) \in i\mathbb{R}$ (respectively if there exists some Cartan subalgebra $\mathfrak{c} \subset \mathfrak{g}^{\mathbb{C}}$ such that $\alpha(X) \in \mathbb{R}$ for all $X \in \mathfrak{c}$ and all roots). See [48], Chapter VI for a proof of this fact. Yet another characterization is given by Helgason (see [40], §IX.5), where a real form $\mathfrak{g}$ of a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is defined to be split if and only if for any Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ there exists a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$.*

Besides compact and split subagebras, there is a whole range of real forms with intermediate compact and non-compact (or split) rank. We will be interested in maximally non-compact Cartan subalgebras.

Another important class of Lie algebras are **quasi-split** real Lie algebras.

Definition 1.1.18. A quasi-split real Lie algebra is a real Lie algebra $\mathfrak{g}$ such that there exists a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$ which is σ-invariant for the involution σ defining $\mathfrak{g}$ inside $\mathfrak{g}^{\mathbb{C}}$.

Remark 1.1.19. *An equivalent condition to $\mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ being abelian is to ask that $\mathfrak{g}$ be a quasi-split real form of $\mathfrak{g}^{\mathbb{C}}$. This is easy to see by expressing a σ-stable Borel subgroups in terms of a system of σ-stable root. See [48] for details.*

Another useful characterization of quasi-split forms is in terms of compatibility of regularity notions in $\mathfrak{m}^{\mathbb{C}}$ and in $\mathfrak{g}^{\mathbb{C}}$. Namely, a real form is quasi-split if and only if $\mathfrak{m}_{reg} \subset \mathfrak{g}_{reg}$.

1.1.2 Maximal split subalgebras and restricted root systems

Let $\mathfrak{g}$ be a real reductive Lie algebra with a Cartan involution θ decomposing

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Given a maximal subalgebra $\mathfrak{a} \subset \mathfrak{m}$ it follows from the definitions that it must be abelian, and one can easily prove that its elements are semisimple and diagonable over the real numbers (cf. [48], Chapter VI; note that Knapp proves it for semisimple groups, but for reductive groups it suffices to use stability of the center and the semisimple part of $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$.). Considering the adjoint representation ad : $\mathfrak{a} \to \operatorname{End} \mathfrak{g}$ we obtain a decomposition into $\mathfrak{a}$ eigenspaces

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda(\mathfrak{a})} \mathfrak{g}_\lambda,$$

where $\Lambda(\mathfrak{a}) \subset \mathfrak{a}^*$ is called the set of **restricted roots** of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Extending by $\mathbb{C}$-linearity, one may consider the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into $\mathfrak{a}^{\mathbb{C}}$-eigenspaces

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}})} \mathfrak{g}^{\mathbb{C}}{}_\lambda.$$

Definition 1.1.20. A subalgebra $\mathfrak{a}$ as defined above is called a **maximal anisotropic Cartan subalgebra** of $\mathfrak{g}$. Its complexification $\mathfrak{a}^{\mathbb{C}}$ is also called a maximal anisotropic Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Remark 1.1.21. *Let $\Lambda(\mathfrak{a}^{\mathbb{C}}) \subset \mathfrak{a}^{\mathbb{C}^*}$ be the set of all the elements of $\Lambda(\mathfrak{a})$ extended to $\mathfrak{a}^{\mathbb{C}}$ by $\mathbb{C}$- linearity. We will refer to $\Lambda(\mathfrak{a}^{\mathbb{C}})$ as the set of **restricted roots** of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$; when speaking of restricted roots we will be referring to elements in $\Lambda(\mathfrak{a}^{\mathbb{C}})$ rather than $\Lambda(\mathfrak{a})$, but bearing in mind that the former is just an extension of the latter.*

Definition 1.1.22. A root system R is said to be reduced if given $r \in R \Rightarrow 2r \notin R$.

$\Lambda(\mathfrak{a}^{\mathbb{C}})$ forms indeed a root system in the sense of [2], which may not be reduced. The name restricted roots is due to the following fact.

Proposition 1.1.23. *Given a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a})$, where*

$$\mathfrak{c}_{\mathfrak{h}}(\mathfrak{a}) = \{x \in \mathfrak{h} \ : \ [x, y] = 0 \textit{ for all } y \in \mathfrak{a}\},$$

the subalgebra $\mathfrak{c} = (\mathfrak{a} \oplus i\mathfrak{t}^{\mathbb{C}}$ is a θ- invariant Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ satisfying that the set of roots $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$ associated to $\mathfrak{c}$ satisfies

$$\mathfrak{g}^{\mathbb{C}}{}_\lambda = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{c}, \mathfrak{g}^{\mathbb{C}}) \\ \alpha|_{\mathfrak{a}^{\mathbb{C}}} \equiv \lambda}} \mathfrak{g}^{\mathbb{C}}{}_\alpha \qquad \textit{for any } \lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}}).$$

Also $\alpha = \lambda + i\beta, \ \lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}}), i\beta \in i\mathfrak{t}^*$ for any $\alpha \in \Delta(\mathfrak{c}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$.

Remark 1.1.24. *With the above we have*

$$\mathfrak{g}_\lambda = \mathfrak{g} \cap \bigoplus_{\substack{\alpha\in\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{c})\\ \alpha|_{\mathfrak{a}^{\mathbb{C}}}\equiv\lambda}} \mathfrak{g}^{\mathbb{C}}{}_\alpha.$$

Note that $\mathfrak{g}\cap\mathfrak{g}^{\mathbb{C}}{}_\alpha \neq 0$ if and only $\alpha \in \mathfrak{a}^{\mathbb{C}*}$ is a restricted root. In general, this will not be the case, and the smallest positive dimensional subspaces of $\mathfrak{g}\cap\mathfrak{g}^{\mathbb{C}}{}_\lambda$ will have the form $\left(\mathfrak{g}^{\mathbb{C}}{}_\alpha \oplus \mathfrak{g}^{\mathbb{C}}{}_{\sigma\alpha}\right)\cap\mathfrak{g}$ for some $\alpha|_{\mathfrak{a}^{\mathbb{C}}} = \lambda$, where by abuse of notation we will write $\sigma\alpha$ instead of $\sigma^t\alpha$, where σ is the involution defining $\mathfrak{g}$ inside $\mathfrak{g}^{\mathbb{C}}$.

In [51], Kostant and Rallis give a procedure to construct a θ-invariant subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$ such $\widetilde{\mathfrak{g}} \subset (\widetilde{\mathfrak{g}})^{\mathbb{C}}$ is a split real form, and it is maximal for the latter property (by maximality, we mean that $\mathfrak{a}^{\mathbb{C}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}^{\mathbb{C}}$.

Definition 1.1.25. A subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$ satisfying the above conditions is called a **maximal split subalgebra**.

Remark 1.1.26. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex reductive Lie algebra, and let $\mathfrak{g}^{\mathbb{C}}{}_{\mathbb{R}}$ be its underlying real reductive algebra. Then, the maximal split subalgebra of $\mathfrak{g}^{\mathbb{C}}{}_{\mathbb{R}}$ is isomorphic to the split real form of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{g}_{split}$.*

It is clearly split within its complexification and it is maximal within $\mathfrak{g}^{\mathbb{C}}{}_{\mathbb{R}}$ with this property, which can be easily checked by identifying $\mathfrak{g}^{\mathbb{C}}{}_{\mathbb{R}} \cong \mathfrak{g}_{split} \oplus i\mathfrak{g}_{split}$.

To construct this split subalgebra, consider

$$\widetilde{\Lambda}(\mathfrak{a}) = \{\lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}}) \mid \lambda/2 \notin \Lambda(\mathfrak{a}^{\mathbb{C}})\},$$

and let $\{\lambda_1, \ldots, \lambda_a\} = \Sigma(\mathfrak{a}^{\mathbb{C}}) \subset \Lambda(\mathfrak{a}^{\mathbb{C}})$ be a system of simple restricted roots, which is also a system of simple roots for the reduced root system $\widetilde{\Lambda}(\mathfrak{a})$. Let $h_i \in \mathfrak{a}$ be the dual to λ_i with respect to some θ and $\mathrm{Ad}(\exp(\mathfrak{g}^{\mathbb{C}}))$-invariant bilinear form B satisfying that B is positive definite on $\mathfrak{h}$ and negative definite on $\mathfrak{m}$. Strictly speaking, in [51] they take B to be the Cartan-Killing form on $\mathfrak{g}$; however, the above assumptions are enough to obtain the necessary results hereby quoted. Now, for each $\lambda_i \in \Sigma(\mathfrak{a}^{\mathbb{C}})$ choose $y_i \in \mathfrak{g}^{\mathbb{C}}{}_{\lambda_i}$. We have that

$$[y_i, \theta y_i] = b_i h_i,$$

where $b_i = B(y_i, \theta y_i)$. Indeed, $[y_i, \theta y_i] \in \mathfrak{a}^{\mathbb{C}} \cap [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$, so it is enough to prove that $B([y_i, \theta y_i], x) = B(y_i, \theta y_i)\lambda_i(x)$ for all $x \in \mathfrak{a}^{\mathbb{C}}$, which is a simple calculation.

Consider

$$z_i = \frac{2}{\lambda_i(h_i)b_i}\theta y_i, \qquad w_i = [y_i, z_i] = \frac{2}{\lambda_i(h_i)}h_i.$$

Definition 1.1.27. Define $\widetilde{\mathfrak{g}}$ to be the subalgebra generated by all the $y_i,\ z_i,\ w_i$'s and $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$, where $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ denotes the centraliser of $\mathfrak{a}$ in $\mathfrak{g}$.

We have the following (cf. Propositions 21 and 23 and Remark 13 in [51]).

Proposition 1.1.28. *Let $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ be a real form, and let σ be the antilinear involution on $\mathfrak{g}^{\mathbb{C}}$ defining $\mathfrak{g}$. Let $\widetilde{\mathfrak{g}}$ be as defined above, and let $\widetilde{\mathfrak{g}}^{\mathbb{C}} = \widetilde{\mathfrak{g}} \otimes \mathbb{C}$. Then*

1. *$\widetilde{\mathfrak{g}}^{\mathbb{C}}$ is a σ and θ invariant reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$.*

2. *$\widetilde{\mathfrak{g}}$ is the split form of $\widetilde{\mathfrak{g}}^{\mathbb{C}}$; the Cartan subalgebra of $\widetilde{\mathfrak{g}}$ is $\mathfrak{a}$. Moreover, the subsystem of $\Lambda(\mathfrak{a}^{\mathbb{C}})$ defined by*

$$\widetilde{\Lambda}(\mathfrak{a}^{\mathbb{C}}) = \{\lambda \in \Lambda(\mathfrak{a}^{\mathbb{C}}) \mid \frac{\lambda}{2} \notin \Lambda(\mathfrak{a}^{\mathbb{C}})\}$$

   is the root system of $\widetilde{\mathfrak{g}}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$.

3.

Proof. 1 see Proposition 23 in [51].

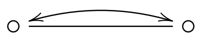
2 follows by construction and Proposition 23 in [51].

□

Remark 1.1.29. *Note that by construction $\mathfrak{s}^{\mathbb{C}}$we have a morphism $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$*

Note that $\widetilde{\Lambda}(\mathfrak{a})$ being a reduced root system, we can uniquely assign to it a complex semisimple Lie algebra $\widetilde{\mathfrak{g}}^{\mathbb{C}}$. In [2] Araki gives the details necessary to obtain $\widetilde{\mathfrak{g}}^{\mathbb{C}}$ (or its Dynkin diagram) from the Satake diagram of $\mathfrak{g}$ whenever the latter is a simple Lie algebra.

Recall that the Satake diagram of a real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ defined by σ is obtained from the Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$ by coloring a vertex in black $\bullet$ anytime the corresponding simple root $\alpha = i\beta \in i\mathfrak{t}^*$, where $i\mathfrak{t} \oplus \mathfrak{a}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. The vertices such that $\alpha \notin \mathfrak{t}^{\mathbb{C}^*}$ are left blank, but adding an arrow between any two vertices corresponding to roots related by $\sigma\alpha_i = \alpha_j + \beta$ where β is a compact root

 .

For example, the Satake diagram for the split real form has only blank vertices and no arrows (as $\sigma\alpha = \alpha$ for all $\alpha \in \mathfrak{a}^*$) and the Satake diagram of the maximal compact form has only black vertices.

Note that for the Satake diagram to be well defined, it is necessary to choose subsets of simple roots $\Sigma \subset \Lambda,\ S \subset \Delta$ in such a way that $\lambda + i\beta = \alpha > \alpha' = \lambda' + i\beta' \iff \lambda > \lambda'$.

Remark 1.1.30. *A way to obtain this is as follows (see [2]): take* $\dim \mathfrak{t}$ *independent imaginary roots* $i\beta$*; complete these to a* $\mathbb{Z}$*-basis of* $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$*, say* $S = \{\alpha_1, \ldots, \alpha_t, i\beta_1, \cdots, i\beta_t\}$ *where* $\alpha_i|_{\mathfrak{a}^{\mathbb{C}}} \not\equiv 0$*, and finally take any ordering for which* $\mathfrak{a}^* > i\mathfrak{t}^*$*.*

The advantage of Araki's procedure is that it allows identifying the isomorphism class of $\widetilde{\mathfrak{g}}$ easily. However, unlike Kostant and Rallis' method, it does not provide the embedding

$$(\widetilde{\mathfrak{g}}^{\mathbb{C}})^{\widetilde{\sigma}} \hookrightarrow \mathfrak{g}^{\mathbb{C}\sigma}$$

(where $\widetilde{\sigma} := \sigma|_{\widetilde{\mathfrak{g}}^{\mathbb{C}}}$).

Let us examine a few examples. For details on restricted roots see for example [38].

Example 1.1.31. In this simple case one can follow the procedure explained in [51] easily and build $\widetilde{\mathfrak{g}}$ directly.

Recall that $\mathfrak{g} = \mathfrak{su}(2,1) \subset \mathfrak{sl}(3,\mathbb{C})$ is defined as the set of fixed points for the involution

$$\sigma(X) = -\mathrm{Ad}(I_{2,1})({}^t\overline{X})$$

where

$$I_{2,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, we have $\theta = \mathrm{Ad}(I_{2,1})$ and

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{s}\left(\mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})\right), \qquad \mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ z & w & 0 \end{pmatrix} \;:\; x,y,z,w \in \mathbb{C} \right\}.$$

We can set

$$\mathfrak{a}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \;:\; x \in \mathbb{C} \right\}.$$

The restricted roots in this case are $\pm\lambda, \pm 2\lambda$ with $\alpha(M) = x$ the only non-zero entry of $M \in \mathfrak{a}^{\mathbb{C}}$. We have that

$$\mathfrak{g}^{\mathbb{C}}{}_{\lambda} = \left\{ \begin{pmatrix} 0 & b & -b \\ c & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \;:\; b,c \in \mathbb{C} \right\}, \qquad \mathfrak{g}^{\mathbb{C}}{}_{-\lambda} = \left\{ \begin{pmatrix} 0 & b & b \\ c & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} \;:\; b,c \in \mathbb{C} \right\},$$

$$\mathfrak{g}^{\mathbb{C}}{}_{2\lambda} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & -s \\ 0 & s & -s \end{pmatrix} \;:\; s \in \mathbb{C} \right\}, \qquad \mathfrak{g}^{\mathbb{C}}{}_{-2\lambda} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s & -s \\ 0 & s & s \end{pmatrix} \;:\; s \in \mathbb{C} \right\}.$$

Now, a set of simple restricted roots is given simply by λ, so the Lie algebra $(\widetilde{\mathfrak{su}}(2,1))^{\mathbb{C}}$ will simply be the three dimensional subalgebra $\mathfrak{s} = \langle x, y, w\rangle$ in the theorem above, namely

$$(\widetilde{\mathfrak{su}}(2,1))^{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}),\ \widetilde{\mathfrak{su}}(2,1) \cong \mathfrak{sl}(2,\mathbb{R}).$$

Indeed, choose generators for $\widetilde{\mathfrak{su}}(2,1)$ as follows:

$$e = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Namely

$$\widetilde{\mathfrak{su}}(2,1) = \left\{ \begin{pmatrix} 0 & b & c \\ -b & 0 & a \\ c & a & 0 \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\} \cong \mathfrak{so}(2,1)$$

and

$$(\widetilde{\mathfrak{su}}(2,1))^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & b & c \\ -b & 0 & a \\ c & a & 0 \end{pmatrix} \mid a,b,c \in \mathbb{C} \right\} \cong \mathfrak{so}(3,\mathbb{C}).$$

The isomorphism

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

also respects the Cartan involution. Indeed

$$\mathfrak{sl}(2,\mathbb{R}) \cap \mathfrak{su}(2) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \mathrm{Im}\left(\langle e - f\rangle\right) = \mathrm{Im}(\widetilde{\mathfrak{su}}(2,1) \cap \mathfrak{h}),$$

$$\mathfrak{sl}(2,\mathbb{R}) \cap i\mathfrak{su}(2) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle = \mathrm{Im}\left(\langle e + f, x\rangle\right) = \mathrm{Im}(\widetilde{\mathfrak{su}}(2,1) \cap \mathfrak{m}).$$

Example 1.1.32. In this higher rank case, a more convenient (yet less explicit) procedure is given by Araki [2].

For this, we realise $\mathfrak{su}(p,q)$ as the $(p+q) \times (p+q)$ matrices fixed under the involution

$$\sigma : X \mapsto -\mathrm{Ad}\ J_{p,q}{}^t\overline{X}$$

where

$$J_{p,q} = \begin{pmatrix} 0 & 0 & s_p \\ 0 & Id_{q-p} & \\ s_p & 0 & 0 \end{pmatrix}$$

where s_p is the $p \times p$ matrix with antidiagonal entries 1 and zero elsewhere. With this realisation, we have that the diagonal matrices are θ-stable for $\theta = \mathrm{Ad}(J_{p,q})$ and that

$$\mathfrak{a}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Ad}(s_p)A \end{pmatrix} : A = \mathrm{diag}(a_1, \ldots, a_p), -\mathrm{Ad}(s_p)A = \mathrm{diag}(-a_p, \ldots, -a_1) \right\},$$

$$\mathfrak{t}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \mathrm{Ad}(s_p)A \end{pmatrix} : \begin{matrix} A = \mathrm{diag}(a_1, \ldots, a_p) \\ B = \mathrm{diag}(b_1, \ldots, b_{q-p}) \\ \mathrm{Ad}(s_p)A = \mathrm{diag}(a_p, \ldots, a_1) \end{matrix} \right\}$$

gives a maximally non-compact Cartan subalgebra $\mathfrak{c} = \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{sl}(p+q, \mathbb{C})$. Choose now a system of simple roots

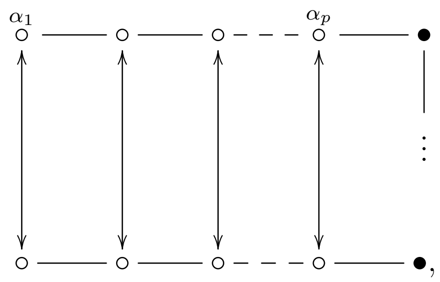
$$\alpha_i = L_i - L_{i+1} i = 1, \ldots, p+q-1$$

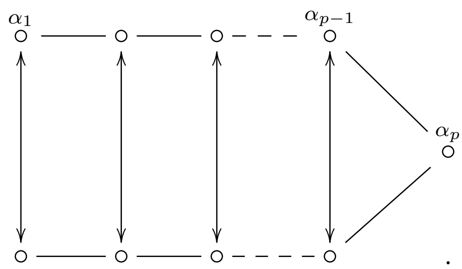
where L_i applied to a diagonal matrix returns the i-th entry and $\alpha_i < \alpha_{i+1}$. Note that

$$\alpha_i|_{\mathfrak{a}^{\mathbb{C}}} = \alpha_{p+q-i+1}|_{\alpha_a} \; i \leq p,$$

$$\alpha_i|_{\mathfrak{a}^{\mathbb{C}}} \equiv 0 \; p+1 \leq i \leq q.$$

In fact $\sigma(\alpha_i) = \alpha_{p+q-i+1}$, all of which yields the following Satake diagrams, depending on whether $p < q$ or $p = q$





Let $\epsilon_i \in \mathfrak{a}^{\mathbb{C}^*}$, $i = 1, \ldots, p$ give the i-th entry of the diagonal matrix $A \in \mathfrak{a}^{\mathbb{C}}$. Clearly

$$L_i - L_{i+1}|_{\mathfrak{a}^{\mathbb{C}}} = \epsilon_i - \epsilon_{i+1} \; i \leq p-1,$$

$$L_p - L_{p+1}|_{\mathfrak{a}^{\mathbb{C}}} = \begin{cases} \epsilon_p & if\ p < q \\ 2\epsilon_p & if\ p = q \end{cases}$$

Thus, the correponding Dynkin diagrams are the one for $\mathfrak{so}(2p+1)$ if $p \neq q$ (as $||\epsilon_p|| < ||\epsilon_i - \epsilon_{i+1}|| = ||\epsilon_j - \epsilon_{j+1}||$ for any $i, j < p$) and the one for $\mathfrak{sp}(2p, \mathbb{C})$ whenever $p = q$ (as $||2\epsilon_p|| > ||\epsilon_i - \epsilon_{i+1}|| = ||\epsilon_j - \epsilon_{j+1}||$ for any $i, j < p$). Hence

$$\widetilde{\mathfrak{su}}(p,q) \cong \mathfrak{so}(p, p+1), \qquad \widetilde{\mathfrak{su}}(p,p) \cong \mathfrak{sp}(2p, \mathbb{R}).$$

Example 1.1.33. Consider $\mathfrak{so}(2n, \mathbb{C})$ the algebra of $2n \times 2n$ matrices antisymmetric with respect to the bilinear form whose matrix has 1's in the antidiagonal and 0's elsewhere. These matrices have the form

$$\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \qquad \text{with } B = -{}^tB,\ C = -{}^tC.$$

Let

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Gamma_n = i \begin{pmatrix} \gamma & & & & & & \\ & \ddots & & & & & \\ & & \gamma & & & & \\ & & & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & & & \\ & & & & -\gamma & & \\ & & & & & \ddots & \\ & & & & & & -\gamma \end{pmatrix}$$

where the middle matrix appears only if $l = 2p+1$. Define the linear involution

$$\theta : X \mapsto \Gamma_n X \Gamma_n^{-1}.$$

A maximal non-compact Cartan subalgebra is given by diagonal matrices. We have

$$\mathfrak{a}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -{}^tA \end{pmatrix} \ :\ A = \mathrm{diag}(a_1, -a_1, \ldots, a_p, -a_p) \right\},$$

$$\mathfrak{t}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -{}^tA \end{pmatrix} \ :\ A = \mathrm{diag}(a_1, a_1, \ldots, a_p, a_p) \right\}.$$

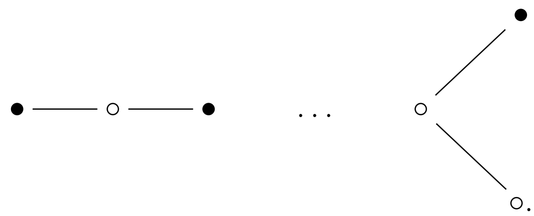
<u>Case 1: $l = 2p$</u> In this case, the set of simple restricted roots is given by

$$\alpha_1 = L_1 + L_2, \qquad \alpha_i = -L_1 - L_{i+1}\ i = 2, \ldots, 2p-2, \qquad \alpha_{2p-1} = L_{2p-1} + L_{2p}, \qquad \alpha_{2p} = L_{2p-1} - L_{2p}.$$

Note that

$$\alpha_{2i+1} \in \mathfrak{t}^{\mathbb{C}*} \;\text{ for any } i = 0, \ldots, p-1 \qquad \alpha_{2i} \notin \mathfrak{t}^{\mathbb{C}*}$$

whence the Satake diagram



Define $\epsilon_i \in \mathfrak{a}^{\mathbb{C}*}$ to assign the $(2i-1)$-th entry to the diagonal matrix A. Then

$$\alpha_{2i}|_{\mathfrak{a}^{\mathbb{C}}} = \epsilon_i - \epsilon_{i+1} \;\text{ for any }\; i \leq p-1 \qquad \alpha_{2p}|_{\mathfrak{a}^{\mathbb{C}}} = 2\epsilon_p.$$

So the restricted Dynkin diagram reads

and

$$\widetilde{\mathfrak{so}}^*(4p) \cong \mathfrak{sp}(2p, \mathbb{R}).$$

<u>Case 2: $l = 2p+1$</u> The set of simple restricted roots is

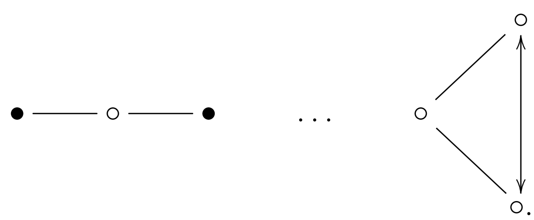
$$\alpha_1 = L_1 + L_2, \qquad \alpha_i = -L_1 - L_{i+1}\; i = 2, \ldots, 2p-2, \qquad \alpha_{2p-1} = L_{2p-1} + L_{2p}, \qquad \alpha_{2p} = -L_{2p} - L_{2p+1}$$

$$\alpha_{2p+1} = -L_{2p} + L_{2p+1}.$$

So we obtain that

$$\alpha_{2i+1} \in \mathfrak{t}^{\mathbb{C}*} \;\text{ for any } i < p-1 \ldots \qquad \alpha_{2i}, \alpha_l \notin \mathfrak{t}^{\mathbb{C}*},\; \sigma\alpha_{l-1} = \alpha_l$$

so we have the following Satake diagram



With the same notation as above,

$$\alpha_{2i}|_{\mathfrak{a}^{\mathbb{C}}} = \epsilon_i - \epsilon_{i+1} \;\text{ for any }\; i \leq p-1 \qquad \alpha_{2p}|_{\mathfrak{a}^{\mathbb{C}}} = \epsilon_p.$$

So the restricted Dynkin diagram reads

$$\circ \!-\!\!-\!\!-\! \circ \!-\!\!-\!\!-\! \ldots \qquad \circ \Longrightarrow \circ$$

and

$$\widetilde{\mathfrak{so}}^*(4p+2) \cong \mathfrak{so}(p, p+1).$$

Remark 1.1.34. *A natural question that arises is to what extent up to conjugation the maximal split subalgebra $\widetilde{\mathfrak{g}}$ is an intersection of $\mathfrak{g}$ with $\mathfrak{g}_{split}$, the split form of $\mathfrak{g}^{\mathbb{C}}$.*

We examine some concrete cases.

Example 1.1.35. $\mathfrak{su}^*(2p)$ is defined as the fixed point set of the involution

$$\sigma(X) = \mathrm{Ad}\overbrace{\begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}}^{J_p}\overline{X}.$$

Its maximal split real form is $\mathfrak{sl}(p, \mathbb{R})$, but $\mathfrak{su}^*(2p) \cap \mathfrak{sl}(2p, \mathbb{R})$ strictly contains$\mathfrak{sl}(p, \mathbb{R})$. Indeed,

$$\mathfrak{su}^*(2p) \cap \mathfrak{sl}(2p, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;:\; \begin{array}{c} A, B, C, D \in \mathfrak{gl}(p, \mathbb{R}) \\ A = D, \ tr(A) = 0, \ B = -C \end{array} \right\}.$$

Example 1.1.36. $\mathfrak{su}(p, q)$ $p < q$. This form is defined as the fixed point set of the involution

$$\sigma(X) = -\mathrm{Ad}\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}^t \overline{X}.$$

The intersection $\mathfrak{su}(p, q) \cap \mathfrak{sl}(p+q, \mathbb{R})$ is by definition the real form $\mathfrak{so}(p, q)$, which is the maximal split real form only in the case $q = p + 1$. This is due to thefact that there are $2 \dim \mathfrak{a}^{\mathbb{C}} = 2p$ restricted root spaces of dimension $2(q - p)$.

Table of maximal split real forms of non compact simple real Lie algebras The information on the following table has been extracted from [2], [62] and [40]. Real forms of exceptional Lie algebras are classified by their character $\delta = \dim \mathfrak{m} - \dim \mathfrak{h}$, which on the table appears in brackets by the real rank of the algebra (for instance $\mathfrak{e}_{6(6)}$ is the real form of $\mathfrak{e}_6$ with character 6). Note that the character reaches its maximal value $\delta = \mathrm{rk}(\mathfrak{g})$ whenever the form is split, and its minimal value $\delta = -\dim \mathfrak{g}$ when the form is compact. For us $\mathrm{Sp}(2m, \mathbb{C}) = \mathrm{Sp}(\mathbb{C}^{2m})$.

Type	$\mathfrak{g}$	$\widetilde{\mathfrak{g}}$
AI	$\mathfrak{sl}(n, \mathbb{R}))$	$\mathfrak{sl}(n, \mathbb{R})$
AII	$\mathfrak{su}^*(2n)$	$\mathfrak{sl}(n, \mathbb{R})$
AIII	$\mathfrak{su}(p, q),\ p < q$	$\mathfrak{so}(p, p+1)$
AIII	$\mathfrak{su}(p, p)$	$\mathfrak{sp}(2p, \mathbb{R})$
BI	$\mathfrak{so}(2p, 2q+1),\ p \leq q$	$\mathfrak{so}(p, p)$
BII	$\mathfrak{so}(1, 2n)$	$\mathfrak{sl}(2, \mathbb{R})$
CI	$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R})$
CII	$\mathfrak{sp}(2p, 2q)\ p < q$	$\mathfrak{so}(p, p+1)$
CII	$\mathfrak{sp}(2p, 2p)$	$\mathfrak{sp}(2p, \mathbb{R})$
BDI	$\mathfrak{so}(p, 2n-p)\ p \leq n-2$	$\mathfrak{so}(p, p+1)$
BDI	$\mathfrak{so}(p-1, p+1)$	$\mathfrak{so}(p-1, p)$
BDI	$\mathfrak{so}(p, p)$	$\mathfrak{so}(p, p)$
DIII	$\mathfrak{so}^*(4p+2)$	$\mathfrak{so}(p, p+1)$
DIII	$\mathfrak{so}^*(4p)$	$\mathfrak{sp}(2p, \mathbb{R})$
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(6)}$
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(1, 2)$
EIV	$\mathfrak{e}_{6(-26)}$	$\mathfrak{sl}(3, \mathbb{R})$
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(7)}$
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{f}_{4(4)}$
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{sp}(6, \mathbb{R})$
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{8(8)}$
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{f}_{4(4)}$
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{f}_{4(4)}$
FII	$\mathfrak{f}_{4(-20)}$	$\mathfrak{sl}(2, \mathbb{R})$
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{g}_{2(2)}$

1.2 Reductive Lie groups

Following Knapp [48], we define reductivity of a Lie group as follows.

Definition 1.2.1. A **reductive group** is a 4-tuple (G, H, θ, B) where

1. G is a Lie group with reductive Lie algebra $\mathfrak{g}$.
2. $H < G$ is a maximal compact subgroup.
3. θ is a Lie algebra involution on $\mathfrak{g}$ inducing an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

   where $\mathfrak{h} = \mathrm{Lie}(H)$ is the $(+1)$-eigenspace for the action of θ, and $\mathfrak{m}$ is the (-1)-eigenspace.
4. B is a θ and $\mathrm{Ad}(G)$-invariant non-degenerate bilinear form, with respect to which $\mathfrak{h} \perp_B \mathfrak{m}$ and B is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$.
5. The multiplication map $H \times \exp(\mathfrak{m}) \to G$ is a diffeomorphism.

If furthermore (G, H, θ, B) satisfies

- G acts by inner automorphisms on its Lie algebra via the adjoint representation (1.1)

then the group will be called **strongly reductive**.

Remark 1.2.2. *Condition 1.1 is very strong, as it leaves out groups such as $O(2n)$ $n \in \mathbb{N}$. However, we will need to use it at times, and so we will be careful distinguishing reductive and strongly reductive groups.*

Given a Lie group G with reductive Lie algebra $\mathfrak{g}$, a piece of extra data (H, θ, B) making a reductive group out of G will be refered to as **Cartan data** for G.

Definition 1.2.3. A morphism of reductive Lie groups

$$(G', H', \theta', B') \to (G, H, \theta, B)$$

is a morphism of Lie groups $G' \to G$ which respects the respective Cartan data in the obvious way. In particular, a reductive Lie subgroup of a reductive Lie group (G, H, θ, B) is a reductive Lie group (G', H', θ', B') such that $G' \leq G$ is a Lie subgroup and the Cartan data (H', θ', B') is obtained by intersection and restriction.

Let us examine some examples.

Example 1.2.4. A compact Lie group U can be endowed with a Cartan data in the following way: take $H = U$, $\theta = id$ and any $\mathrm{Ad}(U)$-invariant bilinear form B defined from an arbitrary non-degenerate bilinear form B' by choosing an invariant measure dg on the group and integrating

$$B(X,Y) = \int_G B'(Ad(g)X, Ad(g)Y)dg.$$

Moreover, if U satisfies Condition (1.1), the group is strongly reductive.

Example 1.2.5. Classically, by a complex reductive Lie group one means a complex Lie group G which is the complexification of a compact Lie group U. One may ask whether a complex Lie group G satisfying this property is reductive in the sense of Definition 1.2.1. In fact, such groups admit a reductive structure in the sense of Knapp, determined by the choice of a Cartan data on U. Indeed, once a non degenerate metric B on $\mathfrak{u} := \mathrm{Lie}(U)$ has been fixed, identify $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ with the approapriate Lie bracket and extend B to $\mathfrak{g}$ in such a way that both summands are orthogonal and the restriction to each of the summands is B. Finally, take $\tau(x,y) = (x,-y)$ where $(x,y) \in \mathfrak{u} \oplus \mathfrak{u}$. Clearly (G, U, τ, B) is reductive, and it is strongly reductive if and only if G satisfies (1.1).

Remark 1.2.6. *What the above points out is that complex Lie groups which are reductive in the sense of Definition 1.2.1 are complexifications of compact Lie groups together with some extra data. In particular, most theorems for complex reductive Lie groups in the classical sense apply.*

However, our definition of reductivity is more convenient for our purposes, as on the one hand it emphasizes the essential ingredients necessary to manipulate reductive Lie groups with no ambiguity. On the other hand, in the case of real Lie groups, the Cartan data is even more significative, as, for example, polar decomposition and compactness of H leave out groups with infinitely many connected components. Furthermore, we will see in Chapter 2 that the Cartan data plays an important role in the theory of Higgs pairs.

Example 1.2.7. When G is connected and semisimple with finite center, all the information amounts to choosing a maximal compact subgroup $H < G$. Indeed, one may take B to be the Killing form and $\mathfrak{m}$ a B-orthogonal complement to $\mathfrak{h}$. Finally, define θ to be constantly 1 on $\mathfrak{h}$ and -1 on $\mathfrak{m}$.

Proposition 1.2.8. *Any connected reductive Lie group (G, H, θ, B) contained in a complex reductive Lie group is strongly reductive. Furthermore, G acts by automorphisms of the Lie algebra on any Lie algebra of a complex Lie group containing G.*

Proof. Indeed, by Condition 5 in Definition 1.2.1, we have $G = e^{\mathfrak{h}} \cdot e^{\mathfrak{m}}$, since H being compact and connected it must be $H = e^{\mathfrak{h}}$. Finally, a simple computation shows that in the case of matrix groups $\mathrm{Ad}_{e^X} \circ \mathrm{Ad}_{e^Y} \equiv \mathrm{Ad}_{e^{X+Y}} \in \mathrm{Aut}\ \mathfrak{g}$. Since the group is contained in a complex reductive Lie group, it follows that $\mathrm{Ad}\ G = \mathrm{Ad}\ G'$ where G' is now a group of matrices, whence the result.

The same kind of arguments prove the second statement. □

We have the following.

Lemma 1.2.9. *The class of reductive Lie groups is closed by exact sequences*

$$1 \to K \to G \xrightarrow{f} G' \to F \to 1$$

for some finite groups F, K. In particular, it is closed by isogeny and finite quotients.

Moreover, a choice of Cartan data for G induces one for G' and viceversa.

Proof. We will prove the statement for groups admitting a piece of Cartan data, without fixing the latter. This will automatically imply the Lemma, together with the last statement, as if there are fewer morphisms respecting the Cartan data than morphisms of Lie groups.

Now, observe that if G, G' are groups as in the hypothesis, they have the same Lie algebra, and so conditions *1.*, *3.* and *4.* in Definition 1.2.1 hold automatically.

As for conditions 2. and *4.*, first assume that $G = He^{\mathfrak{m}}$ be reductive. Then $f(H)$ is compact in G', and $f(e^{\mathfrak{m}}) \subseteq e^{\mathfrak{m}}$; on the other hand, we must have

$$f(G) = f(H) \cdot e^{df(\mathfrak{m})}.$$

Since the preimage of F inside G', say F', must be finite, by finiteness of F and K, it follows that G' is some finite extension of the connected component of $f(H)e^{\mathfrak{m}}$, which is $f(H)_0 e^{\mathfrak{m}}$. Namely, the subgroup $F' f(H)_0$ is a maximal compact subgroup of G' and

$$G' = F' \cdot f(H)_0 e^{\mathfrak{m}}.$$

Suppose now $G' = H' e^{\mathfrak{m}}$ is a reductive group. Then, $f^{-1}(H')$ is compact in G by finiteness of K. Also, $f^{-1}(e^{\mathfrak{m}})$ is a finite extension of $e^{\mathfrak{m}}$. Thus, the same reasoning as before yields a maximal compact subgroup H and $G = H \cdot e^{\mathfrak{m}}$. □

Remark 1.2.10. *It is false in general that a group isogenous to a strongly reductive Lie group be strongly reductive. This is the case of the groups $O(2n)$ and $SO(2n)$. The latter is strongly reductive, but not the former.*

Remark 1.2.11. *When the group G is semisimple, we will omit the Cartan data to make the notation less cumbersome.*

1.2.1 Real forms of complex reductive Lie groups.

Definition 1.2.12. Let $(G^{\mathbb{C}}, U, \tau, B)$ be a complex (strongly) reductive Lie group. We say that $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ is a real form if there exists an antiholomorphic involution

$$\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$$

such that $G = \left(G^{\mathbb{C}}\right)^{\sigma}$ is the subgroup of fixed points by the given involution and furthermore

1. $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$.
2. The compatible involutions τ, σ defining $\mathfrak{u}$ and $\mathfrak{g}$ inside $\mathfrak{g}^{\mathbb{C}}$ lift to $G^{\mathbb{C}}$ in such a way that $\left(G^{\mathbb{C}}\right)^{\tau} = U$, $\sigma\tau = \tau\sigma$ on $G^{\mathbb{C}}$.
3. The extension of θ to $\mathfrak{g}^{\mathbb{C}}$ lifts to a holomorphic involution on $G^{\mathbb{C}}$ such that all three lifts commute and $(G^{\mathbb{C}})^{\theta} = H^{\mathbb{C}}$.

Remark 1.2.13. *An obvious example of a real form is the maximal compact subgroup of a reductive Lie group.*

A great variety of examples of real reductive Lie groups is provided by real forms of complex reductive Lie groups.

Definition 1.2.14. Given a complex or real Lie group G and an involution $\iota : G \to G$ (holomorphic or antiholomorphic), we define

$$G_{\iota} = \{g \in G \ : \ g^{-1}g^{\iota} \in Z(G)\}.$$

Lemma 1.2.15. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ be a real form of a connected complex reductive Lie group and let σ be the antiholomorphic involution defining G. Then*

1. Ad $G^{\mathbb{C}}$ *is reductive.*

2. *If we let* $(\mathrm{Ad}\ G^{\mathbb{C}})_\sigma$ *be as in Definition 1.2.14, then*

$$(\mathrm{Ad}\ G^{\mathbb{C}})_\sigma = \mathrm{Ad}\ F \cdot \mathrm{Ad}\ G,$$

where

$$F = \{e^{iX}\ :\ X \in \mathfrak{g},\ e^{2iX} \in Z(G^{\mathbb{C}})\}.$$

In particular, all elements of $\mathrm{Ad}\ F$ *have order* 2.

3. $(\mathrm{Ad}\ G^{\mathbb{C}})_\sigma = \mathrm{Ad}\ \left((G^{\mathbb{C}})_\sigma\right)$

4. *If* $G^{\mathbb{C}}$ *and* G *are strongly reductive,* $(\mathrm{Ad}\ G^{\mathbb{C}})_\sigma$ *is strongly reductive if and only if* $F \subseteq Z(G^{\mathbb{C}})$.

Remark 1.2.16. *In the adjoint group case,* $\mathrm{Ad}\ F \subset e^{i\mathfrak{g}} \cap G$*, as* $Z(G^{\mathbb{C}})$ *being trivial* $e^{2iX} = 1$*, so* $\sigma e^{iX} = e^{-iX} = e^{iX}$*. In particular,* $\mathrm{Ad}\ G = \left(\mathrm{Ad}\ G^{\mathbb{C}}\right)^\sigma$*. For semisimple groups,* F *is finite.*

Proof. The first statement is straightforward to prove by noticing that the center is stable by involutions. Let us prove 2.

Let $g \in G^{\mathbb{C}}$ be such that $\mathrm{Ad}\ g = \mathrm{Ad}\ g^\sigma$. Express $g = ue^s$ for some $u \in U$, $s \in i\mathfrak{u}$. It follows that $\mathrm{Ad}\ u^\sigma e^{\sigma s} = \mathrm{Ad}\ ue^s$, which implies

$$\mathrm{Ad}\ u^{-1}u^\sigma, \mathrm{Ad}\ e^{-s}e^{\sigma s} \in \mathrm{Ad}\ U \cap \mathrm{Ad}\ e^{i\mathfrak{u}} = id.$$

Hence $\mathrm{Ad}\ u \in (\mathrm{Ad}\ U)^\sigma$, $\mathrm{Ad}\ e^{-s+\sigma s} \in (\mathrm{Ad}\ e^{i\mathfrak{u}})^\sigma$. By connectedness of U, $u = e^V$. Now, expressing both $V \in \mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$ and $s \in i\mathfrak{u} = i\mathfrak{h} \oplus \mathfrak{m}$ as a sum of elements in $\mathfrak{g}$ and $i\mathfrak{g}$, we see that the imaginary part acts by elements of order 2 in $\mathrm{Ad}\ G^{\mathbb{C}}$.

3. is straightforward.

Finally, the fact that $(\mathrm{Ad}\ G^{\mathbb{C}})_\sigma = \mathrm{Ad}\ F \cdot \mathrm{Ad}\ G^{\mathbb{C}}$ follows from the arguments above together with the fact that $\mathrm{Ad}\ (e^X e^Y) = \mathrm{Ad}\ (e^{X+Y})$, which is easily proved for matrix groups, in particular, adjoint groups. □

Remark 1.2.17. *In [51], Kostant and Rallis observe a similar phenomenon. They work with the groups* $H^{\mathbb{C}} = \mathrm{Ad}\ \mathfrak{h}^{\mathbb{C}}$ *and* $G^{\mathbb{C}} = \mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}}$*, and notice that* $\left(G^{\mathbb{C}}\right)^\theta$ *(denoted by* K_θ *in [51]) is a finite extension of* $H^{\mathbb{C}}$ *by the finite group* $F = \{e^{iX}\ :\ X \in \mathfrak{g},\ e^{2iX} = 1\}$.

Now, with our definitions, given (G, H, θ, B) *a group of the adjoint type, we readily have that*

$$H^{\mathbb{C}}_\theta = H^{\mathbb{C}}.$$

We see this just as in Lemma 1.2.15 substituting σ by θ, and using Remark 1.2.16).

Note that our situation differs from theirs, as they were interested in invariant theory, whence the choice of the adjoint group. We need to work with more general groups our goal being a theory suitable for Higgs pairs. The meaningful group for us will be $(\mathrm{Ad}\ H^{\mathbb{C}})_\theta := \left(\mathrm{Ad}\ G^{\mathbb{C}}\right)^\theta$, which is the image by the adjoint representation of the group $H^{\mathbb{C}}_\theta$. This can again be checked just as in Lemma 1.2.15.

