SHEAF COHOMOLOGY

1 Exact sequences

For this section we will be working with manifolds over $\mathbb{C}$ or $\mathbb{R}$.

Def: let X be a manifold, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ sheaves on it. Let

$$\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C}$$

be morphisms of sheaves. The sequence is exact at $\mathcal{B}$ if the induced sequence on stalks

$$\mathcal{A}_x \xrightarrow{g_x} \mathcal{B}_x \xrightarrow{h_x} \mathcal{C}$$

is exact, namely $\operatorname{Im} g_x = \operatorname{Ker}(h_x)$

Rk $0 \longrightarrow \mathcal{B} \xrightarrow{h} \mathcal{C}$ is exact at $\mathcal{B}$ $\Leftrightarrow$ h is injective

$\mathcal{A} \xrightarrow{g} \mathcal{B} \longrightarrow 0$ is exact at $\mathcal{B}$ $\Leftrightarrow$ g is surjective

Def: a short exact sequence is a sequence of the form

$$0 \longrightarrow \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \longrightarrow 0$$

(with 3 non trivial sheaves) wich is exact at $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

This is equivalent to having $\mathcal{A} \hookrightarrow \mathcal{B}$ injective

$\mathcal{A} \cong \operatorname{Ker} h$

$\mathcal{B} \twoheadrightarrow \mathcal{C}$ surjective

Examples

1) We already studied

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_X \xrightarrow{e^{2i\pi(\cdot)}} \mathcal{O}_X^{\times} \to 0$$

and proved its exactness.

2) Let $X = \mathbb{C}$, and consider $\mathcal{J} \subset \mathcal{O}_X$ the subsheaf of holomorphic sections vanishing at 0. Then

$$0 \to \mathcal{J} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{J} \to 0$$

Note that $\left(\mathcal{O}/\mathcal{J}\right)_x = \begin{cases} \mathbb{C} & \text{on } x = 0 \\ 0 & \text{on } x \neq 0 \end{cases}$

In the same way, we can consider any ideal $\mathcal{J}$ of the ring $\mathcal{O}_X$ (for any manifold X), which define subspaces of X (possibly singular) whose structure sheaf is $\left(\mathcal{O}_X/\mathcal{J}\right)\Big|_{\mathcal{J}=0}$

3) X manifold $/\mathbb{R}$, $\Omega^p_X \to$ sheaf of differential forms. Then

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim X}_X \to 0$$

is exact at every term. It is an example of a resolution of $\underline{\mathbb{R}}$

4) X manifold $/\mathbb{R}$

Let $S_p(\mathcal{U}) = \{\sigma_a : \Delta^p \longrightarrow \mathcal{U}$ continuous $\}$ and sums of them $/\mathbb{Z}$ singular chains

Δ^0 $\quad\quad$ Δ^2:

Δ^1 $\quad\quad$ Δ^3

etc.

Consider $S^p(\mathcal{U}, \mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(S_p(\mathcal{U}), \mathbb{R})$ singular cochains

Let $\partial : S_p(\mathcal{U}) \longrightarrow S_{p-1}(\mathcal{U})$

$$\sigma \longmapsto \underset{\substack{\text{faces of } \sigma \\ \text{labelled} \\ \text{by integers}}}{\sum} (-1)^i \, \sigma|_{F_i}$$

Then $\partial^* : S^p(\mathcal{U}, \mathbb{R}) \longrightarrow S^{p+1}(\mathcal{U}, \mathbb{R})$

Let $\mathcal{S}^p_{\mathbb{R}}$ be the sheaf generated by the presheaf $S^p(\mathcal{U}, \mathbb{R})$.

We have

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^0_{\mathbb{R}} \longrightarrow \mathcal{S}^1_{\mathbb{R}} \longrightarrow \cdots$$

is everywhere exact.

5) X manifold over $\mathbb{C}$, $\dim X = n$

Let $\Omega^p = \mathrm{Ker}(\Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1})$ where $\Omega^{p,i}$ are differential forms of type (p,i), namely, of the form $\sum f_{I,J} \, dz^I \wedge d\bar z^J$, where $I, J \subseteq \{1, \dots, n\}$ and $|I| = p$ $\quad |J| = i$

Then

$$0 \to \Omega^p \longrightarrow \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \cdots \Omega^{p,n} \longrightarrow 0$$

is exact, and notice that $\Omega^0 = \mathcal{O}_X$.