Proposition 1.2.18. *Let $(G^{\mathbb{C}}, U, \tau, B)$ be a connected complex reductive Lie group, and let σ be an antilinear involution on $G^{\mathbb{C}}$ defining $G = \left(G^{\mathbb{C}}\right)^\sigma$. Then, there exists an inner conjugate involution $\sigma' = \mathrm{Ad}_g \circ \sigma \circ \mathrm{Ad}_{g^{-1}}$ such that for any Cartan data on $G^{\mathbb{C}}$ $(G^{\mathbb{C}}, U, \tau, B)$, $G' = gGg^{-1}$ can be endowed with Cartan data (G', H', θ', B') making it a reductive subgroup of $(G^{\mathbb{C}}, U, \tau, B)$ in the sense of Definition 1.2.1.*

Furthermore, if $(G^{\mathbb{C}}, U, \tau, B)$ is strongly reductive, (G, H, θ, B) is strongly reductive if and only if $\left(\mathrm{Ad}\ G^{\mathbb{C}}\right)^\sigma = \mathrm{Ad}\ G$.

Proof. By Proposition 1.1.14 and conjugacy of all compact Lie subalgebras, we have that on the level of the Lie algebras there is always an inner conjugate of $d\sigma$ that commutes with τ (where by abuse of notation we will denote by τ both the involution defining $U = \left(G^{\mathbb{C}}\right)^\tau$ and its differential), say $(d\sigma)' = \mathrm{Ad}_g \circ d\sigma \circ \mathrm{Ad}_{g^{-1}}$. We notice that $(d\sigma)' = d\sigma'$ where $\sigma' = \mathrm{Ad}_g \circ \sigma \circ \mathrm{Ad}_{g^{-1}}$.

Now, by connectedness of $G^{\mathbb{C}}$ and compactness of U, $U = \exp(\mathfrak{u})$, and so it is σ' stable, as

$$\mathfrak{u} = \mathfrak{h}' \oplus i\mathfrak{m}'.$$

Similarly $\exp i\mathfrak{u}$ is σ' stable. Now, since G admits a polar decomposition (see Remark 1.2.6), say $G = He^{\mathfrak{m}}$, it follows that $G' = H' \times \exp \mathfrak{m}' = G^{\mathbb{C}^{\sigma'}}$, where $H' = \mathrm{Ad}_g H \mathrm{Ad}_{g^{-1}}$. We need to prove that $H' = U^{\sigma'}$, $\exp \mathfrak{m}' = \exp \mathfrak{u}^{\sigma'}$. Take $h \in G$, and write its translate by Ad_g as ue^V, $u \in U$, $V \in i\mathfrak{u}$. Then,

$$u^{\sigma'} e^{\sigma' V} = ue^V \iff u^{-1}u^{\sigma'} = e^{-\sigma' V} e^V \in U \cap \exp i\mathfrak{u} = \{1\}$$

and so the result follows.

Non degeneracy of $B|_{\mathfrak{g}}$ follows easily: for any element $X \in \mathfrak{g}$ there exists $Y = Y_1 + iY_2 \in \mathfrak{g}^{\mathbb{C}}$ such that $0 \neq B(X, Y) = B(X, Y_1) + iB(X, Y_2)$. In particular $B(X, Y_1) \neq 0$. Clearly $\mathfrak{h}' \perp_B \mathfrak{m}'$, and all other properties are staightforward to check.

As for the second statement, it follows from Lemma 1.2.15. □

Remark 1.2.19. *What Proposition 1.2.18 points out is that to recover a real form of a complex reductive Lie group $G^{\mathbb{C}}$ it is enough to choose a maximal compact subgroup of $G^{\mathbb{C}}$ and a holomorphic involution stabilising it, just as in the semisimple case.*

Remark 1.2.20. *As an example to the above remark, given a complex reductive Lie group* $(G = U^{\mathbb{C}}, U, \tau, B)$ *we can recover its underlying real Lie group* $G_{\mathbb{R}}$ *as a real form of* $(G \times G, U \times U, \tau \oplus \tau, B \oplus B)$ *by fixing the involution* $\theta : G \times G \to G \times G : (g, h) \mapsto (h, g)$*, due to this correspondence.*

Remark 1.2.21. *Note that there are more real reductive subgroups of a complex reductive group* $(G^{\mathbb{C}}, U, \tau, B)$ *than real forms. For example,*

$$\big(SL(2,\mathbb{R}), SO(2), -(\cdot)^t, B_{Killing}\big)$$

appears as a real form, whereas

$$\big(N_{SL(2,\mathbb{C})}(SL(2,\mathbb{R})), N_{SU(2)}SO(2), -(\cdot)^t, B_{Killing}\big)$$

does not. Recall that $N_{SL(2,\mathbb{C})}(SL(2,\mathbb{R}))$ *fits into the exact sequence*

$$1 \to SL(2,\mathbb{R}) \to N_{SL(2,\mathbb{C})}(SL(2,\mathbb{R})) \to \mathbb{Z}_2 \to 1$$

where $\mathbb{Z}_2$ *is generated by the image of*

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

inside the quotient. The importance of these normalising subgroups will be made clear in Chapter 2.

Proposition 1.2.22. *Let* $(G, H, \theta, B) \subseteq (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ *be a real form of a connected complex reductive Lie group, and let* $N := N_{G^{\mathbb{C}}}(G)$*,*

$$N_U := N_U(G) := \{u \in U \ : \ Ad_u(g) \in G \ \ \textit{for all } g \in G\}.$$

Then, if $[N : G] < \infty$*,* $(N, N_U, \tau|_N, B_{\mathbb{C}}|_N)$ *is reductive. Furthermore, it fits into an exact sequence:*

$$1 \to G \to N \to F \to 1$$

where

$$F = \{e^{iX} \ : \ X \in \mathfrak{g} \ : \ e^{2iX} \in Z(G^{\mathbb{C}})\}.$$

In particular, if (G, H, θ, B) *and* $(G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ *are strongly reductive,* $(N, N_U, \tau|_N, B_{\mathbb{C}}|_N)$ *is strongly reductive if and only if* $\left(\mathrm{Ad}\ G^{\mathbb{C}}\right)^{\sigma} = \mathrm{Ad}\ G$

Remark 1.2.23. *When* $G^{\mathbb{C}}$ *is semisimple, the hypothesis* $[N : G] < \infty$ *is automatically met, by definition of* F*.*

Proof. By Lemma 1.2.9, it is enough to prove that

$$1 \to G \to N \to F \to 1$$

is exact and that $N_U = FH$. Note that we have the following characterisation of N:

$$n \in N \iff n^{-1}n^\sigma \in Z_{G^{\mathbb{C}}}(G).$$

So we easily see $\left(\mathrm{Ad}\ G^{\mathbb{C}}\right)^\sigma = \mathrm{Ad}\ N$. The same proof as in Lemma 1.2.15 applies to check that N is indeed a finite extension of G by the group F.

Now, N_U is compact being a finite extension of H by the finite group $F \cap U$. □

Remark 1.2.24. *Note that $N_U(G) \subseteq N_U(H)$ and the connected component of both groups is the same. These subgroups need not be the same, as $e^{\mathfrak{z} i\mathfrak{u}(\mathfrak{g}^{\mathbb{C}})} \subseteq F \setminus U$. In the semisimple case, they are the same.*

Remark 1.2.25. *We have* $\mathrm{Ad}\ N = \mathrm{Ad}\ (G^{\mathbb{C}})_\sigma$*, but the groups* $(G^{\mathbb{C}})_\sigma \subseteq N = \{g \in G\ :\ g^{-1}g^\sigma \in Z_{G^{\mathbb{C}}}(G)\}$ *need not be the same.*

1.2.2 Connected maximal split subgroup

Just as there is a maximal split subalgebra of a real reductive Lie algebra, we can define the connected maximal split subgroup of a reductive Lie group (G, H, θ, B).

Definition 1.2.26. Let (G, H, θ, B) be a reductive Lie group. The connected maximal Lie subgroup is defined to be the analytic subgroup $\widetilde{G} \leq G$ with Lie algebra $\widetilde{\mathfrak{g}}$.

Consider the tuple $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B})$ where $\widetilde{H} := \exp(\widetilde{\mathfrak{h}}) \leq H$, and $\widetilde{\theta}$ and $\widetilde{B}$ are obtained by restriction.

Proposition 1.2.27. *If (G, H, θ, B) is a reductive Lie group, then, so is the tuple $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B})$.*

Remark 1.2.28. *Note that by Proposition 1.2.8, $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B})$ is strongly reductive whenever G is contained in a complex group.*

Proof. By construction of $\widetilde{\mathfrak{g}}$ we have that $B|_{\widetilde{\mathfrak{g}}}$ is non-degenerate if B is non-degenerated (observe that $B(y_i, \theta y_i) < 0$).

Reductivity of $\widetilde{\mathfrak{g}}$ is clear, and connectivity of $\widetilde{G}$ implies that $\mathrm{Ad}\ \widetilde{G} = \mathrm{Ad}\ \widetilde{\mathfrak{g}}$.

So all there is left to prove is that $\widetilde{G} \cong \widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$. Note that if $Im(\widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$ defines a subgroup of G, it must be connected, as so is $\widetilde{H}$. It's Lie algebra is clearly $\widetilde{\mathfrak{g}}$, so it must equal $\widetilde{G}$.

Thus, let us check $\widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$ is indeed a subgroup. First of all, notice that $\widetilde{H}$ acts on $\widetilde{\mathfrak{m}}$ by the isotropy representation, so that $he^M \in \widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$ whenever $h \in \widetilde{H},\ M \in \widetilde{\mathfrak{m}}$. The same reasoning proves that any element in $e^{\widetilde{\mathfrak{m}}}$ sends $e^{\widetilde{\mathfrak{m}}}$ to $e^{\widetilde{\mathfrak{g}}} \subseteq \widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$ via the adjoint action. Finally, $\mathrm{Ad}_{e^{\widetilde{\mathfrak{m}}}} : \widetilde{H} \to e^{\widetilde{\mathfrak{g}}}$. Since $H \cap e^{\mathfrak{m}} = \{1\}$, it follows that $\widetilde{H} \times e^{\widetilde{\mathfrak{m}}}$ defines a diffeomorphism onto its image, which must then be $\widetilde{G}$. □

Remark 1.2.29. *Assume $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ is a real form defined by the involution σ. Then, $\widetilde{G}$ is naturally contained in a complex subgroup $\widetilde{G}^{\mathbb{C}}$ defined to be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\widetilde{\mathfrak{g}}^{\mathbb{C}}$. The involution σ restricts to $\widetilde{G}^{\mathbb{C}}$ in an obvious way, but note that $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B})$ need not be the real form of $\widetilde{G}^{\mathbb{C}}$ defined by σ, but only the connected component of it. We can prove in the usual way that $\left(\widetilde{G}^{\mathbb{C}}\right)^{\sigma}$ is a finite extension of $\widetilde{G}$. As usual, $\left(\widetilde{G}^{\mathbb{C}}\right)^{\sigma}$ need not be strongly reductive even if G, $G^{\mathbb{C}}$ are.*

Remark 1.2.30. *Note that even when G is connected, there may be more than one candidate to the (not necessarily connected) maximal split subgroup. For example, the simple case $G = SL(2, \mathbb{C})$ already presents such a phenomenon, as Example 1.2.21 shows. So the notion of maximal split subgroup is not so obvious and needs to be studied in more detail.*

Given a reductive Lie group, we would like to determine its maximal connected split subgroup. This is studied in work by Borel and Tits [11] in the case of real forms of complex semisimple algebraic groups. It is important to note that over $\mathbb{R}$, the category of semisimple algebraic groups differs from the category of semisimple Lie groups. For example, the semisimple algebraic group $\mathrm{Sp}(2n, \mathbb{R})$ has a finite cover of any given degree, all of which are semisimple Lie groups, but none of them is a matrix group. So although their results do not apply to real Lie groups in general, they do apply to real forms of complex semisimple Lie groups.

In former work [10], both authors build, in the context of reductive algebraic groups (which they consider functorially), a maximal connected split subgroup, unique up to the choice of a maximal split subtorus $\mathcal{A}$ and a choice of one unipotent generator of an $\mathcal{A}$-invariant three dimensional subgroup corresponding to each root $\alpha \in \Delta$ such that $2\alpha \notin \Delta$.

Let $\mathcal{G}$ be a reductive algebraic group, and let $\widetilde{\mathcal{G}}_0$ be the maximal connected split subgroup. In case $\widetilde{\mathcal{G}}$ has a complexification $\widetilde{\mathcal{G}}^{\mathbb{C}}$, it is well known that the map that to a group assigns its complex points

$$\widetilde{\mathcal{G}}^{\mathbb{C}} \mapsto \widetilde{\mathcal{G}}(\mathbb{C})$$

establishes an equivalence of categories between the categories $\mathcal{AG}$ of complex semisimple algebraic groups and $\mathcal{LG}$ of (holomorphic) complex semisimple Lie groups (also reductive, but on the holomorphic side we get a subcategory). This yields:

Proposition 1.2.31. *Let $G^{\mathbb{C}}$ be a complex semisimple Lie group, and let $\mathcal{G}^{\mathbb{C}}$ be the corresponding algebraic group, so that $G^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}(\mathbb{C})$. Let $G < G^{\mathbb{C}}$ be a real form. Then, there exists a real linear algebraic group $\mathcal{G}$ such that $\mathcal{G}(\mathbb{C}) = G$ and moreover $\widetilde{\mathcal{G}}_0(\mathbb{C}) = \widetilde{G}_0$.*

Proof. The equivalence between $\mathcal{AG}$ and $\mathcal{LG}$ implies that the holomorphic involution $\theta \curvearrowright G^{\mathbb{C}}$ corresponding to G via Corollary 1.2.20 is algebraic. Thus, both τ and σ are real algebraic, that it, defined by polynomial equations over the real numbers. This implies they induce involutions (that we denote by the same letters) on $\mathcal{G}^{\mathbb{C}}$. Let $\mathcal{G} = \left(\mathcal{G}^{\mathbb{C}}\right)^{\sigma}$. Then, $\mathcal{G}(\mathbb{C}) = \left(\mathcal{G}^{\mathbb{C}}(\mathbb{C})\right)^{\sigma} = G$. By construction of $\widetilde{G}_0$, the choices required for the uniqueness of Borel-Tits' maximal connected split subgroup are met. So there is a unique algebraic group $\widetilde{\mathcal{G}}_0$ such that $\widetilde{\mathcal{G}}_0(\mathbb{C}) = \widetilde{G}_0$. □

The following lemma gives a necessay condition for a subgroup to be the maximal connected split subgroup.

Lemma 1.2.32. *Let $\mathcal{G}$ be a real semisimple algebraic group, $\widetilde{\mathcal{G}}$ a semisimple subgroup such that there exist maximal tori $\mathcal{T}, \widetilde{\mathcal{T}}$ of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ with $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$. Suppose that:*

1. *$r(\Delta)^{\times} = r(\Delta) \setminus 0$ is a root system such that $\widetilde{\Delta}$ is the associated reduced system,*
2. *$\widetilde{S} \subset \widetilde{\Delta}$ is such that $\widetilde{\Delta} \cap r(\Delta)^{\times}$ generates $r(\Delta)^{\times}$.*

Then if $\mathcal{G}$ is simply connected or has a non reduced root system, then $\widetilde{\mathcal{G}}$ is simply connected.

Remark 1.2.33. *In the above corollary, simple connectedness is meant in the algebraic sense: namely, the lattice of inverse roots is maximal within the lattice of weights of the group. Note that the algebraic fundamental group for compact linear algebraic groups and the topological fundamental group of their corresponding groups of matrices of complex points are the same (see [19] for details). The polar decomposition implies the same for the class of reductive Lie groups. However, algebraic simple connectedness does not mean that the fundamental group be trivial.*

Lemma 1.2.32 has the following consequence:

Corollary 1.2.34. *Let $G^{\mathbb{C}}$ be a complex semisimple Lie group, and let $G < G^{\mathbb{C}}$ be a real form that is either simply connected or of type BC. Then the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\widetilde{\mathfrak{g}}^{\mathbb{C}}$, say $\widetilde{G}^{\mathbb{C}}$, is (topologically) simply connected.*

Proof. By Proposition 1.2.31, we have algebraic groups $\mathcal{G}^{\mathbb{C}}$, $\widetilde{\mathcal{G}}^{\mathbb{C}}$ and real forms $\mathcal{G}$, $\widetilde{\mathcal{G}}$ to which the results of Borel and Tits may be applied. In particular $\widetilde{\mathcal{G}}$ is simply connected. Assume $\widetilde{\mathcal{G}}^{\mathbb{C}}$ was not. Then, it would have a finite cover $(\widetilde{\mathcal{G}}^{\mathbb{C}})'$, which in turn would contain a real form $(\widetilde{\mathcal{G}})'$ (defined by a lifting σ) that would be a finite cover of $\widetilde{\mathcal{G}}$. □

Example 1.2.35. Take the real form $\mathrm{SU}(p,q) < \mathrm{SL}(p+q,\mathbb{C})$. Its fundamental group is

$$\pi_1(\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(q))) = \mathbb{Z}.$$

We know from [2] that the maximal split Lie subalgebra of $\mathfrak{su}(p,q)$ $p>q$ is $\mathfrak{so}(q+1,q)$, whereas the maximal split subalgebra of $\mathfrak{su}(p,p)$ is $\mathfrak{sp}(2p,\mathbb{R})$.

• $p > q$. Since the root system is non-reduced (see [48], VI§4), Lemma 1.2.32 and Corollary 1.2.34 imply that the maximal split subgroup is the algebraic universal cover of $\mathrm{SO}(q+1,q)_0$. We have the following table of fundamental groups of the connected component of $\mathrm{SO}(p+1,p)$:

$\pi_1(\mathrm{SO}(q+1,q)_0)$	$q=1$	$\mathbb{Z}$
	$q=2$	$\mathbb{Z}\times\mathbb{Z}_2$
	$q\geq 3$	$\mathbb{Z}_2\times\mathbb{Z}_2$

For $q=1$, we have the exact sequence

$$1\to\mathbb{Z}_2\to\mathrm{Sp}(2,\mathbb{R})\to\mathrm{SO}(2,1)_0\to 1$$

Note that the topological universal cover $\widetilde{\mathrm{SO}(2,1)_0}^{u}$ is the topological universal cover of $\mathrm{Sp}(2,\mathbb{R})$ which is not a matrix group (see [78] for details on this). Hence it cannot be a subgroup of $\mathrm{SU}(2,1)$. This illustrates Remark 1.2.33.

Nevertheless, as pointed out in Remark 1.2.33, the appropriate group should embed in the universal cover of $\mathrm{SO}(3,\mathbb{C})$, which is $\mathrm{Spin}(3,\mathbb{C})$. In particular $\widetilde{\mathrm{SO}(2,1)_0} = \mathrm{Spin}(2,1)_0$ and $\widetilde{\mathrm{SO}(2)} = \mathrm{Spin}(2)$.

When $q=2$, the maximal split subgroup is again the algebraic universal cover of $\mathrm{SO}(3,2)_0$, which is a two cover in sight of the fundamental group. It is well known that $\mathfrak{so}(2,3)\cong\mathfrak{sp}(4,\mathbb{R})$. Thus, the algebraic universal cover is $\mathrm{Sp}(4,\mathbb{R})\cong \mathrm{Spin}(3,2)_0/\mathbb{Z}_2$, where we must quotient by the canonical bundle since $\mathrm{Spin}_0(3,2)_0$ is a 4 cover of $\mathrm{SO}(2,3)_0$.

As for $q \geq 3$, the universal covering group of $\mathrm{SO}(q,q+1)_0$ is the connected component of $\mathrm{Spin}(q,q+1)$. This group is a 4 cover of $\mathrm{SO}(q,q+1)_0$, which is thus simply connected.

• $p = q$. Since $\mathrm{Sp}(2n,\mathbb{R}) \subseteq \mathrm{SU}(n,n)$, the candidate to the maximal split subgroup is a finite cover of $\mathrm{Sp}(2n,\mathbb{R})$ embedding into $\mathrm{Sp}(2n,\mathbb{C})$ (which is simply connected). Thus $\widetilde{\mathrm{SU}(n,n)} = \mathrm{Sp}(2n,\mathbb{R})$.

Remark 1.2.36. *There are many embeddings of* $\mathfrak{so}(2,1)$ *into* $\mathfrak{su}(2,1)$*, but not all of them give the maximal split form, as regularity need not be respected by Lie algebra morphisms. This is the same reason why not all TDS are principal in spite of* $\mathfrak{sl}(2,\mathbb{C})$ *being so. See Section 3.4.1 for details on this.*

1.2.3 Groups of Hermitian type

Definition 1.2.37. A reductive Lie group (G,H,θ,B) is said to be of Hermitian type if G/H is a symmetric space that admits a complex structure compatible with the metric making each global involution s_p and isometry.

When the symmetric pair is irreducible (that is G/H cannot be expressed as a direct product of two or more Riemannian symmetric spaces) we have the following characterization:

Proposition 1.2.38. *The following conditions are equivalent for a simple Lie group* G *with maximal compact subgroup* H*:*

1. $\pi_1(G)$ *has a* $\mathbb{Z}$*-factor.*

2. *The symmetric space* G/H *admits a complex structure.*

3. $\mathfrak{z}(\mathfrak{h}) \neq 0$

1.3 The Kostant–Rallis section

Let (G,H,θ,B) be a reductive Lie group, and consider the decomposition induced by θ

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Let $\mathfrak{a} \subseteq \mathfrak{m}$ be a maximal anisotropic Cartan subalgebra, and let $H^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}$... denote the complexifications of the respective groups and algebras. Note that we do not assume for $G^{\mathbb{C}}$ to exist. In [51], Kostant and Rallis study the orbit structure of the $H^{\mathbb{C}}$ module

$\mathfrak{m}^{\mathbb{C}}$ in the case when $G^{\mathbb{C}}$ is the adjoint group of a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (namely, $G^{\mathbb{C}} = \mathrm{Inn}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})^0$ In this section, we study a generalization of their result to reductive Lie groups in the sense of Definition 1.2.1.

The first result we will be concerned about is the Chevalley restriction theorem, which is classical for Lie groups of the adjoint type. Recall that given a complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$, its **adjoint group**, denoted by $\mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$, is the connected component of its automorphism group $\mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$. It coincides with the connected component of the image via the adjoint representation of any Lie group $G^{\mathbb{C}}$ such that $Lie(G^{\mathbb{C}}) = \mathfrak{g}^{\mathbb{C}}$. We need the following.

Definition 1.3.1. We define the **restricted Weyl group** of $\mathfrak{g}$ ($\mathfrak{g}^{\mathbb{C}}$) associated to $\mathfrak{a}$ ($\mathfrak{a}^{\mathbb{C}}$), $W(\mathfrak{a})$ ($W(\mathfrak{a}^{\mathbb{C}})$), to be the group of automorphisms of $\mathfrak{a}$ ($\mathfrak{a}^{\mathbb{C}}$) generated by reflections on the hyperplanes defined by the restricted roots $\lambda \in \Lambda(\mathfrak{a})$ ($\Lambda(\mathfrak{a}^{\mathbb{C}})$).

The Chevalley restriction theorem asserts that, given a group G of the adjoint type, we have that restriction $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}] \to \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]$ induces an isomorphism

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \to \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}.$$

See for example [40].

The restricted Weyl group admits other useful characterizations in the case of strongly reductive Lie groups.

Lemma 1.3.2. *Let (G, H, θ, B) be a strongly reductive Lie group. We have*
1. $W(\mathfrak{a}) = N_H(\mathfrak{a})/C_H(\mathfrak{a})$, where

$$N_H(\mathfrak{a}) = \{h \in H \ : \ \mathrm{Ad}_h(x) \in \mathfrak{a} \ \mathrm{forall} x \in \mathfrak{a}\},$$

$$C_H(\mathfrak{a}) = \{h \in H \ : \ \mathrm{Ad}_h(x) = x \ \mathrm{forall} x \in \mathfrak{a}\}.$$

2. Moreover, as automorphism groups of $\mathfrak{a}^{\mathbb{C}}$, we have

$$W(\mathfrak{a}^{\mathbb{C}}) = N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})/C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) = W(\mathfrak{a}).$$

Proof. The first statement follows from Proposition 7.24 in [48].

As for *2.*, it is enough to prove that $W(\mathfrak{a}) = N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})/C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$, as by definition of restricted roots, the action of $W(\mathfrak{a}^{\mathbb{C}})$ is the extension of th action of $W(\mathfrak{a})$ on $\mathfrak{a}$ to $\mathfrak{a}^{\mathbb{C}}$ by complex linearity. Strong reductivity of G implies the same for $H^{\mathbb{C}}$. Then, letting $h = xe^Y \in N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ be the polar decomposition of an element normalising $\mathfrak{a}^{\mathbb{C}}$, we have, by τ-invariance of $\mathfrak{a}^{\mathbb{C}}$, that Lemma 7.22 in [48] applies to h as an element of $N_{G^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$. In particular, both x and Y normalise $\mathfrak{a}^{\mathbb{C}}$. This means that $x \in N_H(\mathfrak{a}^{\mathbb{C}}) = N_H(\mathfrak{a})$, and $Y \in \mathfrak{n}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) = \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ (Lemma 6.56 in [48]), so the statement is proved. □

With this, we have:

Proposition 1.3.3. *Let (G, H, θ, B) be a strongly reductive Lie group and $(\widetilde{G}_0, \widetilde{H}_0, \widetilde{\theta}, \widetilde{B})$ be the maximal connected split subgroup. Then, restriction induces an isomorphism*

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widetilde{\mathfrak{m}}^{\mathbb{C}}]^{(\widetilde{H}_0)^{\mathbb{C}}}.$$

Here, the superscript $^{\mathbb{C}}$ denotes complexification.

If moreover $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ is a real form, recall that it makes sense to consider the maximal split subgroup $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B}) < (G, H, \theta, B)$, and we have

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\widetilde{\mathfrak{m}}^{\mathbb{C}}]^{\widetilde{H}^{\mathbb{C}}}.$$

Proof. By Lemma 7.24 in [48],

$$\mathrm{Ad}(H) \subseteq \mathrm{Inn}\mathfrak{h} \oplus i\mathfrak{m}.$$

Then, given that $H^{\mathbb{C}} = He^{i\mathfrak{h}}$, $H^{\mathbb{C}}$ clearly acts on $\mathfrak{g}^{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}^{\mathbb{C}}$. So $\mathrm{Ad}(\mathfrak{h}^{\mathbb{C}}) = \mathrm{Ad}(H^{\mathbb{C}}) \subseteq (\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}$, which implies

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ \mathfrak{h}^{\mathbb{C}}} = \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ H^{\mathbb{C}}} \supseteq \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{(\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}}. \tag{1.2}$$

Now, Proposition 10 in [51] implies that

$$\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{(\mathrm{Ad}\ \mathfrak{g}^{\mathbb{C}})^{\theta}} = \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{\mathrm{Ad}\ \mathfrak{h}^{\mathbb{C}}}$$

and so we obtain equalities in Equation (1.2) above.

Since $W(\mathfrak{a}^{\mathbb{C}}) = N_{\mathrm{Ad}\ \mathfrak{h}}(\mathfrak{a})/C_{\mathrm{Ad}\ \mathfrak{h}}(\mathfrak{a})$, the isomorphism $\mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$ follows from the adjoint group case and (1.2).

As for the split subgroup, by the adjoint case and Proposition 1.2.27, we have $\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widetilde{\mathfrak{m}}^{\mathbb{C}}]^{\widetilde{H}_0^{\mathbb{C}}}$. Also, by definition of $\widetilde{H}$, $\mathrm{Ad}(\widetilde{H}_0) \subset \mathrm{Ad}(\widetilde{H}) \subset \mathrm{Ad}(\widetilde{G}_0)_{\theta}$, which by Lemma 1.2.15 and Proposition 10 in [51] implies that $\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})} \cong \mathbb{C}[\widetilde{\mathfrak{m}}^{\mathbb{C}}]^{\widetilde{H}^{\mathbb{C}}}$. □

Proposition 1.3.4.

1. *$\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W}(\mathfrak{a}^{\mathbb{C}})$ is generated by homogeneous polynomials of fixed degrees $d_1, \ldots, d_a$, called the **exponents** of the group G. Here $a = \dim \mathfrak{a}^{\mathbb{C}}$.*

2. *If $(\widetilde{G}, \widetilde{H}, \theta, B) < (G, H, \theta, B)$ is the maximal split subgroup, the exponents are the same for both groups.*

Proof. 1. is classical and follows from Proposition 1.1.28 2.

2. Follows by construction of the maximal split subgroup Proposition 1.1.28 2. □

So we have an algebraic morphism

$$\chi : \mathfrak{m}^{\mathbb{C}} \twoheadrightarrow \mathfrak{m}^{\mathbb{C}} // H^{\mathbb{C}} \cong \mathfrak{a}^{\mathbb{C}} / W(\mathfrak{a}^{\mathbb{C}}) \tag{1.3}$$

where the double quotient sign // stands for the affine GIT quotient.

Next we want to build a section of the above surjection. This is done by Kostant and Rallis in the case $G^{\mathbb{C}} = \mathrm{Ad}(\mathfrak{g}^{\mathbb{C}})$ for some complex reductive Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let us start by some preliminary definitions.

Definition 1.3.5. An element $x \in \mathfrak{m}^{\mathbb{C}}$ is said to be **regular** whenever $\dim \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \dim \mathfrak{a}^{\mathbb{C}} = a$. Here $\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \{y \in \mathfrak{m}^{\mathbb{C}} \ : \ [y, x] = 0\}$. Denote the subset of regular elements of $\mathfrak{m}^{\mathbb{C}}$

$$\mathfrak{m}_{reg} = \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \dim(\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = \dim(\mathfrak{a}^{\mathbb{C}}) =: a\}.$$

Regular elements are those whose $H^{\mathbb{C}}$-orbits are maximal dimensional, so this notion generalises the classical notion of regularity of reductive complex Lie algebras. We can now state the following.

Remark 1.3.6. *Note that the intersection $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}^{\mathbb{C}}_{reg}$ (where $\mathfrak{g}_{reg}$ are points with maximal dimensional $G^{\mathbb{C}}$ orbit) is either empty or the whole of $\mathfrak{m}_{reg}$.*

The following definitions follow naturally from the preceeding remark.

Definition 1.3.7.

1. A three dimensional subalgebra (TDS) $\mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is the image of a morphism $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$.

2. A TDS is called **normal** if $\dim \mathfrak{s}^{\mathbb{C}} \cap \mathfrak{h}^{\mathbb{C}} = 1$ and $\dim \mathfrak{s}^{\mathbb{C}} \cap \mathfrak{m}^{\mathbb{C}} = 2$. It is called **principal** if it is generated by elements of the form

$$\{e, f, x\}$$

where $e, f \in \mathfrak{m}_{reg}$ are nilpotent elements, $x \in \mathfrak{t}$, where $\mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{t}$ is a θ and σ stable Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The generators satisfying such relations are called a normal basis.

Definition 1.3.8. A real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is **quasi-split** if $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{g}^{\mathbb{C}}_{reg}$. These include split real forms, $\mathfrak{su}(p, p)$, $\mathfrak{su}(p, p+1)$, $\mathfrak{so}(p, p+2)$, $\mathfrak{e}_{6(2)}$. Quasi-split real forms admit several equivalent characterizations: $\mathfrak{g}$ is quasi-split if and only if $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ is abelian, which holds if and only if $\mathfrak{g}^{\mathbb{C}}$ contains a θ-invariant borel subalgebra.

We can now state the following.

Theorem 1.3.9. *Let (G, H, θ, B) be a strongly reductive Lie group. Let $\mathfrak{s}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ be a principal normal TDS with normal basis $\{x, e, f\}$. Then*

1. *The affine subspace $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is isomorphic to $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ as an affine variety.*
2. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is contained in the open subset $\mathfrak{m}_{reg}$*
3. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ intersects each $(\mathrm{Ad}(G_\theta))^{\mathbb{C}}$-orbit at exactly one point. Here G_θ is as in Definition 1.2.14.*
4. *$f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is a section for the Chevalley morphism (1.3).*
5. *Let $(\widetilde{G}_0, \widetilde{H}_0, \widetilde{\theta}, \widetilde{B}) < (G, H, \theta, B)$ be the maximal connected split subgroup. Then, s can be chosen so that $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e) \subseteq \widetilde{\mathfrak{m}}^{\mathbb{C}}$. If moreover G is a real form of $G^{\mathbb{C}}$, say, then $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$ is the image of Kostant's section for $\widetilde{G}^{\mathbb{C}}$ [50].*

Proof. We hereby follow the proof due to Kostant and Rallis (See Theorems 11, 12 and 13 in [51]) adapting their arguments to our setting when necessary. This will be helpful setting notation for Section 3.2.

First note that Proposition 1.3.3 implies we do have a surjection

$$\mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}).$$

In [51], the authors consider the element

$$e_c = i \sum_j d_j y_j \in i\mathfrak{g}, \tag{1.4}$$

where $y_i \in \mathfrak{g}_{\lambda_i}$ as in Section 1.1.2 and

$$d_j = \sqrt{\frac{-c_j}{b_j}}. \tag{1.5}$$

Here the elements c_j are defined so that

$$w = \sum_i c_i h_i \in \mathfrak{a} \tag{1.6}$$

is the only element in $\mathfrak{a}$ such that $\lambda(w) = 2$ for any $\lambda \in \Lambda(\mathfrak{a})$. Here h_i is the $B_{Killing}$ dual to λ_i.

Note that in order for e_c to belong to $i\mathfrak{g}$, we must prove that $c_i/b_i < 0$. In the semisimple case, this follows from the fact that B_{CK} is definite negative on $\mathfrak{h}^{\mathbb{C}}$ and positive definite on $\mathfrak{m}^{\mathbb{C}}$. Now, following the proof of Proposition 18 in [51], we have that for any form B satisfying the hypothesis in Definition 1.2.1, any $y \in \mathfrak{g}^{\mathbb{C}}$ we have $2B(y, \theta y) = B(y+\theta y, y+\theta y) < 0$ since $y+\theta y \in \mathfrak{h}$. Hence, if $b_i = B(y_i, \theta y_i)$ it must be a negative real number. Also the fact that $c_i > 0$ follows from general considerations on the representations of three dimensional subalgebras (see Lemma 15 in [51]) and so does not depend on the choice of pairing B.

Once we have that, taking

$$f_c = \theta e_c,$$

it follows by the same arguments found in [51] that $\{e_c, f_c, w\}$ generate a principal normal TDS $\mathfrak{s}^{\mathbb{C}}$ stable by σ and θ (Proposition 22 in [51]). In particular, $\mathfrak{s}^{\mathbb{C}}$ has a normal basis, say $\{e, f, x\}$. By construction, it is clear that $f + \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e) \subseteq \widetilde{\mathfrak{m}}_{reg}$, where $\widetilde{\mathfrak{m}}_{reg}$ is as in Section 1.1.2. It is furthermore a section, which is proved as in [51], as groups act by inner automorphisms of the Lie algebra, together with Lemma 1.3.10 following this theorem. This proves (1), (2) and (4).

As for (3), it follows from [51] together with Lemma 1.2.15 (see Remark 1.2.17) and strong reductivity.

Statement (5) follows from the fact that by definition $\widetilde{G}_0$ is connected, thus strongly reductive, hence the statement follows from Theorem 7 in [50] together with Remark 19 in [51] and its proof. □

Lemma 1.3.10. *The Lie algebra $\mathfrak{s}^{\mathbb{C}}$ is the image of a σ and θ-equivariant morphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$ where σ on $\mathfrak{sl}(2,\mathbb{C})$ is complex conjugation and θ on $\mathfrak{sl}(2,\mathbb{C})$ is defined by $X \mapsto -\mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}({}^tX)$.*

Proof. Consider the basis of $\mathfrak{sl}(2,\mathbb{R})$

$$E = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and note that $H \in \mathfrak{m}_{\mathfrak{sl}}$, $E = \theta F$, so that $E + F \in \mathfrak{so}(2,\mathbb{R})$.

Consider e_c, f_c, w as described in the preceeding proposition. Then the map defined by

$$\rho' : E \mapsto ie_c,\ F \mapsto if_c,\ W \mapsto -w \tag{1.7}$$

is the desired morphism. Indeed, it is σ-invariant by definition. Furthermore, $\mathfrak{so}(2,\mathbb{R}) \ni E + F \mapsto ie_c + if_c \in \mathfrak{h}$ by construction. Finally, $\mathfrak{m}_{\mathfrak{sl}}$ is generated by W and $E - F$, and so is $\mathfrak{s} \cap \mathfrak{m}$. Indeed, we must only prove that $ie_c - if_c$ is not a multiple of w. But this follows from simplicity of $\mathfrak{sl}(2,\mathbb{C})$, the fact that $\mathfrak{s}^{\mathbb{C}}$ is homomorphic to it and $w \neq 0$, which forces to S-triples to be independent. □

Remark 1.3.11. *Theorem 1.3.9 implies that the GIT quotient $\mathfrak{m}^{\mathbb{C}}//H^{\mathbb{C}}$ does not parameterize $H^{\mathbb{C}}$ orbits or regular elements, but rather $\mathrm{Ad}(H^{\mathbb{C}})_\theta$ orbits, each of which contains finitely many $H^{\mathbb{C}}$-orbits. This is a consequence of the fact that not all normal principal TDS's are $H^{\mathbb{C}}$ conjugate, which yields different sections for different choices of a TDS. See [51] for more details.*

By the above remark, we will need to keep track of conjugacy classes of principal normal TDS's.

Proposition 1.3.12. *Let $\mathfrak{s}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ be a principal normal TDS, and let (e, f, x) be a normal triple generating it. Then:*

1. *The triple is principal if and only if $e + f = \pm w$, where w is defined by (1.6).*
2. *There exist e', f' such that (e', f', w) is a TDS generating $\mathfrak{s}^{\mathbb{C}}$ and $e' = \theta f'$. Under these hypothesis, e' is uniquely defined up to sign.*

Proof. See Lemma 5 and Proposition 13 in [51]. □

1.4 Regular elements and their centralisers

We continue using the notation of the previous section: the lack of notation will be reserved for real groups and algebras, and the superscript $^{\mathbb{C}}$ will denote complexification when applicable.

Lemma 1.4.1. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \theta, B)$ be a real form of a complex Lie group of the adjoint type. Let $e \in \mathfrak{m}_{reg}$ be a principal nilpotent element. Then, $C_{H^{\mathbb{C}}}(e)$ is connected.*

Proof. We know ([51]) that principal nilpotent elements form a unique orbit under ${H^{\mathbb{C}}}_\theta$. By Remark 1.2.17, in the adjoint group case, ${H^{\mathbb{C}}}_\theta = H^{\mathbb{C}}$, so that all regular nilpotent elements are conjugate by $H^{\mathbb{C}}$. This implies that the centraliser must be connected. □

Proposition 1.4.2. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \theta, B)$ be a real quasi-split form of a complex reductive Lie group. Let*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

be the decomposition of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ *induced by the Cartan decomposition of* $\mathfrak{g}$*, and consider* $e \in \mathfrak{m}_{reg}$ *a regular nilpotent element. Consider* $\mathfrak{a}^{\mathbb{C}} \subseteq \mathfrak{m}^{\mathbb{C}}$ *a maximal anisotropic Cartan subalgebra. Then*

$$C_{H^{\mathbb{C}}}(e) \cong C_{H^{\mathbb{C}}}(G^{\mathbb{C}}) \times C_{\mathrm{Ad}\ H^{\mathbb{C}}_{\theta}}(e)$$

is the Jordan decomposition of $C_{H^{\mathbb{C}}}(e)$*, with*

$$C_{H^{\mathbb{C}}}(G^{\mathbb{C}}) = \{h \in H^{\mathbb{C}} \ : \ \mathrm{Ad}_h(g) = g \text{ for all } g \in G^{\mathbb{C}}\}$$

the semisimple part of $C_{H^{\mathbb{C}}}(e)$ *and* $C_{H^{\mathbb{C}}_{ss}}(e) = C_{H^{\mathbb{C}}}(e)_{uni}$ *its unipotent part.*

Proof. One easily sees that $\mathfrak{m}_{reg} \subseteq \mathfrak{g}_{reg}$ if and only if $\mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ is abelian. So since $e \in \mathfrak{g}_{reg}$, we know that

$$C_{G^{\mathbb{C}}}(e) = Z(G^{\mathbb{C}}) \cdot C_{\mathrm{Ad}\ G^{\mathbb{C}}}(e)$$

is the Jordan decomposition (see Proposition 12.7 in [25]). Then:

$$C_{H^{\mathbb{C}}}(e) = Z(G^{\mathbb{C}}) \cap H^{\mathbb{C}} \cdot C_{\mathrm{Ad}\ H^{\mathbb{C}}_{\theta}}(e)$$

and the result follows. □

Proposition 1.4.3. *Let* $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ *be a quasi-split real form of a complex reductive Lie group, and let* $w \in \mathfrak{m}_{reg}$ *be a regular nilpotent element. Then, if* $w_i \in \mathfrak{m}_{reg,ss}$ *are regular and semisimple elements tending to* w *in the analytic topology, it follows that any element in* $C_{H^{\mathbb{C}}}(w)$ *is a limit in the analytic topology of a sequence* $\{g_i\}$ *where* $g_i \in C_{H^{\mathbb{C}}}(w_i)$ *.*

Proof. It is clear that any converging sequence of such elements g_i is in $C_{H^{\mathbb{C}}}(w)$, since $\iota : H^{\mathbb{C}} \to \mathrm{Aut}\ \mathfrak{m}_{reg}$ is continuous.

To prove the converse, note that by Lemma 1.4.1 and Proposition 1.4.2 it is enough to assume the group $G^{\mathbb{C}}$ to be of the adjoint type. In this case, the result follows from Proposition 1.4.11. □

Proposition 1.4.4. *Let* $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ *be a quasi-split real form of a complex reductive Lie group. Let* $x \in \mathfrak{m}_{reg,ss}$ *be a regular semisimple element. Then its centraliser in* $H^{\mathbb{C}}$ *is abelian. Conversely, if the centraliser of such an element is abelian, then the real form involved is quasi split.*

Proof. By Remark 1.1.19, if the form is quasi-split, it follows that $x \in \mathfrak{g}_{reg}$. Thus, its centraliser in $G^{\mathbb{C}}$ is abelian, and so the first statement is a consequence of the commutativity of $C_{G^{\mathbb{C}}}(x)$.

As for the second statement, if the form is not quasi-split, then $\mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(x)$ is not abelian, so neither is the connected component of $C_{H^{\mathbb{C}}}(x)$, so the centraliser cannot be abelian. □

Corollary 1.4.5. $(G^{\mathbb{C}}_{\mathbb{R}}, U, \tau \times -\tau, B_{CK}) < (G^{\mathbb{C}} \times G^{\mathbb{C}}, U \times U, \tau \times \tau, B_{CK})$ *is a quasi-split form.*

Table of quasi-split real forms of complex simple Lie algebras The information on the following table has been extracted from [48].

Type	$\mathfrak{g}$	$\mathfrak{g}^{\mathbb{C}}$
AI	$\mathfrak{sl}(n,\mathbb{R}))$	$\mathfrak{sl}(n,\mathbb{C})$
AIII	$\mathfrak{su}(p,p+1)$	$\mathfrak{sl}(2p+1,\mathbb{C})$
AIII	$\mathfrak{su}(p,p)$	$\mathfrak{sp}(2p,\mathbb{C})$
BI	$\mathfrak{so}(2p,2p+1),\ p \leq q$	$\mathfrak{so}(2p+1,\mathbb{C})$
CI	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{C})$
BDI	$\mathfrak{so}(p-1,p+1)$	$\mathfrak{so}(2p,\mathbb{C})$
BDI	$\mathfrak{so}(p,p)$	$\mathfrak{so}(2p,\mathbb{C})$
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{6)}$

Next, we introduce some geometrical objects which will be useful in Chapter 4

Let $a = \dim \mathfrak{a}$. Denote by $Ab^a(\mathfrak{m})$ the closed subvariety of $Gr(a,\mathfrak{m})$ whose points are abelian subalgebras of $\mathfrak{m}$. Define the incidence variety

$$\mu_{reg} = \{(x,\mathfrak{c}) \in \mathfrak{m}_{reg} \times Ab^a(\mathfrak{m}) \ : \ x \in \mathfrak{c}\}. \tag{1.8}$$

We have the following.

Proposition 1.4.6. *The map*

$$\psi : \mathfrak{m}_{reg} \to Ab^a(\mathfrak{m}) \ x \mapsto \mathfrak{z}_{\mathfrak{m}}(x)$$

is smooth with smooth image and its graph is μ_{reg}.

Proof. First of all, note that the map is well defined: indeed, it is clear for regular and semsimimple elements in $\mathfrak{m}$. By theorem 20 and lemma 21 of [51], it extends to the whole of $\mathfrak{m}_{reg}$. As for smoothness, the proof given in [25] adapts.

We check that ψ is well defined and has graph μ_{reg} by proving that $\mu_{reg} \to \mathfrak{m}_{reg}$ is an embedding (hence, by properness and surjectivity, an isomorphism). To see this, as $\mathfrak{m}_{reg}$ is reduced and irreducible (being a dense open set of a vector space), if the fibers are reduced points we will be done. We have that

$$T_{(x,\mathfrak{b}_\mathfrak{m})}(\mu_{reg}\cap\{x\}\times Ab^a(\mathfrak{m})) \cong \left\{T : \mathfrak{b} \to \mathfrak{m}/\mathfrak{b} \;\middle|\; \begin{array}{l}[T(y), x] = 0 \text{ for any } y \in \mathfrak{b} \\ T[y,z] = [Ty, z] + [y, Tz] \text{ for any } y, z \in \mathfrak{b}\end{array}\right\}.$$

By definition, the only T satisfying those conditions is $T \equiv 0$, so the map is well defined.

For smoothness, given a closed point $x \in \mathfrak{m}_{reg}$, as $\mathfrak{m}_{reg} \subset \mathfrak{m}$ is open and dense, it follows that $T_x\mathfrak{m}_{reg} \cong \mathfrak{m}$. Consider

$$T_x\mathfrak{m}_{reg} \cong \mathfrak{m} \xrightarrow{d_x\psi} T_{\mathfrak{z}_\mathfrak{m}}Ab^a(\mathfrak{m}) \xrightarrow{ev_x} \mathfrak{m}/\mathfrak{z}_\mathfrak{m}(x)$$

$$y \longrightarrow \{T : [T(z), x] = [-z, y]\} \longrightarrow T(x) = [y].$$

Namely, $d_x\psi$ sends y to the only map satisfying $[T(z), x] = [-z, y]$. Now, clearly $ev_x \circ d_x\psi$ is the projection map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{z}_\mathfrak{m}(x)$. Also, ev_x is surjective. We will prove it is injective, so it will follow that $Im(\psi)$ is contained in the smooth locus of $Ab^a(\mathfrak{m})$. The same fact proves that $d_x\psi$ must be surjective, and so we will be done.
Suppose $T(x) = T'(x)$ for some $T,\ T' \in T_{\mathfrak{z}_\mathfrak{m}(x)}Ab^a(\mathfrak{m})$. Then:

$$0 = [T(x) - T'(x), y] = [-x, T(y)] - [-x, T'(y)] = [-x, T - T'(y)] \text{ for all } y \in \mathfrak{z}_\mathfrak{m}(x)$$

and hence ev is injective. □

Definition 1.4.7. We will call the image of ψ the variety of regular centralisers, and denote it by $\overline{H/N_H(\mathfrak{a})}$.

Remark 1.4.8. *$H/N_H(\mathfrak{a}) \subset \overline{H/N_H(\mathfrak{a})}$ is an open subvariety consisting of the image of $\mathfrak{m}_{reg,ss}$.*

Remark 1.4.9. *Note that ψ is $H^\mathbb{C}$-equivariant for the isotropy representation on $\mathfrak{m}_{reg}$ and conjugation on $Ab^a(\mathfrak{m}_{reg})$.*

Definition 1.4.10. We let $C_\mathfrak{m} \to \mathfrak{m}$ be the group scheme over $\mathfrak{m}$ defined by

$$C_\mathfrak{m} = \{(m, h) \in \mathfrak{m} \times H \mid h \cdot m = m\}. \tag{1.9}$$

Similarly, we define the $\overline{H/N_H(\mathfrak{a})}$-group scheme

$$C_{\overline{H/N_H(\mathfrak{a})}} = \{(\mathfrak{c}, h) \in \overline{H/N_H(\mathfrak{a})}m \times H \mid \mathrm{Ad}_h\mathfrak{c} = \mathfrak{c}\}. \tag{1.10}$$

Both $C_{\mathfrak{m}}$ and $C_{\overline{HN_H(\mathfrak{a})}}$ are endowed with an action of $H^{\mathbb{C}}$ lifting the action on the corresponding base scheme.

Proposition 1.4.11. *$\psi^* Lie(C_{\overline{H/N_H(\mathfrak{a})}}) \cong Lie(C_{\mathfrak{m}})$.*

Proof. We have a morphism $f : \psi^*\mathrm{Lie}(C_{\overline{H/N_H(\mathfrak{a})}}) \to \mathrm{Lie}(C_{\mathfrak{m}})$, which is clearly an isomorphism whenever $\mathfrak{g} \subset \mathfrak{g}$ is quasi split or $\dim \mathfrak{c}_{\mathfrak{m}}(x) = 1$.

In all other cases, for $x \in \mathfrak{m}_{reg,ss}$, $\mathfrak{c}_{\mathfrak{h}}(x) = \mathfrak{c}_{\mathfrak{h}}(\mathfrak{c}_{\mathfrak{m}}(x))$ (cf. [51]). Now, we claim that the codimension of the nilpotent locus is at least two.

Indeed, by assumption $\dim \mathfrak{a} \geq 2$. Recall that the Kostant–Rallis section gives a transversal subvariety of $\mathfrak{m}_{reg}$. The only H_θ-nilpotent orbit maps to $0 \in \mathfrak{a}/W(\mathfrak{a})$. So the codimension of $\mathfrak{m}_{reg,nilp}$ is at least two, as $\dim H_\theta \cdot x = \dim H \cdot x$. Thus, for $\dim \mathfrak{a} \geq 2$, we have that f is an isomorphism in codimension two. Thus it extends uniquely to an isomorphism over $\mathfrak{m}_{reg}$. □

Corollary 1.4.12. $\mathfrak{c}_{\mathfrak{h}}(x) = \mathfrak{c}_{\mathfrak{h}}(\mathfrak{c}_{\mathfrak{m}}(x))$ *for all* $x \in \mathfrak{m}_{reg}$.

Following Vust ([77]), asuming $G^{\mathbb{C}}$ is a connected reductive algebraic group, we define

Definition 1.4.13. Let $(G, H, \theta, B) \leq (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ be a real form defined by an involution σ. A σ-stable parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}^{\mathbb{C}}$ is called minimal θ-anisotropic if $\theta(\mathfrak{p}) \cap \mathfrak{p} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$

We have

Proposition 1.4.14. *The variety $H^{\mathbb{C}}/C_H(\mathfrak{a})$ parameterises minimal θ anisotropic subalgebras of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. By Proposition 5 in [77], $H^{\mathbb{C}}$ acts transitively on the subset of minimal θ-anisotropic subgroups. Now, this implies that any such subalgebra is conjugate to one of the form:

$$\mathfrak{p} = \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a}) \oplus \bigoplus_{\lambda \in \Lambda(\mathfrak{a})^+} \mathfrak{g}_\lambda^{\mathbb{C}}$$

for some choice of positivity of the set of restricted roots $\Lambda(\mathfrak{a})$. The normaliser in $G^{\mathbb{C}}$ of such a subalgebra is $C_H(\mathfrak{a})AN$, where $N = \exp(\mathfrak{n})$, by proposition 7.83 in [48]. Since $K \cap N = \{id\}$ by the Iwasawa decomposition theorem (see for example, Theorem 6.46 in [48]), we are done. □

Proposition 1.4.15. *Any element $x \in \mathfrak{m}_{reg}$ has $\mathfrak{c}_{\mathfrak{m}}(x) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in H/C_H(\mathfrak{a})$.*

Proof. By [51], any such element satisfies that for some subalgebra $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$ $x \in \widetilde{\mathfrak{g}}_{reg}$ and furthermore $\mathfrak{c}_{\widetilde{\mathfrak{g}}}(x) = \mathfrak{c}_{\mathfrak{m}}(x)$. Hence, for some Borel subalgebra $\mathfrak{b} \subseteq \widetilde{\mathfrak{g}}$, $\mathfrak{c}_{\mathfrak{m}}(x) \subseteq \mathfrak{b}$. But any such is contained in some θ-anisotropic minimal parabolic subalgebra. □

Definition 1.4.16. Define $\overline{H/C_H(\mathfrak{a})} \subset \mathfrak{m}_{reg} \times H/C_H(\mathfrak{a})$ to be the incidence variety. Namely, its points are pairs $(x, \mathfrak{p})$ with $x \in \mathfrak{m}_{reg}$, $\mathfrak{p} \in H/C_H(\mathfrak{a})$.

Lemma 1.4.17. *The projection* $\overline{H/C_H(\mathfrak{a})} \to \overline{H/N_H(\mathfrak{a})}$ *is a* $W(\mathfrak{a})$*-cover.*

Proof. Fix a point $(\mathfrak{a}^{\mathbb{C}}, \mathfrak{p}_0) \in \overline{H/C_H(\mathfrak{a})}$. Then, any other parabolic subgroup in the fiber is conjugate by $N_H(\mathfrak{a}^{\mathbb{C}})$. This action needs to be quotiented by $C_H(\mathfrak{a})$, whence the result. □

Lemma 1.4.18. *Let* $(G, H, \theta, B) \leq (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ *be a quasi-split real form. Then minimal* θ*-anisotropic parabolic subalgebras are Borel subalgebras. Moreover, for any* $\mathfrak{b}, \mathfrak{b}'$ *such subalgebras there are canonical isomorphisms*

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{a} \oplus \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a}) \cong \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'].$$

Proof. The firs statement follows by definition. The second is Lemma 3.1.26 in [20]. □

Remark 1.4.19. *Note that for any minimal* θ*-anisotropic Borel subgroup, the quotient* $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ *is* θ*-invariant. It follows from the proof that the isomorphism* $\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}] \cong \mathfrak{c}_{\mathfrak{g}}(\mathfrak{a})$ *is* θ *equivariant, so that the morphism respects the Cartan decompositions.*

Definition 1.4.20. Let $\widetilde{\mathfrak{m}}_{reg} := \mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})} \mathfrak{a}^{\mathbb{C}}$.

We have

Proposition 1.4.21. *If* $G \leq G^{\mathbb{C}}$ *is quasi split, the choice of a pair* $(\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{b}_0)$ *determines* $\widetilde{\mathfrak{m}}_{reg} \cong \mathfrak{m}_{reg} \times_{\overline{H/N_H(\mathfrak{a})}} \overline{H/C_H(\mathfrak{a})}$

Proof. Recall that by definition $\widetilde{\mathfrak{m}}_{reg} = \mathfrak{m}_{reg} \times_{\mathfrak{a}/W(\mathfrak{a})} \mathfrak{a}$. Now, $\mathfrak{m}_{reg}$ being regular (it is an open subset of a vector space), it suffices to prove the existence of a morphism

$$\mathfrak{m}_{reg} \times_{\overline{H/N_H(\mathfrak{a})}} \overline{H/C_H(\mathfrak{a})} \to \mathfrak{m}_{reg} \times_{\mathfrak{a}/W(\mathfrak{a})} \mathfrak{a}$$

which is an isomorphism over an open subset.

By Lemma 1.4.18, we may define the morphism that sends a pair $(x, \mathfrak{b}) \in \mathfrak{m}_{reg} \times_{\overline{H/N_H(\mathfrak{a})}} \overline{H/C_H(\mathfrak{a})}$ to $(x, \pi_{\mathfrak{b}}(x))$. Note that by Remark 1.4.19, the map induces an isomorphism over $\mathfrak{m}_{reg,ss}$. Indeed, over $\mathfrak{m}_{reg,ss}$ both $\mathfrak{m}_{reg,ss} \times_{\mathfrak{a}_{reg}/W(\mathfrak{a}_{reg})} \mathfrak{a}_{reg}$ and $\mathfrak{m}_{reg,ss} \times_{\overline{H/N_H(\mathfrak{a})}} \overline{H/C_H(\mathfrak{a})}$ are $W(\mathfrak{a})$-principal bundles over $\mathfrak{m}_{reg,ss}$ with a morphism between them, hence isomorphic. □

Let us fix some notation that will be useful for the sequel: we have the commutative diagram

$$
\begin{array}{ccccc}
\overline{H/C_H(\mathfrak{a})} & \xleftarrow{q} & \widetilde{\mathfrak{m}}_{reg} & \xrightarrow{p_2} & \mathfrak{a} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \chi_{\mathfrak{a}}} \\
\overline{H/N_H(\mathfrak{a})} & \xleftarrow[p]{} & \mathfrak{m}_{reg} & \xrightarrow{\chi} & \mathfrak{a}/W(\mathfrak{a}).
\end{array}
\tag{1.11}
$$

Chapter 2

G-Higgs pairs

Notation and conventions

We fix once and for all X a smooth projective curve over the complex numbers, $L \to X$ a holomorphic line bundle and (G, H, θ, B) a connected reductive Lie group. Complexifications will be denoted by the superscript $^{\mathbb{C}}$. The subscript ss denotes the semisimple part (of a Lie algebra or a group).

We will denote bundles associated to a principal bundle either by specifying the representation or by putting the fiber in brackets.

2.1 L-twisted Higgs pairs

For this section, we follow the approach in [32].

Definition 2.1.1. An L-twisted G-Higgs pair is a pair (E, ϕ) where E is a holomorphic $H^{\mathbb{C}}$-principal bundle on X and $\phi \in H^0(X, E \times_\iota \mathfrak{m}^{\mathbb{C}} \otimes L)$. Here, ι denotes the isotropy representation and $E \times_\iota \mathfrak{m}^{\mathbb{C}}$ is the vector bundle associated to E via the isotropy representation. We will also denote this bundle by $E(\mathfrak{m}^{\mathbb{C}})$. When $L = K$ is the canonical bundle of X, and a pair (E, ϕ) is referred as a G-**Higgs bundle**.

Remark 2.1.2. *Note that the above definition uses all the Cartan data of G except for B. Its use will become apparent in the definition of stability conditions, as well as the Hitchin equations for G-Higgs pairs.*

2.2 Parabolic subgroups and antidominant characters

Definition 2.2.1. Given $s \in i\mathfrak{h}$, we define:

$$\mathfrak{p}_s = \{x \in \mathfrak{h}^{\mathbb{C}} \mid \mathrm{Ad} e^{ts} x \text{ exists as } t \to \infty\},$$

$$P_s = \{g \in H^{\mathbb{C}} \mid \mathrm{Ad} e^{ts} x \text{ exists as } t \to \infty\},$$

$$\mathfrak{l}_s = \{x \in \mathfrak{h}^{\mathbb{C}} \mid [x, s] = 0\} = \mathfrak{c}_{\mathfrak{h}}(x),$$

$$L_s = \{g \in H^{\mathbb{C}} \mid \mathrm{Ad} e^{ts} g = g\} = C_{H^{\mathbb{C}}}(e^{\mathbb{R}s}).$$

We call P_s and $\mathfrak{p}_s$ (respectively $L_s, \mathfrak{l}_s$) the parabolic (respectively Levi) subgroup and subalgebra associated to s.

In the same fashion, we introduce the following

Definition 2.2.2.

$$\mathfrak{m}_s = \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \lim_{t \to 0} \iota(e^{ts}) x \text{ exists}\},$$

$$\mathfrak{m}_s^0 = \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts}) x = x\}.$$

Remark 2.2.3. *Note that for any $s \in i\mathfrak{h}$, we can define a parabolic and a Levi subalgebra of $\mathfrak{g}^{\mathbb{C}}$, say $\mathfrak{p}'_s$, $\mathfrak{l}'_s$, in an analogous way to Definition 2.2.1. Since $s \in i\mathfrak{h}$, it follows that both $\mathfrak{p}'_s$ and $\mathfrak{l}'_s$ are θ and σ invariant (where σ is the conjugation on $\mathfrak{g}^{\mathbb{C}}$ defining $\mathfrak{g}$). In particular*

$$\mathfrak{p}'_s = \mathfrak{p}'_s \cap \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{p}'_s \cap \mathfrak{m}^{\mathbb{C}}, \qquad \mathfrak{l}'_s = \mathfrak{l}'_s \cap \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{l}'_s \cap \mathfrak{m}^{\mathbb{C}}.$$

But by definition

$$\mathfrak{p}'_s \cap \mathfrak{h}^{\mathbb{C}} = \mathfrak{p}_s, \quad \mathfrak{p}'_s \cap \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_s, \quad \mathfrak{l}'_s \cap \mathfrak{h}^{\mathbb{C}} = \mathfrak{l}_s, \quad \mathfrak{l}'_s \cap \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_s^0.$$

The isotropy representation restricts to actions

$$P_s \curvearrowright \mathfrak{m}_s, \qquad L_s \curvearrowright \mathfrak{m}_s^0.$$

By Remark 2.2.3, if furthermore G has a common connected component with a real form of a complex reductive Lie group $G^{\mathbb{C}}$, then also the groups P_s, L_s are the intersection with $H^{\mathbb{C}}$ of parabolic and Levi subgroups of $G^{\mathbb{C}}$.

Recall that a character of a complex Lie algebra $\mathfrak{h}^{\mathbb{C}}$ is a complex linear map $\mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ which factors through the quotient map $\mathfrak{h}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$. Let $\mathfrak{z}^{\mathbb{C}}$ be the center of $\mathfrak{h}^{\mathbb{C}}$, and denote by $\mathfrak{z}$ the center of $\mathfrak{h}$. For a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{h}^{\mathbb{C}}$ let $\mathfrak{l}$ be a corresponding Levi subalgebra with center $\mathfrak{z}_{\mathfrak{l}}$. One shows that $(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^* \cong \mathfrak{z}_{\mathfrak{l}}^*$, and then a character χ of $\mathfrak{p}$ comes from an element in $\mathfrak{z}_{\mathfrak{l}}^*$. Using an Ad-invariant non-degenerate bilinear form of $\mathfrak{h}^{\mathbb{C}}$ (for example, the Killing form), from $\chi \in \mathfrak{z}_{\mathfrak{l}}^*$ we get an element $s_\chi \in \mathfrak{z}_{\mathfrak{l}}$. Conversely, any $s \in \mathfrak{z}_{\mathfrak{l}}$ yields a character χ_s of $\mathfrak{p}_s$ since $B(s, [\mathfrak{p}_s, \mathfrak{p}_s]) = 0$. When $\mathfrak{p} \subseteq \mathfrak{p}_{s_\chi}$, we say that χ is an antidominant character of $\mathfrak{p}$. When the equality is attained, $\mathfrak{p} = \mathfrak{p}_{s_\chi}$, we say that χ is a strictly antidominant character. Note that for $s \in i\mathfrak{h}$, χ_s is a strictly antidominant character of $\mathfrak{p}_s$.

Remark 2.2.4. *Note that $\mathfrak{z}_{\mathfrak{l}} = \mathfrak{z} \oplus \mathfrak{c}_{\mathfrak{l}}$ where $\mathfrak{c}_{\mathfrak{l}} \subseteq \mathfrak{c}$ is a piece of a Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}}$ that becomes central within $\mathfrak{l}$.*

Then, any character has the form

$$\chi = z + \sum z_j \omega_j$$

where $z \in \mathfrak{z}^*$ and ω_j are the fundamental weights of $\mathfrak{h}$. It turns out that χ is antidominant if and only if:

1. *$z \in i\mathfrak{z}^*$,*

2. *$z_i \in \mathbb{R}_{\leq 0}$ are non-positive real numbers.*

Furthermore, χ is strictly antidominant if and only if $z_i < 0$.

2.3 α-stability and moduli spaces

Consider an L-twisted G-Higgs pair (E, ϕ).

Given any parabolic subgroup $P_s \leq H^{\mathbb{C}}$ and a reduction of the structure group $\sigma \in H^0(X, E(H^{\mathbb{C}}/P_s))$, let E_σ denote the corresponding principal bundle. Then, it makes sense to consider $E_\sigma(\mathfrak{m}_s)$. Similarly, any reduction $\sigma_L \in H^0(X, P_s/L_s)$ allows to take $E_{\sigma_L}(\mathfrak{m}_s^0)$.

Definition 2.3.1. Let F_h be the curvature of the Chern connection of E with respect to a metric $h \in H^0(X, E(H^{\mathbb{C}}/H))$. Let $s \in i\mathfrak{h}$, and let $\sigma \in H^0(X, E(H^{\mathbb{C}}/P_s))$. We define the degree of E with respect to s (or, equivalently, χ_s) and the reduction σ as follows:

$$\deg E(s, \sigma) = \int_X \chi_s(F_h).$$

Remark 2.3.2. *When a multiple of χ_s, say $m\chi_s$, exponentiates to a character of the group P_s, say $\widetilde{\chi}$, an alternative way to define the degree is by setting*

$$\deg E(s,\sigma) = \frac{1}{m}\deg L_{\widetilde{\chi}},$$

where $L_{\widetilde{\chi}} = E_\sigma \times_{\widetilde{\chi}} \mathbb{C}^\times$ is the line bundle associated to E via $\widetilde{\chi}$.

This is not always possible, but using the decomposition of χ_s explained in Remark 2.2.4 as a sum of characters of the centre and multiples of fundamental weights

$$\chi_s = \sum_j a_j z_j + \sum_k b_k \omega_k,$$

we have that for some $n \in \mathbb{Z}$ all of the characters of the centre and the fundamental weights exponentiate, so that we can define the degree as

$$\deg E(s,\sigma) = \frac{1}{m}\left(\sum_j a_j \deg E_\sigma \times_{nz_j} \mathbb{C}^\times + \sum_k b_k \deg E_\sigma \times_{n\omega_k} \mathbb{C}^\times\right)$$

This value is independent of the expression of χ_s as sum of characters and the integer n.

See [32] for details.

We can now define the stability of a G-Higgs pair. This notion naturally depends on an element in $i\mathfrak{z}(\mathfrak{h})$ which has a specific significance when G is a group of Hermitian type.

Definition 2.3.3. Let $\alpha \in i\mathfrak{z}$. We say that the pair (E,ϕ) is:

1. α-semistable if for any $s \in i\mathfrak{h}$ and any reduction $\sigma \in H^0(X, E(H^{\mathbb{C}}/P_s))$ such that $\phi \in H^0(X, E(\mathfrak{m}_s) \otimes L)$, we have

$$\deg E(s,\sigma) - B(\alpha, s) \geq 0.$$

2. α-stable if it is semistable and moreover, for any $s \in i\mathfrak{h} \setminus \mathrm{Ker}(d\iota)$, given a reduction $\sigma \in H^0(X, E(H^{\mathbb{C}}/P_s))$ we have

$$\deg E(s,\sigma) - B(\alpha, s) > 0.$$

3. α-polystable if it is α-semistable and anytime

$$\deg E(s,\sigma) - B(\alpha, s) = 0$$

for χ_s strictly antidominant of $\mathfrak{p}_s$ and σ as above, there exists a reduction σ' to the corresponding Levi subgroup L_s such that ϕ takes values in $H^0(X, E_{\sigma'}(\mathfrak{m}^0_s) \otimes L)$.

For a more detailed account of these notions we refer the reader to [32].

Definition 2.3.4. Let $F : (G', H', \theta', B') \to (G, H, \theta, B)$ be a morphism of reductive Lie groups. Given a G'-Higgs pair (E', ϕ'), we defined the extended Higgs pair (by the morphism F) to be the pair $(E' \times_F H^{\mathbb{C}}, dF(\phi))$.

Parameters appear naturally when studying the moduli problem from the gauge theoretical point of view: Higgs pairs are related to solutions to Hermite–Yang–Mills-type equations. This relation is established by the Hitchin–Kobayashi correspondence stated below following [32].

Theorem 2.3.5. *Let $\alpha \in i\mathfrak{z}$. Let $L \to X$ be a line bundle, and let h_L be a Hermitian metric on L. Fix ω a Kähler form on X. An L-twisted Higgs pair (E, ϕ) is α-polystable if and only if there exists $h \in \Omega^0(X, E(H^{\mathbb{C}}/H))$ satisfying:*

$$F_h - [\phi, \tau_h(\phi)]\omega = -i\alpha\omega \tag{2.1}$$

In the above:

1. *F_h is the curvature of the Chern connection on E corresponding to h,*
2. *$\tau_h : \Omega^0\left(E(\mathfrak{m}^{\mathbb{C}} \otimes L)\right) \to \Omega^0\left(E(\mathfrak{m}^{\mathbb{C}}) \otimes L\right)$ is the conjugation on $\Omega^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes L)$ determined by h and h_L. Let E_h be the principal H-bundle determined by h. Then we have an isomorphism, depending on h,*

$$E(\mathfrak{m}^{\mathbb{C}}) \cong_h E_h(\mathfrak{m}^{\mathbb{C}}) \cong E_h(\mathfrak{m}) \otimes E_h(i\mathfrak{m}).$$

Equivalently, the involutive isomorphisms $\tau'_h : \overline{E} \cong E^*$ determined by h by $\tau'_h(s) = h(s, \cdot)$, induces one on $E(\mathfrak{m}^{\mathbb{C}})$, that we will denote by the same letter: $\tau'_h : E(\mathfrak{m}^{\mathbb{C}}) \to E(\mathfrak{m}^{\mathbb{C}})$ whose fixed point bundle is $E_h(i\mathfrak{m})$.

Now, the choice of h and h_L allows to define the bracket on $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$ by

$$r := [s \otimes l, \tau_h(s \otimes l] = [s, \tau'_h(s)]\mathfrak{h}_L(l, \tau_L(l))$$

where $s \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}))$, $l \in H^0(X, L)$ are (possibly local) sections. Note that $r \in E_h(\mathfrak{h})$, as for all $Y \in \mathfrak{m}^{\mathbb{C}}$ $\theta[Y, \tau(Y)] = [-Y, -\tau Y] = [Y, \tau Y]$.

In the above theorem, we fix a holomorphic Higgs pair and look for a solution of equation (2.1), whose existence determines polystability.

From a different perspective, we can construct the gauge moduli space associated to equation (2.1). Fix a C^∞ principal $H^{\mathbb{C}}$-bundle $\mathbb{E}$. Given a reduction $h \in$

$\Omega^0(X, \mathbb{E}(H^{\mathbb{C}}/H))$, let $\mathbb{E}_h$ be the corresponding principal H-bundle. Now, the gauge moduli space consists of pairs (A, ϕ) where A is a connection on $\mathbb{E}_h$, $\phi \in \Omega^0(X, \mathbb{E}_h \otimes L)$ is holomorphic for the structure defined by A and both satisfy (2.1). The gauge group

$$\mathcal{H} = \Omega^0(X, \mathrm{Ad}\ \mathbb{E}_h). \tag{2.2}$$

acts on solutions to (2.1). The moduli space obtained by quotenting by this action is called the gauge moduli space, and we denote it by $\mathcal{M}_L^{gauge,\alpha}(G)$.

Remark 2.3.6. *In the case of complex groups, stability and simplicity of the Higgs pair implies uniqueness of the corresponding $\mathcal{H}$-orbit of metrics. This is due to the fact that the center of the group fixes both the principal bundle and the Higgs field. When dealing with real groups, however, this statement fails to be true.*

Proposition 2.3.7. *Let (E, ϕ) be an α-polystable G-Higgs pair. Let $M_{(E,\phi)}$ be the space of solutions $h \in \Omega^0(X, E(H^{\mathbb{C}}/H))$ to the equation $F_h - [\phi, \tau_h(\phi)]\omega = \alpha\omega$. Fix a solution h, and let E_h be the corresponding reduction of E to an principal H-bundle. Then $M_{(E,\phi)} \cong \mathrm{Aut}(E, \phi) \cap C_{H^{\mathbb{C}}}(\alpha)/\mathrm{Aut}(E_h, \phi) \cap C_H(\alpha)$.*

In particular, if (E, ϕ) is stable and simple, namely, its automorphisms are $Ker(\iota) \cap Z(H^{\mathbb{C}})$, then $M_{(E,\phi)} \cong Ker(\iota) \cap Z(H^{\mathbb{C}})/Ker(\iota) \cap Z(H)$.

Proof. Note that $z \in H^{\mathbb{C}}$ leaves h invariant if and only if it belongs to $Z(H)$. On the other hand: $F_{zh} = \mathrm{Ad}_z F_h$. Let $z \in H^{\mathbb{C}}$. Applying Ad_z to equation (2.1), we obtain that h solves the equation for (E, ϕ) if and only if zh solves the equation for the parameter $\mathrm{Ad}_z(\alpha)$ and $(\mathrm{Ad}_z E, \mathrm{Ad}_z \phi)$. Indeed:

$$\begin{aligned} &\mathrm{Ad}_z(F_h) - \mathrm{Ad}_z[\phi, \tau_h \phi]\omega = \mathrm{Ad}_z(\alpha)\omega \\ \iff\ & F_{zh} - [\mathrm{Ad}_z\phi, \mathrm{Ad}_z \tau_h \phi]\omega = \mathrm{Ad}_z(\alpha)\omega \\ \iff\ & F_{zh} - [\mathrm{Ad}_z\phi, \tau_{zh} \mathrm{Ad}_z \phi]\omega = \mathrm{Ad}_z(\alpha)\omega \end{aligned}$$

where the last equality follows from the fact that $\tau_{hz} = \mathrm{Ad}_z \circ \tau_h \circ \mathrm{Ad}_{z^{-1}}$. Thus, zh will be a solution for (E, ϕ), α if and only if $z \in \mathrm{Aut}(E_h, \phi) \cap C_{H^{\mathbb{C}}}(\alpha)$. Finally, zh will be different from h whenever z is not in H □

Fixing the topological type d of the principal bundle (equivalently, fixing $\mathbb{E}$), we obtain a subset $\mathcal{M}_d^{\alpha}(G)$ of all degree d α-polystable Higgs pairs with underlying topological bundle $\mathbb{E}$. Similarly, let $\mathcal{M}_d^{gauge}(G)$ denote the moduli space of pairs (A, ϕ) (where $A = A_h$ is a solution to (2.1) and ϕ satisfies $\overline{\partial}_A \phi = 0$) with the equivalence relation given by the action of $\mathcal{H}$.

Proposition 2.3.8. *There exists a homeomorphism*

$$\mathcal{M}_L^{\alpha}(G) \cong \mathcal{M}_L^{gauge,\alpha}(G)$$

2.3.1 Higgs bundles and surface group representations

In the case $L = K$, for $\alpha = 0$, there is a third moduli space that can be considered. We let

$$\mathcal{R}ep_X(G) = \mathrm{Hom}^+(\pi_1(X), G)/G$$

be the quotient of the set of reductive homomorphism $\rho : \pi_1(X) \to G$, quotiented by the conjugation action of G. Reductive means that the Zariski closure of the image $\overline{\rho(\pi_1(G))}^{Zar} < G$ is a reductive Lie group. The way to produce a Higgs bundle is as follows: ρ induces a flat principal G-bundle $(\mathbb{E} := \widetilde{X} \times_\rho, D_\rho)$, where the connection is induced from exterior differentiation on $\widetilde{X} \times G$. Donaldson and Corlette proved (in different contexts) that ρ is reductive if and only if there exists a reduction of the structure group $h \in \Omega^0(X, E(H^{\mathbb{C}}/H))$ satisfying the harmonicity equations

$$F_h = -[\psi, \psi^*], \qquad \nabla_h^* \psi = 0.$$

In the above: $D_\rho = D_h + \psi$ is the decomposition into an h-unitary connection D_h, $\psi \in \Omega^1(X, E(i\mathfrak{m}))$ and ∇_h is the covariant derivative associated to D_h.

Now, if D_h, ψ satisfy the harmonicity equations, then $(A, \phi) = (D_h, \psi^{1,0})$ is, up to gauge transformations, a solution to Hitchin's equations. The inverse direction, which we explained in the preceeding section, are due to Hitchin, Simpson and García-Prada–Gothen–Mundet. We note that irreducible homomorphism (namely, such that $Z_G(\rho(\pi_1)) = Z(G)$) correspond to stable Higgs bundles and viceversa. This can be summarised in the following.

Proposition 2.3.9. *There is a homeomorphism*

$$\mathcal{R}ep_X(G) \cong \mathcal{M}_K^0(G).$$

Propositions 2.3.9 and 2.3.8 are known as the **non-abelian Hodge correspondence**.

Remark 2.3.10. *From Proposition 1.2.38 we see that a non zero parameter $\alpha \neq 0$ makes sense either if the group is $(G^{\mathbb{C}})_{\mathbb{R}}$ for some complex Lie group $G^{\mathbb{C}}$ or if the form is of Hermitian type, as otherwise $\mathfrak{z}(\mathfrak{h}) = 0$.*

In general, the 0-moduli space corresponds, when $L = K$, to the moduli space of representations of the fundamental group $\pi_1(X)$ into G. Due to the importance of this case, omission of the parameter from the notation will refer to the 0-moduli space for any real form.

Remark 2.3.11. *Remark. (Topological type and parameters). By Remark 1.2.20, given $G^{\mathbb{C}}$ a complex reductive Lie group, we have a natural real Lie group associated to it $((G^{\mathbb{C}})_{\mathbb{R}}, U, \tau, B)$. Then*

$$\mathcal{M}((G^{\mathbb{C}})_{\mathbb{R}})_L \cong \mathcal{M}(G^{\mathbb{C}})_L.$$

Note that in this case, when $G^{\mathbb{C}}$ has a positive dimensional center, the topology of the bundle fully determines the parameter, and the torsion free piece of the topological type is also determined by the parameter. Indeed, it suffices to evaluate characters on both sides of equation 2.1, to recover α with no ambiguity from the topological type $d \in \pi_1(G)$. We also see that in the opposite direction we can only recover the torsion free part of d.

The above argument fails for general real groups, as $Char(\mathfrak{g}) \neq Char(\mathfrak{h})$. Thus, not all characters $\chi \in Char(\mathfrak{h})$ will vanish on $[\phi, \tau_h \phi]$ for any $\phi \in [\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}] \cap \mathfrak{z}(\mathfrak{h})$. Thus, the method fails to determine α from d and viceversa.

For groups of non-hermitian type, since $\alpha = 0$, there are no considerations to make.

Example 2.3.12. Let us analyse $\mathcal{M}_d^{\alpha}(\mathrm{SL}(2,\mathbb{R}))_L$ for an arbitrary line bundle L of degree d_L. Note that $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$, so in particular it is of Hermitian type.

An $\mathrm{SL}(2,\mathbb{R})$-Higgs pair on a curve X is a line bundle $F \to X$ together with a section

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : F \oplus F^{-1} \to (F^{-1} \oplus F) \otimes L.$$

In particular: $b \in H^0(X, F^2 \otimes L)$, $c \in H^0(X, F^{-2} \otimes L)$. So the first condition we get on F is $|d_L/2| \geq |\deg F|$.

Since $H^{\mathbb{C}} \cong \mathbb{C}^{\times}$ the only parabolic subgroup within $H^{\mathbb{C}}$ is $H^{\mathbb{C}}$ itself, so the only reduction is the identity. Now, $\mathfrak{h} = \mathbb{R}$, so $i\alpha \in \mathfrak{z}\mathfrak{h} = \mathbb{R}$. Since the only antidominant character is the identity, and $B(\alpha, id) = \alpha||id||_B$, for a bundle to be α-semistable it must happen

$$\deg F \geq \alpha||id||_B.$$

So after normalising $||id||_B = 1$, we find that there will be no α-semistable bundles for $\alpha > d_L/2$, and for $\alpha \leq d_L/2$ we get bundles whose degree is at least $\lceil\alpha\rceil$ (where $\lceil\alpha\rceil$ is the lowest integer greater that real number α) and at most $[d_L/2]$.

Conditions for polystability are empty, as the Levi is again $H^{\mathbb{C}}$ itself.

Namely, $\mathcal{M}_d^{\alpha}(\mathrm{SL}(2,\mathbb{R}))$ is empty whenever $d < \alpha$ or $\alpha > |d_L/2|$, whereas it consists of all S-equivalence classes of $\mathrm{SL}(2,\mathbb{R})$-Higgs pairs of degree d otherwise.

Morphisms induced from group homomorphisms Consider a morphism of reductive Lie groups $f : (G', H', \theta', B') \to (G, H, \theta, B)$.

Definition 2.3.13. Given a G'-Higgs pair (E', ϕ'), we define the **extended G-Higgs pair** (by the morphism f) to be the pair

$$(E' \times_f H^{\mathbb{C}}, df(\phi)) =: (E'(H^{\mathbb{C}}), df(\phi)).$$

These pairs satisfy the following.

Proposition 2.3.14. *With the above notation, we have that if the G-Higgs pair (E', ϕ') is α-polystable, and $df(\alpha) \in i\mathfrak{z}(\mathfrak{h})$, then the corresponding extended G-Higgs bundle, say (E, ϕ), is $df(\alpha)$-polystable*

Proof. By Theorem 2.3.5, polystability of (E', ϕ') is equivalent to the existence of a solution to the Hitchin equation (2.1). Let h' be the corresponding solution. Now, h' extends to a Hermitian metric on E, as f defines a map

$$\Omega^0(E((H')^{\mathbb{C}}/H')) \to \Omega^0(E(H^{\mathbb{C}}/H)).$$

Let $h \in \Omega^0(E(H^{\mathbb{C}}/H))$ be the image of h' via that map. Clearly $F_{h'}$ is a two form with values in $\mathfrak{h}$. But $F_h = df(F_{h'})$, where df is evaluated on the coefficients of the 2 form $F_{h'}$, as the canonical connection ∇_h is defined by

$$dh = \langle \nabla_h \cdot, \cdot \rangle + \langle \cdot, \nabla_h \cdot \rangle.$$

Since $dh = df(dh')$, it follows that $\nabla_h = df(\nabla_{h'})$. Namely, we obtain the equation for (E, ϕ) by applying df to the equation for (E', ϕ').

□

As a corollary we have the following.

Corollary 2.3.15. *With the above notation, if $\alpha \in i\mathfrak{z}'$ is such that $df(\alpha) \in i\mathfrak{z}$, then the map*

$$(E, \phi) \mapsto \big(E(H^{\mathbb{C}}), df(\phi)\big)$$

induces a morphism

$$\mathcal{M}_d^{\alpha}(G') \to \mathcal{M}_{f_*d}^{df\alpha}(G),$$

where $f_*\alpha$ is the topological type of $E(H^{\mathbb{C}})$. In case G is connected, this corresponds to the image via the mp $f_*\pi_1(H') \to \pi_1(H)$ induced by the group homomorphism.

Remark 2.3.16. *Although polystability is preserved, stability is not necessarily. For example, under the morphism*

$$i : \mathcal{M}(SL(2,\mathbb{R})) \to \mathcal{M}(SL(2,\mathbb{C})), \tag{2.3}$$

the polystability condition is not trivial anymore, as

$$(F, \beta, \gamma) \mapsto (F \oplus F^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).$$

In particular, for $\deg F = 0$ *and* $\gamma = \beta = 0$*, the bundle is strictly polystable.*

Remark 2.3.17. *In the particular case when the morphism of groups is an embedding of a real form inside its complexification* $G \leq G^{\mathbb{C}}$*, one has that the involution* θ *induces an involution* $[\theta]$ *(depending on the inner conjugacy class of* θ*) on* $\mathcal{M}(G^{\mathbb{C}})$*, whose fixed points contain the image of* $\mathcal{M}(G)$*. See [30] for an analysis for* $SL(n,\mathbb{C})$ *and [33] for the general case.*

However, for some special kinds of morphisms, and stability is preserved in an approapriate sense.

Proposition 2.3.18. *Let* $j : (G', H', \theta', B') \to (G, H, \theta, B)$ *be an isogeny of real forms of complex reductive Lie groups. Let* (E', ϕ') *be* G'*-Higgs pair, and let* (E, ϕ) *be its extended* G*-Higgs pair. Then* (E', ϕ') *is* α*-(semi,poly)stable for some* $\alpha \in i\mathfrak{z}(\mathfrak{h}) = i\mathfrak{z}(\mathfrak{h}')$ *if and only if there exists* $n(j) \in \mathbb{N}$ *(depending on the isogeny* j*) such that* (E, ϕ) *is* $n\alpha$*-(semi,poly)stable.*

Proof. The condition on the group morphism ensures that both Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic. In particular,

1. H' is isogenous to H
2. The preimage of the identity $j^{-1}(e)$ is a finite group of order, say, $o \in \mathbb{N}$.
3. For any $s \in i\mathfrak{h} = i\mathfrak{h}'$, the parabolic subgroup $P'_s \leq H'^{\mathbb{C}}$ maps to $P_s \leq H^{\mathbb{C}}$ (by (2) above and the fact that for all $n \in \mathbb{N}$, $\mathfrak{p}_s = \mathfrak{p}_{ns}$), and similarly for the corresponding Levi subgroups.
4. If $\widetilde{P'}_s$ denotes the parabolic subgroup in $G'^{\mathbb{C}}$ associated to s (and similarly for $\widetilde{L'}_s$, $\widetilde{P}_s$, $\widetilde{L}_s$ and the corresponding Lie algebras) we have that $\mathfrak{m}_s = \widetilde{\mathfrak{p}}_s \cap \mathfrak{m}^{\mathbb{C}}$, $\mathfrak{m}_0 = \widetilde{\mathfrak{l}}_s \cap \mathfrak{m}^{\mathbb{C}}$, where we recall that $\widetilde{\mathfrak{p}}_s \cong \widetilde{\mathfrak{p}'}_s$.

There is a one to one correspondence between $\sigma' \in H^0(X, E'(H'^{\mathbb{C}}/P'_s))$ (respectively $\eta' \in H^0(X, E(H'^{\mathbb{C}}/L'_s)))$ and $\sigma \in H^0(X, E(H^{\mathbb{C}}/P_s))$ (respectively $\eta \in H^0(X, E(H^{\mathbb{C}}/L_s)))$. Indeed, by definition of (E, ϕ), any reduction to a parabolic subgroup $P \leq G$ must be a reduction to the image $j(P')$ of a parabolic subgroup $P' \leq G'$.

Now, let χ'_s be the character of $\mathfrak{p}'_s = \mathfrak{p}_s$ B-dual to s. Recall from [32] that

$$\chi'_s = \sum_j a_j z_j + \sum_k b_k \omega_k$$

where the z_j's are central characters and the ω_k's are fundamental weights. Take $m \in \mathbb{N}$ be as in Definition 2.7 of [32]. Then $m\chi_s$ lifts to a character δ' of P'_s which fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \delta'} & & & & \\
1 & \longrightarrow & K & \longrightarrow & P'_s & \longrightarrow & j(P'_s) & \longrightarrow & 1
\end{array}$$

for some $n \in \mathbb{N}$. It follows that $\delta = (\delta')^n$ is a lift to P_s of $nm\chi_s$.

Note furthermore that δ' factors through $P'_s/[L'_s, L'_s]$, which in turn maps to $P_s/[L_s, L_s]$ by point (3) above. Hence also δ factors through $j(P'_s/[L'_s, L'_s]) \subseteq P_s/[L_s, L_s]$.

With that, assume E'_σ (E_σ) is a reduction to a P'_s (P_s) principal bundle. Then notice that if $V' = E_{\sigma'} \times_{\delta'} \mathbb{C}^\times$, $V = E_\sigma \times_\delta \mathbb{C}^\times$, we have in fact that

$$V \cong E' \times_{\delta'^n} \mathbb{C}^\times$$

so that $\deg V(\mathbb{C}) = n \deg V'(\mathbb{C})$.

This implies that $\deg E'(s, \sigma') - B(\alpha, s) \geq 0 \iff \deg E(s, \sigma) - B(n\alpha, s) \geq 0$, with strict inequalities and equalities respected by the equivalence.

This proves the case of (semi)stability, the remaining case following in a similar way. $\square$

2.4 The role of normalisers

We next investigate injectivity of morphisms of moduli spaces of Higgs pairs.