6) Let X be a manifold $/\mathbb{C}$. Let

- $\mathcal{M}_X^*$ be the sheaf of non zero meromorphic functions on X
- $\mathcal{O}_X^*$ " " " of nowhere vanishing "

Clearly, $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X^*$ is an embedding.

The quotient

$$0 \to \mathcal{O}_X^* \hookrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^* / \mathcal{O}_X^* \longrightarrow 0$$

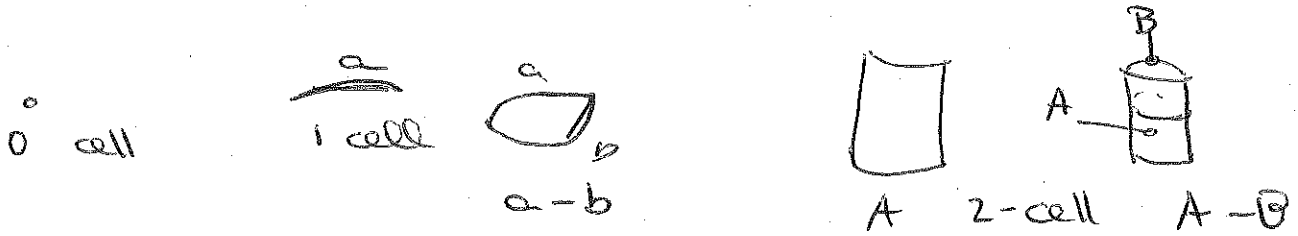
is called the sheaf of Cartier divisors (that is, divisors locally defined by the vanishing of a meromorphic function). We will denote it by $\mathcal{D}$.

The sequence is exact by definition.

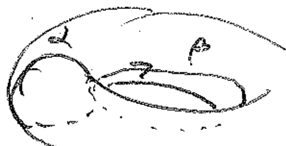
2 Cohomology

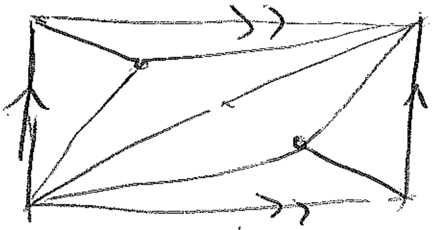
Motivation: (co)homology is an useful tool to study the topology of a manifold.

• Simplicial / singular cohomology proceed by counting holes encircled by integral combinations of n-cells



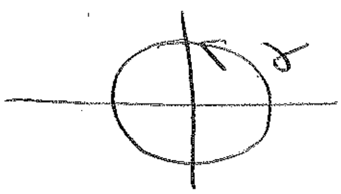
For example, the torus has 2 holes. A 3D hole "encircled" by a loop β, and a 3D hole (inside) encircled by a simplex of dimension 2.





This translates into non triviality of homology in degree 1 and 2.

• De Rham cohomology. When the topological space has a C^∞ structure $\rightsquigarrow$ differential forms detect wholes and give representatives for cohomology classes. For example, on $\mathbb{R}^2 \setminus \{0\}$, take



$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x\,dy}{x^2+y^2}$$

Then $\omega \neq df$ $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, as otherwise it'd happen $\oint_\gamma \omega = 0$ (but it is $\oint_\gamma \omega = 2\pi$.)

DeRham cohomology is calculated as follows: given a manifold M we define

$$\Omega^i(M) = \text{diff. forms of degree } i \text{ on } M$$

Recall that $\Omega^i(M)$ is a $\mathcal{C}^\infty(M)$-module generated by $dx_{j_1} \wedge \cdots \wedge dx_{j_i}$ $\{j_1, \ldots, j_i\} \subseteq \{1 \ldots \dim M\}$

Differentiation induces a map

$$\begin{aligned} \Omega^i(M) &\xrightarrow{d} \Omega^{i+1}(M) \\ \omega &\longmapsto \sum_{\ell=1}^{\dim M} \frac{\partial \omega}{\partial x_\ell} \wedge dx_\ell \end{aligned}$$

Since $d^2 = 0$, we have a chain complex

$$\underset{\mathcal{C}^\infty(M)}{\underset{\parallel}{\Omega^0(M)}} \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^{\dim M}(M) \to 0$$

In particular $\operatorname{Im} d \subseteq \operatorname{Ker} d$. We define

$$H^i_{dR}(M,\mathbb{R}) = \frac{\operatorname{Ker}(\Omega^i \xrightarrow{d} \Omega^{i+1})}{\operatorname{Im}(d : \Omega^{i-1} \to \Omega^i)}$$

the ith DeRham cohomology group.