Example 2.4.1. Consider $\mathrm{SL}(2,\mathbb{R}) \subseteq \mathrm{SL}(2,\mathbb{C})$. The induced morphism (2.3) has non trivial kernel, as points in $\mathcal{M}_d(\mathrm{SL}(2,\mathbb{R}))$ and their duals in $\mathcal{M}_{-d}(\mathrm{SL}(2,\mathbb{R}))$ will be identified within $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$, due to the existence of the complex gauge transformation $\mathrm{Ad}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. However, i factors through $\mathcal{M}(N)$, where $N := N_{\mathrm{SL}(2,\mathbb{C})}\mathrm{SL}(2,\mathbb{R})$ and

$$\mathcal{M}(N) \hookrightarrow \mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$$

is a morphism. Note that the gauge transformation causing the lack of injectivity of i belongs to the gauge group of $\mathcal{M}(N_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{SL}(2,\mathbb{R})))$.

The above example is a particular case of a general fact.

Lemma 2.4.2. *Let $G' \subseteq G$ be two Lie groups. Let E, $\widetilde{E}$ be two G' principal bundles over X, and suppose there exists a morphism*

$$F: \ E(G) \to \widetilde{E}(G)$$

of principal G-bundles. Then there exists an isomorphism of principal $N_G(G')$-bundles $E(N_G(G')) \cong \widetilde{E}(N_G(G'))$.

Proof. By Theorem 10.3 in [75], F is an isomorphism. Denote $N_G(G')$ by N. Choose common trivialising neighbourhoods $U_i \to X$ such that

$$E|_{U_i} \cong U_i \times G' \qquad \widetilde{E}|_{U_i} \cong U_i \times G'.$$

Let $g_{ij}, \widetilde{g_{ij}}$ be the transition functions for E and $\widetilde{E}$ respectively and define $F_i := F|_{E(G)|_{U_i}}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
U_j \times G & \longrightarrow & U_j \times G \, . \\
\Big\downarrow & \begin{array}{ccc} (x,g) & \longmapsto & (x, F_j(g)) \\ \downarrow & & \downarrow \\ (x, g g_{ij}) & \longmapsto & (x, F_i(g g_{ij})) = (x, F_j(g)\widetilde{g}_{ij}) \end{array} & \Big\downarrow \\
U_i \times G & \longrightarrow & U_i \times G
\end{array}$$

Now, since for any $n \in N,\ g \in G'$ we have that $ng \in N$, it follows that for all i, j's $F_i(N) = F_j(N)\widetilde{g_{ij}}$. Namely, the image bundle of $E(N)$ is isomorphic to $\widetilde{E}(N)$. □

Proposition 2.4.3. *Let $i : (G', H', \theta', B') \subseteq (G, H, \theta, B)$ be an embedding of reductive Lie groups; assume that $N_G(G')$ be reductive. Then, the morphism*

$$\mathcal{M}(G') \to \mathcal{M}(G)$$

induced by the embedding i factors through the moduli space of $N_G(G')$-Higgs bundles, and furthermore

$$\mathcal{M}(N_G(G')) \hookrightarrow \mathcal{M}(G)$$

is injective.

Proof. Clear from 2.4.2 and 2.3.15. □

2.5 Deformation theory

The general setup for the study of deformations was established by Schlessinger ([68]). In our context, it was systematically studied by several authors, amongst which A. Beauville [5], N. Nitsure [61], I. Biswas and S. Ramanan [9], and later developped in particular cases by several authors (see for example [13, 32] amongst others).

Let us recall the basics, following the discussion in [32] and [15].

The deformation complex of a G-Higgs pair $(E,\phi)\to X$ is:

$$C^\bullet : [d\phi,\ \cdot\ ] : E(\mathfrak{h}^{\mathbb{C}}) \to E(\mathfrak{m}^{\mathbb{C}})\otimes L \tag{2.4}$$

whose hypercohomology sets fit into the exact sequence

$$\begin{aligned} 0 \to \mathbb{H}^0(X,C^\bullet) \to H^0(X,E(\mathfrak{h}^{\mathbb{C}})) \to H^0(X,E(\mathfrak{m}^{\mathbb{C}})\otimes L) \to \mathbb{H}^1(X,C^\bullet) \\ \to H^1(E(\mathfrak{h}^{\mathbb{C}})) \to H^1(X,E(\mathfrak{m}^{\mathbb{C}})\otimes L) \to \mathbb{H}^2(X,C^\bullet) \end{aligned} \tag{2.5}$$

In particular, we see that

$$\mathbb{H}^0(X,C^\bullet) = \mathfrak{aut}(E,\phi). \tag{2.6}$$

Proposition 2.5.1. *The space of infinitesimal deformations of a pair (E,ϕ) is canonically isomorphic to $\mathbb{H}^1(X,C^\bullet)$, where $C^\bullet$ is the complex (2.4).*

Remark 2.5.2. *In case the moduli functor is representable by some scheme S, the space of deformations is actually the tangent space TS. However, this will not be the case in general (see for example, [55, 56] for the case of sheaves.)*

Definition 2.5.3. A G-Higgs pair (E,ϕ) is said to be **simple** if

$$\mathrm{Aut}(E,\phi) = H^0(X,\mathrm{Ker}(\iota)\cap Z(H^{\mathbb{C}})).$$

(E,ϕ) is said to be **infinitesimally simple** if

$$\mathbb{H}^0(X,C^\bullet) \cong H^0(X,(\mathrm{Ker}(d\iota)\cap \mathfrak{z}(\mathfrak{h}^{\mathbb{C}}))).$$

Given $(G,H,\theta,B) < (G^{\mathbb{C}},U,\tau,B)$ a real form if a complex reducitve Lie group we have the following.

Proposition 2.5.4. *Let (E,ϕ) be a G-Higgs pair. Consider the complex:*

$$C^\bullet_{G^{\mathbb{C}}} : E(\mathfrak{g}^{\mathbb{C}}) \stackrel{[\phi,\ \cdot\ ]}{\to} E(\mathfrak{g}^{\mathbb{C}})\otimes L,$$

the deformation complex of the corresponding complexified Higgs pair $(E\times_{H^{\mathbb{C}}} G^{\mathbb{C}},\phi)$. Then, there is an isomorphism of complexes:

$$C^\bullet_{G^{\mathbb{C}}} \cong C^\bullet_G \oplus (C^\bullet_G)^*\otimes K.$$

Proposition 2.5.5. *Let* (E,ϕ) *be a stable and simple* G*-Higgs bundle such that* $\mathbb{H}^2(X,\mathbb{C}^\bullet)=0$. *Then* (E,ϕ) *is a smooth point of the moduli space. In particular, if the* G*-Higgs pair is stable and simple and the associated* $G^{\mathbb{C}}$*-Higgs pair is stable, then it is a smooth point of the moduli space.*

Proposition 2.5.6. *Given an* α*-polystable Higgs pair* (E,ϕ), (E,ϕ) *is* α*-stable iff*

$$\mathfrak{aut}^{ss}(E,\phi)\subseteq H^0(X,E(\mathfrak{z}))$$

where

$$\mathfrak{aut}^{ss}(E,\phi)=\{s:X\to\mathfrak{aut}(E,\phi)\mid s(x)\ is\ semisimple\ for\ all\ x\in X\}$$

The above has its counterpart in terms of the gauge moduli space. This is done in full detail in [32] in the case $\alpha=0$, $L=K_X$. We extend it here to the deformation complex of an arbitrary pair. In the general situation we have the complex

$$C^\bullet(A,\phi):\ \Omega^0(X,E_h(\mathfrak{h}))\xrightarrow{d_0}\Omega^1(X,E_h(\mathfrak{h}))\oplus\Omega^0(X,E_h(\mathfrak{m}^{\mathbb{C}})\otimes L)$$

$$\xrightarrow[d_1]{}\Omega^2(X,E_h(\mathfrak{h}))\oplus\Omega^{0,1}(X,E_h(\mathfrak{m}^{\mathbb{C}})\otimes L),$$

where E_h is the reduction of E to an H-principal bundle given by h, and the maps are defined by

$$d_0(\psi)=(d_A\,\dot{\psi},[\phi,\psi]),\quad d_1(\dot{A},\dot{\phi})=(d_A(\dot{A})-[\dot{A},\tau\phi]\omega+[\phi,\tau\,\dot{\phi}]\omega,\overline{\partial}_A\,\dot{\phi}+[\dot{A}^{0,1},\phi]) \tag{2.7}$$

Definition 2.5.7. A pair (A,ϕ) is said to be **irreducible** if its group of automorphisms

$$\mathrm{Aut}(A,\phi)=\{h\in\mathcal{H}\ :\ h^*A=A,\ \iota(h)(\phi)=\phi\}=Z(H)\cap\mathrm{Ker}\iota. \tag{2.8}$$

It is said to be **infinitesimally irreducible** if

$$\mathfrak{aut}(A,\phi)=\mathfrak{z}(\mathfrak{h})\cap\mathrm{Ker}d\iota.$$

The following two propositions are explained in full detail in [32] for 0-moduli spaces of Higgs bundles. For the general case, arguments are also standard and consist in resolving the hypercohomology complex $\mathbb{H}^1(C^\bullet(E,\phi))$ and choosing harmonic representatives (see for example [49], Chapter VI, §8).

Proposition 2.5.8. *Let* $(E, \phi) \in \mathcal{M}_d^\alpha(G)$ *correspond to* $(A, \phi) \in \mathcal{M}_d^{gauge}(G)$*. Assume they are both smooth points of their respective moduli. Then*

$$\mathbb{H}^0(C^\bullet(E, \phi)) \cong \mathbb{H}^0(C^\bullet(A, \phi))$$

Proposition 2.5.9. *Let* $(E, \phi) \in \mathcal{M}_d^\alpha(G)$ *correspond to* $(A, \phi) \in \mathcal{M}_d^{gauge}(G)$*. Then*

$$\mathbb{H}^1(C^\bullet(E, \phi)) \cong \mathbb{H}^1(C^\bullet(A, \phi))$$

Proposition 2.5.10. *Under the correspondence established by Theorem 2.3.5, stable Higgs pairs correspond to infinitesimally irreducible solutions to 2.1. On the other hand, simple and stable pairs correspind to irreducible solutions.*

Chapter 3

The Hitchin map and the Hitchin–Kostant–Rallis section

Throughout this chapter, we fix the following notation: $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ denotes a strongly reductive real form of a complex reductive Lie group. The corresponding Lie algebras will be denoted by gothic letters. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be the Cartan decomposition induced by θ and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ be the decomposition induced on $\mathfrak{g}^{\mathbb{C}}$. Fix $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$ a maximal anisotropic Cartan subalgebra (cf. Definition 1.1.20) with Weyl group $W(\mathfrak{a}^{\mathbb{C}})$. Let

$$\mathfrak{c}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}, \qquad \mathfrak{t}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} \cap \mathfrak{h}^{\mathbb{C}}$$

be a θ-invariant Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, where by abuse of notation we denote by θ the extension to $\mathfrak{g}^{\mathbb{C}}$ by linearity of the involution on $\mathfrak{g}$ with the same name.

We fix X a smooth complex projective curve of genus $g \geq 2$, and a line bundle $L \to X$.

3.1 The Hitchin map

Let $\alpha \in i\mathfrak{z}(\mathfrak{h})$ and let $\mathcal{M}_L^{\alpha}(G)$ be the moduli space of G-Higgs bundles for the parameter α. Let

$$\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$$

be the Chevalley morphism. Note that one has a $\mathbb{C}^{\times}$ action both on $\mathfrak{m}^{\mathbb{C}}$ and on $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$: the former is the usual action on the $\mathbb{C}$-vector space $\mathfrak{m}^{\mathbb{C}}$, whereas the latter is induced from the natural weighted action on $\mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$. Note that the

Chevalley map is both $H^{\mathbb{C}}$-invariant and $\mathbb{C}^{\times}$-equivariant. Due to this, the Chevalley morphism induces a map

$$h_{G,L} : \mathcal{M}(G)^{\alpha}_{L} \to \mathcal{A}_{G,L} := H^0(X, \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})), \tag{3.1}$$

which sends a pair (E, ϕ) to the corresponding conjugacy class of ϕ. More concretely, a choice of a homogeneous basis $p_1, \dots, p_a \in \mathbb{C}[\mathfrak{a}^{\mathbb{C}}]^{W(\mathfrak{a}^{\mathbb{C}})}$ of degrees $d_1, \dots, d_a$ (where $a = \dim\ \mathfrak{a}^{\mathbb{C}}$), defines an isomorphism

$$\mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}}) \cong \oplus_{i=1}^{a} L^{d_i}.$$

In these terms, the Hitchin map is defined by

$$(E, \phi) \mapsto (p_1(\phi), \dots, p_a(\phi)).$$

Lemma 3.1.1. *Let* $F : (\widetilde{G}, \widetilde{H}, \theta, B) \hookrightarrow (G, H, \theta, B)$ *be the maximal split subgroup of a reductive Lie group. Let* $\alpha \in i\mathfrak{z}_{\widetilde{\mathfrak{h}}}(\mathfrak{h}) \subseteq \mathfrak{z}(\widetilde{\mathfrak{h}}_0)$. *Then, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}^{\alpha}_{L}(\widetilde{G}) & \xrightarrow{F} & \mathcal{M}^{F(\alpha)}_{L}(G) \\ \downarrow & & \downarrow \\ \mathcal{A}_{\widetilde{G},L} & = & \mathcal{A}_{G,L} \end{array}$$

where $\widetilde{G} \subset G$ *is the maximal split subgroup (cf. Definition 1.2.26) whose Lie algebra is* $\widetilde{\mathfrak{g}}$.

Proof. Note that by assumption $F(\alpha) \in \mathfrak{z}(\mathfrak{h}_0) \cap \mathfrak{z}_{\mathfrak{h}_0}(\widetilde{\mathfrak{h}}_0)$, so Corollary 2.3.15 applies to define a morphism between both moduli spaces. Commutativity of the diagram follows from Proposition 1.3.4, by noticing that the basis of polynomials generating the algebra of invariants is the same for both groups. □

3.2 Construction of the Hitchin–Kostant–Rallis section

The ideas in this paper differ from Hitchin's [45] in essentially two ways:

1. Instead of building the section to the moduli space of $G^{\mathbb{C}}$-Higgs pairs, then checking points are fixed under the action of an involution, we work directly with the moduli space for the real groups $\mathrm{SL}(2, \mathbb{R})$, G. In order to do this, we will use the results from Section 1.3 to find the appropriate three dimensional subalgebras.

2. The presence of parameters and arbitrary twistings implies we have no ressource to the moduli of representations. Thus, new arguments are needed in order to prove smoothness of the section.

See Section 3.2.6 for details on this.

We will consider a real form $(G, H, \theta, B) \leq (G^{\mathbb{C}}, U, \tau, B_{\mathbb{C}})$ such that the semisimple part of G, $G_{ss} := [G, G]$, be simple.

For the sake of clarity, we will distinguish two cases: on the one hand, non-hermitian real forms. These have parameter $\alpha = 0$, which simplifies arguments. In particular, in this case it will be enough to build the section for split real forms, that is, Hitchin's section, then use the morphism $\mathcal{M}_L^0(\widetilde{G}) \to \mathcal{M}_L^0(G)$ where $(\widetilde{G}, \widetilde{H}, \widetilde{\theta}, \widetilde{B}) \leq (G, H, \theta, B)$ is the maximal split subgroup (cf. Proposition 2.3.15). On the other hand, we will consider hermitian real forms. In this situation, $\alpha \in i\mathfrak{h}$ need not be zero, so the α moduli for both $\widetilde{G}$ and G will only make sense for $\alpha \in i\mathfrak{z}_{\widetilde{\mathfrak{h}}}(\mathfrak{h})$. In this situation, it may happen for the α-Hitchin–Kostant–Rallis section to exist for the group but not its maximal split subgroup, and viceversa.

3.2.1 Reminder on representation theory

Choose $\mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ a principal normal TDS, defined by the σ and θ-equivariant homomorphism 1.7 of Lemma 1.3.10

$$\rho' : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{s}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}. \tag{3.2}$$

Here, σ and θ on $\mathfrak{sl}(2, \mathbb{C})$ are defined as in Proposition 1.3.10. Recall from (**??**) that the Cartan decomposition of $\mathfrak{sl}(2, \mathbb{R})$ under θ reads

$$\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2) \oplus \mathfrak{sym}_0(2, \mathbb{R}), \tag{3.3}$$

which identifies $\mathfrak{so}(2)$ to trace cero diagonal matrices, and $\mathfrak{sym}_0(2, \mathbb{R})$ to real antidiagonal matrices.

The image under ρ' of the standard basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto x$$

is a standard principal normal triple (e, f, x).

By θ-equivariance, $\rho' = \rho'_+ \oplus \rho'_-$ where

$$\rho'_+ : \mathfrak{so}(2, \mathbb{C}) \cong \mathbb{C} \to \mathfrak{h}^{\mathbb{C}}, \qquad \rho'_+ : \mathfrak{sym}_0(2, \mathbb{C}) \cong \mathbb{C}^2 \to \mathfrak{m}^{\mathbb{C}}. \tag{3.4}$$

In particular, ρ'_+ fits into a commutative diagram

$$\begin{CD} \mathbb{C} @>{\rho'_+}>> \mathfrak{h}^{\mathbb{C}} \\ @V{\iota}VV @VV{\iota}V \\ \mathfrak{gl}(\mathbb{C}^2) @>>> \mathfrak{gl}(\mathfrak{m}^{\mathbb{C}}). \end{CD} \tag{3.5}$$

We claim that the restriction of ρ' to $\mathfrak{sl}(2,\mathbb{R})$ lifts to a θ-equivariant group homomorphism

$$\rho : \mathrm{SL}(2,\mathbb{R}) \to G. \tag{3.6}$$

taking SO(2) to H. Indeed, by connectedness of $\mathrm{SL}(2,\mathbb{R})$ and the polar decomposition, we can define $\rho(e^U e^V) = e^{\rho' X} e^{\rho' V}$ for given $U \in \mathfrak{so}(2,\mathbb{C})$, $V \in i\mathfrak{so}(2,\mathbb{C})$. We will abuse notation and use ρ_+ both for the restriction $\rho|_{\mathrm{SO}(2)}$ and its complexification. That is

$$\rho_+ : \mathrm{SO}(2,\mathbb{C}) \to H^{\mathbb{C}}. \tag{3.7}$$

Now, by simple connectedness of $\mathrm{SL}(2,\mathbb{C})$, ρ' lifts to

$$\mathrm{Ad}(\rho) : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Ad}(G)^{\mathbb{C}} \tag{3.8}$$

where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ is the adjoint representation and $\mathrm{Ad}(\rho)|_{\mathrm{SL}(2,\mathbb{R})} = \mathrm{Ad} \circ \rho$. Note that

$$\mathrm{Ker}(\mathrm{Ad}) = Z_G(\mathfrak{g}) \supseteq Z(G). \tag{3.9}$$

The adjoint representation induces a splitting

$$\mathfrak{g}^{\mathbb{C}} \cong \oplus M_k \tag{3.10}$$

into irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules. Since such an irreducible module M_k is generated by the translated highest weight vector e_k by the action of f, and such highest weight vectors are annihilated by the action of e, it follows that $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e) = \oplus_k \mathbb{C} \cdot e_k$. Since the centraliser splits into a direct sum $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(e) = \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(e) \oplus \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(e)$, it follows that each of the submodules M_k are invariant by the action of θ. In particular, due to the way the modules are generated, we have that if $m_k - 1 \neq 0$ is the eigenvalue of e_k via the action of x, then

$$\begin{aligned} m_k - 1 &:= \frac{\dim M_k - 1}{2} \quad \text{if } e_k \in \mathfrak{m}^{\mathbb{C}}, \\ m_k - 1 &:= \frac{\dim M_k - 3}{2} \quad \text{if } e_k \in \mathfrak{h}^{\mathbb{C}}. \end{aligned} \tag{3.11}$$

Note that m_k is an exponent of G whenever $e_k \in \mathfrak{m}^{\mathbb{C}}$. We have that, if the eigenvalue $m_k - 1 = 0$ happens with multiplicity n then if $\mathfrak{g}$ is quasi-split, $n = \dim \mathfrak{z}(\mathfrak{g})$ (cf. Definition 1.3.8). As a corollary we have:

Corollary 3.2.1. *Let $i : S \hookrightarrow G$ be a three dimensional subgroup corresponding to a normal principal TDS $\mathfrak{s} \subset \mathfrak{g}$. Then, if G is quasi split, i is irreducible into the component of the identity G_0 (namely, $Z_{G_0}(S) = Z(G_0)$).*

3.2.2 SL$(2, \mathbb{R})$-Higgs pairs

Our base case is $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$, so in particular it is a Hermitian real form.

Following [45], fix a line bundle $L \to X$ of positive even degree, and consider

$$L^{1/2} \oplus L^{-1/2}, \qquad \phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H^0(X, \mathrm{Hom}(L^{1/2}, L^{-1/2} \otimes L)). \tag{3.12}$$

Note that $(L^{1/2} \oplus L^{-1/2}, \phi)$ is an L-twisted $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle (cf. Example 2.3.12). Furthermore, if $\deg L \neq 0$, the Higgs pair is α-stable for all $\alpha \in \mathfrak{z}(\mathfrak{h}_0) \cong i\mathbb{R}$ satisfying that $i\alpha \leq d_L/2$.

From now on we will assume that

$$d_L > 0, \qquad i\alpha \leq d_L/2. \tag{3.13}$$

Note that for $\alpha = 0$, the hypothesis $i\alpha \leq d_L/2$ is trivially met, whence the conditions do not vary from non-hermitian to hermitian forms.

Proposition 3.2.2. *Given $L \to X$ of strictly positive even degree d_L, then:*

1. *There exists a Hitchin–Kostant–Rallis section for the Hitchin map*

$$\mathcal{M}_L^\alpha(SL(2,\mathbb{R})) \to H^0(X, L^2)$$

   if and only if $i\alpha \in \mathbb{R} = \mathfrak{z}(\mathfrak{so}(2))$ satisfies that

$$i\alpha \leq d_L/2.$$

   We also call this section HKR section for short.

2. *In the above ciscumstances, the section is defined by*

$$H^0(X, L^2) \ni \omega \mapsto L^{1/2} \oplus L^{-1/2}, \phi_\omega := \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}.$$

Remark 3.2.3. *We understand by a HKR section one built using the results of Kostant–Rallis [51]. So the if and only if in the above propostion does not mean there cannot be other different sections to the Hitchin map.*

Proof. Both statements follow from the considerations preceeding this proposition as well as the usual construction of the Hitchin section (cf.[45]). Note that it is enough to check the statement for $\alpha = id_L/2$, as for $i\beta \leq i\alpha$, $\mathcal{M}^\alpha(\mathrm{SL}(2,\mathbb{R})) \subseteq \mathcal{M}^\beta(\mathrm{SL}(2,\mathbb{R}))$. □

3.2.3 Definition of the basic $SL(2,\mathbb{R})$-Higgs pair

Let V be the principal $\mathbb{C}^\times$-bundle of frames of $L^{1/2}$, and note its structure group is $\mathrm{SO}(2,\mathbb{C})\cong\mathbb{C}^\times$. By the preceeding discussion, we can consider the associated bundle

$$E = V\times_\rho H^{\mathbb{C}} \tag{3.14}$$

and Higgs field

$$d\rho(\phi)\in H^0(X, E(\mathfrak{m}^{\mathbb{C}})\otimes L) \tag{3.15}$$

where ϕ is as in (3.12).

Since E is a $\mathbb{C}^\times$-bundle, the structure of $E(\mathfrak{m}^{\mathbb{C}})\otimes L$ is determined by the action of $\mathrm{ad}(x)$. Furthermore, Proposition 1.3.10 implies that e is a principal nilpotent element of $\mathfrak{m}$.

Note that $V(\mathfrak{s}\mathfrak{y}\mathfrak{m}_0(2,\mathbb{C}))\cong L\oplus L^{-1}\cong E\times_\iota(V_1\cap\mathfrak{m}^{\mathbb{C}})$ where we identify $L=\mathrm{Hom}(L^{-1/2},L^{1/2})$. In particular f can be seen as a section of $L^{-1}\otimes L=\mathrm{Hom}(L^{1/2},L^{-1/2})\otimes L$. It follows that

$$E(V_k\cap\mathfrak{m}^{\mathbb{C}})\cong\begin{cases}\oplus_{i=0}^{m_k}L^{m_k-2i} & \text{if } e_k\in\mathfrak{m}^{\mathbb{C}}\\ \oplus_{i=0}^{m_k-1-2i}L^i & \text{if } e_k\in\mathfrak{h}^{\mathbb{C}}\end{cases}. \tag{3.16}$$

In particular, the image of ϕ inside $E\times_\iota\mathfrak{m}^{\mathbb{C}}\otimes L$, say Φ, is the element $f\in\mathfrak{m}^{\mathbb{C}}$ considered as a section of

$${\mathfrak{m}^{\mathbb{C}}}_{-1}\otimes L^{-1}\otimes L\overset{(3.16)}{\subset}E(\mathfrak{m}^{\mathbb{C}})\otimes L.$$

Definition 3.2.4. We call the pair (E,Φ) the basic Higgs pair.

3.2.4 Groups of non-Hermitian type

Let us focus on non-hermitian real forms. As we already pointed out at the beginning of this section, it is enough to consider (G,H,θ,B) a split form of its complexification.

Lemma 3.2.5. *Let (E,Φ) be as in (3.14),(3.15). Then $(E,\Phi)\in\mathcal{M}_L^0(G)$.*

Proof. By θ-equivariance of the morphism $\mathrm{SL}(2,\mathbb{C})\to G^{\mathbb{C}}$ we obtain an $H^{\mathbb{C}}$-principal bundle and a Higgs field taking values in $\mathfrak{m}^{\mathbb{C}}$. Corollary 2.3.15 gives the rest. □

We have even more.

Proposition 3.2.6. *Let G be a quasi-split real form. Then the pair (E,Φ) is stable and simple.*

Proof. (E, Φ) is the Higgs pair associated to (V, ϕ) via the σ and θ-equivariant irreducible morphism ρ. We know by Theorem 2.3.5 that (V, ϕ) yields a solution to α-Hitchin's equations (2.1) for $\mathrm{SL}(2, \mathbb{R})$. Let (A, ϕ) be the connection and field corresponding to (V, ϕ) (see the discussion following Theorem 2.3.5). Then, to (E, ϕ) correspond $(A', \phi') = (d\rho(A), d\rho\phi)$, which is a well defined solution to Hitchin's equations 2.1 for G by σ-equivariance of ρ and (*). By Proposition 2.5.10, we need to check that (A, ϕ) is an irreducible solution (cf. Definition 2.5.7).

Locally, write

$$A = d + M_A$$

where $M_A \in H^1(X, \mathbb{C})$. Then M_A is generically non zero, as otherwise L would be flat, which by assumption is not the case (cf. 3.13). Now, an automorphism of (A', ϕ') will generically take values on $Z(H) \cap \mathrm{Ker}\iota$. Indeed, on each generic point $x \in X$, $g_x \in \mathcal{H}_x$ must satisfy that

$$\mathrm{Ad}_{g_x} d\rho(M_A) = \mathrm{Ad}_{g_x} M'_{A,x} = M'_{A,x} = d\rho(M_A)$$

and

$$\mathrm{Ad}_{g_x} \phi_x = \phi_x.$$

That is, g_x must centralise $d\rho(\mathfrak{so}(2))$ and $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{s}^{\mathbb{C}}$. In particular, g_x centralises $\rho(\mathrm{SO}(2)) = e^{d\rho\mathfrak{so}(2)}$ and $\mathfrak{m}^{\mathbb{C}}$. Since the three dimensional subgroup

$$S^{\mathbb{C}} = \rho(\mathrm{SL}(2, \mathbb{C})) = \rho(\mathrm{SU}(2)) e^{d\rho(\mathfrak{m}_{\mathfrak{sl}} + i\mathfrak{so}(2))},$$

we have that $g_x \in Z_H(S^{\mathbb{C}}) = H \cap Z_{G^{\mathbb{C}}}(S^{\mathbb{C}}) = H \cap Z(G^{\mathbb{C}}) \subseteq Z(H) \cap \mathrm{Ker}\iota$. Now, by closedness of $Z(H) \cap \mathrm{Ker}\iota$ inside of H, it follows that $g_x \in Z(H) \cap \mathrm{Ker}\iota$ for arbitrary $x \in X$. Thus

$$\mathrm{Aut}(A', \phi') \subseteq H^0(X, Z(H) \cap \mathrm{Ker}\iota) \subseteq \mathrm{Aut}(A', \phi').$$

Hence we have equality and so (A', ϕ') is irreducible. Equivalently, the pair (E, ϕ) is stable and simple. □

Proposition 3.2.7.

$$\mathbb{H}^2(C^\bullet(E, \Phi)) = \mathfrak{z}(\mathfrak{h}) \cap \mathit{Ker}\ d\iota$$

Proof. By Proposition 2.5.5, it is enought to check that the associated $G^{\mathbb{C}}$-Higgs pair (E', ϕ') is stable. This is done as in the previous proposition, since $\mathrm{SL}(2, \mathbb{C}) \to G^{\mathbb{C}}$ is irreducible. □

Corollary 3.2.8. (E, Φ) *is a smooth point of* $\mathcal{M}^0_L(G)$.

Given the above corollary, the deformation argument used by Hitchin in [45] adapts: for each $\overline{\gamma} \in \oplus_{i=1}^a H^0(X, L^{m_i})$, take the field

$$\Phi_{\overline{\gamma}} = f + \sum_{i=1}^{a} \gamma_i e_i,$$

where $e_i, i = 1, \ldots, a$ generate $\mathfrak{z}_{\mathfrak{m}^{\mathbb{C}}}(e)$ and $e_1 = e$. Note that e_i can be considered as a section of the bundle $E \times_H \mathfrak{m}^{\mathbb{C}}$ simply by taking

$$e_i \in \mathfrak{m}^{\mathbb{C}}{}_i = H^0(X, \mathfrak{m}^{\mathbb{C}}{}_{m_i} \otimes L^{-m_i})$$

so that

$$\gamma_i e_i \in H^0(X, \mathfrak{m}^{\mathbb{C}}{}_{m_i} \otimes L^{m_i}).$$

So we get a section

$$\Phi_\gamma \overset{\mathrm{Ad}(\exp(x))}{\mapsto} \Psi_\gamma = f + \gamma_1 e_1 + c \cdots + \gamma_a e_a.$$

By openness of $\mathcal{M}_L^{d\rho(\alpha)}(G)^{smooth}$, if $\overline{\gamma} \in H^0(X, \oplus_i K^{d_i})$ is such that $|\gamma_i|$ are small enough, we have

$$(E, \Phi_{\overline{\gamma}}) \in \mathcal{M}_L^{d\rho(\alpha)}(G)^{smooth}.$$

Namely, the basic solution (E, Φ) can be deformed to a section from an open neighbourhood of $0 \in \mathcal{A}_{G,L}$ into $\mathcal{M}_L^{d\rho(\alpha)}(G)^{smooth}$.

Then, using the $\mathbb{C}^\times$ action on $\mathfrak{m}^{\mathbb{C}}$ and positivity of the eigenvalues for e_k under the action of x, we can deform it to the whole of the base hitting only smooth points.

Let us retake the case of a general non-hermitian form case:

Theorem 3.2.9. *Under the hypothesis (3.13), for any non-hermitian strongly reductive real form (G, H, θ, B), there exists a section s of the map*

$$h_L : \mathcal{M}_L^0(G) \to \oplus_{i=1}^a H^0(X, L^{d_i}).$$

Furthermore, the image of s is contained in the smooth locus of $\mathcal{M}_L^0(G)$ and factors through $\mathcal{M}_L^0(\widetilde{G})^{smooth}$, where $\widetilde{G}$ is the connected maximal split subgroup of G.

Proof. By Corollary 2.3.15 and Theorem 1.3.9, the Hitchin–Kostant–Rallis section for the split subgroup induces a section for G. Note that for the section to be a section of the Hitchin map we can omit the hypothesis if strong reductivity from the cited theorem. The difference is that for non-strongly reductive forms the section will not be transversal.

All that is left to prove is that the image $\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \to \mathcal{M}(\widetilde{G}) \to \mathcal{M}(G)$ lands in $\mathcal{M}(G)^{smooth}$. This is so because by construction, $\mathrm{SL}(2,\mathbb{R}) \to \widetilde{G} \to G$ is irreducible, so the same arguments as in Proposition 3.2.6 and Corollary 3.2.8 apply. □

3.2.5 Arbitrary real forms

The extra parameter α in this case only makes the discussion heavier, without affecting the essetial ideas of the results in the previous section. We have indeed the following.

Theorem 3.2.10. *Under the hypothesis (??), assuming $d\rho(\alpha) \in i\mathfrak{z}(\mathfrak{h})$, there exists a section s of the map*

$$h_L : \mathcal{M}_L^{d\rho(\alpha)}(G) \to \mathcal{A}_L(G)$$

for G a strongly reductive Lie group. This section takes values in the smooth locus of the moduli space.

Moreover, the section factors through $\mathcal{M}_L^{d\rho(\alpha)}(\widetilde{G})^{smooth}$, where $\widetilde{G}$ is the connected maximal split subgroup of G, if and only if $d\rho(\alpha) \in i\mathfrak{z}(\widetilde{\mathfrak{h}})$.

Remark 3.2.11. *Note that in the special case $\alpha = 0$, we recover Theorem 3.2.9 as a corollary to Theorem 3.2.10.*

Corollary 3.2.12. *With the notation of the above theorem, suppose that $\alpha = 0$; then, there exists a section of the map*

$$h_L : \mathcal{M}_L^0(G) \to \oplus_{i=1}^a H^0(X, L^{d_i})$$

that factors through $\mathcal{M}_L^0(\widetilde{G})$.

Proof. When $\alpha = 0$, conditions (3.13) are trivially met, so the result follows. □

Hitchin [45] proved that the image of the Hitchin section is a connected component of $\mathcal{M}_K(G_{split})$. For more general real forms, this fails to be true, as dimensions of the base and of the moduli space fail to coincide.

Proposition 3.2.13. *Let $G \leq G^{\mathbb{C}}$ be a quasi-split real form of a simple Lie group. Then the HKR section covers a connected component of the moduli space of (K-twisted) Higgs bundles if and only if G is the split real form.*

Proof. The if direction is in Hitchin's [45]. As for the converse, if $G \leq G^{\mathbb{C}}$ is non split, it contains a maximal split real form of strictly lower dimension. Since there is a smooth point in the image of the section, the dimension of the connected component it falls into has the expected dimension. Hence, it cannot cover a component. □

3.2.6 Some remarks

Classical moduli When $L = K$, $\alpha = 0$, the proof simplifies, as we have a correspondence between the moduli space of G-Higgs pairs and that of representations $\rho : \pi_1(X) \to G$. In particular, smoothness of the basic Higgs pair is immediate from the irreducibility of $\mathrm{SL}(2,\mathbb{C}) \to G^{\mathbb{C}}$. This is the argument used by Hitchin [45] to prove smoothness of the section.

The involutory argument. An alternative argument is to observe that normal TDS's exist, and they are contained in some maximal split subalgebra. So when $\alpha = 0$, we can construct a particular section to $\mathcal{M}_L(\widetilde{G}^{\mathbb{C}})$, which will be fixed by the involution and map to $\mathcal{M}_L(G)$. These arguments underly Hitchin's proof (cf. Proposition 6.1 in [45]). In order to work with arbitrary real forms, however, the intrinsic approach is more appropriate.

Hitchin's section In the particular case $L = K$, $\alpha = 0$, $(G^{\mathbb{C}})_{\mathbb{R}}$ is the real group underlying a complex one, the Hitchin–Kostant–Rallis section gives the factorization of Hitchin's section through $\mathcal{M}(G^{\mathbb{C}}_{split})$.

3.3 Topological type of the HKR section

In this section we calculate the topological invariant for the HKR section for two cases: $\mathrm{SU}(2,1)$-Higgs pairs, and $\mathrm{SU}(p,p)$-Higgs pairs. We find that for the non-tube type $\mathrm{SU}(2,1)$, the image fails to fill a connected component.

In the case when (G, H, θ, B) is semisimple, Lemma 1.2.32 allows us to deduce topological informacion of the component in which the Hitchin-Kostan-Rallis section lies from the corresponding information for $\widetilde{G}^{\mathbb{C}}$-Higgs pairs.

Let us examine some examples of the map $\mathcal{M}(\widetilde{G}^{\mathbb{C}}, \widetilde{\theta}) \to \mathcal{M}(G^{\mathbb{C}}, \theta)$:

Example 3.3.1. An $\mathrm{SU}(p,q)$-Higgs bundle is given by the following piece of data: (V, W, β, γ) where V is a rank p vector bundle, W is a rank q vector bundle, $detV \otimes detW = \mathcal{O}$,

$$\beta : W \to V \otimes K \qquad \gamma : W \to V \otimes K$$

The Toledo invariant is given by

$$\tau_{\mathrm{SU}(p,q)} = 2\frac{p \deg V - q \deg W}{p+q} \overset{\deg V = -\deg W}{=} 2\deg V$$

and bounded by

$$|\tau_{\mathrm{SU}(p,q)}| \leq 2q(g-1).$$

Namely, the Toledo invariant is determined by he degree of V, which is bounded by $\pm p(g-1)$. See [13] for details.

Case 1: $p > q$. Recall that a $\mathrm{SO}(q+1,q)$-Higgs bundle tuple $(V, W(\beta,\gamma))$

1. V is an $\mathrm{SO}(q+1,\mathbb{C})$-bundle and W is an $\mathrm{SO}(q,\mathbb{C})$-bundle
2. $\beta \in H^0(X, W^* \otimes V \otimes K)$
3. $-{}^t\beta = \gamma \in H^0(X, V^* \otimes W \otimes K)$

Let us neglect the Higgs field, which will be taken care of in next section.

In [1], Aparicio-Arroyo calculates the principal bundle corresponding to the Hitchin section for this real form to be

$$K^{-q} \oplus K^{-q+1} \oplus \cdots \oplus K^{-1} \oplus \mathcal{O} \oplus K \oplus \cdots \oplus K^{q-1} \oplus K^q$$

which has topological class 0 (as the W piece is $\oplus_{j=1}^{[q/2]} K^{2j+1} \oplus K^{-(2j+1)}$).

$q = 1$. By Example 1.2.35, $\mathrm{Spin}(2,1)_0 \cong \mathrm{SL}(2,\mathbb{R})$ is the maximal split form. Its fundamental group is $\mathbb{Z}$. Note that a $\mathrm{Spin}(2,1)_0$-Higgs bundle is described by an $\mathrm{SO}(2,1)_0$-Higgs bundle with vanishing Stiefel-Whitney class. In particular, the corresponding principal $H^{\mathbb{C}}$-bundle is an $\mathrm{SO}(2,\mathbb{C})$ bundle with even degree, since the map $\mathcal{M}(\mathrm{SL}(2,\mathbb{R})) \to \mathcal{M}(\mathrm{SO}(2,1)_0)$ induces the square map on the level of $\mathrm{SO}(2,\mathbb{C})$ bundles, it sends $K^{-1/2} \oplus K^{1/2}$ to $K^{-1} \oplus K$. Now, the embedding $\mathrm{SO}(2,1)_0 \to \mathrm{SU}(2,1)$ is induced by the embedding specified in Fact 3.4.3.

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives indeed the right principal bundle of the HKR section.

$q = 2$. In this case $\pi_1(\mathrm{SO}(3,2)_0) = \mathbb{Z} \times \mathbb{Z}_2$, the $\mathbb{Z}_2$ factor corresponding to the Stiefel-Whitney class. So in order to obtain an embedding of fundamental groups, we must consider $\mathrm{Spin}(3,2)_0/\mathbb{Z}_2$-Higgs pairs. These map to $\mathrm{SO}(3,2)_0$ pairs with 0 Stiefel-Whitney class.

Indeed, the map $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,2)$ factors through $\mathrm{SO}(2,1)_0$, thus annihilating the Stiefel-Whitney class by squaring the $\mathbb{C}^\times$ bundle $L^{1/2} \oplus L^{-1/2}$. Finally, we compose with the map $\mathrm{SO}(3,2)_0 \to \mathrm{SU}(3,2)$ given by

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}.$$

$q \geq 3$**.** In this case, the structure group is $\mathrm{Spin}(q, q+1)$.

Case 2: $p = q$**.** By 1.2.35, we have that $\mathrm{Sp}(2p,\mathbb{R}) < \mathrm{SU}(n,n)$ is the maximal split subgroup. Recall that a $\mathrm{Sp}(2p,\mathbb{R})$-Higgs bundle is given by a pair $(E,(\beta,\gamma))$ where

1. E is a $\mathrm{SL}(p,\mathbb{C})$ bundle
2. $\beta \in H^0(X, S^2E \otimes K)$
3. $\gamma \in H^0(X, S^2E^* \otimes K)$

So to know the topological type of the HKR section for $\mathrm{SU}(p,p)$ it is enough to compute Hitchin section's topological type for $\mathrm{Sp}(2p,\mathbb{R})$. Following Kostant and Rallis ([51]), we compute a principal S-triple using the results and notation specified in the proof of Theorem 1.3.9, together with Section B.2. We easily compute that

$$w = \sum_i (2(p-i)+1)(E_{i,p+i} + E_{p+i,i}).$$

We know $w = \sum_{i \leq p-1} c_i(E_{i,p+i} - E_{i+1,p+i+1} + E_{p+i,i} - E_{p+i+1,i+1}) + c_p(E_{p,2p} + E_{2p,p})$, from which we deduce $c_j = j(2p-j)$. In particular, we choose e_c

$$e_c = i \sum_{i=1}^{p-1} \sqrt{-\frac{c_j}{b_j}} y_j,$$

where the y_j are as in Section B.2 and we recall that b_j is defined to be $[y_j, \theta y_j] = b_j h_j$. We have $b_j = -4$ for all j. Namely

$$e_c = \frac{i}{2} \sum_{i=1}^{p-1} \sqrt{j(2p-j)}(E_{j,j+1} - E_{j+1,j} + E_{p+j,p+j+1} - E_{p+j+1,p+j}) +$$

$$+ \frac{i}{2} \sum_{i=1}^{p-1} \sqrt{j(2p-j)}(E_{j,p+j+1} + E_{j+1,p+j} + E_{p+j,j+1} + E_{p+j+1,j}) +$$

$$+\frac{p}{2}(-E_{p,p}+E_{2p,2p}+E_{p,2p}-E_{2p,p}),$$

and so

$$f_c=\theta e_c=\frac{i}{2}\sum_{i=1}^{p-1}\sqrt{j(2p-j)}(E_{j,j+1}-E_{j+1,j}+E_{p+j,p+j+1}-E_{p+j+1,p+j})+$$

$$-\frac{i}{2}\sum_{i=1}^{p-1}\sqrt{j(2p-j)}(E_{j,p+j+1}-E_{j+1,p+j}-E_{p+j,j+1}-E_{p+j+1,j})+$$

$$+\frac{p}{2}(-E_{p,p}+E_{2p,2p}-E_{p,2p}+E_{2p,p}).$$

The TDS $\mathfrak{s}$ generated by the above elements is normal by the proof of Theorem 1.3.9, and in particular, it contains principal nilpotent generators $e,f\in\mathfrak{m}^{\mathbb{C}}$. Let:

$$e=\frac{-e_c+f_c+w}{2},\qquad f=\frac{e_c-f_c+w}{2}.$$

They belong to $\mathfrak{m}^{\mathbb{C}}$, and furthermore $x:=[e,f]=e_c+f_c\in\mathfrak{h}^{\mathbb{C}}$ is semisimple. By Proposition 1.3.12, it is enough to check that $e,\ f$ are nilpotent to deduce that e,f,x is a normal principal triple genetaring $\mathfrak{s}$. We compute:

$$[e,\cdot]\begin{cases} f\mapsto x\\ x\mapsto -e\end{cases}$$

which implies nilpotency. Similarly for f. Readjusting the constants so $[x,f]=-2f,\quad [x,e]=2e,\quad [e,f]=x$ we set:

$$x=2(e_c+f_c),\qquad e=\frac{\sqrt{2}}{2}(-e_c+f_c+w),\qquad f=\frac{\sqrt{2}}{2}(e_c-f_c+w).\tag{3.17}$$

Now, by definition x has off diagonal blocks equally zero. The diagonal blocks are matrices whose eigenvalues are $\epsilon_j\cdot(2(p-j)+1)$ for the upper diagonal block, where $\epsilon_j^2=1$ and $\epsilon_j\epsilon_{j+1}=-1$, and $-\epsilon_j\cdot(2(p-j)+1)$ for the lower diagonal block:

$$x=\begin{pmatrix}
2p-1 & 0 & \dots & 0 & 0 & \dots & \dots & 0\\
0 & -2p+3 & & \vdots & \vdots & \ddots & & \vdots\\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots\\
0 & \dots & \dots & (-1)^{p+1} & 0 & \dots & \dots & 0\\
0 & \dots & \dots & 0 & -2p+1 & 0 & \dots & \\
\vdots & \ddots & & \vdots & 0 & 2p-3 & & \vdots\\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots\\
0 & \dots & \dots & 0 & 0 & \dots & \dots & (-1)^p
\end{pmatrix}$$

It follows that the corresponding principal bundle is $E\oplus E^*$ with

$$E=\oplus_{i=0}^{p-1}K^{(-1)^i(2i+1)/2}$$

and so the topological type is $p(g-1)$.

3.4 Examples

3.4.1 $\mathbf{SU}(2,1)$

Lie theoretical facts $\mathrm{SU}(2,1)$ is the subgroup of $\mathrm{SL}(3,\mathbb{C})$ defined as the subgroup of fixed points of the involution

$$\sigma(X) = \mathrm{Ad}(J_{1,2})\ {}^t\overline{X}^{-1}$$

where

$$J_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So we get a decomposition

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{s}(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C})) \oplus \mathfrak{m}^{\mathbb{C}}$$

where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C})\ \ |b \in\ Mat_{2\times 1}(\mathbb{C}),\ C \in\ Mat_{1\times 2}(\mathbb{C}) \right\}$$

One calculates easily that

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C})\ \ |a \in \mathbb{C} \right\}$$

is the maximal anisotropic Cartan subalgebra.

In this case H is realised as the subgroup of matrices of $\mathrm{SL}(3,\mathbb{C})$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}$$

Fact 3.4.1. Nilpotent elements in $\mathfrak{m}^{\mathbb{C}}$ are elements of the form

$$\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ w & z & 0 \end{pmatrix}$$

satisfying $uw + vz = 0$

Fact 3.4.2.

$$\mathfrak{m}_{reg} = \left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ w & z & 0 \end{pmatrix}\ \mid uw \neq 0\ or\ uz \neq 0\ or\ vw \neq 0\ or\ vz \neq 0 \right\}$$

Fact 3.4.3. The map $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{su}(2,1)$ defined by choosing generators $x \in \mathfrak{so}(2,\mathbb{C})$, e,f $\in$ $\mathfrak{sym}_0(2,\mathbb{C})$ generating a three dimensional subalgebra satisfying $[x,e] = e$, $[x,f] = -f$,$[e,f] = x$; say

$$x := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: x',$$

$$e := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} =: e',$$

$$f := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} =: f'$$

defines a θ-equivariant morphism $\mathfrak{sl}(2,\mathbb{C})$ into a θ and σ invariant TDS $\mathfrak{s}^{\mathbb{C}} \cong \mathfrak{so}(2,1) \subseteq \mathfrak{su}(2,1)$.

Fact 3.4.4. The connected maximal split subgroup of $\mathrm{SU}(2,1)$ is

$$\mathrm{Spin}(2,1)_0 \cong \mathrm{SL}(2,\mathbb{R}).$$

The section An $\mathrm{SU}(2,1)$-Higgs pair is a pair $(W \oplus V, \phi)$ consisting of a rank 2 bundle W and a line bundle V such that $\det(V \oplus W) = \mathcal{O}_X$, together with a Higgs field

$$\phi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{pmatrix}$$

Since $\mathfrak{a} \otimes L \cong L$, $\mathfrak{a} \otimes L/\mathbb{Z}_2 \cong L^2$, the Chevalley map induces the square map

$$p : L \to L^2$$

Hence, the Hitchin map is:

$$\begin{array}{rcl} h_{\mathbb{R}} : \mathcal{M}_L^{\alpha}(\mathrm{SU}(2,1)) & \to & H^2(X, L^2) \\ (E,\phi) & \mapsto & bd + ad. \end{array}$$

Now, consider the $\mathrm{SL}(2,\mathbb{R})$-bundle $E = L^{1/2}$ and let V be the frame bundle of E.

In this particular case, by Fact 3.4.4, there is no harm in building the section first to $\mathcal{M}_L^{\alpha}(\mathrm{SO}(2,1)_0)$ and then composing with the morphism $\mathfrak{so}(2,1) \to \mathfrak{su}(2,1)$, since $\mathfrak{so}(2,1) = \mathfrak{sl}(2,\mathbb{R})$. Now, Facts 3.4.4 and 3.4.3 imply that

$$\mathcal{M}_L^{\alpha}(\mathrm{SL}(2,\mathbb{R})) \to \mathcal{M}_L^{\alpha}(\mathrm{SO}(2,1)_0)$$

sends

$$E \mapsto E^2 \oplus E^{-2} \oplus \mathcal{O},$$

as $0 \to \mathbb{Z}_2 \to \mathrm{SL}(2,\mathbb{R}) \to P\mathrm{SL}(2,\mathbb{R}) \to 0$ induces the square map on line bundles.

As for the Higgs field, for $\alpha \leq d_L$, we can directly consider the Hitchin–Kostant–Rallis section for $\mathrm{SL}(2,\mathbb{R})$ (cf. Section 3.2)

$$\begin{array}{rcl} s : H^0(X, L^2) & \to & \mathcal{M}_L^\alpha(\mathrm{SL}(2,\mathbb{R})) \\ \omega & \mapsto & \left[L^{1/2}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \right] \end{array}$$

Now, we have that $\alpha \mapsto x \in i\mathfrak{z}(\mathfrak{u}(2)) = \mathrm{diag}(\mathbb{R})$ via the morphism specified in Fact 3.4.3 if and only if $\alpha = 0$. So since $\mathcal{M}_L^\beta(SO(2,1)) \subseteq \mathcal{M}_L^\gamma(SO(2,1))$ if and only if $i\beta \geq i\gamma$, we have that, for any $i\alpha \geq 0$,

$$\mathcal{M}_L^\alpha \mathrm{SO}(2,1)_0 \hookrightarrow \mathcal{M}_L^0 \mathrm{SO}(2,1)_0 \to \mathcal{M}_L^0 \mathrm{SU}(2,1).$$

Whence we have the following.

Proposition 3.4.5. *1. The Hitchin–Kostant–Rallis section for $\mathcal{M}_L^\beta(SU(2,1))$, $\beta \in \mathfrak{z}(\mathfrak{su}(2)) = i\mathbb{R}$ exists if and only if $i\beta \leq 0$.*

If so, it can be explicitely written as

$$\begin{array}{rcl} s : H^0(X, L^2) & \to & \mathcal{M}_L^0 SU(2,1) \\ \omega & \mapsto & \left[L \oplus L^{-1} \oplus \mathcal{O}, \begin{pmatrix} 0 & 0 & \omega \\ 0 & 0 & 1 \\ 1 & \omega & 0 \end{pmatrix} \right], \end{array}$$

and its topological type is $\tau = 0$.

2. *The image of the HKR section is contained in the strictly stable locus.*

3. *The HKR section factors through $\mathcal{M}_L^\alpha(SO(2,1))$ for any $i\alpha \leq 0$.*

Proof. Everything is proved from 1. except the sufficiency of negativity of β. We already know the HKR section exists for $\beta = 0$. Moreover, we can easily prove stability, either by a direct calculation, or arguing as follows: since $\mathrm{SO}(2,\mathbb{C})$ is conjugate to $\mathbb{C}^\times = Z(\mathrm{GL}(2,\mathbb{C}))$, it follows that

$$\mathfrak{aut}(E \times_F \mathrm{GL}(2,\mathbb{C}), dF\phi) = \mathfrak{aut}(E,\phi) \times_F \mathfrak{n}_{\mathfrak{gl}(2,\mathbb{C})}(\mathbb{C}^\times) \subseteq H^0(X, E(\mathfrak{z}(\mathfrak{h}^{\mathbb{C}}))).$$

So by Proposition 2.5.6 we are done.

This means in particular that negativity of β is enough to obtain a section which will then be stable. ∎

3.4.2 $\mathbf{SU}(p,p)$

Lie theoretical facts $\mathrm{SU}(p,p)$ is the subgroup of $\mathrm{SL}(2p,\mathbb{C})$ defined as the subgroup of fixed points of the involution

$$\sigma(X) = \mathrm{Ad}(J_{p,p})\, {}^t\overline{X}^{-1}$$

where

$$J_{p,p} = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$$

where I_p is the identity matrix.

So we get a decomposition

$$\mathfrak{sl}(2p,\mathbb{C}) = \mathfrak{s}(\mathfrak{gl}(p,\mathbb{C}) \oplus \mathfrak{gl}(p,\mathbb{C})) \oplus \mathfrak{m}^{\mathbb{C}}$$

where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{sl}(2p,\mathbb{C}) \ |B, C \in \ Mat_{p\times p}(\mathbb{C}) \right\}$$

One calculates easily that

$$\mathfrak{a}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \in \mathfrak{sl}(2p,\mathbb{C}) \ |A \in M_{p\times p}(\mathbb{C}) \text{ is a diagonal matrix} \right\}$$

complexifies to the maximal anisotropic Cartan subalgebra.

In this case $H^{\mathbb{C}}$ is realised as the subgroup of matrices of $\mathrm{SL}(2p,\mathbb{C})$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with $\det B = \det A^{-1}$

Fact 3.4.6. The map $\mathfrak{sp}(2p,\mathbb{R}) \to \mathfrak{su}(p,p)$, where $\mathfrak{sp}(2p,\mathbb{R})$ is realised as in Section B.2 is the identity. Indeed, one readily checks that $\mathfrak{sp}(2n,\mathbb{R})$ is contained in $\mathfrak{su}(p,p)$ and the embedding respects regularity.

Similarly for the embedding $\mathrm{Sp}(2n,\mathbb{R}) \hookrightarrow \mathrm{SU}(p,p)$.

Fact 3.4.7. The elements

$$x = \sum_i (-1)^i (E_{i,i} - E_{p+i,p+i}),\ e = \frac{-e_c + f_c + w}{2},\ e = \frac{e_c - f_c + w}{2}$$

form a normal basis of a normal TDS. Here $e_c,\ f_c,\ w$ are as in Example 3.3.1 Case 2.

Fact 3.4.8. $\mathfrak{z}(\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(p))) = i\mathbb{R}$

The section

Proposition 3.4.9. *1. The Hitchin–Kostant–Rallis section for $\mathcal{M}_L^{\beta}(SU(2,1))$, $\beta \in \mathfrak{z}(\mathfrak{su}(2)) = i\mathbb{R}$ exists if and only if $i\beta \leq 0$.*

If so, it can be explicitely written as

$$\begin{array}{rcl} s : H^0(X, L^2) & \to & \mathcal{M}_L^0 SU(2,1) \\ \omega & \mapsto & \left[L \oplus L^{-1} \oplus \mathcal{O}, \begin{pmatrix} 0 & 0 & \omega \\ 0 & 0 & 1 \\ 1 & \omega & 0 \end{pmatrix} \right], \end{array}$$

and its topological type is $\tau = 0$.

2. *The image of the HKR section is contained in the strictly stable locus.*

3. *The HKR section factors through $\mathcal{M}_L^{\alpha}(SO(2,1))$ for any $i\alpha \leq 0$.*

Proposition 3.4.10. *There exists a section for the Hitchin map*

$$h_L : \mathcal{M}_L^{\alpha}(SU(p,p)) \to \mathcal{A}_L(SU(p,p)) = \bigoplus_{i=0}^{p-1} H^0(X, L^{2i})$$

where $\alpha \in \mathfrak{z}(\mathfrak{u}(1)) \cong i\mathbb{R}$ if and only if $i\alpha \leq 0$. Under these circumstances it factors through $\mathcal{M}_L^{\alpha}(Sp(2p,\mathbb{R}))$ and

$$s : \oplus_{i=0}^{p-1} H^0(X, L^{2i}) \to \mathcal{M}_L^{\alpha}(G)$$

sending $\oplus \omega_i$ to

$$\sum_{\substack{2\,\not|\,i,\ 2|j \\ j \geq i-1}} \omega_{\frac{j-i+1}{2}} E_{ip+j} + \delta \cdot \sum_{\substack{2\,\not|\,i,\ 2\,\not|\,j \\ p-2 \geq j+i \geq 2}} \omega_{\frac{p-(i+j)+2}{2}} E_{ip+j} + \sum_{\substack{2|i,\ 2\,\not|\,j \\ i \geq j-1}} \omega_{\frac{i-j+1}{2}} E_{ip+j} +$$

$$\sum_{\substack{2|i,\ 2\,\not|\,j \\ j \geq i-1}} \omega_{\frac{1+j-i}{2}} E_{p+i,j} + \delta \sum_{\substack{2|i,\ 2|j \\ p-2 \geq j+i \geq 2}} \omega_{\frac{p-(j+i)+2}{2}} E_{p+i,j} + \sum_{\substack{2\,\not|\,i\ 2|j \\ p \leq j \geq i-1}} \omega_{\frac{1+j-i}{2}} E_{p+i,j}$$

where $\delta = 1$ if p is even and zero otherwise. The Toledo invariant of the image is $\tau = p(g-1)$, and the section is contained in the strictly stable locus.

Proof. The fact that the basis consists of differential forms of even degree follows from the same fact for $\mathrm{Sp}(2p,\mathbb{C})$, as the latter group has split form $\mathrm{Sp}(2p,\mathbb{R})$ (see [40, 41]).

Now, recall that given V_λ a λ-eigenspace for x, letting e_k $k = 1, \ldots, a$ generate $\mathfrak{c}_{\mathfrak{m}}(e)$, we have that e_k takes V_λ to $V_{\lambda+d_k}$. Similarly, f takes V_λ to $V_{\lambda-2}$. Note that the eigenbundles for x are precisely $K^{\pm(2(p-j)+1)/2}$, with eigenvalues $\pm(2(p-j)+1)$. So instead of calculating the generators we may just as well directly compute the section by moving eigenbundles around.

So on the one hand we know that the matrices e_k are sums of multiples of basic matrices of the form $E_{i,p+j}$ or $E_p + i, j$, where $i, j \leq p$. Now, $E_{i,p+j}$ takes V_λ, where $\lambda = (-1)^j(2(p-j)+1)$ to V_μ, with $\mu = (-1)^{i+1}(2(p-i)+1)$. So in order to define a HKR section, it must be multiplied by a form of degree $\frac{\mu+1-\lambda}{2}$. A similar argument for $E_{p+i,j}$ gives the form of the section.

We know that any subbundle stable by any Higgs field in the image of the section will be fixed by f (in fact, by any of the terms in the expression of the section). But f fixes no bundles, and so we have that the section is in fact stable.

□

Chapter 4

Higgs pairs and cameral data.

The aim of this chapter is to give a description of the Hitchin fibration. The case of reductive complex Lie groups has been studied by several authors in different contexts, starting by Hitchin ([41]), who established a correspondence between the generic fiber of the Hitchin map for classical complex Lie groups and certain abelian subvarieties (Prym varieties) of the Jacobian of the so called spectral curve. This is a consequence of a deeper fact: the moduli space endowed with the Hitchin map has the structure of an algebraically completely integrable system.

This perspective is the one taken by L.P. Schaposnik to study spectral data for G-Higgs bundles, where G is a real form of a classical Lie group (cf. [66, 67]). In joint work with Hitchin [43], they show that the fibration is non-abelian for linear groups defined over the quaternions.

In a more general context, Donagi–Gaitsgory [25], Faltings ([27], Ngô ([60] and Scognamillo ([70]) have tackled the problem from different points of view, giving a spectral (cameral) construction of the fibers and an abelianization procedure.

We have already pointed out that the language of stacks is more appropriate when studying universal properties of moduli problems. Furthermore, we aim at a global description of the fibration. This problem was solved in full generality in the complex case by Donagi and Gaitsgory ([25]), whence the choice of the stacky language.

We will follow both Donagi and Gaitsgory and Ngô in our description of the Hitchin fibration for real forms of complex algebraic groups.

4.1 The stack of Higgs pairs and the Hitchin map

Let us start by reformulating Definition 2.1.1.

All sites considered in this section will be analytic sites over schemes.

Definition 4.1.1. Given a complex scheme X, the small analytic site on X, denoted by $(X)_{an}$, to be the category whose objects are open embeddings $U \hookrightarrow X(\mathbb{C})$ for the usual complex topology, and whose morphisms are open embeddings over X, on which we distingush collections of open embeddings to be open covers if they are jointly surjective (cf. Appendix A for definitions).

We can similarly define the big analytic site $(X)_{AN}$ on X. We consider the full category Sch/X, with the same open subsets and coverings.

We can consider the following transformation stack on $(X)_{an/AN}$ (see Appendix A):

Definition 4.1.2. $\mathcal{H}iggs_L(G) := \left[\mathfrak{m}^{\mathbb{C}} \otimes L/H^{\mathbb{C}}\right]$ is the stack that to each open $U \hookrightarrow X$ associates the category of G-Higgs pairs over U.

We have, the following definition, equivalent to Definition 2.1.1.

Definition 4.1.3. An L-twisted G-Higgs pair on X is a morphism

$$[P, \phi] : X \to \mathcal{H}iggs_L(G).$$

Consider now the Chevalley morphism

$$\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}}). \tag{4.1}$$

As we argued in Section 3.1, this morphism is $\mathbb{C}^{\times}$-equivariant. This implies that for any complex scheme U and a line bundle $L \to U$ on it, the map (4.1) induces a morphism

$$\chi_L : \mathfrak{m}^{\mathbb{C}} \otimes L \to \mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a}^{\mathbb{C}}),$$

which in turn, due to $H^{\mathbb{C}}$-equivariance, yields a morphism of stacks over $(U)_{an}$

$$[\chi]_L : \left[\mathfrak{m}^{\mathbb{C}} \otimes L/H^{\mathbb{C}}\right] \to \mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a}^{\mathbb{C}}).$$

Definition 4.1.4. The map

$$[\chi]_L : \left[\mathfrak{m}^{\mathbb{C}} \otimes L/H^{\mathbb{C}}\right] \to \mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a}^{\mathbb{C}}) \tag{4.2}$$

is called the (L-twisted) Hitchin map. The scheme $(\mathfrak{a}^{\mathbb{C}} \otimes L)//W(\mathfrak{a}^{\mathbb{C}})$ is called the Hitchin base and it will be denoted by $\mathcal{A}_L$ or $\mathcal{A}_{G,L}$ when necessary.

Note that this map can be defined in more generality. Following Ngô (cf. [60]), on the big analytic site over Spec $\mathbb{C}$, denoted by $(pt)_{AN}$, we can consider the stack

$$\left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right].$$

The same arguments as before imply that the Chevalley morphism induces

$$\widetilde{\chi} : \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] \to \left[\left(\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})\right/\mathbb{C}^{\times}\right]. \tag{4.3}$$

Furthermore, by mapping each of the above stacks to $B\mathbb{C}^{\times}$ via the respective forgetful morphisms, we obtain a commutative diagram:

$$\begin{array}{ccc} \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] & \longrightarrow & \left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/\mathbb{C}^{\times}\right] \\ & \searrow \quad \swarrow & \\ & B\mathbb{C}^{\times} & \end{array}$$

where all the stacks above are seen as sheaves over the analytic site of complex schemes $(pt)_{AN}$.

Fixing a holomorphic line bundle on X is considering a map $[L] : X \to B\mathbb{C}^{\times}$; so one recovers $[\chi]_L$ by looking at the restriction of $\widetilde{\chi}$ to $X \xrightarrow{[L]} B\mathbb{C}^{\times}$.

Lemma 4.1.5. *There are Cartesian diagrams*

$$\begin{array}{ccc} \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}}\right] & \longrightarrow & \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) \\ \downarrow & & \downarrow \\ \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] & \longrightarrow & \left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/\mathbb{C}^{\times}\right], \end{array} \tag{4.4}$$

$$\begin{array}{ccc} \left[\mathfrak{m}^{\mathbb{C}} \otimes L/H^{\mathbb{C}}\right] & \longrightarrow & \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] \\ \downarrow & & \downarrow \\ X & \xrightarrow[{[L]}]{} & B\mathbb{C}^{\times}, \end{array} \tag{4.5}$$

$$\begin{array}{ccc} \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}}) & \longrightarrow & \left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/\mathbb{C}^{\times}\right] \\ \downarrow & & \downarrow \\ X & \xrightarrow[{[L]}]{} & B\mathbb{C}^{\times}. \end{array} \tag{4.6}$$

Proof. By definition. □

In particular, the description of the **abstract Hitchin map**

$$[\chi] : \mathcal{H}iggs^{abs}(G) \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}}) \tag{4.7}$$

yields a description of (4.3). Indeed, the vertical arrows in (4.4) are surjective (as $\mathfrak{a}^{\mathbb{C}} \to \left[(\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}))/\mathbb{C}^{\times}\right]$ is an atlas). On the other hand, the stack of L-twisted G-Higgs pairs on X is a fibered product of $\left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right]$ by $[L]$ by (4.5), and similarly for the twisted base by (4.6). Thus we can certainly deduce information of the twisted Hitchin map by studying $[\chi]$ and $\widetilde{\chi}$. In fact, we will see that the gerby nature of the map allows to fully describe the twisted fibration from both untwisted ones.

Before we proceed to the study of the Hitchin map, let us summarise some preliminary results.

4.2 Reminder of the complex group case.

In this section we briefly review Donagi–Gaitsgory's results from [25], following Ngô's formulation from [60].

Let $G^{\mathbb{C}}$ be a complex reductive algebraic group (the complexification of a compact algebraic group). Let $\mathfrak{g}^{\mathbb{C}}$ be its Lie algebra, and denote by $\mathfrak{g}_{reg}$ the subset of regular elements of the Lie algebra. Fix $\mathfrak{d}^{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, $D^{\mathbb{C}} \leq G^{\mathbb{C}}$ the corresponding maximal torus, and W the Weyl group.

Definition 4.2.1. Given X a complex scheme, and $L \to X$ a line bundle on it, we define the stack of L-twisted $G^{\mathbb{C}}$-Higgs pairs to be

$$\mathcal{H}iggs_L(G^{\mathbb{C}}) = \left[\mathfrak{g}^{\mathbb{C}} \otimes L/G^{\mathbb{C}}\right].$$

We define the substack of regular Higgs pairs $\mathcal{H}iggs_L(G^{\mathbb{C}})^{reg}$ to be the open substack of Higgs pairs with everywhere regular Higgs field.

Remark 4.2.2. *$\mathcal{H}iggs_L(G^{\mathbb{C}}) \cong \mathcal{H}iggs_L(\left(G^{\mathbb{C}}\right)_{\mathbb{R}})$, where $\mathcal{H}iggs_L(\left(G^{\mathbb{C}}\right)_{\mathbb{R}})$ is as in the previous section and we consider $\left(G^{\mathbb{C}}\right)_{\mathbb{R}}$ as a real form of $G^{\mathbb{C}} \times G^{\mathbb{C}}$ (cf. Remark 1.2.20).*

This is the usual stack of L-twisted $G^{\mathbb{C}}$-Higgs bundles on S. In particular there is a correspondence between $G^{\mathbb{C}}$-Higgs pairs (E, ϕ) on S and morphisms

$$[(E, \phi)] : S \to \mathcal{H}iggs_L(G^{\mathbb{C}}).$$

We also have a Hitchin map

$$[\chi]_{\mathbb{C}} : \mathcal{H}iggs_L(G^{\mathbb{C}}) \to \mathfrak{d}^{\mathbb{C}} \otimes L/W \tag{4.8}$$

induced from the Chevalley morphism by $\mathbb{C}^\times$ and $G^\mathbb{C}$-equivariance. Denote $\mathfrak{d}^\mathbb{C}\otimes L/W$ by $\mathcal{A}_{G^\mathbb{C},L}$ or $\mathcal{A}_L^\mathbb{C}$ the corresponding Hitchin base. The corresponding fiber of the Hitchin map $[\chi]_L^{-1}(b)$ fits into the Cartesian diagram

$$\begin{array}{ccc} [\chi]_L^{-1}(b) & \longrightarrow & \left[\mathfrak{g}^\mathbb{C}\otimes L/G^\mathbb{C}\right] \\ \downarrow & & \downarrow \\ S & \xrightarrow[b]{} & \mathfrak{d}^\mathbb{C}\otimes L/W \end{array} .$$

Definition 4.2.3. Let $b : U \to \mathfrak{d}^\mathbb{C}\otimes L/W$ be a $\mathfrak{d}^\mathbb{C}\otimes L/W$-scheme. Its associated cameral cover $\widehat{U}_b$ is defined by the Cartesian diagram

$$\begin{array}{ccc} \widehat{U} & \longrightarrow & \mathfrak{d}^\mathbb{C}\otimes L \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathfrak{d}^\mathbb{C}\otimes L/W. \end{array} \tag{4.9}$$

Theorem 4.2.4 (Donagi-Gaitsgory, Ngô). *The map $[\chi]_L : \mathcal{H}iggs_L(G^\mathbb{C})^{reg} \to \mathcal{A}_L^\mathbb{C}$ is a $\mathcal{D}_W$-banded gerbe, where $\mathcal{D}_W \to \mathfrak{d}^\mathbb{C}\otimes L/W$ is a sheaf of groups whose sections on $b : U \to \mathfrak{d}^\mathbb{C}\otimes L/W$ are*

$$\mathcal{D}_W(U) = \{s : \widehat{U}_b \overset{W-equivariant}{\longrightarrow} D^\mathbb{C} \mid \alpha(s(x)) \neq -1 \text{ for all } x \in \widehat{U} \text{ such that } s_\alpha(x) = x\}.$$

This means that the moduli stack is locally isomorphic to $B\mathcal{D}_W$. In particular, the fibers are categories of abelian torsors over X. Moreover, there is a simple transitive action of $B\mathcal{D}_W$ on $\mathcal{H}iggs_L(G^\mathbb{C})$.

As a corollary to Theorem 4.2.4 and Hitchin's construction of a section to the moduli space of $G^\mathbb{C}$-Higgs bundles (cf. [45]) we have the following.

Corollary 4.2.5. *If X is a projective curve and $L \to X$ is a line bundle of even degree, then*

$$\mathcal{H}iggs_L(G^\mathbb{C})^{reg} \cong B\mathcal{D}_W.$$

Proof. The existence of the Hitchin section (cf. [45]) yields triviality of the gerbe. □

The authors also characterise the fibers to be categories of $D^\mathbb{C}$ torsors over the cameral cover satisfying certain equivariance conditions. Any such torsor is called a cameral datum, in analogy with Hitchin's spectral data.

Fix a system of roots Δ, and consider a system of simple roots $\{\alpha_1, \ldots, \alpha_r\}$, with corresponding elements of the Weyl group w_i. Let D_α be the ramification locus in $\mathfrak{d}^\mathbb{C}\otimes L \to \mathfrak{d}^\mathbb{C}\otimes L//W$ corresponding to w_α for any root $\alpha \in \Delta$. Let $\mathcal{R}_\alpha = \check{\alpha}\left[\mathcal{O}(D_\alpha)\right]$

be the $D^{\mathbb{C}}$-principal bundle obtained from $\mathcal{O}(D_\alpha)$ via the coroot $\check{\alpha}$. Now, for any $w \in W$, we define

$$\mathcal{R}_w = \bigotimes_{\alpha \in \Delta \;:\; w(\alpha) \in \Delta^-} \mathcal{R}_\alpha. \tag{4.10}$$

We denote the fibered product of all these sheaves/schemes with X over $X \to \mathfrak{d}^{\mathbb{C}} \otimes L//W$ by a superscript ^{X}. Now, let us define another stack over $\mathfrak{d}^{\mathbb{C}} \otimes L/W$, the stack of cameral data $\mathcal{C}am$, associating to each $X \to \mathfrak{d}^{\mathbb{C}} \otimes L//W$ a tuple $(P, \underline{\gamma}, \underline{\beta})$ where

1. P is a $D^{\mathbb{C}}$-principal bundle on $\widehat{X}_b$ such that $P \cong w_\alpha^* P^{(w_\alpha)} \otimes \mathcal{R}_\alpha^X$ for any $\alpha \in \Delta$. Here, $P^w := P \times_w D^{\mathbb{C}}$.

2. The tuple $\underline{\gamma}$ fits in a map of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & D^{\mathbb{C}} & \longrightarrow & N_{G^{\mathbb{C}}}(D^{\mathbb{C}}) & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \gamma & & \downarrow id & & \\ 1 & \longrightarrow & \mathrm{Hom}(\widehat{X}_b, D^{\mathbb{C}}) & \longrightarrow & \mathrm{Aut}_{\mathcal{R}}(P) & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

   Here

$$\mathrm{Aut}_{\mathcal{R}}(P) = \{(w, \gamma_w) \;:\; w \in W,\ \gamma_w : P \cong w^* P^w \otimes \mathcal{R}_w\}$$

   and the leftmost vertical arrow is an embedding (an isomorphism whenever X is compact).

3. The tuple $\underline{\beta} = (\beta_1, \ldots, \beta_r)$ where for any simple root α_i, β_i is a morphism associating to each $n \in N_{G^{\mathbb{C}}}(D^{\mathbb{C}})$ lifting w_i an isomorphism $\beta_i(n) : \alpha_i(P)|_{D_{\alpha_i}^X} \cong \mathcal{O}(D_{\alpha_i}^X)$.

All the above must satisfy the natural compatibility conditions:

1. For all n lifting w_i, $z \in \mathbb{C}^\times$. $\beta_i(\check{\alpha}_i(z) \cdot n) = z \cdot \beta_i(n)$.

2. With the same notation, $w_i^* P^{w_i} \overset{\alpha_i \gamma(n)}{\cong} \alpha_i(P)$.

3. For any $n \in N_{G^{\mathbb{C}}}(D^{\mathbb{C}})$ lifting w_i, note that $\gamma(n) : w_{\alpha_i} P^{w_{\alpha_i}} \otimes \mathcal{R}_{w_{\alpha_i}}^X$ induces an isomorphism

$$\check{\alpha}_i \left(\alpha \left(P|_{D_{\alpha_i}^X} \right) \right) \cong \check{\alpha}_i \left(\mathcal{O}(D_{\alpha_i}^X)|_{D_{\alpha_i}^X} \right).$$

   We impose the condition that it be exactly $\check{\alpha}_i(\beta(n))$.

4. Given two simple roots α_i, α_j and an element of the Weyl group interchanging them $w(\alpha_i) = \alpha_j$, any $n \in N$ lifting w induces an isomorphism

$$\gamma(w) : \alpha_i(P)|_{D^X_{\alpha_i}} \cong \alpha_i(w^* P^w)|_{D^X_{\alpha_i}} \otimes \alpha_i \left(\mathcal{R}^X_w\right)|_{D^X_{\alpha_i}}$$

It is also proved in [25] that the sheaf on the RHS is canonically isomorphic to $\alpha_i(w^*(P))|_{D^X_{\alpha_i}}$. We ask for the composition of both morphisms to coincide with $\beta(\mathrm{Ad}_{n^{-1}} n_j)$ for some n_j lifting w_{α_j}.

Theorem 4.2.6 (Donagi-Gaitsgory)**.** *The stack $\mathcal{C}am$ is a gerbe over the Hitchin base, equivalent to $\mathcal{H}iggs_L(G^{\mathbb{C}})^{reg}$. In particular, the fiber $[\chi]_{\mathbb{C}}^{-1}(b)$ is in correspondence with the category $\mathcal{C}am_b$ whose objects are the tuples above describe for a fixed b.*

So the $\mathcal{D}_W$-gerbe restricts to a relatively simple category over a given cameral cover. In particular, it is a $D^{\mathbb{C}}$-gerbe away from ramification of the cover. The philosophy of the proof is based on this remark, from which the hard technical work consists in extending the band $D^{\mathbb{C}}$ to ramification.

4.3 The gerbe of Higgs pairs

A necessary condition for a stack in groupoids (in particular, $\mathcal{H}iggs_L(G)$) to be a gerbe is for inertia to be flat (cf. Proposition A.2.2). Now, one cannot expect to have any "nice" structure of the abstract Hitchin map as a whole, as dimensions of inertia jump.

So we will have to restrict attention to a substack ensuring this essential condition.

4.3.1 Abstract Higgs pairs. The local situation.

Consider the atlas

$$\mathfrak{m}^{\mathbb{C}} \to \left[\mathfrak{m}^{\mathbb{C}}/H^{\mathbb{C}}\right].$$

On $\mathfrak{m}^{\mathbb{C}}$ we can define the following group scheme.

Definition 4.3.1. We let $C_{\mathfrak{m}^{\mathbb{C}}} \to \mathfrak{m}^{\mathbb{C}}$ be the group scheme over $\mathfrak{m}^{\mathbb{C}}$ defined by

$$C_{\mathfrak{m}^{\mathbb{C}}} = \{(m, h) \in \mathfrak{m}^{\mathbb{C}} \times H^{\mathbb{C}} \mid h \cdot m = m\} \tag{4.11}$$

where $H^{\mathbb{C}}$ acts on X via the isotropy representation. Here, $\mathcal{H}iggs^{abs}(G)$ is considered as a stack on $(pt)_{AN}$.

Note that there is an action of $H^{\mathbb{C}}$ on $C_{\mathfrak{m}^{\mathbb{C}}}$ (namely, the adjoint action) lifting the isotropy action on $\mathfrak{m}^{\mathbb{C}}$. This means that the inertia stack of $\mathcal{H}iggs^{abs}(G)$ descends from the latter sheaf. Indeed:

Lemma 4.3.2. *Let $U \to Spec\ \mathbb{C}$, and consider an abstract Higgs pair (P, ϕ) on it. Then, the sheaf of automorphisms of the Higgs pair over U is*

$$Aut(P, \phi) = \phi^* P \times_{H^{\mathbb{C}}} C_{\mathfrak{m}^{\mathbb{C}}}$$

where ϕ is considered as an $H^{\mathbb{C}}$-equivariant map $P \to \mathfrak{m}^{\mathbb{C}}$ and $H^{\mathbb{C}}$ acts on $C_{\mathfrak{m}^{\mathbb{C}}}$ by conjugation.

Proof. This is a tautological statement. □

So clearly the inertia stack is not flat. The natural way to solve this is to impose regularity on the Higgs field. That is, to look at abstract Higgs pairs which are in the image of the subatlas $\mathfrak{m}_{reg} \hookrightarrow \mathfrak{m}^{\mathbb{C}}$.

Definition 4.3.3. We will call the stack $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$ the stack of everywhere regular abstract $G^{\mathbb{C}}$-Higgs pairs, and denote it by $\mathcal{H}iggs^{abs,reg}(G)$. This substack of $\mathcal{H}iggs^{abs}(G)$ corresponds to Higgs pairs whose Higgs field is everywhere regular in the sense of Kostant and Rallis' [51].

We have the following.

Lemma 4.3.4. *$C_{\mathfrak{m}^{\mathbb{C}}} \to \mathfrak{m}_{reg}$ is smooth.*

Proof. The proof is similar to the one of Proposition 11.2 in [25]. A new element that appears in the real group case is the non-abelian property of the centralisers. Given a complex point $(x, h) \in C_{\mathfrak{m}^{\mathbb{C}}}(\mathbb{C})$, we have that the tangent space $T_{(x,h)}C_{\mathfrak{m}^{\mathbb{C}}}(\mathbb{C})$ is defined inside $T_{(x,h)}\mathfrak{m}_{reg} \times H^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \times \mathfrak{h}^{\mathbb{C}}$ by the equation $d_{(x,h)}f(y, \xi) = 0$ where

$$f(x, h) = \mathrm{Ad}(h)(x) - x.$$

Now:

$$\frac{\partial}{\partial h}|_{h,x} f(y, \xi) = \frac{\partial}{\partial h}|_{h,x} \mathrm{Ad}(h) \circ ev_x(y, \xi) = h \cdot [\xi, x].$$

Hence $d_{(x,h)}f(y, \xi) = \mathrm{Ad}(h)([\xi, x]) + h \cdot y - y$. Clearly, the differential of the map $C_{\mathfrak{m}^{\mathbb{C}}} \to \mathfrak{m}_{reg}$ sends $(y, \xi) \mapsto y$. So all we need to check is that

$$\{y \in \mathfrak{m}^{\mathbb{C}} \mid y - h^{-1}(y) \in \mathrm{ad}(x)(\mathfrak{h}^{\mathbb{C}})\} = \mathfrak{m}^{\mathbb{C}}.$$

One inclusion is clear, so let's see that any $z \in \mathfrak{m}^{\mathbb{C}}$ satisfies the condition. First note that

$$\mathfrak{g}^{\mathbb{C}} \cong [x, \mathfrak{g}^{\mathbb{C}}] \oplus \mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(x) \cong [x, \mathfrak{h}^{\mathbb{C}}] \oplus \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) \oplus [x, \mathfrak{m}^{\mathbb{C}}] \oplus \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(x)$$

so that

$$\mathfrak{m}^{\mathbb{C}} \cong [x, \mathfrak{h}^{\mathbb{C}}] \oplus \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x). \tag{4.12}$$

Since the action of any $h \in C_{H^{\mathbb{C}}}(x)$ respects the direct sum (4.12), it is enough to check that

$$\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) \subseteq \{y \in \mathfrak{m}^{\mathbb{C}} \mid y - \mathrm{Ad}_{h^{-1}}(y) \in \mathrm{ad}(x)(\mathfrak{h}^{\mathbb{C}})\}.$$

This is possible if and only if $y = \mathrm{Ad}_{h^{-1}}(y)$, as

$$y - \mathrm{Ad}_h y \in \mathrm{ad}(x)(\mathfrak{h}^{\mathbb{C}}) \cap \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x) = 0$$

whenever $y \in \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x)$. First, suppose that $C_{H^{\mathbb{C}}}(x)$ is connected. We have $\mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(x) = \mathfrak{c}_{\mathfrak{h}^{\mathbb{C}}}(\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x))$ by Corollary 1.4.12. So $\mathrm{Ad}_h y = y$ for $y \in \mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(x)$.

Now, for the non connected case, since fibers are algebraic groups in zero characteristic, they are smooth. Thus independently of the component h is in, the dimension of the tangent bundle will not vary. □

With this we have the following.

Proposition 4.3.5. *The abstract Hitchin map endows $\mathcal{H}iggs^{abs,reg}(G)$ with a gerbe structure over the analytic site of $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$.*

Proof. By definition, we need to check that for some open cover of $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ we have an abstract Higgs pair over each of the open subsets, and that any such are locally isomorphic. This is ensured by Luna's slice theorem (cf. [53]), since the étale topology is weaker than the analytic. As for local connectedness, note that flatness of inertia ensures that any two objects over $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ will be isomorphic by the sheaf $\mathrm{Hom}_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})}((P, \phi), (Q, \psi))$. □

Corollary 4.3.6. *$\mathcal{H}iggs^{abs,reg}(G) \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ is a neutral gerbe.*

Proof. The Kostant–Rallis section $s_{KR} : \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) \to \mathfrak{m}_{reg}$ (cf. Section 1.3) provides a well defined section $s : \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) \to \mathcal{H}iggs^{abs,reg}(G)$.

Indeed, let $a : S \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ be a scheme over $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$. Consider the trivial $H^{\mathbb{C}}$-bundle on S and define

$$\phi : S \times H^{\mathbb{C}} \to \mathfrak{m}_{reg}, \qquad (x, h) \mapsto \mathrm{Ad}_h s_{KR}(a(x)).$$

□

Remark 4.3.7. *Note that in spite of the Kostant–Rallis section being transversal to the orbits only for the action of H_θ (cf. Theorem 1.3.9* 3.*), it is still a section for the Chevalley map (1.3) under the action of $H^{\mathbb{C}}$ (Theorem 1.3.9* 1.*).*

4.3.1.1 The abelian case

When the sheaf $C_{\mathfrak{m}^{\mathbb{C}}}$ is abelian, much of the approach described in Section 4.2 is valid in the case of real forms. In particular, a finer description of the gerbe can be given by specifying its band.

Recall the following.

Proposition 4.3.8. *The sheaf $C_{\mathfrak{m}^{\mathbb{C}}}$ is abelian if and only if the real form $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ is quasi-split.*

Proof. It follows from Propositions 1.4.2 and 1.4.4. □

Corollary 4.3.9. *The Hitchin map $[\chi] : \mathcal{H}iggs^{abs}(G) \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$ has abelian fibers if and only if G is quasi-split*

With this, we obtain the following.

Proposition 4.3.10. *If $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ is a quasi-split real form of a complex reductive algebraic group, then the sheaf $C_{\mathfrak{m}^{\mathbb{C}}} \to \mathfrak{m}_{reg}$ descends uniquely to a sheaf of abelian groups*

$$J_{\mathfrak{m}^{\mathbb{C}}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}). \tag{4.13}$$

Proof. The proof in [60] works on the nose. By abelianity of $C_{\mathfrak{m}^{\mathbb{C}},x}$, $x \in \mathfrak{m}_{reg}$, we can define the fiber over $\chi(x) \in \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ to be $C_{\mathfrak{m}^{\mathbb{C}},x}$ itself. Any other choice will be uniquely isomorphic over $\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$, so that the fiber is well defined.

As for the sheaf itself, it can be defined by descent of $C_{\mathfrak{m}^{\mathbb{C}}}$ along the flat morphism $\mathfrak{m}_{reg} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$.

For $C_{\mathfrak{m}^{\mathbb{C}}}$ to descend, it must happen that both pullbacks to $\mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg}$ be isomorphic. Consider both projections

$$p_1, p_2 : \mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg} \to \mathfrak{m}_{reg},$$

and let $C_i = p_i^* C_{\mathfrak{m}^{\mathbb{C}}}$. Consider $f : H^{\mathbb{C}} \times \mathfrak{m}_{reg} \to \mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg}$ given by $(h, x) \mapsto (x, h \cdot x)$. We will proceed by proving that there exists an isomorphisms $f^*C_1 \cong f^*C_2$ over $H^{\mathbb{C}} \times \mathfrak{m}_{reg}$, and then check it descends to an isomorphism over $\mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg}$.