- Sheaf cohomology: we saw in a former example that $d : \mathcal{C}^\infty_{\mathbb{R}\setminus\{0\}} \longrightarrow \Omega^1$ has $\operatorname{Im}(d) = \text{closed}$ forms (as all closed forms are locally exact) but certainly it is false that

$$\mathcal{E}^\infty(\mathbb{R}^2\setminus\{0\}) \twoheadrightarrow Z^1(\mathbb{R}^2\setminus\{0\}) := \text{closed forms}$$

Cas $\frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$ is closed but not exact.

In other words

$$0 \to \underline{\mathbb{R}} \to \mathcal{E}^\infty_{\mathbb{R}^2\setminus\{0\}} \longrightarrow \Omega^1 \longrightarrow \Omega^2 \to \cdots$$

Is exact but, letting $X = \mathbb{R}^2\setminus\{0\}$,

$$0 \to \underset{\underset{\mathbb{R}}{\parallel}}{\underline{\mathbb{R}}(X)} \longrightarrow \mathcal{E}^\infty(X) \longrightarrow \Omega^1(X) \longrightarrow \Omega^2(X) \cdots$$

may not be. This is the formal reason for sheaf cohomology.

<u>Definition</u>: a resolution of a sheaf $\mathcal{F}$ is an exact sequence of sheaves

$$0 \to \mathcal{F} \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \cdots$$

We will denote it by $0 \to \mathcal{F} \to \mathcal{F}^\bullet$

<u>Examples</u>: any of the examples of §1 on exact sequences are resolutions of the 1st non zero term

a) $0 \to \underline{\mathbb{R}} \to \mathcal{E}^\bullet_{\mathbb{R}}$ $\qquad$ $0 \to \underline{\mathbb{R}} \to \Omega^\bullet$

b) $\mathcal{O}^p \to \Omega^p \xrightarrow{\bar\partial} \Omega^{p,\bullet}$ $\qquad$ $0 \to \bar\Omega^p \xrightarrow{\partial} \Omega^{\bullet,p}$

c) $\underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^*$

The canonical resolution

$\mathcal{F}$ sheaf $\mathbb{F} \xrightarrow{\pi} X$ étalé space associated to $\mathcal{F}$. Consider

$$\mathcal{C}^0(\mathcal{F})(U) = \{ f : U \to \mathbb{F} \quad \pi \circ f = 1 \}$$

This is the sheaf of discontinuous sections of $\mathcal{F}$.

Clearly

$$0 \to \mathcal{F} \hookrightarrow \mathcal{C}^0(\mathcal{F})$$

continuous sections   any section

Proceed inductively by letting $\mathcal{F}^1(\mathcal{F}) = \mathcal{C}^0(\mathcal{F})/\mathcal{F}$

$$\mathcal{C}^1(\mathcal{F}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{F})), \quad \mathcal{F}^i(\mathcal{F}) = \mathcal{C}^{i-1}(\mathcal{F})/\mathcal{F}^{i-1}(\mathcal{F})$$

$$\mathcal{C}^i(\mathcal{F}) = \mathcal{C}^0(\mathcal{F}^{i-1}(\mathcal{F}))$$

<u>Note</u>: we have SES

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{F}) \to \mathcal{F}^1(\mathcal{F}) \to 0$$

$$0 \to \mathcal{F}^i(\mathcal{F}) \to \mathcal{C}^i(\mathcal{F}) \to \mathcal{F}^{i+1}(\mathcal{F}) \to 0$$

which pastes to a LES

$$0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{F})$$

This is the canonical resolution.

<u>Rk</u> by construction $\mathcal{C}^0(\mathcal{F})$ satisfies that $\forall V \subset X$ closed subspace $\mathcal{C}^0(\mathcal{F})(X) \twoheadrightarrow \mathcal{C}^0(\mathcal{F}(V)) = \varinjlim_{V \subset U \text{ open}} \mathcal{F}(U)$

A sheaf satisfying this is called soft.