Consider

$$F : f^*C_1 \longrightarrow f^*C_2$$

$$((m,h),g) \longmapsto ((m,h), \mathrm{Ad}_h g)\,.$$

It defines an isomorphism over $H^{\mathbb{C}} \times \mathfrak{m}_{reg}$. To see whether F descends to $\mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg}$, we need to check that $F(m,h,g)$ depends only on $(m, h \cdot m, g)$ and not on the particular element $h \in H^{\mathbb{C}}$. To see that, it must happen that the restriction of F to

$$S := \left(H^{\mathbb{C}} \times \mathfrak{m}_{reg}\right) \times_{\mathfrak{m}_{reg} \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{m}_{reg}} \left(H^{\mathbb{C}} \times \mathfrak{m}_{reg}\right)$$

in both possibe ways (by each of the projections $\pi_1,\ \pi_2 : S \to H^{\mathbb{C}} \times \mathfrak{m}_{reg}$ to $H^{\mathbb{C}} \times \mathfrak{m}_{reg}$) fits into the commutative square

$$\begin{array}{ccc} \pi_1^* f^* C_1 & \longrightarrow & \pi_1^* f^* C_2 \\ \downarrow & & \downarrow \\ \pi_2^* f^* C_1 & \longrightarrow & \pi_2^* f^* C_2. \end{array}$$

To do this, note that $S \cong H^{\mathbb{C}} \times C_1$ by the map

$$((m,h),(m,h')) \mapsto (h, (m, hm), h^{-1}h').$$

Then, in these terms,

$$\pi_1 : (h, (m, hm), z) \mapsto (m, h) \qquad \pi_2 : (h, (m, hm), z) \mapsto (m, hz)$$

$$F_{21} : [(h, (m, hm), z), g] \mapsto [(h, (m, hm), z), \mathrm{Ad}_h g],$$

so that the above square reads

$$\begin{array}{ccc} [(h,(m,hm),z),g] & \longmapsto & [(h,(m,hm),z),\mathrm{Ad}_h g] \\ \big\downarrow & & \big\downarrow \\ & & [(h,(m,hm),z),\mathrm{Ad}_h g] \\ & & \big\| ? \\ [(hz,(m,hm),z),g] & \longmapsto & [(h,(m,hm),z),\mathrm{Ad}_{hz} g], \end{array}$$

where the question mark means that is the equality we need to prove, and follows from abelianity of $C_{H^{\mathbb{C}}}(\mathfrak{m})$. $\square$

Corollary 4.3.11. *If $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ is a quasi-split real form of a complex reductive algebraic group, then the stack of abstract G-Higgs pairs is locally isomorphic to $BJ_{\mathfrak{m}}$.*

Remark 4.3.12. *Note that in particular, when $G = \left(G^{\mathbb{C}}\right)_{\mathbb{R}}$ for some complex reductive algebraic group $G^{\mathbb{C}}$, we recover the complex case, by Corollary 1.4.5.*

4.3.1.2 Abelian versus non abelian case.

The main fact about $G^{\mathbb{C}}$-Higgs pairs, which needs not be true for G-Higgs pairs in general, is abelianity of the group scheme of centralisers $C_{\mathfrak{g}^{\mathbb{C}}} \to \mathfrak{g}^{\mathbb{C}}$, where

$$C_{\mathfrak{g}^{\mathbb{C}}} = \{(x, g) \in \mathfrak{g}^{\mathbb{C}} \times G^{\mathbb{C}} \ : \ \mathrm{Ad}_g x = x\}. \tag{4.14}$$

Now, by Kempf's lemma about descent of sheaves to a GIT quotient, the sheaf $C_{\mathfrak{m}}$ descends to the scheme $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}})$ if and only if it is abelian. Indeed, let us prove it by a simple direct argument. Suppose $C_{\mathfrak{m}}$ descends, that is, there exists some sheaf $J_{\mathfrak{m}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}})$ such that the diagram

$$\begin{array}{ccc} C_{\mathfrak{m}} & \longrightarrow & J_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ \mathfrak{m}_{reg} & \longrightarrow & \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) \end{array}$$

be Cartesian (note that we need to restrict attention to semistable elements for the action of $H^{\mathbb{C}}$, that is, regular elements). Then, given that $J_{\mathfrak{m}}$ has as stalks of the orbit $\mathbb{O}_x = H^{\mathbb{C}}x$ the maps $\mathrm{Hom}_{H^{\mathbb{C}}}(H^{\mathbb{C}} \cdot x, C_{H^{\mathbb{C}}}(x))$, it follows that any such $f \in J_{\mathfrak{m},x}$ will satisfy that for $h \in C_{H^{\mathbb{C}}}(x)$ and any $y \in \mathbb{O}_x$ $\mathrm{Ad}_h(f(y)) = f(y)$. Namely, $f(y) \in Z(C_H(\mathfrak{a}))$.

This means that in the non abelian case the band is not trivial over the base, and so we cannot conclude that the gerbe be a category of torsors over the base, just locally so. Despite this fact, a notion of cameral data can still exist if the band lifts to a sheaf of groups over the cameral cover. We will discuss this in section 4.5.

4.3.2 Twisted Higgs pairs.

The results in the preceeding section together with Lemma 4.1.5 yield the following.

Theorem 4.3.13. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ be a real form of a connected complex reductive algebraic group. Let X be a complex projective scheme, and fix a holomorphic line bundle $L \to X$. Then*

1. *The stack of everywhere regular L-twisted G-Higgs pairs over X is a gerbe over $\mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a}^{\mathbb{C}})$ which is abelian if and only if the form G is quasi-split.*

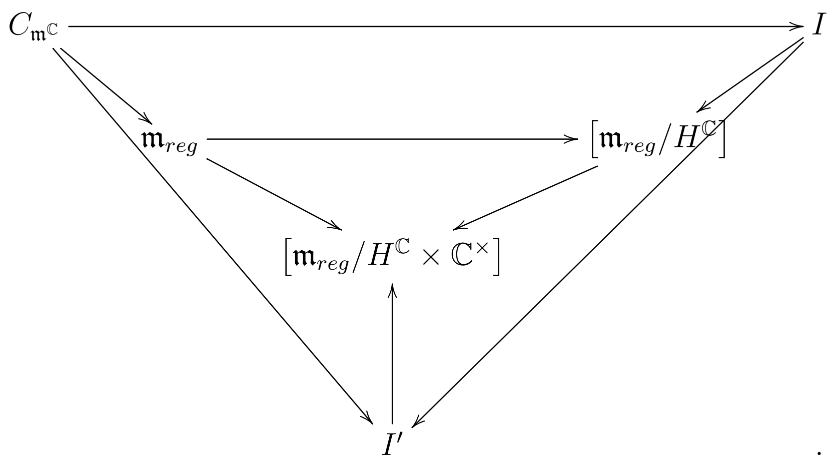
2. *If the form is quasi-split, the gerbe is banded by $J^L_{\mathfrak{m}} \to \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})$ where given $s : U \to \mathcal{A}_{L,G}$, we have $J^L_{\mathfrak{m}}(U) = Hom_{H^{\mathbb{C}}}(L \times_s \mathfrak{m}_{reg}, C_{\mathfrak{m}}|_{L \times_s \mathfrak{m}_{reg}})$. Note that in the latter expression we interpret L to be a principal $\mathbb{C}^\times$-bundle, and s a $\mathbb{C}^\times$ equivariant morphism $L \to \mathfrak{a}_{reg}//W(\mathfrak{a}^{\mathbb{C}})$.*

3. *If X is a curve, and $L \to X$ is a line bundle of even degree,* $\left[\mathfrak{m}_{reg} \otimes L/H^{\mathbb{C}}\right] \cong BJ^L_{\mathfrak{m}}$.

Proof. The first statement follows from Proposition 4.3.5 and Lemma 4.1.5. Indeed, the stack $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$ is locally non empty over $\left[\mathfrak{a}^{\mathbb{C}}/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$, since any covering of $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}})$ over which $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$ is non-empty (which exists by Proposition 4.3.5) is a cover for $\left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$ over which $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$ is non-empty, by Lemma 4.1.5. Local connectedness follows in the same way.

So (4.3) is a gerbe. In particular, for any choice of a line bundle $L \to S$, the Hitchin map (4.2) will endow $\left[\mathfrak{m}_{reg} \otimes L/H^{\mathbb{C}}\right] \to \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})$ with a gerbe structure as long as it satisfies local non emptiness (as the rest of the structure is induced from the one of (4.3)). Local non emptiness is a consequence of local triviality of L and Proposition 4.3.5.

For 2. note that the action of $\mathbb{C}^\times$ on $\mathfrak{m}_{reg}$ lifts to $C_{\mathfrak{m}^{\mathbb{C}}}$ in an equivariant way, and commutes to the action of $H^{\mathbb{C}}$. Thus $C_{\mathfrak{m}^{\mathbb{C}}}$ descends to the inertia stack of $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$. Commutativity of both actions implies that we have Cartesian diagrams of sheaves



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In the above I, I' are the inertia stacks of $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$ and $\left[\mathfrak{m}_{reg}/H^{\mathbb{C}} \times \mathbb{C}^\times\right]$ respectively.

By the same reason, $J_{\mathfrak{m}}$ descends to $\left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/\mathbb{C}^\times\right]$, so that so does $C_{\mathfrak{m}^{\mathbb{C}}}$, and there is a Cartesian diagram

$$\begin{array}{ccccc}
I & & \longrightarrow & & J_{\mathfrak{m}} \\
\downarrow & \searrow & & \swarrow & \downarrow \\
 & \left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right] & \longrightarrow & \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) & \\
 & \downarrow & & \downarrow & \\
 & \left[\mathfrak{m}_{reg}/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] & \longrightarrow & \left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})/H^{\mathbb{C}} \times \mathbb{C}^{\times}\right] & \\
\downarrow & \nearrow & & \nwarrow & \downarrow \\
I' & & \longrightarrow & & J'_{\mathfrak{m}}
\end{array}$$

Now, once a particular $L \to S$ has been fixed, it is straighforward to check that $J'_{\mathfrak{m}}|_{\mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}})}$ has the appropriate definition.

3. is just a remark, as the Hitchin-Kostant–Rallis section from Chapter 3 induces a morphism

$$s_L : \mathfrak{a}^{\mathbb{C}} \otimes L/W(\mathfrak{a}^{\mathbb{C}}) \to \mathcal{H}iggs_L(G)$$

contained in the regular (and polystable) locus. □

4.4 Cameral data for quasi-split forms.

In this section we give a different, more workable characterization of the band of the gerbe in the abelian case. We will prove the band to be a sheaf of tori, in the spirit of Donagi and Gaitsgory's Theorem 4.2.4. At this point, cameral covers come into the picture, as well as the notion of cameral data: we will prove the fiber to be equivalent to a category of twisted bundles on the cameral cover. The latter will be taken care of in Section 4.4.2.

As in the previous section, we start by examining the abstract setting, moving on to the stack of Higgs pairs. The strategy followed is to study the image of the real fiber inside the complex one, then use the complex characterization of the fibers to extract information about the real Hitchin fibers.

In this section $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ will denote a quasi-split real form of a complex connected reductive algebraic group.

4.4.1 Characterization of the band: relation with the complex group case

Let us start by recalling the notation: given a real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ and a Cartan involution θ on it, we have the corresponding eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Complexifying, we get a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. Let $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$ be a maximal θ-anisotropic Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, where the action of θ on $\mathfrak{g}^{\mathbb{C}}$ is obtained by linearization of the action on $\mathfrak{g}$. Namely, $\mathfrak{d}^{\mathbb{C}} = (\mathfrak{t} \oplus \mathfrak{a})^{\mathbb{C}}$ where $\mathfrak{a} \subseteq \mathfrak{m}$ is a maximal abelian subalgebra, and so is $\mathfrak{t} \subseteq \mathfrak{c}_{\mathfrak{h}}(\mathfrak{a})$. Let $W = W(\mathfrak{g}^{\mathbb{C}}, \mathfrak{d}^{\mathbb{C}})$, $W(\mathfrak{a}^{\mathbb{C}})$ be the respective Weyl and restricted Weyl groups. Detone $D^{\mathbb{C}} = \exp(\mathfrak{d}^{\mathbb{C}})$, and consider $D^{\mathbb{C}} = T^{\mathbb{C}} A^{\mathbb{C}}$ where

$$T^{\mathbb{C}} = \left(D^{\mathbb{C}}\right)^{\theta}, \qquad A^{\mathbb{C}} = \exp(\mathfrak{a}^{\mathbb{C}}). \tag{4.15}$$

We set

$$\chi_{\mathbb{C}} : \mathfrak{g}_{reg} \to \mathfrak{d}^{\mathbb{C}}/W \tag{4.16}$$

the complex Chevalley map, and

$$\chi : \mathfrak{m}_{reg} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}) \tag{4.17}$$

the one corresponding to $\mathfrak{g}$. Also, let

$$\pi_{\mathfrak{d}^{\mathbb{C}}} : \mathfrak{d}^{\mathbb{C}} \to \mathfrak{d}^{\mathbb{C}}/W \tag{4.18}$$

and

$$\pi_{\mathfrak{a}^{\mathbb{C}}} : \mathfrak{a}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}}). \tag{4.19}$$

Denote by

$$\mathfrak{r} := \pi_{\mathfrak{d}^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) \tag{4.20}$$

the image of $\mathfrak{a}^{\mathbb{C}}$ in $\mathfrak{d}^{\mathbb{C}}/W$. Given a $\mathfrak{d}^{\mathbb{C}}/W$ scheme $b : U \to \mathfrak{d}^{\mathbb{C}}/W$, we will denote by

$$\widehat{U}_b = U \times_{\mathfrak{d}^{\mathbb{C}}/W} \mathfrak{d}^{\mathbb{C}} \tag{4.21}$$

the **abstract cameral cover**. Similarly, for $a : U \to \mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})$ we set

$$\widetilde{U}_a = U \times_{\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})} \mathfrak{a}^{\mathbb{C}} \tag{4.22}$$

the **abstract real cameral cover**.

Lemma 4.4.1. *Let $(G, H, \theta, B) < (G^{\mathbb{C}}, U, \tau, B)$ be a quasi-split real form of a complex strongly reductive algebraic group. Assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{a}^{\mathbb{C}} \subseteq \mathfrak{m}^{C}$ a maximal anisotropic Cartan subalgebra. We have that:*

1. $i : \mathfrak{m}_{reg} \hookrightarrow \mathfrak{g}_{reg}$.
2. $W(\mathfrak{a}^{\mathbb{C}}) \leq W$.
3. $x, y \in \mathfrak{m}_{reg}$ *are conjugate by* $g \in G^{\mathbb{C}}$, *say* $y = \mathrm{Ad}_g x$ *if and only if* $g^{-1}g^{\theta} \in C_{H^{\mathbb{C}}}(x)$.
4. $x, y \in \mathfrak{a}^{\mathbb{C}}$ *are conjugate by* $w \in W$ *if and only if* $w^{-1}w^{\theta} \in C_W(y)$, *where the action of* θ *on* W *is induced by the action of* θ *on* $N(D^{\mathbb{C}})$.
5. $N_{G^{\mathbb{C}}}(\mathfrak{m}_{reg})$ *is a central extension of* $H^{\mathbb{C}}$;
6. $\chi_{\mathbb{C}}^{-1}(\mathfrak{r}) = \{y \in \mathfrak{g}_{reg} \ : \ \chi_{\mathbb{C}}(y) = \chi_{\mathbb{C}}(-\theta y)\} =$
   $\{y \in \mathfrak{g}_{reg} \ : \ $ *there exists* $x \in \mathfrak{m}_{reg}$ *such that* $\chi_{\mathbb{C}}(x) = \chi_{\mathbb{C}}(g)\}$.

Proof. 1 By Remark 1.1.19, for any semisimple element $x \in \mathfrak{m}_{reg,ss}$ $\mathfrak{c}_{\mathfrak{g}^{\mathbb{C}}}(x)$ is a Cartan subalgebra, which implies the fact.

2 By strong reductivity $W(\mathfrak{a}^{\mathbb{C}}) \cong N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})/C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$. Similarly, $W \cong N(D^{\mathbb{C}})/D^{\mathbb{C}}$. By Proposition 7.49 in [48] we have that $C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) \curvearrowright \mathfrak{t}$ acts by elements of the connected component $(C_H(\mathfrak{a}))^0 = T$. On the other hand, $N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) \leq N(D^{\mathbb{C}})$. Hence, the action of any $n \in N_H(\mathfrak{a})$ on $\mathfrak{d}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$ depends only on its coset inside $N_H(\mathfrak{a})/C_H(\mathfrak{a})$.

3 and 4 are proved in a similar fashion.

5 Let $n \in N_{G^{\mathbb{C}}}(\mathfrak{m}_{reg})$. This is equivalent to having

$$n^{-1}n^{\theta} \in C_{G^{\mathbb{C}}}(\mathfrak{m}_{reg}).$$

1 and Proposition 12.7 in [25] imply that $n^{-1}n^{\theta}$ is central. Clearly, $N_{G^{\mathbb{C}}}(\mathfrak{m}_{reg})^0 = H^{\mathbb{C}0}$, so the result follows.

6 is an easy calculation. □

Lemma 4.4.1 implies the existence of a morphism

$$\kappa : \mathcal{H}iggs^{abs}(G) \to \mathcal{H}iggs^{abs}((G^{\mathbb{C}})_{\mathbb{R}}) \tag{4.23}$$

whenever $G \leq G^{\mathbb{C}}$ is quasi-split. This morphism sends a pair (E, ϕ) to the extended pair $(E \times_{H^{\mathbb{C}}} G^{\mathbb{C}}, di(\phi))$.

On $\mathfrak{g}^{\mathbb{C}} \times G^{\mathbb{C}}$ we define the involution Θ by

$$\Theta : (x, g) \mapsto (-\theta x, g^{\theta}). \tag{4.24}$$

Proposition 4.4.2. *Θ defines an involution on $\mathcal{H}iggs^{abs}(G^{\mathbb{C}})$ whose stack of fixed points is $Im(\kappa)$.*

Proof. Indeed, consider the map of presentations corresponding to the morphism 4.23

$$\begin{array}{ccccc} C_{\mathfrak{m}} & \longrightarrow & \mathfrak{m}_{reg} \times H^{\mathbb{C}} & \rightrightarrows & \mathfrak{m}_{reg} \\ \downarrow & & \downarrow & & \downarrow \\ C_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}_{reg} \times G^{\mathbb{C}} & \rightrightarrows & \mathfrak{g}_{reg}. \end{array}$$

where $C_{\mathfrak{m}}$ is as in (4.11) and $C_{\mathfrak{g}}$ is defined in (4.14). We readily check that $C_{\mathfrak{g}^{\mathbb{C}}}$ is left invariant by Θ. Hence, the action of Θ on $\mathfrak{g}_{reg} \times G^{\mathbb{C}}$ restricts to $C_{\mathfrak{g}}$ and induces the action $-\theta$ on $\mathfrak{g}_{reg}$, making all horizontal arrows Θ-equivariant on the downmost row of the above diagram. Furthermore, the fixed points are exactly the subschemes on the upmost row. □

Remark 4.4.3. *At the level of Higgs pairs, the involution reads as follows: $(E, \phi) \mapsto (E \times_{\theta} G, -\theta\phi)$. This is already known and it was first described by García-Prada–Ramanan [33, 30, 29], then used in Schaposnik's thesis [66]. The above gives a different proof of this fact.*

So we have the image of a gerbe inside another gerbe. In what follows, we characterise the image as a subgerbe giving its band. Recall from Proposition 4.3.10 that $C_{\mathfrak{m}}$ is a pullback of the sheaf $J_{\mathfrak{m}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$. Similarly, it is proved in [25] that $C_{\mathfrak{g}}$ is pullbacked from $J_{\mathfrak{g}} \to \mathfrak{d}^{\mathbb{C}}//W$ and that the latter sheaf is isomorphic to $\mathcal{D}_W$ (this is Theorem 4.2.4). Consider the action of $\theta \curvearrowright W$ induced from the action on the normaliser $N(D^{\mathbb{C}})$.

Definition 4.4.4. We define the group $\widetilde{W} = W \rtimes \mathbb{Z}_2$, where we identify the cyclic group generated by θ to $\mathbb{Z}_2$.

Remark 4.4.5. *$\widetilde{W}$ fits into an exact sequence: $1 \to \widetilde{D} \to \widetilde{N} \to \widetilde{W}$, where $\widetilde{D} = D^{\mathbb{C}} \rtimes \mathbb{Z}_2$, $\widetilde{N} = N \rtimes \mathbb{Z}_2$ with the natural action of θ on the groups.*

Definition 4.4.6. Consider the following actions of $\widetilde{W}$:

1. on $D^{\mathbb{C}}$ by $(w, \theta) \cdot g = w \cdot g^{\theta}$,

2. on $\mathfrak{d}^{\mathbb{C}} \to \mathfrak{d}^{\mathbb{C}}//W$ (seen as a cameral cover) by $(w,\theta)\widehat{x} = -\theta(\widehat{x}) \cdot w$.

Note that (w,θ) acts as $\theta \circ w^{\theta}$ on $D^{\mathbb{C}}$ and as $-\theta \circ w^{\theta}$ on $\mathfrak{d}^{\mathbb{C}} \to \mathfrak{d}^{\mathbb{C}}/W$.

Consider the action of θ on $\mathfrak{d}^{\mathbb{C}} \to \mathfrak{d}^{\mathbb{C}}//W$ by $\theta\widehat{x} = -\theta(\widehat{x})$. Note that it takes orbits to orbits, as $-\theta(\widehat{x}w) = -\theta(\widehat{x})w^{\theta}$. Thus, it induces an action on $\mathfrak{d}^{\mathbb{C}}//W$ whose fixed point set is $\mathfrak{r}$. Note that we may quotient $\mathfrak{d}^{\mathbb{C}}$ by $\widetilde{W}$, and the quotient is exactly the quotient of $\mathfrak{d}^{\mathbb{C}}//W$ by the action of θ:

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathfrak{r} & \longrightarrow & \mathfrak{d}^{\mathbb{C}}//W \\ \downarrow & & \downarrow \\ \mathfrak{r}/\theta & \longrightarrow & \mathfrak{d}^{\mathbb{C}}//\widetilde{W} \end{array} \tag{4.25}$$

Remark 4.4.7. *As a stack $\mathfrak{r} \to \mathfrak{r}/\theta$ is the trivial gerbe $[\mathfrak{r}/\theta]$, as the action of θ is trivial. Note that in both cases the underlying schemes are but $\mathfrak{r}$.*

Given $U \to \mathfrak{r}$, denote by

$$\underline{D^{\mathbb{C}}}_{\widetilde{W}}(U) = \{f : \widehat{U} \to D^{\mathbb{C}} \;:\; w \circ f \circ w \equiv f \;\forall\, w \in \widetilde{W}\}. \tag{4.26}$$

Definition 4.4.8. We define the following sheaf of groups on $\mathfrak{r}$:

$$\mathcal{D}_{\widetilde{W}}(U) = \{f \in \underline{D^{\mathbb{C}}}_{\widetilde{W}}(U) \;:\; \breve{\alpha}(f(\widehat{x})) = 1 \text{ for all } \widehat{x} \in D^{U}_{\alpha}\}$$

The main result of this section is the following.

Theorem 4.4.9. *The image of $\mathcal{H}iggs^{abs}(G)$ into $\mathcal{H}iggs^{abs}(G^{\mathbb{C}})$ is a $\mathcal{D}_{\widetilde{W}}$-banded gerbe over the big analytic site on $\mathfrak{r}$.*

Proposition 4.4.10. *The involution $\Theta : C_{\mathfrak{g}^{\mathbb{C}}}|_{G^{\mathbb{C}}\cdot\mathfrak{m}_{reg}} \to C_{\mathfrak{g}^{\mathbb{C}}}|_{G^{\mathbb{C}}\cdot\mathfrak{m}_{reg}}$ descends to an involution $\Theta : J_{\mathfrak{g}^{\mathbb{C}}}|_{\mathfrak{r}} \to J_{\mathfrak{g}^{\mathbb{C}}}|_{\mathfrak{r}}$. Here $G^{\mathbb{C}} \cdot \mathfrak{m}_{reg}$ denotes the orbit of $\mathfrak{m}_{reg}$ under the action of $G^{\mathbb{C}}$.*

By abuse of notation, we will denote the descended action also by Θ.

Proof. Θ induces the following action on global sections: given $s : \mathfrak{g}_{reg} \to C_{\mathfrak{g}}$, we define s^{θ} by $s^{\theta}(x) = \theta \circ s \circ (-\theta)$. Note that this is well defined on any $-\theta$ stable subset of $\mathfrak{g}_{reg}$, in particular on the $G^{\mathbb{C}}$ orbit of any subset of $\mathfrak{m}_{reg}$. Moreover, given $g \in G^{\mathbb{C}}$, we have that

$$s^{\theta}(\mathrm{Ad}_g x) = \theta(s(\mathrm{Ad}_{g^{\theta}})) = \theta(g^{\theta} \cdot s(-\theta(x))) = g \cdot s^{\theta}(x)$$

where the second equality is a consequence of $C_{\mathfrak{g}}$ being a pullback of $J_{\mathfrak{g}}$. Thus, Θ commutes to the action of $G^{\mathbb{C}}$ and so it descends. $\square$

Proposition 4.4.11. *$Im(\kappa) \cong BJ_{\mathfrak{g}}|_{\mathfrak{r}}^{\Theta}$.*

Proof. By Corollary 4.3.11, $\mathcal{H}iggs^{abs}(G) \cong \left[(\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}))/J_{\mathfrak{m}}\right] = BJ_{\mathfrak{m}}$. Similarly, in [25] they prove that $\mathcal{H}iggs^{abs}(G^{\mathbb{C}}) \cong \left[(\mathfrak{d}^{\mathbb{C}}//W)/J_{\mathfrak{g}}\right]$. From the proof of Proposition 4.4.2, we know that the inertia stack of the image is the subsheaf of Θ equivariant sections of $C_{\mathfrak{g}}|_{\mathfrak{m}_{reg}}$. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{m}_{reg} & \xrightarrow{\;i\;} & G^{\mathbb{C}} \cdot \mathfrak{m}_{reg} & \longrightarrow & \mathfrak{g}_{reg} \\
\chi \big\downarrow & & \big\downarrow & & \big\downarrow \chi_{\mathbb{C}} \\
\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}) & \xrightarrow[\;j\;]{} & \mathfrak{r} & \longrightarrow & \mathfrak{d}^{\mathbb{C}}/W.
\end{array}$$

On the one hand we have that $C_{\mathfrak{m}} = i^*\chi_{\mathbb{C}}^*(J_{\mathfrak{g}}^{\Theta}|_{\mathfrak{r}})$. Indeed, $s^{\theta}(x) = s(x)$ for $x \in \mathfrak{m}_{reg} \iff \theta(s(x)) = s(x) \iff s(x) \in H^{\mathbb{C}}$. On the other, since $\chi_{\mathbb{C}} \circ i = j \circ \chi$, uniqueness of descent implies that $j^*(J_{\mathfrak{g}})^{\Theta}|_{\mathfrak{r}} \cong J_{\mathfrak{m}}$. Namely, the inertia stack of the image descends to $J_{\mathfrak{g}}|_{\mathfrak{r}}^{\Theta}$ and the result follows. □

Proposition 4.4.12. *There exists an action of θ on $\mathcal{D}_W|_{\mathfrak{r}}$, that we also denote by Θ. Furthermore, the subsheaf of Θ-equivariant sections of $\mathcal{D}_W|_{\mathfrak{r}}$ is $\mathcal{D}_{\widetilde{W}}$.*

Proof. Given a W-equivariant section $f : \widehat{U} \to D^{\mathbb{C}}$, define $f^{\theta} = \theta \circ f \circ (-\theta)$. This is still a W-equivariant morphism $\widehat{U} \to D^{\mathbb{C}}$. Indeed:

$$f^{\theta}(w \cdot \widehat{x}) = \theta\left(f\left(-\theta(w \cdot \widehat{x})\right)\right) = \theta\left(f\left(w^{\theta} \cdot (-\theta(\widehat{x}))\right)\right) =$$

$$\theta\left(w^{\theta}\left(f\left(\cdot\left(-\theta(\widehat{x})\right)\right)\right)\right) = w\left(\theta\left(f\left(\cdot\left(-\theta(\widehat{x})\right)\right)\right)\right) = w \cdot f(\widehat{x}),$$

where the action $\theta \curvearrowright W$ is as described in Proposition 4.4.1, 4. Furthermore, assume α is a root. Then $\breve{\alpha}(f(\widehat{x})) = 1 \iff \breve{\theta\alpha}(f(-\theta\widehat{x})) = 1$. Note that for any $b : U \to \mathfrak{r}$, $\widehat{x} \in \widehat{U}_b \iff -\theta(\widehat{x}) \in \widehat{U}_b$. Indeed, by Lemma 4.4.1,6, the fibers over $\mathfrak{r}$ are stable by $-\theta$. Thus, $\mathcal{D}_W|_{\mathfrak{r}}$ is stable by the action. By definition $\mathcal{D}_W|_{\mathfrak{r}}^{\Theta} = \mathcal{D}_{\widetilde{W}}$. □

We have all the elements to prove Theorem 4.4.9.

Proof of Theorem 4.4.9. Let us recall how to prove the isomorphism $J_{\mathfrak{g}} \cong \mathcal{D}_W$ (see [60]). Consider the Springer–Grothendieck resolution

$$\widehat{\mathfrak{g}}_{reg} = \mathfrak{g}_{reg} \times_{\mathfrak{d}^{\mathbb{C}}/W} \mathfrak{d}^{\mathbb{C}}. \tag{4.27}$$

Fixing a Borel subgroup B containing $D^{\mathbb{C}}$ we can prove

$$\widehat{\mathfrak{g}}_{reg} \cong \{(x, gB) \ : \ x \in \mathrm{Ad}_g\mathfrak{b}\} \subset \mathfrak{g}_{reg} \times G^{\mathbb{C}}/B. \tag{4.28}$$

See [20] for details on this. By Proposition in [25],

$$\widehat{\mathfrak{g}}_{reg} \cong \{(x, gB) \ : \ \mathfrak{c}_{\mathfrak{g}}(x) \subset \mathrm{Ad}_g \mathfrak{b}\} \subset \mathfrak{g}_{reg} \times G^{\mathbb{C}}/B.$$

So on $\widehat{\mathfrak{g}}_{reg}$ we have a sheaf of Borel subgroups $\underline{B}$ and one checks that

$$C_{\mathfrak{g}^{\mathbb{C}}}|_{\widehat{\mathfrak{g}}_{reg}} \hookrightarrow \underline{B} \twoheadrightarrow \widehat{\mathfrak{g}}_{reg} \times D^{\mathbb{C}} \tag{4.29}$$

is $G^{\mathbb{C}}$-invariant, so it induces a morphism $J_{\mathfrak{g}} \hookrightarrow (\pi_{\mathfrak{d}})_*^W D^{\mathbb{C}}$. In order to finish the proof of the theorem, it is enough to check that the above embedding is Θ-equivariant.

The form being quasi-split it contains a σ-invariant Borel subgroup (cf. 1.1.18). It can be chosen such that $B^\theta = B^{op}$ is the opposed Borel subgroup. Indeed, this is done by choosing an ordering on the roots making $\mathfrak{a}^* > i\mathfrak{t}^*$ (cf. [48]), and noticing that the form being quasi split, there are no purely imaginary roots (cf. 1.1.19). In particular, $B \cap B^\theta = D^{\mathbb{C}}$.

Rewrite the morphism 4.29 as $C_{\mathfrak{g}}|_{\widehat{\mathfrak{g}}_{reg}} \to \underline{B}/[\underline{B}, \underline{B}] \cong \widehat{\mathfrak{g}}_{reg} \times D^{\mathbb{C}}$. Then, we have a commutative diagram

$$\begin{array}{ccccc} C_{\mathfrak{g}}|_{\widehat{\mathfrak{g}}_{reg}} & \longrightarrow & \underline{B}/[\underline{B}, \underline{B}] & \longrightarrow & \widehat{\mathfrak{g}}_{reg} \times D^{\mathbb{C}} \\ \Theta \downarrow & & \Theta \downarrow & & \Theta \downarrow \\ C_{\mathfrak{g}}|_{\widehat{\mathfrak{g}}_{reg}} & \longrightarrow & \underline{B^\theta}/[\underline{B^\theta}, \underline{B^\theta}] & \longrightarrow & \widehat{\mathfrak{g}}_{reg} \times D^{\mathbb{C}}. \end{array}$$

By Proposition 4.4.10, the restriction to the pertinent subscheme descends to a Θ equivariant morphism $J_{\mathfrak{g}}|_{\mathfrak{r}} \to (\pi_{\mathfrak{d}})_*^W D^{\mathbb{C}}|_{\mathfrak{r}}$, and the proof of Theorem 4.4.9 is complete. □

All the above essentially adapts to the twisted case. Fix L a holomorphic line bundle on a complex projective scheme X. Just as in the preceeding discussion, given $a : X \to \mathcal{A}_L(G^{\mathbb{C}})$ factoring through $\mathfrak{r}_L = \mathrm{Im}(\mathcal{A}_L(G) \to \mathcal{A}_L(G^{\mathbb{C}}))$, define the sheaf on $(\mathfrak{r}_L)_{an/AN}$

$$\mathcal{D}^L_{\widetilde{W}}(U) = \left\{ f : \widehat{U} \to D^{\mathbb{C}} \ : \ \begin{array}{c} w \circ f \circ w \equiv f, \\ \breve{\alpha}(f(\widehat{x})) = 1 \text{ for all } x \in D^U_\alpha \end{array} \right\}.$$

Corollary 4.4.13. *The image of the gerbe $\mathcal{H}iggs_L(G)^{reg}$ inside of $\mathcal{H}iggs_L(G^{\mathbb{C}})^{reg}$ is $\mathcal{D}^L_{\widetilde{W}}$-banded over $\mathfrak{r}_L$.*

Proof. The proof is very similar to that of Theorem 4.3.13. We just need to check that the regular locus of (4.3) is $\mathcal{D}_{\widetilde{W}}$-banded (rather, its image inside $\left[\mathfrak{g}_{reg}/G^{\mathbb{C}} \times \mathbb{C}^\times\right] \to \left[\mathfrak{d}^{\mathbb{C}}//W/\mathbb{C}^\times\right]$) and that the preceeding sheaf pullsback to $\mathcal{D}^L_{\widetilde{W}}$ by $[L] : X \to B\mathbb{C}^\times$. The first statement follows from the same arguments as above. As for the second, it is straightforward from the definition of $\mathcal{D}^L_{\widetilde{W}}$. □

Corollary 4.4.14. *If X is a curve and $2|\deg(L)$, $\mathcal{H}iggs_L(G)^{reg} \cong B\mathcal{D}^L_{\widetilde{W}}$.*

Proof. Under the given hypothesis, there is a global section given by the HKR section from Chapter 3.

4.4.2 Cocyclic description: cameral data.

In this section we characterise the fibers of the Hitchin map as categories of principal bundles on the cameral cover, much in the spirit of Theorem 4.2.6.

Lemma 4.4.15. *Let X be an irreducible smooth curve over $\mathfrak{r} \hookrightarrow \mathfrak{d}^{\mathbb{C}}//W$. Then $\widehat{X}$ decomposes as a union of irreducible components $\widehat{X} = \bigcup_{w\in W/W(\mathfrak{a})} w\widehat{X}_0$ where $\widehat{X}_0$ is the image of $\widetilde{X}$ inside $\widehat{X}$.*

Proof. The preimage $\pi_{\mathfrak{d}}^{-1}(\mathfrak{r}) = \bigcup_{w\in W/W(\mathfrak{a})} w\mathfrak{a}^{\mathbb{C}}$, which is clearly a decomposition by irreducible components, which are connected away from ramification. The result follows by smoothness of X over $\mathfrak{r}$. □

Definition 4.4.16. On $X \to \mathfrak{r}$ we define $\mathcal{R}_\theta = \otimes_{\alpha\in\Delta^+\cap\mathfrak{a}^*}\mathcal{R}_\alpha$.

Theorem 4.4.17. *Let $a \in \mathfrak{r}$, and let $Higgs_a(G)$ be the fiber of $[\chi]_{\mathbb{C}}^{-1}(a)$ intersected with the image of κ. Let $\mathcal{C}am_a(G)$ be the category whose objects are tuples $(P, \gamma, \underline{\beta})$ defined as in Theorem 4.2.6 with the extra condition that*

$$P|^{\theta}_{\widehat{U}_0} \cong P|_{\widehat{U}_0} \otimes \mathcal{R}_\theta. \tag{4.30}$$

Then, there is an equivalence of categories between $Higgs_a(G)$ and $\mathcal{C}am_a(G)$.

Lemma 4.4.18. *1. There exists $w_0 \in W$ whose action on $\mathfrak{d}^{\mathbb{C}}$ is equivalent to the action of θ.*

2. $\mathcal{R}_\theta \cong \mathcal{R}_{w_0}$ over any cover $\widehat{X}_a$, $a : X \to \mathfrak{r}_L$.

Proof. 1. Since the form is quasi split, there exists an anisotropic Borel subgroup. The ordering induced on the roots takes $\mathfrak{a}^* > i\mathfrak{t}^*$, so that θ takes a system of simple roots to another one. Whence the result.

2. By definition $\mathcal{R}_{w_0} \cong \otimes_{\alpha\in\Delta^+}\mathcal{R}_\alpha$. Now, every root can be expressed as $\alpha = \lambda + i\beta$, where $\lambda \in \mathfrak{a}^*$, $\beta \in \mathfrak{t}^*$. Note that β may vanish, but not α. Now, the root system is stable by θ, and choosing the appropriate order, we may assume that $\mathfrak{a}^* > i\mathfrak{t}^*$. Then, $\theta : D_\alpha \to D_{\theta\alpha}$ is an isomorphism, so

$$\breve{\theta\alpha}\mathcal{O}(D_{\theta\alpha}) \cong \breve{\theta\alpha}\mathcal{O}(D_\alpha) = \breve{\bar{\alpha}}\mathcal{O}(D_\alpha) \cong \breve{\alpha}(\mathcal{O}(D_\alpha)^{-1}$$

whenever $\alpha \notin \mathfrak{a}^*$. Thus, all terms of $\otimes_{\alpha\in\Delta^+}\mathcal{R}_\alpha$ cancel out except for real roots. $\square$

Proposition 4.4.19. *The cocycle* $\{\mathcal{R}_w \ : \ w \in W\} \in Z^1(W, B_{\widehat{\mathfrak{r}}}D^{\mathbb{C}})$ *extends to a cocycle* $\{\mathcal{R}_w \ : \ w \in \widetilde{W}\} \in Z^1(W, B_{\widehat{\mathfrak{r}}}D^{\mathbb{C}})$ *where*

$$\mathcal{R}_{(w,\theta)} = \bigotimes_{\{\alpha\in\Delta: w\circ\theta(\alpha)\in\Delta^-\}} \mathcal{R}_\alpha \tag{4.31}$$

and $B_{\widehat{\mathfrak{r}}}D^{\mathbb{C}}$ *is the category of* $D^{\mathbb{C}}$ *principal bundles on* $\widehat{\mathfrak{r}} := \mathfrak{g}_{reg} \times_{\mathfrak{r}} \mathfrak{d}^{\mathbb{C}}$.

Proof. The fact that $\{\mathcal{R}_w \ : \ w \in W\}$ is a cocycle is Lemma 5.4 in [25]. We need to check that

$$\mathcal{R}_{(w,\theta)} \cong \theta \cdot \mathcal{R}_w \otimes \mathcal{R}_\theta \tag{4.32}$$

Here, the action of θ on $\mathcal{R}_\alpha = \breve{\alpha}\left[\mathcal{O}(D_\alpha)\right]$ is as follows:

$$\theta \cdot \left(\breve{\alpha}\left[\mathcal{O}(D_\alpha)\right]\right) := -\theta^*\left(\breve{\alpha}\left[\mathcal{O}(D_\alpha)\right]\right)^\theta .$$

Now, let $w_0 \in W$ be the element described in Lemma 4.4.18. By definition and Lemma 5.4 in [25], it follows that $\mathcal{R}_{(w,\theta)} \cong w_0^*\mathcal{R}_w^{w_0} \otimes \mathcal{R}_\theta$. Thus, (4.32) will hold if and only if

$$(-\theta)^*\mathcal{O}(D_\alpha) \cong \theta^*\mathcal{O}(D_{-\theta\alpha})$$

which is certainly true as $-1 : D_\alpha \to D_{-\alpha}$ is an isomorphism. $\square$

4.4.2.1 The universal cameral cover and datum

Recall from [25] that the Springer–Grothendieck resolution $\widehat{\mathfrak{g}}_{reg}$ can be obtained as a fibered product in two ways:

$$\begin{array}{ccccc} \overline{G/T} & \longleftarrow & \widehat{\mathfrak{g}}_{reg} & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{G/N} & \longleftarrow & \mathfrak{g}_{reg} & \longrightarrow & \mathfrak{d}^{\mathbb{C}}//W. \end{array} \tag{4.33}$$

In the above $\overline{G/N} \subset Gr(r, \mathfrak{g}^{\mathbb{C}})$ is the subvariety of centralisers of regular elements in $\mathfrak{g}^{\mathbb{C}}$ and $\overline{G/T}$ is the corresponding incidence variety in $\overline{G/N} \times G^{\mathbb{C}}/B$ for some Borel subgroup B.

On $G^{\mathbb{C}}/B$ we have the bundle $G^{\mathbb{C}}/U \to G^{\mathbb{C}}/B$, where $U := [B, B]$ is the unipotent part of B. Its pullback to $\overline{G/T}$ is a principal $D^{\mathbb{C}}$-bundle P_u which can be endowed with the corresponding $\underline{\gamma} : N \to Aut_{\mathcal{R}}(P_u)$, $\underline{\beta}$ turning the triple $(P_u, \underline{\gamma}, \underline{\beta})$ into a cameral datum (cf. Theorem 4.2.6,[25].) In a similar fashion, its pullback defines a cameral datum on $\widehat{\mathfrak{g}}_{reg}$. The latter is more appropriate when working with twistings.

The importance of this universal datum is reflected in the following.

Proposition 4.4.20. *Consider the tensor product* $\left[\mathfrak{g}_{reg}/G^{\mathbb{C}}\right] \times_{\mathfrak{d}^{\mathbb{C}}/W} \mathfrak{d}^{\mathbb{C}}$*. Then:*

1. $\left[\mathfrak{g}_{reg}/G^{\mathbb{C}}\right] \times_{\mathfrak{d}^{\mathbb{C}}/W} \mathfrak{d}^{\mathbb{C}} \cong \left[\widehat{\mathfrak{g}}_{reg}/G^{\mathbb{C}}\right]$.

2. *There exists a principal* $D^{\mathbb{C}}$ *bundle* $\mathcal{L} \to \left[\widehat{\mathfrak{g}}_{reg}/G^{\mathbb{C}}\right]$ *such that given* $[(E,\phi),b] : X \to \left[\widehat{\mathfrak{g}}_{reg}/G^{\mathbb{C}}\right]$*, corresponding to a pair* $(E,\phi) \to X$*,* $\widehat{X}_b \to X$*, we have that* $\mathcal{L}|_{\widehat{X}_b} \cong P_E$ *is the cameral datum corresponding to* (E,ϕ)*.*

Proof. It follows from Donagi–Gaitsgory's [25]. □

Now, let B be a maximal θ-anisotropic Borel subgroup. These can be characterised as Borels such that B^θ is the opposed Borel subgroup. Denote by $\mathcal{B}_a$ the subset of θ-anisotropic Borel subgroups. It is in fact a subvariety, as it follows from Vust's work [77] that it can be identified with $H^{\mathbb{C}}/C_H(\mathfrak{a})$.

Lemma 4.4.21. *The image* $\widehat{\mathfrak{m}}_{reg} \to G^{\mathbb{C}}/B$ *consists of the subvariety of* θ*-anisotropic Borel subgroups.*

Proof. By definition, $\widehat{\mathfrak{m}}_{reg} \subset \widehat{\mathfrak{g}}_{reg}$ is the subvariety of fixed points by $-\theta$. On the level of Borel subalgebras, $-\theta$ acts by taking $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\theta\alpha}$. Thus, $-$ is the operation consisting in taking the opposed subalgebra, and θ acts as usual. This means that fixed points in $G^{\mathbb{C}}/B$ are exactly θ-anisotropic Borel subgroups. □

Proposition 4.4.22. *Let* $\mathcal{U} \to \mathcal{B}$ *denote the universal* $D^{\mathbb{C}}$*-bundle over* $\mathcal{B}$*. By choosing a Borel subgroup* B*, this is identified with* $G^{\mathbb{C}}/U \to G^{\mathbb{C}}/B$*. We have that*

$$\theta^*(\mathcal{U})|_{\mathcal{B}_a} \cong \mathcal{U}|_{\mathcal{B}_a}. \tag{4.34}$$

Proof. Choose $B \in \mathcal{B}_a$ an anisotropic Borel subgroup. We have a θ-equivariant morphism $G^{\mathbb{C}}/B \to G^{\mathbb{C}}/B^\theta$, which composed with the respective isomorphisms $G^{\mathbb{C}}/B \cong \mathcal{U} \cong G^{\mathbb{C}}/B^\theta$ yields a $G^{\mathbb{C}}$ equivariant morphism $\theta : \mathcal{B} \to \mathcal{B}$. There exists a morphism

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\theta} & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow[\theta]{} & \mathcal{B} \end{array}$$

via which the fiber over $\mathrm{Ad}_g B$ is sent to the fiber over $\mathrm{Ad}_{g^\theta} B^\theta$. We use this to build a $D^{\mathbb{C}}$-equivariant morphism $\mathcal{U} \to \theta^*\mathcal{U}$ over $\mathcal{B}_a$. Note that the action of $D^{\mathbb{C}}$ on $\theta^*\mathcal{U}$ is twisted by θ. So proving the existence of such a morphism is equivalent to proving that the structure group of $\mathcal{U}|_{\mathcal{B}_a}$ reduces to $T^{\mathbb{C}}$. Now, if $gB \in \mathcal{B}_a$ then $gD^{\mathbb{C}}g^{-1}$ is

θ-invariant. Not only that, but the isomorphism $B/[B,B] \to D^{\mathbb{C}}$ is θ-equivariant. So consider the section

$$G^{\mathbb{C}}/B \to G^{\mathbb{C}}/U \times_{D^{\mathbb{C}}} D^{\mathbb{C}}/T^{\mathbb{C}} \cong G^{\mathbb{C}}/B \times D^{\mathbb{C}}/T^{\mathbb{C}}$$

sending $gB \mapsto gD^{\mathbb{C}}g^{-1}/\left(gD^{\mathbb{C}}g^{-1}\right)^{\theta} \cong A^{\mathbb{C}} = e \cdot T^{\mathbb{C}}$. □

4.4.2.2 Proof of Theorem 4.4.17

We can now complete the proof of our theorem. First of all, we must check that over $\mathfrak{r}$, the stack $\mathcal{C}am(G)$ is non-empty. By Proposition 4.4.19 and Theorem 4.2.6, it is enough to check condition (4.31). Consider $w_0 \in W$ as in Lemma 4.4.18. By Theorem 4.2.6, we know that globally over $\widehat{\mathfrak{g}}_{reg}$

$$P_u \cong w_0^* P^{w_0} \otimes \mathcal{R}_{w_0} = \theta^* P^{\theta} \otimes \mathcal{R}_{\theta}.$$

So for (4.31) to hold it suffices to check $\theta^* P_u \cong P_u$. This follows from Proposition 4.4.22.

Now, the above implies in particular that $\mathcal{C}am(G)$ is a subgerbe of $\mathcal{C}am$ (cf. Theorem 4.2.6). The morphism $\mathcal{H}iggs^{abs}(G^{\mathbb{C}})^{reg} \to \mathcal{C}am$ maps $\mathcal{H}iggs^{abs}(G)^{reg} \to \mathcal{C}am(G)$. Indeed, this was built as follows: a Higgs pair (E, ϕ) induces by pullback of $\mathcal{L} = \widehat{\mathfrak{g}}_{reg} \times_{G/B} \mathcal{U} \to \widehat{\mathfrak{g}}_{reg}$ a cameral datum on $\widehat{E}$, which in turn is seen to descend. Now, if ϕ factors through $\mathfrak{m}_{reg}$, Proposition 4.4.22 implies that $\mathcal{L}$, and so also the cameral datum on $\widehat{E}$, satisfy (4.31). So we have a morphism between two subgerbes of the isomorphic gerbes $\mathcal{H}iggs^{abs}(G^{\mathbb{C}})$ and $\mathcal{C}am$. We need to check the automorphisms of a cameral datum for G-Higgs pairs are exactly the automorphisms of the associated complex cameral datum which are θ-equivariant. But Proposition 4.4.22 implies that this is indeed the case.

4.4.2.3 The rank one case

When the real rank of the real form is one, the analysis is particularly simple, and it is worth to include it here.

Proposition 4.4.23. *Theorem 4.4.17 holds for all real forms of real rank one if and only if it holds for split real forms of rank one. Real rank one connected quasi-split forms of strongly reductive groups include $SU(2,1)$, $U(2,1)$, $SO(3,1)_0$, $SO(2,1)_0 \cong PSL(2,\mathbb{R})$, $SL(2,\mathbb{R})$, amongst which the last two are split.*

Proof. Let (P, γ, β) be a cameral datum for a G-Higgs bundles. We want to check that $P|_{\widehat{U}_0} \cong P|^{\theta}_{\widehat{U}_0} \otimes \mathcal{R}_\theta$. Now, consider $P \otimes \left(P^\theta\right)^{-1}$. It is an $A^{\mathbb{C}} \cong \mathbb{C}^\times$ principal bundle, as

$$\left(P \otimes \left(P^\theta\right)^{-1}\right)^\theta \cong \mathcal{R}_\theta{}^\theta \cong \mathcal{R}_\theta^{-1} \cong \left(P \otimes \left(P^\theta\right)^{-1}\right)^{-1}.$$

Note also that $\deg P \otimes \left(P^\theta\right)^{-1}$ is even, so that for some $A^{\mathbb{C}}$ bundle Q, we have that

$$P \otimes \left(P^\theta\right)^{-1} \cong Q \otimes \left(Q^\theta\right)^{-1} = Q^2.$$

Moreover, the only roots involved in the definition of $\mathcal{R}_\theta$ are real roots, so if P is a cameral datum for G, then Q is a cameral datum for $\widetilde{G}$. $\square$

So to check Theorem 4.4.17 for all rank one quasi split forms, i is enough to consider the cases $\mathrm{SL}(2, \mathbb{R})$ and $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$. We recall first some preliminaries from [25].

Rank one groups: the universal cameral cover and datum Let $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$, B the upper triangular matrices, U the upper triangular unipotent matrices, $D^{\mathbb{C}} = \mathbb{C}^\times$ the diagonal torus. We have the following.

- $\overline{G/T} \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\overline{G/N} \cong \mathbb{P}^2$, and $\overline{G/T} \to \overline{G/N}$ is the canonical projection.
- $G^{\mathbb{C}}/B \cong \mathbb{P}^1$, $G^{\mathbb{C}}/U \to G^{\mathbb{C}}/B \cong \mathcal{O}(-1) \to G^{\mathbb{C}}/B$.
- $\overline{G/T} \to G^{\mathbb{C}}/B$ is the projection $p_2 \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Now, the involution of the double cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ exchanges both factors. Note that the form being split, this involution is exactly the action of θ. The pullback of $\mathcal{O}(-1)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is, away from ramification, isomorphic to

$$P'_u := p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}. \tag{4.35}$$

Indeed, $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{P}^1$ consists of all centralisers except the nilpotent, so it maps to $\mathbb{P}^1 \setminus \{0\} \cong \mathbb{C}^2$. Over the latter, $\mathcal{O}(1) \cong f^* \mathcal{O}(-1)$ where $f : \ z \mapsto z^{-1}$; in terms of Borel subalgebras, it takes one to its opposed. But that means that the diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\theta} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1 \end{array}$$

is commutative. Whence the isomorphism $p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O} \cong p_2^* \mathcal{O}(-1)$ away from ramification. Not only that, but clearly $p_2^* \mathcal{O}(-1) \cong p_1^* \mathcal{O} \otimes p_2^* \mathcal{O}(-1)$, so that away from ramification we obtain $P'_u \cong \theta^* P'^{\theta}_u$. Donagi and Gaitsgory prove that the preceeding isomorphism extends to ramification. The corresponding line bundle is then $\mathcal{O}(D) = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$.

Let P_u be the pullback of P'_u to $\widehat{\mathfrak{g}}_{reg}$.

Rank one split real forms

Proposition 4.4.24. *Let G be either $SL(2,\mathbb{R})$ or $\mathbb{P}SL(2,\mathbb{R})$. Then $P_u \cong P_u^{\theta} \otimes \mathcal{R}_{\theta}$ over $\widehat{\mathfrak{m}}_{reg} := \mathfrak{m}_{reg} \times_{\mathfrak{r}} \mathfrak{a}$.*

Proof. From the discussion in the preceeding paragraph, we see that $P_u \cong P_u^{\theta} \otimes \mathcal{R}_{\theta} \iff P_u \cong \theta^* P_u$, as $P \cong \theta^* P_u^{\theta} \otimes \mathcal{R}_{\theta}$ and $\mathcal{R}_{\theta} \cong \mathcal{R}_{\theta}$ by definition. So we need to check that, over $\widehat{\mathfrak{m}}_{reg}$,

$$p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O} \cong p_1^*\mathcal{O} \otimes p_2^*\mathcal{O}(1). \tag{4.36}$$

From Section B.1 we have $H^{\mathbb{C}} \cong \mathrm{SO}(2,\mathbb{C}) \cong \mathbb{C}^{\times}$. We take a different realization taking the embedding by conjugating the involutions by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This identifies $\mathrm{SO}(2,\mathbb{C})$ to the diagonal torus and $\mathfrak{m}^{\mathbb{C}}$ to the antidiagonal matrices. Now, the image of $\mathfrak{m}_{reg} \to \overline{G/N}$ is exactly the scheme $\overline{H/N_H(\mathfrak{a})}$ from Proposition 1.4.6, which in this case is identified to $\mathbb{P}^1$. Regarding the cover $\widehat{\mathfrak{m}}_{reg}$, in this case, $\mathfrak{r} = \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$, as the form is split. Moreover, since the real and the total rank match, we have that $\mathrm{Im}(\widehat{\mathfrak{m}}_{reg} \to \overline{G/T}) = \overline{G/T}|_{\overline{H/N_H(\mathfrak{a})}}$. The latter is a surface inside $\mathbb{P}^1 \times \mathbb{P}^1$, the incidence variety of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$. Its projection q_2 to $G^{\mathbb{C}}/B = \mathbb{P}^1$ consists of Borel subalgebras $\mathfrak{b}$ such that $\theta\mathfrak{b} = \mathfrak{b}^{op}$. All these are conjugate by $\mathrm{SO}(2,\mathbb{C})$ (cf. Section 1.4).

The torus $\mathrm{SO}(2,\mathbb{C})$ acts transitively on $H/N_H(\mathfrak{a}) \times H/C_H(\mathfrak{a})$ with the same weight, and the quotient is precisely $\overline{G/T}|_{\overline{H/N_H(\mathfrak{a})}}$ away from ramification. Compactifying we get a surface of type $(1,1)$ inside $\mathbb{P}^1 \times \mathbb{P}^1$, over which $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O} \cong p_1^*\mathcal{O} \otimes p_2^*\mathcal{O}(1)$ canonically.

A similar argument works for $\mathrm{PSL}(2,\mathbb{R})$, the difference being the lift of . □

Remark 4.4.25. *Note that there are two $D^{\mathbb{C}}$-bundles that descend: on the one hand P_u, on the other $P_u \otimes \mathcal{R}_{\theta}^{-1/2}$. Moreover $P_u \otimes \mathcal{R}_{\theta}^{-1/2}$ is invariant by θ, namely, its structure group reduces to $T^{\mathbb{C}}$. We will make use of these facts in our analysis of $SU(2,1)$-Higgs pairs in Chapter 5.*

So we have a morphism $F_a : \mathcal{H}iggs_a \to \mathcal{C}am_a$.

Proposition 4.4.26. *The above morphism is an equivalence.*

Proof. The proof of Theorem 4.4.17 adapts to this case. $\square$

Corollary 4.4.27. *Let G be a real form of real rank one. Then there exists a principal $D^{\mathbb{C}}$-bundle $\mathcal{L} \to \left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$ such that given $(E, \phi) : X \to \left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$, the corresponding cameral datum is $\phi^*\mathcal{L}$.*

Remark 4.4.28. *For $b \in \mathfrak{r}$, and an object $(P, \gamma, \beta) \in Higgs'_b$, P can be described by the following data: a $D^{\mathbb{C}}$ principal bundle P_w on each of the irreducible components $w\widehat{U}_0 \subset \widehat{U}_0$, and isomorphisms $f_{w,w'} : P_{w'}|_{w\widehat{U}_0 \cap w'\widehat{U}_0} \cong P_{w'}|_{w\widehat{U}_0 \cap w'\widehat{U}_0}$ satisfying the obvious compatibility conditions. We recover P as the kernel*

$$P \hookrightarrow \oplus_{w \in W/W(\mathfrak{a}^{\mathbb{C}})} P_w \stackrel{f-id}{\to} \oplus L_{w,w'}$$

where the last morphism is zero away from intersections of different components and $f - id : P_w|_{w\widehat{U}_0 \cap w'\widehat{U}_0} \oplus P_{w'}|_{w\widehat{U}_0 \cap w'\widehat{U}_0} \to P_{w'}|_{w\widehat{U}_0 \cap w'\widehat{U}_0}$ sends $(g, h) \mapsto (f_{w'w}g - h)$.

Remark 4.4.29. *For real rank one split forms, for $L = K$, the cameral cover is in fact the spectral cover (see [24, 41, 66]). A slight modification of Schaposnik's proof from [66] would yield the desired result for particular twistings for $SL(2, \mathbb{R}), \mathbb{P}SL(2, \mathbb{R})$.*

4.4.3 Cameral data for twisted Higgs pairs

On $\mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a})$ we may define another stack $\mathcal{C}am^L$ just as we :

Theorem 4.4.30. *Consider an $\mathfrak{r}_L$ scheme $a : X \to \mathfrak{r}_L$, and let $\mathcal{H}iggs_L(G)_a$ be the fiber $[\chi]_L^{-1}(a)$ in $\mathcal{H}iggs_L(G)^{reg}$. Define $\mathcal{C}am_a$ as the category whose objects are triples $(P, \underline{\gamma}, \underline{\beta})$ where $P \to \widehat{X}_a$ is a principal $D^{\mathbb{C}}$-bundle, and $\underline{\gamma}, \underline{\beta}$ are defined as in the local (abstract) case. Then, $\mathcal{H}iggs_L(G)_a$ and $\mathcal{C}am_a$ are equivalent categories.*

Proof. Let us check that there is a universal cameral datum for twisted pairs. Consider $\widehat{\mathfrak{m}_{reg} \otimes L}$. Unlike $\widehat{\mathfrak{g}}_{reg}$, The latter scheme is not a tensor product of $\overline{G/N}$ with $G^{\mathbb{C}}/B$, only locally so (over coverings $\{U_i\} \to X$ by trivialising open subsets for L). Now, we may work locally by observing that $\widehat{X}_a \hookrightarrow L \otimes \mathfrak{d}^{\mathbb{C}}$, so that the the identifications $L|_{U_i}|_{U_{ij}} \to L|_{U_j}|_{U_{ij}}$ also induces an identification $\phi_i^*\mathfrak{d}^{\mathbb{C}} =: \widehat{X}_j|_{U_{ij}} \to \widehat{X}_i|_{U_{ij}}$, so that the descent data $\{\widehat{X}_i; f_{ij}\}_i$, where f_{ij} is the corresponding glueing data, yields a cameral cover independently of the choice of a covering. With this, the result follows. $\square$

4.5 The non-abelian case

It is interesting to analyse the semisimple locus

$$\mathcal{H}iggs^{abs;reg,ss}(G) := \left[\mathfrak{m}_{reg,ss}/H^{\mathbb{C}}\right].$$

Note that this case only makes sense in the abstract setting, since as soon as we consider twisting by a non trivial line bundle, the Higgs field is forced to take values in non-semisimple orbits (in other words, the cameral cover must be ramified).

By a theorem of Kostant and Rallis' [51], an element in $\mathfrak{m}^{\mathbb{C}}$ is semisimple if and only if it is conjugate to an element of $\mathfrak{a}^{\mathbb{C}}$. In particular, regular semisimple elements are conjugate to elements of $\mathfrak{a}_{reg}$.

Lemma 4.5.1. *We have an isomorphism of stacks over the analytic site* $\left(\mathfrak{a}_{reg}/W(\mathfrak{a}^{\mathbb{C}})\right)_{an}$

$$[\mathfrak{a}_{reg}/N_H(\mathfrak{a})] \to \left[\mathfrak{m}_{reg,ss}/H^{\mathbb{C}}\right]$$

Proof. Consider the morphism of groupoids

$$\left[\mathfrak{a}^{\mathbb{C}}{}_{reg}/N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})\right] \to \left[\mathfrak{m}_{reg,ss}/H^{\mathbb{C}}\right]$$

given by the map of presentations

$$\begin{array}{ccc}
\mathfrak{a}^{\mathbb{C}}{}_{reg} \times N_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) & \rightrightarrows & \mathfrak{a}^{\mathbb{C}}{}_{reg} \\
\downarrow & & \downarrow \\
\mathfrak{m}_{reg,ss} \times H^{\mathbb{C}} & \rightrightarrows & \mathfrak{m}_{reg,ss}.
\end{array}$$

This morphism is clearly representable. It is also a morphism over the moduli space $\mathfrak{a}^{\mathbb{C}}{}_{reg}/W(\mathfrak{a}^{\mathbb{C}})$, so that it induces a morphism of stacks $[\mathfrak{a}_{reg}/N_H(\mathfrak{a})] \to \left[\mathfrak{m}_{reg}/H^{\mathbb{C}}\right]$ over the site $\left(\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a}^{\mathbb{C}})\right)_{an}$. The fact that inertia restricts to inertia implies that the morphism of fibered categories is essentially surjective. Furthermore, it is also faithfully flat for the same reason. □

Remark 4.5.2. *Note that in this particular case local non-emptiness does not need the use of Luna's slice theorem. By Lemma 4.5.1, it is enough to study the restriction of the Hitchin map* $[\chi]_{\mathfrak{a}^{\mathbb{C}}} : [\mathfrak{a}_{reg}/N_H(\mathfrak{a})] \to \mathfrak{a}_{reg}/W(\mathfrak{a}^{\mathbb{C}})$ *in order to describe it over the whole of the regular semisimple locus. But our question becomes much easier in this context, as the stack will be locally non empty as long as the* $W(\mathfrak{a}^{\mathbb{C}})$ *principal bundle* $\mathfrak{a}_{reg} \to \mathfrak{a}_{reg}/W(\mathfrak{a}^{\mathbb{C}})$ *locally lifts to a principal* $N_H(\mathfrak{a})$ *bundle over* $\mathfrak{a}_{reg}$. *Now,* $\mathfrak{a}_{reg}$ *is locally isomorphic to* $U \times W(\mathfrak{a}^{\mathbb{C}})$ *for some* $U \subset \mathfrak{a}_{reg}/W(\mathfrak{a}^{\mathbb{C}})$, *and so over such open* *subsets we can consider* $U \times N_H(\mathfrak{a}) \to U \times W(\mathfrak{a}^{\mathbb{C}})$*, the map induced by the defining sequence*

$$1 \to C_H(\mathfrak{a}) \to N_H(\mathfrak{a}) \to W(\mathfrak{a}^{\mathbb{C}}) \to 1.$$

The following proposition follows easily from Proposition 4.3.5:

Proposition 4.5.3. *There is an isomorphism of stacks over* $\mathfrak{a}_{reg}$

$$[\mathfrak{a}_{reg}/N_H(\mathfrak{a})] \times_{\mathfrak{a}_{reg}/W(\mathfrak{a})} \mathfrak{a}_{reg} \cong BC_H(\mathfrak{a})$$

Proof. Indeed, elements of the fibered product over $x : X \to \mathfrak{a}_{reg}/W(\mathfrak{a})$ are pairs $((P,\phi), f)$ where:

- $f : \widehat{X} \to \mathfrak{a}_{reg}$ which is a pullback of $x : X \to \mathfrak{a}_{reg}/W(\mathfrak{a})$ via the projection $\pi_{\mathfrak{a}} : \mathfrak{a}_{reg} \to \mathfrak{a}_{reg}/W(\mathfrak{a})$.
- $(P,\phi) \in [\mathfrak{a}_{reg}/N_H(\mathfrak{a})]\,(X)$ satisfies that $\pi_{\mathfrak{a}} \circ \phi = x$.

Now, ϕ induces a principal $C_H(\mathfrak{a})$-bundle $P \to \mathfrak{a}_{reg}$. Thus, the pullback of P to $\widetilde{X} = X \times_x \mathfrak{a}_{reg}$ is a principal $C_H(\mathfrak{a})$-bundle. The construction is functorial since so are fibered products.

For the quasi-inverse, taking $P \to \widetilde{X}$ a principal $C_H(\mathfrak{a})$-bundle over $\widetilde{X} = X \times_x \mathfrak{a}_{reg}$, then $(P \to \widetilde{X} \to X, P \to \widetilde{X} \to \mathfrak{a}_{reg}) \in [\mathfrak{a}_{reg}/N_H(\mathfrak{a})]\,(X)$. □

The above means that over the cameral cover, and away from ramification, the gerbe of abstract Higgs bundles trivialises.

Proposition 4.5.4. *Let* $a : X \to \mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a})$ *be an* $\mathfrak{a}^{\mathbb{C}} \otimes L//W(\mathfrak{a})$*-projective curve. Let* $X_{reg} \subseteq X$ *be the dense open set of points mapping to* $\mathfrak{a}_{reg} \otimes L//W(\mathfrak{a})$*. Let* $\widetilde{X}_{reg} = X_{reg} \times_a \mathfrak{a}^{\mathbb{C}} \otimes L$*. Then the set of isomorphism clases of* L*-twisted* G*-Higgs bundles on* X_{reg} *is isomorphic to* $H^1(\widetilde{X}_{reg}, C_H(\mathfrak{a}))$*.*

Remark 4.5.5. *The same argument works for any real form. The difference in the quasi-split case is that we can descend the category of principal bundles on the cameral cover to X.*

The following tables show the groups of centralisers for all real forms of simple classical Lie groups.

Table 4.1: Quasi-split forms

G	$C_H(\mathfrak{a})$
$\mathrm{SL}(p,\mathbb{R})$	$\mathbb{Z}_2^{p-1}$
$\mathrm{SU}(p,p)$	$\mathbb{Z}_2^{p-1} \ltimes (\mathbb{C}^\times)^{p-1}$
$\mathrm{SU}(p,p+1)$	$\mathbb{Z}_2^{p} \ltimes (\mathbb{C}^\times)^{p}$
$\mathrm{SO}(p+1,p)$	$\mathbb{Z}_2^{p}$
$\mathrm{Sp}(2p,\mathbb{R})$	$\mathbb{Z}_2^{p}$
$\mathrm{SO}(p,p)$	$\mathbb{Z}_2^{p}$
$\mathrm{SO}(p,p+2)$	$\mathbb{Z}_2^{p} \ltimes (\mathbb{C}^\times)^{p}$

Table 4.2: Non quasi-split forms

G	$C_H(\mathfrak{a})$
$\mathrm{SU}^*(2p)$	$\mathrm{Sp}(2,\mathbb{C})^p$
$\mathrm{SU}(p,q),\ p \geq q+2$	$(\mathbb{C}^\times)^q \times \mathrm{SL}(p-q,\mathbb{C})$
$\mathrm{SO}(p,q),\ p+q$ odd, $p \geq q+3$	$(\mathbb{C}^\times)^q \times \mathrm{SL}(p-q,\mathbb{C})$
$\mathrm{Sp}(2p,2q)$	$\mathrm{Sp}(2(p-q),\mathbb{C}) \times (\mathrm{Sp}(2,\mathbb{C}))^q$
$\mathrm{SO}(p,q), p+q$ even, $p \geq q+2$	$(\mathbb{C}^\times)^q \times \mathrm{SL}(p-q,\mathbb{C})$
$\mathrm{SO}^*(2p)$	$\mathrm{Sp}(2,\mathbb{C})^k$ if $p=2k$
	$\mathrm{Sp}(2,\mathbb{C})^k \times \mathrm{SO}(2,\mathbb{C})$ if $p=2k+1$

Remark 4.5.6. *1. For SU(p,q), SO(p,q), that is, non-quasi split forms of non-symplectic nature, $C_H(\mathfrak{a})$ consists of an abelian piece C_a and a non-abelian piece C_{na}. The abelian part is contained in a torus of $H^{\mathbb{C}}$. This implies in particular that cameral data too should consist of an abelian and a non-abelian part.*
2. Note that for groups defined over the quaternions, namely SU$^*(2p)$*, Sp*(p,p) *and SO*$^*(2p)$*, the groups consist of* $\dim\mathfrak{a}$ *copies of SL*$(2,\mathbb{C}) \cong$ *Sp*$(2,\mathbb{C})$*. This is consistant with Hitchin–Schaposnik's results [43], as the (complex) cameral cover for classical simple Lie groups is a finite cover of the spectral curve, an analysis of which induces the same kind of result for real cameral covers and spectral covers considered by Hitchin and Schaposnik (cf. [24, 23, 43]).*

4.6 Future directions

4.6.1 Intrinsic description of the fibration

Using the preceeding results, an intrinsic description of the real fibration $\mathcal{H}iggs^{abs}(G) \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$ is at hand. First of all, by identifying an alternative description of the band in terms of centralisers of semisimple elements. The following shows this is indeed possible:

Proposition 4.6.1. *There exists an embedding $J_{\mathfrak{m}} \to (\pi_{\mathfrak{a}})_* C_H(\mathfrak{a})$.*

Proof. The proof is the same as for the complex case. Indeed, on $\overline{H/C_H(\mathfrak{a})}$ (cf. Section 1.4) we have a sheaf of θ-anisotropic Borel subgroups into whose pullback to $\widetilde{\mathfrak{m}}_{reg}$ $C_{\mathfrak{m}}|_{\widetilde{\mathfrak{m}_{reg}}}$ embeds. The rest of the arguments parallel the complex case. □

So Theorem 4.4.9 implies that it is enough to study the exact sequence of sheaves over $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$

$$1 \longrightarrow \mathcal{K} \longrightarrow (\pi_{\mathfrak{a}})_*^{W(\mathfrak{a})} C_H(\mathfrak{a}) \longrightarrow f^*(\pi_{\mathfrak{d}})_*^W D^{\mathbb{C}}.$$

The question is now to describe $\mathcal{K}$. This involves in particular finding the kernel $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}) \to \mathfrak{r}$. We can answer to this over $\mathfrak{a}_{reg}//W(\mathfrak{a})$, as in this case the group normalising elements in $\mathfrak{a}_{reg}$ is $N_W(\mathfrak{a})$. Note that $\mathfrak{a}^{\mathbb{C}}$ is the gerbe $BW(\mathfrak{a})$ (BN_W) over $\left[\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a})\right]\left(\left[\mathfrak{a}^{\mathbb{C}}/N_W|_{\mathfrak{a}^{\mathbb{C}}}\right]\right)$. Moreover, it is a classical result that $W(\mathfrak{a})$ fits into an exact sequence

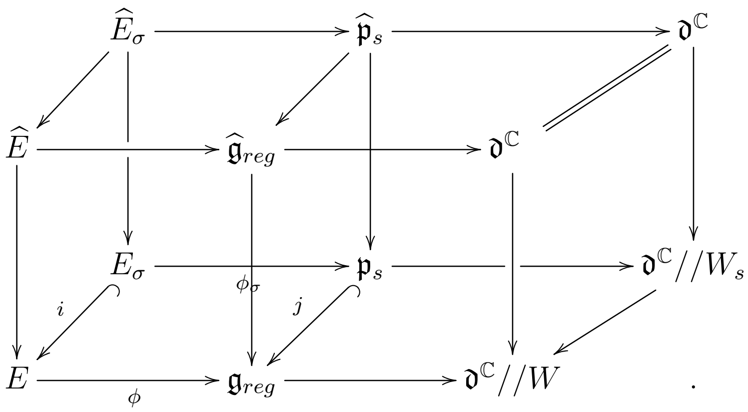
$$1 \to C_W(\mathfrak{a}) \to N_W(\mathfrak{a}) \to W(\mathfrak{a}) \to 1 \tag{4.37}$$

where $N_W(\mathfrak{a})$ ($C_W(\mathfrak{a})$) is the normaliser (centraliser) of $\mathfrak{a}^{\mathbb{C}}$ inside of W. This implies that $[\mathfrak{a}_{reg}/W(\mathfrak{a})] \cong BC_W(\mathfrak{a})$ over $BN_W(\mathfrak{a}) \cong [\mathfrak{a}_{reg}/N_W(\mathfrak{a})] \cong [\mathfrak{a}_{reg}/N_W]$. Since $[\mathfrak{a}_{reg}/W(\mathfrak{a})] \cong \mathfrak{a}_{reg}//W(\mathfrak{a})$, away from the branching locus in $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$ we have that $\mathfrak{a}^{\mathbb{C}}_{reg}//W(\mathfrak{a}) \to \mathfrak{r}$ is the gerbe $BC_W(\mathfrak{a})$.

4.6.2 Moduli spaces

It is known that (semi)stability notions of spectral data and vectorial Higgs bundles coincide. This was proved by Simpson in [74], Corollary 6.9. In our setting, Ngô addressed the question by defining an open subset of $\mathfrak{d}^{\mathbb{C}}//W$ over which the fibers contain but stable bundles.

This is already interesting in the complex group case. Note that it makes sense to consider a notion of (poly,semi)-stability on cameral data, as they consist of a principal abelian torsors with extra data. Moreover, Donagi–Gaitsgory [25] already proved that localization produces a reduction of the cameral cover to a cameral cover related to a Levi subgroup. On the other hand, a reduction to a parabolic produces a reduction of the cameral datum to the corresponding parabolic cameral cover (that is, the quotient of the corresponding cameral cover by the Weyl group of the parabolic) as expressed by the following diagram.



All of the above points to a plausible relation between stability notions of cameral data and Higgs pairs that we will undertake in the near future.

4.6.3 Non-abelian case: extension to ramification

Proposition 4.5.4 describes objects of the fiber away from ramification of the cameral cover. Two main questions arise in order to obtain a global statement:

1. In the first place, we are dealing with a non-abelian gerbe. This means that it need not be banded (see Appendix A). The first step is to elucidate whether there exists some sheaf of groups on the base $\mathcal{C} \to \mathfrak{a}//W(\mathfrak{a})$ such that $Band(\mathcal{H}iggs^{abs,reg}) = \mathrm{Out}(\mathcal{C})$.

Lemma 4.6.2. $\mathrm{Out}(C_{\mathfrak{m}})$ *descends uniquely to* $\mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$.

By definition, the band is an $\mathrm{Out}(C_{\mathfrak{m}})$-torsor, so whatever the sheaf $\mathcal{C}$ is, $\mathrm{Out}\chi^*\mathcal{C} \cong \mathrm{Out}C_H(\mathfrak{a})$.

2. If the answer to the previous point is positive, it makes sense to ask whether the band trivialises over the cameral cover. If the answer is negative, we can still consider the pullback gerbe $\left[\widehat{\mathfrak{m}}_{reg}/H^{\mathbb{C}}\right] \to \mathfrak{a}^{\mathbb{C}}$ and check whether it is banded. Our argument for the semisimple case follows, this line, as we prove that the pullback to the cameral cover $\mathfrak{a}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$ of $\mathcal{H}iggs^{abs;reg,ss}$ is a $C_H(\mathfrak{a})$-banded gerbe.

Chapter 5

Cameral data for SU$(2,1)$-Higgs bundles

5.1 Some Lie theoretical lemmas

The Cartan involution θ of $\mathfrak{su}(2,1)$ (see Section B.3) induces the decomposition

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{s}(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C})) \oplus \mathfrak{m}^{\mathbb{C}}$$

where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C}) \ |B \in \ Mat_{2\times 1}(\mathbb{C}),\ C \in \ Mat_{1\times 2}(\mathbb{C}) \right\}$$

One calculates easily that

$$\mathfrak{a}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C}) \ |a \in \mathbb{C} \right\}$$

is the maximal anisotropic Cartan subalgebra.

In this case $H^{\mathbb{C}}$ is realised as the subgroup of matrices of $\mathrm{SL}(3,\mathbb{C})$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}$$

Lemma 5.1.1. *An element*

$$\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ w & z & 0 \end{pmatrix} \in \mathfrak{m}$$

is nilpotent, if and only if $uw + vz = 0$.

Lemma 5.1.2.

$$\mathfrak{m}_{reg} = \left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ w & z & 0 \end{pmatrix} \ |\ uw \neq 0 \ or\ uz \neq 0 \ or\ vw \neq 0 \ or\ vz \neq 0 \right\}$$

Lemma 5.1.3. *There exists θ' a linear involution of $\mathfrak{sl}(3,\mathbb{C})$ which is* Inn $SL(3,\mathbb{C})$*-conjugate to θ. For the corresponding realization of $\mathfrak{su}(2,1)$ we have that $(\mathfrak{a}')^{\mathbb{C}} \oplus (\mathfrak{t}')^{\mathbb{C}}$ is a θ'-invariant maximally anisotropic Cartan subalgebra. Here*

$$(\mathfrak{a}')^{\mathbb{C}} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \; : \; a \in \mathbb{C} \right\}, \tag{5.1}$$

$$(\mathfrak{t}')^{\mathbb{C}} = \left\{ \begin{pmatrix} b & 0 & 0 \\ 0 & -2b & 0 \\ 0 & 0 & b \end{pmatrix} \; : \; b \in \mathbb{C} \right\}. \tag{5.2}$$

Proof. See [38], Section 12.3.2. □

5.2 SU(2,1)-Higgs bundles and the Hitchin map

Let X be a connected smooth projective curve over $\mathbb{C}$, and let $K = K_X$ be its canonical bundle.

An SU(2,1)-Higgs bundle is a pair $(V \oplus W, \phi)$ consisting of:

- A holomorphic rank 2 bundle V and a holomorphic line bundle W such that $\det(V \oplus W) \cong \mathcal{O}_X$. In particular, $\deg V = -\deg W$.

- A Higgs field $\phi \in H^0(X, V \otimes W^* \otimes K \oplus W \otimes V^* \otimes K)$, whose matrix form is

$$\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$

Definition 5.2.1. (cf. [13]) An SU(2,1) Higgs bundle is **semistable** if for all $V' \oplus W' \subsetneq V \oplus W$ such that $\phi : V' \oplus W' \to V' \oplus W' \otimes K$ it holds that $\deg V' \oplus W \leq 0$. It is **stable** if the inequality is stric, and **polystable** if it is semistable and it decomposes as a direct sum of stable SU(2,1)-bundles of degree 0.

The Hitchin map restricted to the image of $\kappa : \mathcal{M}(\mathrm{SU}(2,1)) \to \mathcal{M}(\mathrm{SL}(3,\mathbb{C}))$ reads

$$h_{\mathbb{C}} : \kappa(\mathcal{M}(\mathrm{SU}(2,1))) \to H^0(X, K^2) \oplus H^0(X, K^3), \quad \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \mapsto (\mathrm{tr}(\beta \wedge \gamma), 0)$$

As for the real Hitchin map, $\mathfrak{a}^{\mathbb{C}} \otimes K \cong K$, $\mathfrak{a}^{\mathbb{C}} \otimes K/\mathbb{Z}_2 \cong K^2$, so that, with the notation from the previous chapter, $\mathfrak{r}_K \cong \mathfrak{a}^{\mathbb{C}} \otimes K/\mathbb{Z}_2 \cong K^2$, so that it takes the form

$$h_{\mathbb{R}} : \mathcal{M}(\mathrm{SU}(2,1)) \to H^0(X, K^2), \quad \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \mapsto \mathrm{tr}(\beta \wedge \gamma)$$

5.3 Smoothness and regularity

The relation between regularity and smoothness of points of the complex Hitchin fiber has been known for a while, and it essentially goes back to Kostant's [50], as it is proved by Biswas and Ramanan ([9],Theorem 5.9). This proof applies to the real case, so we have:

Proposition 5.3.1. *If a pair (E,ϕ) is a smooth point of $h_{\mathbb{R}}^{-1}(\omega)$, then $\phi(x)\in\mathfrak{m}_{reg}$ for all $x\in X$.*

Proof. Let us recall the proof, which will be instructive in what follows: fixing $x\in X$, we have that $\mathrm{ev}_x\circ h(E,\phi)=\chi\phi_x$, where $\chi:\mathfrak{m}^{\mathbb{C}}\to\mathfrak{a}^{\mathbb{C}}/W(\mathfrak{a})$ is the Chevalley map. At a smooth point of the fiber, dh is surjective, and since ev_x is surjective too, it follows that $d(\chi\circ\mathrm{ev}_x)$ is itself surjective. Since $d\mathrm{ev}_x: H^0(X,E(\mathfrak{m}^{\mathbb{C}}\otimes K))\to\mathfrak{m}^{\mathbb{C}}\otimes K_x$ is surjective, and is itself evaluation at x, this implies that $d_{\phi_x}\chi$ is surjective. But Kostant's work implies this happens if and only if ϕ_x is regular. $\square$

Remark 5.3.2. *The proof applies to all twistings by line bundles L other than the canonical.*

Proposition 5.3.3. *Let $(E,\phi)\in\mathcal{M}(SU(2,1))$ be strictly polystable. Then:*
1. $E=V_1\oplus V_2\oplus W$ and the Higgs field has the form

$$\phi=\begin{pmatrix}0&0&0\\0&0&b\\0&d&0\end{pmatrix}.$$

2. (E,ϕ) belongs to the image of $\mathcal{M}(SL(2,\mathbb{R}))$ by a morphism of the following form:

$$I_F(L,(b,c))\mapsto\left[F\oplus F^{-1/2}\otimes L\oplus F^{-1/2}\otimes L^{-1},\begin{pmatrix}0&0&0\\0&0&b\\0&c&0\end{pmatrix}\right],$$

where F is some element in $Pic^0(X)$. The above is a map of $H^0(X,K^2)$-schemes.

Proof. 1. Follows easily from the definitions.

2. By 1., we only need to check that if $E=V_1\oplus V_2\oplus W$, the bundles have the desired form.

Indeed, $\deg V_1=0$ and either $\deg V_2\leq 0$ or $\deg W\leq 0$. Letting $L=W\otimes V_1^{1/2}$, we are done. $\square$

Corollary 5.3.4. *Given any point $\omega\in H^0(X,K^2)$, there exists a strictly polystable $SU(2,1)$-Higgs pair in the corresponding fiber of the Hitchin map.*

Proof. Use the Hitchin section for $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles. $\square$

Remark 5.3.5. *The above corollary constitutes yet another difference between the real and the complex group cases. Indeed, by a result by D. Arinkin (unpublished) and B.C. Ngô [60], over cameral covers corresponding to sections $b : X \to \mathcal{A}_{G^{\mathbb{C}},K}$ which intersect ramification of $\mathfrak{d}^{\mathbb{C}} \otimes K \to \mathfrak{d}^{\mathbb{C}} \otimes K//W$ transversally, all Higgs bundles are stable. However, by the preceeding corollary, no matter how nice the real cameral cover may be, the corresponding fiber of $h_{\mathbb{R}}$ fails to be contained in $\mathcal{M}(SU(2,1))^{stable}$.*

Lemma 5.3.6. *Any regular polystable $SU(2,1)$-Higgs bundle is stable.*

Proof. Let $(E = V \oplus W, \phi)$ be a polystable $\mathrm{SU}(2,1)$-pair, and suppose it is regular at every point. If it were strictly polystable, then Proposition 5.3.3 implies that the Higgs field would have form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & d & 0 \end{pmatrix}.$$

Now, by Lemma 5.1.2, for the field to be everywhere regular, we must have $(b) \cap (d) = \emptyset$. Namely, b, d must be constant and with the notation of Proposition 5.3.3, $V_2 \cong WK$, $W \cong V_2K$, which is not possible by degree considerations. $\square$

Proposition 5.3.7. *Regular points are smooth inside $\mathcal{M}(SU(2,1))$.*

Proof. Regular points are stable by Lemma 5.3.6, with non-vanishing Higgs field by regularity, thus they are simple by A.13 [15]. Also, the corresponding $\mathrm{SL}(3,\mathbb{C})$-Higgs bundle is stable, so by Proposition 2.5.5, the pair is smooth. $\square$

In general smoothness in the Hitchin fiber and inside the moduli space do not match, although for complex groups, the former implies the latter by integrability of the Hitchin system. However, by the preceeding discussion, under the regularity condition, they do coincide.

Corollary 5.3.8. *Given $(E,\phi) \in \mathcal{M}(SU(2,1))^{reg}$, then (E,ϕ) is a smooth point of the moduli space if and only if it is smooth in the corresponding Hitchin fiber.*

5.4 Cameral data

In order to describe the cameral data, we find it more convenient to use the realization of $\mathrm{SU}(2,1)$ given by the involution θ' (see Lemma 5.1.3).

The universal cameral cover is $K \oplus K \cong ((\mathfrak{t}')^{\mathbb{C}} \oplus (\mathfrak{a}')^{\mathbb{C}}) \otimes K \to K^2$ where the isomorphism reads

$$(l, l') \mapsto \begin{pmatrix} l + l' & 0 & 0 \\ 0 & -2l & 0 \\ 0 & 0 & l - l' \end{pmatrix}$$

and the projection sends $(l, l') \mapsto (l^2 - (l')^2 - 4l'l, l'((l')^2 - l^2))$. Thus, any cameral cover corresponding to a real Higgs bundle

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & K \oplus K \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[(\omega,0)]{} & K^2 \oplus K^3 \end{array}$$

consists of three irreducible components:

$$\widehat{X}_1 = \{l^2 = \omega, l' = 0\}, \quad \widehat{X}_2 = \{l' = l, -4l^2 = \omega\}, \quad \widehat{X}_3 = \{l' = -l, 4l^2 = \omega\}.$$

Note that $\widehat{X}_1$ is the real cameral cover (the pullback of $(\mathfrak{a}')^{\mathbb{C}} \otimes K$). All three are double covers, with involutions induced by elements of the Weyl group: $(1,3) \in S_3$ restricts to the cover involution on $\widehat{X}_1$, as so do $(1,2)$ on $\widehat{X}_2$ and $(2,3)$ on $\widehat{X}_3$. Denote the cover involution for $\widehat{X}_i$ by σ_i, and let w_i denote the element of the Weyl group corresponding to σ_i. Also: $\widehat{X}_2 = (2,3) \cdot \widehat{X}_1 = (1,3) \cdot \widehat{X}_3$. Note that, in the notation of Chapter 4, $(1,3) = w_0$, as it acts by θ.

Coroots are

$$\breve{\alpha}_{1,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in (\mathfrak{a}')^{\mathbb{C}}, \quad \breve{\alpha}_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \breve{\alpha}_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence $D_{1,3} \cap \widehat{X}_1 = (\mathfrak{t}')^{\mathbb{C}} \otimes K \cap \{l' = 0, l^2 = \omega\} = \mathrm{Ram}(\widehat{X}_1)$ is the ramification divisor of $\widehat{X}_1$. This is D_{w_0} in the notation of Chapter 4.

Thus, by Theorem 4.4.17, a cameral datum consists of a principal $(D')^{\mathbb{C}}$ bundle on $\widehat{X}$ satisfying

$$P \cong (1,3)^* P^{(1,3)} \otimes \mathcal{R}_{(1,3)}, \tag{5.3}$$

$$P \cong (3,2)^* P^{(3,2)} \otimes \mathcal{R}_{(3,2)}, \tag{5.4}$$

$$\theta^* P|_{\widehat{X}_1} \cong P|_{\widehat{X}_1}. \tag{5.5}$$

Now, the bundle P on the non-irreducible curve $\widehat{X}$ is specified by the following data: $(P_1, P_2, P_3, \overline{f})$, where $P_i \to \widehat{X}_i$ is a principal $(D')^{\mathbb{C}}$-bundle, and $\overline{f} = \{f_{ij} \; : \; i \neq j \in \{1,2,3\}\}$ where $f_{ij} : P_i|_{\widehat{X}_i \cap \widehat{X}_j} \to P_j|_{\widehat{X}_i \cap \widehat{X}_j}$ satisfy the obvious compatibility conditions. Smooth bundles correspond to f_{ij} being an isomorphism, while other bundles correspond to morphisms which are not isomorphisms. To recover a bundle on the curve, we just need to consider the kernel of the Hecke transformation

$$P_1 \times P_2 \times P_3 \to (P_2 \times P_3)|_{\bigcap_i \widehat{X}_i}, \qquad (p_1, p_2, p_2) \mapsto (f_{2,1}(p_1)p_2^{-1}, f_{3,1}(p_1)p_3^{-1}).$$

Thus, consider P as $(P_1, P_2, P_3, f_{12}, f_{23})$, where $f_{12} : P_1|_{\widehat{X}_1} \cong P_2|_{\widehat{X}_2}$ and similarly for f_{23}. We omit other pastings as they can be produced from the latter two given that $\widehat{X}_i \cap \widehat{X}_j$ intersect at the same points for all i, j.

It is easy to produce a cameral datum from a Higgs bundle by a descent argument (see the proof of Theorem 4.4.17). We next analyse how to produce an SU(2,1)-Higgs bundle from a cameral datum.

Consider a slight modification of the bundle P: for each $i = 1, 2, 3$, we define $Q_i = P_i \otimes \mathcal{R}_{nilp}^{-1/2}$. The tuple (Q_1, Q_2, Q_3, f'), with f' a suitably induced from f, defines a bundle on $\widehat{X}$ which is smooth if and only if so is P. We readily check that

$$\theta \cdot Q_1 = \theta^* Q_1^{\theta} \cong Q_1 = \theta^* Q_1, \quad \theta \cdot Q_2 \cong Q_3. \tag{5.6}$$

In particular, $Q_1 = p_1^* M$ descends to X, as θ acts as $\sigma_1 = (1,3)$, the two-cover involution on $\widehat{X}_1$. Moreover, the action of θ leaves the fibers invariant. This means that the $D'^{\mathbb{C}}$ bundle Q_1 reduces its structure group to $T'^{\mathbb{C}} \cong \mathbb{C}^{\times}$. Hence, we may identify M to an element in $\mathrm{Pic}(X)$.

Consider the curve $\widehat{X}/\theta$. It consists of two components X and $\widetilde{X}$, the first being the quotient of $\widehat{X}_1$ by θ, and $\widetilde{X} = (\widehat{X}_2 \cup \widehat{X}_3)/\theta$ a two cover $p : \widetilde{X} \to X$. By (5.6), Q descends to $\widehat{X}/\theta$. Indeed, Q_1 descends to M by the above discussion. As for (Q_2, Q_3), we need to check that over ramification, the identity $\theta^* Q_2 \cong Q_3$ holds. But this is true as the restriction to ramification $Q_2|_{\widehat{X}_2 \cap \widehat{X}_3} = Q_2|_{\widehat{X}_2 \cap \widehat{X}_1} \cong_{f_{12}} Q_1|_{\widehat{X}_2 \cap \widehat{X}_1}$ reduces its structure group to $D'^{\mathbb{C}}$, hence

$$\theta^* Q_2|_{\widehat{X}_2 \cap \widehat{X}_3} \cong \theta^* Q_2|^{\theta}_{\widehat{X}_2 \cap \widehat{X}_3} \cong Q_3. \tag{5.7}$$

So Q descends to $Q' = (M, \widetilde{M}, \widetilde{f})$, where M is as in the previous paragraph, $\widetilde{M}$ is obtained by identifying Q_2 with Q_3, and the glueing data $\widetilde{f} : M|_{Ram(\widetilde{X})} \cong \widetilde{M}|_{Branch(\omega)}$ is induced from f'. The same argument (5.7) implies that the descended bundle also reduces its structure group to $D'^{\mathbb{C}}$, and consequently, moreover, it descends to X, as the involution θ on $\widehat{X}$ induces the double cover involution on $\widetilde{X}$. Namely, $\widetilde{M} = p^*F$ for some principal $\mathbb{C}^\times$-bundle $F \to X$. We may push-forward p^*F to X and use the glueing data $p_*\widetilde{f}$ to obtain a Hecke transform

$$M \times F \times F \otimes P_K^{-1} \to L,$$

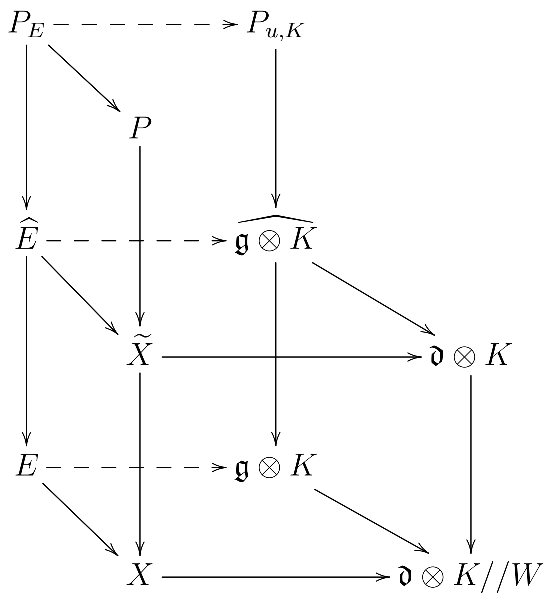
where P_K is the principal $\mathbb{C}^\times$ bundle associated to K and L is a bundle supported on the branching locus of $\widetilde{X}/X$. The kernel of this transform is our principal $\mathrm{GL}(2,\mathbb{C})$ bundle E.

Now relations (5.6) imply that Q has trivial determinant, as

$$\det(Q_1 \times Q_2 \times Q_3) \cong \mathcal{O} \otimes \det(Q_2|_{D_\omega} \times Q_3|_{D_\omega}).$$

and $Q \to Q_1 \times Q_2 \times Q_3 \xrightarrow{f'} \det(Q_2|_{D_\omega} \times Q_3|_{D_\omega})$ is exact.

By Theorem 4.4.30, there always exists a Higgs field completing the following diagram:



(5.8)

and factoring through $\mathfrak{m}^{\mathbb{C}} \otimes K$. In the above $\widehat{E} \to E$ is the pullback of $\widehat{X} \to X$, $\widehat{P} \to \widehat{E}$ be the restriction of P to $\widehat{E}$, where $P_{u,K}$ is the universal principal $D^{\mathbb{C}}$-bundle on $\widehat{\mathfrak{g}_{reg} \otimes K}$ (which is constructed from $P_u \to \widehat{\mathfrak{g}}_{reg}$ by pasting the local pieces, as explained in the proof of Theorem 4.4.30).

More constructively, we can identify the principal bundles to line bundles and note that the $(D')^{\mathbb{C}}$-principal bundle is the semisimple part of the principal bundle E. We thus define $\lambda : p_1^*F \to p_1^*F \otimes K$ which pushes-forward to a morphism $\psi : F \oplus F \otimes K^{-1} \to F \otimes K \oplus F$. This is the sum of the eigenbundles of Higgs field ϕ with eigenvalue different from zero. Now, modulo conjugation, we may assume ϕ

has zero diagonal blocks. A simple analysis proves that the $+\lambda$ eigenvectors have the form $v_\lambda + w_\lambda$, $v_\lambda \in V$, $w_\lambda \in W$, and the $-\lambda$ eigenvetor is of the form $v_\lambda - w_\lambda$. Thus, θ fixes the V part and acts as -1 on the W piece. Clearly F is fixed by θ, so that $W \cong FK^{-1}$.

Up to now we have only defined the identification of the restrictions $P|_{\widehat{X}_i}$ via θ. Now, we have

$$(2,3)^*Q_1^{(2,3)} \cong Q_2 \otimes \mathcal{R}_{\alpha_{(1,2)}}^{-1/2}\mathcal{R}_{\alpha_{(1,3)}}^{-1/2},$$
$$(1,2)^*Q_1^{(1,2)} \cong Q_3 \otimes \mathcal{R}_{\alpha_{(1,2)}}^{-1/2}\mathcal{R}_{\alpha_{(1,3)}}^{-1/2}.$$

We see that the action of $(1,3)$ exchanges the RHS's of the above inequalities, and so does it with the LHS's. In terms of vector bundles on the quotient curve, the above induces an isomorphism:

$$f : p^*MK|_{X\cap\widetilde{X}} \to F|_{X\cap\widetilde{X}}$$

With this, $\phi' : M \oplus F \oplus F \otimes K^{-1} \to M \oplus (F \oplus F \otimes K^{-1}) \otimes K$ can be defined by applying $M \stackrel{p_*f}{\mapsto} F = FK^{-1}K$, $\psi|_F : F \mapsto F \otimes K^{-1}K$, $(\psi|_{FK^{-1}}, p_*f) : FK^{-1} \to FK \oplus MK$. This morphism clearly sends $L \mapsto L \otimes K$, so that it induces a morphism $\phi : V \oplus W \to V \otimes K \oplus W \otimes K$. The Higgs field has to be regular by construction. Sumarising:

$$\begin{array}{ccccccccc}
\mathcal{O} & \longrightarrow & V \oplus W & \longrightarrow & F^{-2}K^3 \oplus F \oplus FK^{-1} & \xrightarrow[p_*(f-id)]{} & F|_{Branch_\omega} & \longrightarrow & \mathcal{O} \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} & & \downarrow & & \\
\mathcal{O} & \longrightarrow & K(V \oplus W) & \longrightarrow & K(F^{-2}K^3 \oplus F \oplus FK^{-1}) & \xrightarrow[p_*(f-id)]{} & KF|_{Branch_\omega} & \longrightarrow & \mathcal{O}
\end{array} \quad (5.9)$$

Remark 5.4.1. *The above discussion shows that we obtain non regular Higgs fields by considering glueing data $M \otimes K|_{Ram(\omega)} \to F|_{Ram(\omega)}$ which are not isomorphisms, namely, which vanish at some points. In particular, when it vanishes identically, we obtain a strictly semistable bundle. Note, however, that given that* $\deg(F^{-2}K^3) > 0$, *the bundle is never polystable.*

All the above yields the following.

Theorem 5.4.2. *Let $\omega \in H^0(X, K^2)$. Define $\mathcal{F}_\omega := \kappa\left(\mathcal{M}(SU(2,1))^{reg}\right) \cap h_{\mathbb{C}}^{-1}(\omega)$. Then, if $\mathcal{F}_\omega$ is non empty, it is a group scheme over a union of connected components of $Pic^{-(g-1)<d<(g-1)}(X)$ with fiber isomorphic to $(\mathbb{C}^\times)^{4g-5}$. Here $Pic^{-(g-1)<d<(g-1)}(X)$ denotes the union $\bigsqcup_{-(g-1)<d<2(g-1)} Pic^d(X)$. In particular, the connected component of the fiber is an abelian variety with operation given by multiplication on $(\mathbb{C}^\times)^{4g-5}$ and tensor product on the base $Pic^0(X)$.*

Proof. By the preceeding discussion, the stacky regular Hitchin fiber can be identified with pairs $(F, f) \in \mathrm{Pic}(X) \times \mathrm{Isom}(F^{-2}K^3|_{Branch(\omega)}\otimes, F|_{Branch(\omega)})$ defining a cameral datum. Note that f is determined by the choice of $4g-4$ non-zero complex numbers. However, we ultimately want to consider the kernel of the associated Hecke transform. Since degrees are fixed, we have that the choice of $4g-5$ such points fully determines the remaining one. Hence, the projection $(F, f) \mapsto FK^{-1}$ has kernel isomorphic to $(\mathbb{C}^\times)^{4g-5}$ with its natural group structure. As for surjectivity of $\mathcal{F}_\omega \to \bigsqcup_{-(g-1)<d<(g-1)} \mathrm{Pic}^d(X)$, the bound on the degree is determined by the Milnor-Wood inequality for SU(2,1)-Higgs bundles (cf. [13]). We need to check that there are no regular polystable Higgs bundles with Toledo invariant $\pm(g-1)$ and there are regular polystable elements for all other values of τ. The first statement follows from Proposition 5.3.7 and Corollary **??**, but here we give a direct proof.

Regular elements with $\deg W = \tau_M$ over $\omega \in \mathcal{A} \setminus \{0\}$ project onto $\mathrm{Pic}^{\pm\tau_M}(X)$. Not only that, but any value $f \in (\mathbb{C}^\times)^{4g-5}$ corresponds to some strictly semistable bundle. Indeed, we may consider the element

$$F \oplus F^{-1/2} \otimes K^{1/2} \oplus F^{-1/2} \otimes K^{-1/2}, \qquad \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega & 0 \end{pmatrix}$$

It is strictly polystable by Proposition 5.3.3, so it is not regular by Lemma 5.3.6. We need to find a section $c \in H^0(X, F^{-3/2} \otimes K^{1/2})$ such that $c\omega$ never vanishes. But this is an open condition on $c \in H^0(X, F^{-3/2} \otimes K^{1/2})$, so that it is always possible to find one such. Note that f corresponds to $c|_{Ram(\widetilde{X}/X)}$, so that over an open subset of $(\mathbb{C}^\times)^{4g-5}$ any choice for f is valid. As for the complement of this open subset, closedness of semistable pairs implies that what remains is also strictly semistable. Note that these regular elements are not polystable.

As for other values of τ, we prove that there exist regular polystable elements. To do this, given τ, we define a polystable $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle over any ω. Next, we prove it is a limit of a sequence of regular elements over different fibers. Next, we check that $Pic^0(X)$ acts simply transitively on the fiber intersected with $\mathcal{M}(\mathrm{SU}(2,1))_\tau$, which implies the result.

Let $(\omega) = \sum_{i=1}^{4g-4} x_i$ be the divisor defined by ω, possibly with multiple zeroes. We need to find L and sections b of $L^{-2}K$ and d of L^2K such that $bc = \omega$. Note that the degree of L makes sure this is possible for $|d| < g-1$. Choose $x_1, \cdots x_{2(d+g-1)} \subset \{x_i\}_{i=1}^{4g-4}$. The corresponding divisor B satisfies that for any square root

$$L = \left[\mathcal{O}(B) \otimes K_X^{-1}\right]^{1/2}$$

there exist $b \in H^0(X, L^2K), d \in H^0(X, L^{-2}K)$ with $(bc) = (\omega)$ (we have simply $B = (b)$, $(c) = (\omega) - (b)$). Now, the associated SU(2, 1)-Higgs bundle $(E = \mathcal{O} \oplus L \oplus L^{-1}, \phi)$ is strictly polystable. We need to modify the Higgs field by finding $a \in H^0(X, LK)$ such that $a(x_i) \neq 0$ for all $i \leq 4(g-1)$, which is always possible for the τ's under consideration. Once this is done, we find $d \in H^0(L^{-1}K)$ not vanishing at any point ar which a or ω vanish. We thus have a regular Higgs bundle of each topological class.

To see the action of $Pic^0(X)$ we go back to the construction of the Higgs bundle: consider the diagram (5.9). Given $L \in Pic^0(X)$, it still makes sense to consider:

$$\begin{array}{ccccc}
(LF)^{-2}K^3 \oplus LF \oplus LFK^{-1} & \xrightarrow[p_*(f'-id)]{} & LF|_{Branch_\omega} & \longrightarrow & \mathcal{O}\,. \\
\downarrow{\scriptstyle \phi''} & & \downarrow & & \\
K((LF)^{-2}K^3 \oplus LF \oplus LFK^{-1}) & \xrightarrow[p_*(f'-id)]{} & KLF|_{Branch_\omega} & \longrightarrow & \mathcal{O}
\end{array}$$

Note that the morphism $LF \oplus LFK^{-1} \oplus (LF \oplus LFK^{-1})K$ induced by ϕ' needs not be modified. As for $f \in \mathrm{Hom}(p^*F^{-2}K^3|_{\mathrm{Ram}(\widetilde{X}/X)}, p^*F|_{\mathrm{Ram}(\widehat{X}/X)})$, for any four line bundles A_i, $i = 1, \ldots, 4$, and any divisor $D \subset \widetilde{X}$, there is an isomorphism $\mathrm{Hom}(A_1|_D, A_2|_D) \cong \mathrm{Hom}(A_3|_D, A_4|_D)$. This implies that polystability of $(\mathrm{Ker}p_*(f - id), \phi'|_{\mathrm{Ker}p_*(f-id)})$ (cf. 5.9) is equivalent to polystablity of $(\mathrm{Ker}p_*(f'-id), \phi''|_{\mathrm{Ker}p_*(f'-id)})$.

We finish by a dimensional count: the dimension of $\mathcal{M}(G)$ is $8(g-1)$, so that smooth opens sets of fibers have dimension $5g-5$. Closedness of polystable bundles does the rest.

□

Remark 5.4.3. *In the course of the above demonstration, we actually prove that strictly polystable bundles are limits of regular bundles. This means in particular that the generic fiber is regular.*

5.5 Spectral data

It is interesting to compare the results of the previous section with the spectral techniques developed by Hitchin and Schaposnik ([66, 67, 43]).

The spectral cover for $(\omega, 0)$ is the curve $\overline{X}_\omega \subset \mathrm{tot}(K)$ defined by the ideal $\lambda(\lambda^2 - \omega)$, where λ is the tautological section of π^*K, $\pi : K \to X$ is the usual submersion and $\mathrm{tot}(K)$ denotes the total space of K. Note that is is isomorphic to $\widehat{X}/\theta$.

Now, to define a spectral datum for a bundle (E, ϕ), we consider the kernel of the exact sequence

$$\mathcal{O} \to A \to \pi^*E \overset{\pi^*\phi - \lambda Id}{\to} \pi^*E \otimes K \to AK^{-1} \to \mathcal{O}$$

(see [6] for details). Applying the involution θ to the sequence we see that A is θ-invariant. Moreover, θ induces the involution of the 2-cover $\lambda^2 - \omega = 0$ so we get $\theta^* A \cong A$, and A descends to $M \to X$. Pushing forward we obtain $M \oplus MK^{-1}$ and the same analysis as in the previous section yields $W = MK^{-1}$. The final piece of data is the kernel, which is identified with a line bundle B on $\lambda = 0$, which obviously descends. The Higgs field induces a morphism $f : B|_{\mathrm{Ram}(\overline{X}_\omega/X)} \to A|_{\mathrm{Ram}(\overline{X}_\omega/X)}$, which together with the identification of V as the kernel of $A \oplus B \stackrel{f-id}{\to} L$ gives a point of $\mathcal{F}_\omega$.

As for the converse direction, a line bundle on $\overline{X}_\omega$ is but a line bundle B on $\lambda = 0$, a line bundle A on $\lambda^2 = \omega$ and a morphism $f : B|_{\{\lambda=0\}\cap\{\lambda^2=\omega\}} \to A|_{\mathrm{Ram}(\overline{X}_\omega/X)}$. Smooth points in $\mathrm{Pic}(\overline{X}_b)$ correspond to triples (A, B, f) for which f is an isomorphism, that is, a regular point of the fiber.

Appendix A

Stacks and gerbes

Stacks appear naturally in the study of moduli problems: as soon as we require that the moduli satisfy certain functorial properties, moduli spaces become insufficient. Suppose that we have a functor

$$F : Sch \to Sets$$

assigning to each $B \in Sch$ a family of objects parameterised by B. If there exists a scheme M representing F (that is, such that $F \cong Hom_{Sch}(\ \cdot\ , M)$, we say that M is a fine moduli space for F, and that $id \in Hom(M, M)$ is a universal family for the corresponding moduli problem. Namely, any $x \in F(B)$ is the pullback of the universal family over M by the morphism $f_x : B \to M$ corresponding to x via the natural isomorphism

$$F(B) \cong Hom(B, M).$$

It is seldom true that the fine moduli space be a scheme. We need to consider the bigger category of stacks in order to find representing objects to moduli problems.

A.1 A primer on stacks

A.1.1 Basic definitions

Stacks are categorical sheaves. In order to define them, we need a topological notion on the category under consideration. These topologies are called **Grothendieck topologies** and were introduced by A. Grothendieck in the 1950's.

Definition A.1.1. Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ consists on the assignment to each object $U \in Ob(C)$ of a collection of objects over U $\mathcal{U} = \{f_i : U_i \to U\ :\ i \in I\} \subseteq Ob(\mathcal{C})$. The collection $\mathcal{U}$ is called an open covering of U, and the pairs (U_i, f_i) are called open subobjects of U. Open coverings must satisfy that:

1. if $V \cong U$, then $\{V \to U\}$ is a covering.
2. If $U_1 \to U,\ U_2 \to U \in \mathcal{U}$ then $U_1 \times_U U_2 \in \mathcal{U}$ (namely, finite intersections of open objects are open).
3. If $\{f_i : U_i \to U\}_j$ is a covering and for each i we have a covering $\{f_{ji} : V_{ij} \to U_i\}_j$, then $\{f_{ji} \circ f_i : V_{ij} \to U_i\}$ is a coverning (namely, refinements of coverings are coverings).

We will denote a topology on a category $\mathcal{C}$ by $\mathcal{T}$, and call the pair $(\mathcal{C}, \mathcal{T})$ a **site**.

Example A.1.2. Consider the category of complex schemes $Sch/\mathbb{C}$. We can endow it with different topologies, amongst which:

• The Zariski topology, in which open subschemes are Zariski open subschemes and coverings are families $\{U_i \to U\}$ such that $\sqcup_i U_i \to U$ is a surjective map.

• The analytic (or usual) topology, in which open subschemes are given by the manifold structure of $S(\mathbb{C})$ for any $S \in Sch/\mathbb{C}$.

• The flat locally finitely presented topology (fppf topology from *fidèlement plat et de présentation finie*), in which open sets are flat morphism which are locally finitely presented.

• The flat locally finitely presented topology (fppf topology from *fidèlement plat et de présentation finie*), in which open sets are flat morphism which are locally finitely presented.

We can compare Grothendieck topologies on the same category: $\mathcal{T}$ is said to be finer than $\mathcal{T}'$ (equivalently $\mathcal{T}'$ coarser than $\mathcal{T}$, written $\mathcal{T}' \leq \mathcal{T}$) if any covering in $\mathcal{T}'$ has a refinement in $\mathcal{T}$. For instance, on $Sch/\mathbb{C}$:

$$\mathcal{T}_{Zar} < \mathcal{T}_{?etale} \equiv \mathcal{T}_{usual} \equiv \mathcal{T}_{smooth} \tag{A.1}$$

Definition A.1.3. A stack on a site $\mathcal{C}$ (we omit the topology in the notation) is a category fibered in groupoids (cf. [76]) $\mathcal{F} \to \mathcal{C}$ such that "descent is effective". Namely, given any covering $\mathcal{U} \in \mathcal{T}$ of U and any collection of objects over U_i $\xi_i \in \mathcal{F}(U_i)$ satisfying the appropriate cocycle conditions (cf. [76] for details), there exists an object $x \in \mathcal{F}(U)$ whose restriction to U_i is isomorphic to ξ_i.

Remark A.1.4. *The notation $\mathcal{F}(\ \cdot\ )$ responds to the fact that a category fibered in groupoids can be seen as a pseudo-functor (cf. [76], Section 3.1.2) from the site to the 2-category of groupoids. Another reference for the algebraic case [37].*

Remark A.1.5. *If $\mathcal{T} \leq \mathcal{T}'$, then any stack on $(\mathcal{C}, \mathcal{T})$ is a stack on $(\mathcal{C}, \mathcal{T}')$ (see [76] Proposition 2.49).*

In moduli problems, the presence of automorphisms prevents the moduli from being fine. Stacks, on their side, "remember" automorphisms through what is known as the inertia stack. Before we can define it we need a preliminary notion.

A.1.2 Morphisms of stacks

Definition A.1.6. Given $\mathcal{F}$, $\mathcal{G}$ two stacks over a site $\mathcal{C}$, a morphims between them is a morphism of fibered categories, namely, a morphism between the corresponding categories that commutes to projections.

Definition A.1.7. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be three stacks on a site $\mathcal{C}$, and assume we have morphisms $g : \mathcal{G} \to \mathcal{F}$, $h : \mathcal{H} \to \mathcal{F}$. We define the **fibered product** of $\mathcal{H}$ and $\mathcal{G}$ over $\mathcal{F}$ to be:

$$\mathcal{G} \times_{\mathcal{F}} \mathcal{H}(U) = \left\{ (a, b, \alpha) \ : \ \begin{array}{l} a \in \mathcal{G}(U), \ a \mapsto a' \in \mathcal{F}(U) \\ b \in \mathcal{H}, \ b \mapsto b' \in \mathcal{F}(U) \\ \alpha \in Hom_{\mathcal{F}(U)}(a', b') \end{array} \right\}.$$

Remark A.1.8. *Note that groupoids are categories in which all morphisms are isomorphisms, so in particular $\alpha : a' \cong b'$ is an isomorphism.*

Remark A.1.9. *The above definition is somehow deceitful. One needs to prove that a pre-stack defined locally satisfies the descent condition. This is a result for fibered products.*

Definition A.1.10. Let $\mathcal{X} \to \mathcal{C}$ be a stack on a site. The inertia stack of $\mathcal{X}$ (not. $\mathcal{I}_{\mathcal{X}}$) is defined to be

$$\mathcal{X} \times_{\mathcal{X}} \mathcal{X}$$

where $\mathcal{X} \to \mathcal{X}$ is taken to be the identity.

Proposition A.1.11. *Consider the natural morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, and let $U \to \mathcal{X}$ be an $\mathcal{X}$-object of thr site $\mathcal{C}$. Then, $\mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} U \cong \mathrm{Aut}(\mathcal{X}|_U) \to \mathcal{X}|_U$ is an equivalence of categories.*

Sketch of proof. Let $(A, B, \alpha), (A, B', \beta) \in \mathcal{I}_{\mathcal{X}}(V)_A, \ A \in \mathcal{X}(V)$. Then $\beta\alpha^{-1}$ is an automorphism of A, so we obtain an automorphism of A. One sees that this defines an equivalence. □

A.1.3 Algebraic stacks

Geometric stacks are those that locally look like schemes. We fix once and for all $\mathcal{S} = (Sch/S)_{\mathcal{T}}$ the site of schemes over S with a given topology.

Definition A.1.12. A stack $\mathcal{F} \to \mathcal{S}$ is said to be representable (by a scheme) if there exists some $Z \in \mathcal{S}$ $\mathcal{F}(U) \cong \mathrm{Hom}(\cdot, Z)$.

A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is said to be representable (by a scheme) if for every $U \to \mathcal{G}$, the fibered product $\mathcal{F} \times_{\mathcal{G}} U$ is representable by a scheme.

When a morphism is representable, it makes sense to consider local properties for it.

Definition A.1.13. Let P be a local property of morphisms of schemes. Given a representable $f : \mathcal{F} \to \mathcal{G}$, we say that f has property P if for any $U \to \mathcal{G}$ and $\mathcal{F} \times_{\mathcal{G}} U \to U$ satisfies P.

Definition A.1.14. An algebraic or Artin stack is a stack $\mathcal{X}$ over $\mathcal{S}$ such that:

1. There exists a scheme X together with a smooth surjective morphism
$$X \to \mathcal{X}.$$
This scheme is called an atlas for $\mathcal{X}$.

2. The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by schemes.

Remark A.1.15. *Strictly speaking, the diagonal can be represented by algebraic spaces, but for our needs, the above definition is enough.*

Example A.1.16. A scheme S is an algebraic stack. The associated 2-functor is $\mathrm{Hom}(\,\cdot\,, S)$, which is in fact a functor. In fact, a stack is representable if and only if its associated 2-functor is a functor.

Example A.1.17. Let X be a scheme, and let G be an algebraic group acting on S. We define the quotient stack of X by the action of G, denoted by $[X/G]$, as
$$[X/G]\,(U) = \left\{ (P, \phi) \ : \ \begin{array}{l} P \to U \text{ is a } G- \text{ principal bundle} \\ \phi : P \to X \text{ is a } G\text{-equivariant morphism.} \end{array} \right\}$$
In this case, X is the atlas, and the projection $\mathcal{I} \to [X/G]$ is the descent of the morphism $\mathcal{C} \subseteq X \times G$, where $\mathcal{C}$ is the group scheme defined by: $\mathcal{C} = \{(x, g) \ : \ g{\cdot}x = x\}$. Indeed, an automorphism of an object $(P, \phi) \to U$ (see Remark A.1.11) is a G-bundle automorphism $f : P \to P$ such that $f(\phi) = \phi$. Namely, a section of $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} G$ centralising ϕ which is exactly what the objects of the descended sheaf $\underline{\mathcal{C}} \to [X/G]$ are over (P, ϕ).

A.2 Gerbes and G-gerbes

References for this section are [18, 17, 58, 35, 22].

Let S be a scheme. Consider the site $\mathcal{S} = (Sch/S)_{\mathcal{T}}$ for some topology $\mathcal{T}$.

Definition A.2.1. A stack in groupoids $\mathbb{G}$ over $\mathcal{S}$ is a gerbe over $\mathcal{S}$ whenever it is locally non-empty and locally connected. Namely,

1. (locally non empty) there exists a covering $\mathcal{U}$ of S and objects $x_i \in \mathbb{G}U_i$.

2. (locally connected) for any $x \in FU$ there exists a covering $\mathcal{U}$ of S and an arrow $x_i|_{U_i \times_S U} \to x|_{U_i \times_S U}$.

With this, it is easy to prove that if $\mathbb{G} \to \mathcal{S}$ is a gerbe, a choice of local objects $x_i \in \mathbb{G}(U_i)$ allows identifying $\mathbb{G}|_{U_i} \cong B_{U_i}\mathrm{Aut}(x_i)$.

Proposition A.2.2. *In the analytic topology, local connectedness of a stack is equivalent to flatness of inertia.*

Proof. Given two smooth complex schemes $f : Z \to Y$, flatness of f is equivalent to fibers having the same dimension (see [57] Theorem 23.1). Now, local connectedness implies flatness of inertia, as for any two local objects the stacks of automorphisms are isomorphic.

As for the converse, note that the inertia stack codifies isomorphisms between objects. Thus, since $id \in \mathrm{Aut}(x)$ for all x, then any two objects have to be locally connected. $\square$

Example A.2.3. *BG* Consider $BG \to \mathcal{S}$. Note that S is an atlas and that

$$BG \cong [S/G]$$

as stacks over $\mathcal{S}$, where the action of G on S is the trivial action. Indeed, note that under the given hypothesis, an object in $[S/G]\,(U)$ is but a G-principal bundle on U together with a G equivariant morphism to S. Namely, a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & S \\ \downarrow & \nearrow & \\ U & & \end{array} .$$

This is clearly equivalent to specifying a G-principal bundle on U[1].

[1] Another way to prove this is to check that $S \to BG$ defined by $U \mapsto U \times G$ is surjective and that the inertia stack of BG is given by G-equivariant morphisms

Given a gerbe $\mathbb{G}$, we have that for some covering $\{U_i\}$ of S, there exist sheaves of groups $G_i \to U_i$ such that $\mathbb{G}|_{U_i} \cong BG_i$. The stack that controls how these local pieces fit together is caller the band of the gerbe.

Definition A.2.4. Let $\mathbb{G}$,$\{U_i\}$ and $G_i \to U_i$ be as above. Let $G \to \sqcup_i U_i$ be defined by $G|_{U_i} = G_i$. Then, the band of $\mathbb{G}$, is defined by $Band(\mathbb{G}) = \mathrm{Isom}(\mathbb{G}, BG) \times_G \mathrm{Out}(G)$.

Remark A.2.5. *The above sheaf of isomorphisms is to be understood as a locally defined sheaf. In particular, the objects can be locally represented by (f_j, λ_{ij}) where $f_j : \mathbb{G}|_{U_j} \to BG_j$ and $\lambda_{ij} : G_j \to G_i$ is defined up to inner automorphisms and determines $BG_j \to BG_i$. See [58] for more details.*

Definition A.2.6. Given $G \to X$ be a sheaf of groups on a scheme X. A G-gerbe over X is a gerbe that is locally isomorphic to BG.

Remark A.2.7. *When a gerbe $\mathbb{G}$ is actually a G-gerbe for some group scheme G, the band is more easily defined as $Isom(\mathbb{G}, BG) \times_G \mathrm{Out}(G)$.*

Definition A.2.8. Given the prestack of group schemes $\mathcal{GS}$ on $\mathcal{S}$, its band is defined to be the stack associated to the prestack whose objects over $U \in \mathcal{S}$ are group schemes over U, with morphisms given by $Hom_{\mathcal{GS}}(G_1, G_2)/Inn(G_1)$.

Definition A.2.9. A gerbe $\mathbb{G}$ is banded by a group scheme $G \to S$ if $Band(\mathbb{G}) \cong Band(G)$.

A.2.1 Cocyclic description of a G-gerbe

In [18], Breen proves that G-gerbes are classified up to equivalence by the cohomology group $H^1(X, G \to \mathrm{Aut}(G))$, where the complex $G \to \mathrm{Aut}G$ is concentrated in degrees -1 and 0. The idea is the following: given a trivialising cover U_i of a G-gerbe $\mathbb{G}$, we have equivalences

$$\psi_{ij} : B(G|_{U_j})|_{U_{ij}} \to B(G|_{U_i})|_{U_{ij}}.$$

Proposition A.2.10. *Given two categories of torsors BH, BG, any equivalence between them is of the form ψ_Q for some (H, G)-bitorsor Q, where*

$$\psi_Q(P) = Q \wedge^H P.$$

Proof. See [18].

So bitorsors are to gerbes what groups are to torsors. Furthermore, bitorsors are 0-cocycles for the same complex, namely, they are classified by $H^0(X, G \to \mathrm{Aut}(G))$. In these terms, 1-cocycles with values in $G \to \mathrm{Aut}(G)$ classify gerbes.

A.2.1.1 Banding revisited

Note that the above cohomological description is a local one. In particular, given G' a twisted form of G (that is, a sheaf of groups locally isomorphic to G) we have that any G-gerbe is a G'-gerbe and viceversa. But certainly $B_{\mathcal{S}}G'$ is by no means equivalent to $B_{\mathcal{S}}G$, just locally so. It is at this stage that the notion of a band comes into the picture, getting us a step further towards the right notion of trivial gerbe.

Remark A.2.11. *In terms of cocycle, to the gerbe BG there corresponds the trivial cocycle in* $H^1(X, G \to \mathrm{Aut}G)$*. Similarly, it can be associated an element in* $H^1(X, G' \to \mathrm{Aut}G')$ *not corresponding to the trivial object.*

It is a well known fact that there is a one to one correspondence between isomorphism classes of twisted forms of G and $\mathrm{Aut}(G)$ torsors (see [18]). Now, the short exact sequence

$$1 \to \mathrm{Inn}(G) \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

induces a long cohomology sequence

$$\cdots \to H^1(X, \mathrm{Inn}(G)) \to H^1(X, \mathrm{Aut}(G)) \to H^1(X, \mathrm{Out}(G)) \to \ldots$$

The rightmost map associates to a principal $\mathrm{Aut}(G)$-bundle P an $\mathrm{Out}(G)$-bundle called **the band** of P. The band of the trivial bundle $X \times \mathrm{Aut}(G)$ maps to zero, so that the band of G (or any inner twisted form of it) maps to $0 \in H^1(X, \mathrm{Out}(G))$.

So now note that we have a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & G & \longrightarrow & 1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1
\end{array}$$

Inducing the short exact sequence of complexes

$$1 \to [G \to \mathrm{Inn}(G)] \to [G \to \mathrm{Aut}(G)] \to [1 \to \mathrm{Out}(G)] \to 1$$

yielding

$$\cdots \to H^1(X, G \to \mathrm{Inn}(G)) \to H^1(X, G \to \mathrm{Aut}(G)) \overset{band_G}{\to} H^1(X, \mathrm{Out}(G)) \to \ldots.$$

Proposition A.2.12. *The isomorphism class of the G-band of a gerbe* $\mathbb{G}$ *is the* $\mathrm{Out}(G)$ *principal bundle corresponding to the image of* $[\mathbb{G}] \in H^1(X, G \to \mathrm{Aut}(G))$*.*

Corollary A.2.13. *A G-gerbe* $\mathbb{G}$ *is G-banded if its band* $band_G(\mathbb{G}) = 0$ *is the trivial* $\mathrm{Out}(G)$*-bundle.*

Note that by the above discussion $H^1(X, G \to \mathrm{Inn}(G))$ classifies G-banded G-gerbes.

Definition A.2.14. A gerbe $\mathbb{G}$ over $\mathcal{S}$ is said to be neutral if there exists a global section $s : \mathcal{S} \to \mathbb{G}$.

Example A.2.15. BG is neutral, as the assignment $U \mapsto U \times G|_U$ is a global section.

This is the final ingrediet, to properly define triviality in global terms.

A.2.1.2 Abelian banded gerbes

Assume G is a sheaf of abelian groups:

Proposition A.2.16. *Let $\mathbb{G}$ be a G-gerbe. Then there exists a twisted form of G, G', such that $\mathbb{G}$ is G'-banded. Furthermore, $\mathbb{G} \cong BG'$ if and only if it is neutral.*

Proof. Let $\eta := band_G(\mathbb{G}) \in H^1(X, \mathrm{Out}(G))$. Assume G is abelian. Then $\mathrm{Inn}G = id$, so that $\mathrm{Out}(G) \cong \mathrm{Aut}(G)$. In particular, the cohomology class $\eta \in H^1(X, \mathrm{Out}(G)) \cong H^1(X, \mathrm{Aut}(G))$ defines a twisted form G' of G. We claim $\mathbb{G}$ is G'-banded. To see this, consider the pairs (g_{ij}, λ_i) representing the cocycle on an open set U_k. Modulo refining the cover, we may assume the sections take values in $\mathrm{Aut}(G')$ and $\mathrm{Out}(G')$. The fact that this maps to the trivial class in $H^1(X, \mathrm{Out}G')$ means the cocycle equations are trivial modulo $\mathrm{Inn}G'$. That is, the gerbe is G'-banded.

As for triviality, assume s is a global section. Taking a fine enough refinement, we may assume $s(U_{ij})$ is a G'-gerbe. By definition of a band, $s(U_{ij})$ paste to a G' torsor. □

Remark A.2.17. *Note that if $\mathbb{G}$ is G'-banded, it follows that G' (rather, the stack of groups it generates over $\mathcal{S}$) is in fact the descent of inertia to $\mathcal{S}$ (indeed, by abelianity of the group inertia admits a unique descent). So if there exists a global section $s : \mathcal{S} \to \mathbb{G}$, the pullback by s of inertia identifies $\mathbb{G}$ to BG' (by uniqueness of the descent).*

A.2.1.3 Non-Abelian banded gerbes

Now, suppose we are given a non abelian scheme of algebraic groups $G \to S$. It is in general false that any G-gerbe is G'-banded for a suitable twisted form G' of G. This is due to the lack of triviality of Inn G, which implies that there exists an obstruction for a band to be induced by a twisted form of G.

However, if the gerbe is G-banded and neutral, it holds that it can be globally identified with BG. Indeed, the descent argument is not valid anymore, but we see that a section produces a global object, whose automorphisms locally restrict to the automorphisms of any object. The gerbe being banded, these can be globally pasted to G.

Appendix B

Lie theoretical computations for some classical Lie groups

B.1 SL$(n,\mathbb{R})$

The form $\mathrm{SL}(n,\mathbb{R}) < \mathrm{SL}(n,\mathbb{C})$ is defined by the antiholomorphic involution $X \mapsto \overline{X}$. Its associated holomorphic involution is

$$\theta(X) = {}^tX^{-1},$$

We find $H^{\mathbb{C}} \cong \mathrm{SO}(n,\mathbb{C}))$, $\mathfrak{m}^{\mathbb{C}} \cong \mathfrak{sym}_0(n,\mathbb{C})$ the subalgebra of symmetric matrices with 0-trace. Note also that the maximal compact subgroup of $H^{\mathbb{C}}$ is $\mathrm{SO}(n)$.

The form is split, as the Cartan subalgebra of real diagonal matrices is contained in $\mathfrak{m}$.

B.2 Sp$(2p,\mathbb{R})$

Consider $\mathrm{Sp}(\Omega,\mathbb{C}^{2p})$ where

$$\Omega = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}.$$

Its subgroup of matrices with real coefficients is $\mathrm{Sp}(2p,\mathbb{R})$, whose associated holomorphic involution is

$$\theta(X) = \mathrm{Ad}\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} {}^tX^{-1}.$$

It is compatible with the involution defining the maximal compact subgroup $\mathrm{Sp}(2p)$, namely

$$\tau(X) = {}^t\overline{X}^{-1},$$

so that the involution defining $\mathrm{Sp}(2p,\mathbb{R})$ is

$$\sigma(X) = \mathrm{Ad}\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}\overline{X}.$$

We find $H^{\mathbb{C}} \cong \mathrm{GL}(p,\mathbb{C})$ with Lie algebra:

$$\mathfrak{h}^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \ : \ A \in \mathfrak{gl}(p,\mathbb{C}) \right\}$$

and

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \ : \ B,\ C \in \mathfrak{gl}(p,\mathbb{C}),\ B -^t B = 0 = C -^t C \right\}.$$

Note also that the maximal compact subgroup of $\mathfrak{h}^{\mathbb{C}}$ is $\mathfrak{u}(p)$.

A maximal anisotropic Cartan subalgebra is generated by the matrices

$$h_j = E_{j,p+j} - E_{j+1,p+j+1} + E_{p+j,j} - E_{p+j+1,j+1}$$

for $j \leq p-1$ and also by

$$h_p = E_{p,2p} + E_{2p,p}$$

Letting $L_i((a_{ij})) = a_{i,p+i}$, we have that a system of simple roots is given by

$$S = \{L_i - L_{i+1}, 2L_p \ : \ i = 1, \dots, p-1\},$$

and h_j satisfies that $L_j - L_{j+1}(h_j) = 2$. The corresponding eigenvectors are, for $j \leq p-1$

$$y_j = E_{j,j+1} - E_{j+1,j} - E_{p+j,p+j+1} + E_{p+j+1,p+j} +$$

$$E_{j,p+j+1} + E_{j+1,p+j} - E_{p+j,j+1} - E_{p+j+1,j}$$

and

$$y_p = i(E_{p,p} - E_{2p,2p} - E_{p,2p} + E_{2p,p}).$$

B.3 SU(p,q)

The real form $\mathrm{SU}(p,q) \leq \mathrm{SL}(p+q,\mathbb{C})$ is defined as the fixed point set of the antiliniear involution

$$\sigma : X \mapsto \mathrm{Ad}_{I_{p,q}}\, {}^t\overline{X}^{-1}$$

where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

The decomposition of $\mathfrak{sl}(p+q,\mathbb{C})$ corresponding to the linearization of Cartan involution for $\mathfrak{su}(p,q)$ is $\mathfrak{sl}(p+q,\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{m}$ where

$$\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(p) \times \mathfrak{gl}(p)) \qquad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} : \begin{array}{c} B \in Mat_{p\times q}(\mathbb{C}) \\ C \in Mat_{q\times p}(\mathbb{C}) \end{array} \right\}$$

The maximal anisotropic subalgebra is

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \right\} \qquad A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_1 \\ \vdots & . & \vdots \\ a_q & \dots & 0 \end{pmatrix} \tag{B.1}$$

Restricted root system and spaces Following [48], let $f_i(x) = a_i$ for any $i = 1,\dots,q$ any $x \in \mathfrak{a}$ as in (B.1). To define the restricted roots and their eigenspaces are, let

$$J(z) = \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix}, \quad I_+(z) = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}, \quad I_-(z) = \begin{pmatrix} z & 0 \\ 0 & -\overline{z} \end{pmatrix}.$$

Now, the root spaces are

$$\mathfrak{g}_{f_i-f_j} = \left\{ \begin{pmatrix} J(z) & -I_+(z) \\ -I_+(\overline{z}) & -J(\overline{z}) \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i+f_j} = \left\{ \begin{pmatrix} J(z) & I_+(z) \\ I_+(\overline{z}) & -J(\overline{z}) \end{pmatrix} \right\}$$

$$\mathfrak{g}_{f_i+f_j} = \left\{ \begin{pmatrix} J(z) & -I_-(z) \\ -I_-(\overline{z}) & J(\overline{z}) \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i-f_j} = \left\{ \begin{pmatrix} J(z) & I_-(z) \\ I_-(\overline{z}) & J(\overline{z}) \end{pmatrix} \right\}.$$

The entries are those of the submatrices $J,\ I_+,\ I_-$ at row and column indices $p-j+1$, $p-i+1$, $p+j$, $p+i$.

There is also a root space

$$\mathfrak{g}_{2f_i} = \mathbb{R}\begin{pmatrix} i & -i \\ i & -i \end{pmatrix}, \qquad \mathfrak{g}_{-2f_i} = \mathbb{R}\begin{pmatrix} i & i \\ -i & -i \end{pmatrix},$$

where the entries correspond to indices $p-i+1,\ p+i$.

When $p \gneq q$, there exist

$$\mathfrak{g}_{f_i} = \left\{ \begin{pmatrix} 0 & v & -v \\ -\overline{v}^t & 0 & 0 \\ -\overline{v}^t & 0 & 0 \end{pmatrix} \right\}, \qquad \mathfrak{g}_{-f_i} = \left\{ \begin{pmatrix} 0 & v & -v \\ -\overline{v}^t & 0 & 0 \\ \overline{v}^t & 0 & 0 \end{pmatrix} \right\},$$

where $v \in \mathbb{C}^{p-q}$ and the 9 entries speacified above are placed at row and column indices $p-q,\ p-i+1,\ p+i$.

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