Def
We define the qth cohomology group of $\mathcal{F}$ to the qth cohomology group of the chain complex

$$\mathcal{C}^0(\mathcal{F})(X) \xrightarrow{d_0} \mathcal{C}^1(\mathcal{F})(X) \xrightarrow{d_1} \cdots$$

Namely $H^q(X, \mathcal{F}) = \dfrac{\operatorname{Ker} d_q}{\operatorname{Im} d_{q-1}}$, where $d_{-1}: 0 \to \mathcal{C}^0$

*Very nice, but what does the canonical sheaf look like? Mistery... luckily, some tools allow some intuition:

Thm: Let $\mathcal{F} \to \mathcal{R}^\circ$ be a resolution of a sheaf $\mathcal{F}$. Then $\exists$ a homomorphism

$$\gamma_p : H^p(\mathcal{R}^\circ(X)) \longrightarrow H^p(X, \mathcal{F})$$

chain complex

If moreover
$$H^q(X, \mathcal{R}^p) = 0 \quad \forall \; q > 0 \quad p \geq 0$$
then γ_p is an isomorphism.

Examples

1) De Rham's theorem proves that on a $\mathcal{C}^\infty$ manifold X, we have isomorphisms
$$H^p_{dR}(X, \mathbb{R}) \simeq H^p_{sing}(X, \mathbb{R})$$

given by $I[\omega](\gamma) = \int_\gamma \omega$ ← any representative

p-chain

The way to proceed with sheaf theory is by checking that both

$$H^p(\mathcal{S}^\bullet(X)) \cong H^p(X, \mathbb{R}) \cong H^p(\Omega^\bullet(X))$$

This requires extra machinery that we won't explore for now.

③ Applications of sheaf cohomology.

Thm: let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a SES of sheaves. Then, we have a LES

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to$$
$$\to H^1(X, \mathcal{A}) \to \cdots$$

With this, we can transform obstruction problems into geometric information modulo the following result.

___: let $\mathcal{U} = \{\mathcal{U}_\alpha\} \to X$ be a covering by open sets. Let $\mathcal{G}$ be a sheaf of abelian groups, and consider the cochains $C^q(\mathcal{U}, \mathcal{G}) = \prod_{i_0 < \cdots < i_q} \mathcal{G}(\mathcal{U}_{i_0 \cdots i_q})$.

Let $\partial^q : C^q(\mathcal{U}, \mathcal{G}) \to C^{q+1}(\mathcal{U}, \mathcal{G})$

$$(\sigma_I)_I \to \left((\partial^q \sigma)_J = \sum_{k=0}^{q+1} (-1)^k \sigma_{j_0 \cdots \hat{j_k} \cdots j_q j_{q+1}} \right)_J$$

We define the qth Čech cohomology group associated w/ $\mathcal{U}$ & $\mathcal{G}$ $\quad H^q(\mathcal{U}, \mathcal{G}) = \dfrac{\operatorname{Ker} \partial^q}{\operatorname{Im} \partial^{q-1}}$

<u>Example</u>: X complex manifold. $\mathcal{U} \to X$ covering $\mathcal{G} = \mathcal{O}_X^\times$. Then

$$C^1(\mathcal{U}, \mathcal{O}_X^\times) = \left\{ (g_{\alpha\beta})_{\alpha\beta} \quad \begin{array}{l} g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \to \mathbb{C} \\ \text{holomorphic} \end{array} \right\}$$

$$\operatorname{Ker} \partial^1 = \{ (g_{\alpha\beta})_{\alpha\beta} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \}$$

Note that $g_{\alpha\alpha}$ is $g(\emptyset) = 1$. This also implies $g_{\alpha\beta} g_{\beta\alpha} g_{\alpha\alpha} = 1$

It follows that $\operatorname{Ker} \partial^1 = \left\{ \begin{array}{l} \text{line bundle on } X + \\ \text{trivialization on } \mathcal{U} \end{array} \right\}$

$$\operatorname{Im} \partial^0 = \{ (g_\alpha g_\beta^{-1}) \quad \}$$

$$H^1(\mathcal{U}, \mathcal{O}_X^\times) = \left\{ \begin{array}{l} \text{line bundles up to isomorphism} \\ \text{with a specified trivialization.} \end{array} \right\}$$

<u>Thm</u> let X be a smooth manifold / $\mathbb{C}$, and let $\mathcal{U}$ be a covering. For $\mathcal{F}$ coherent sheaf on X we have

$$H^q(\mathcal{U}, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

So sheaf cohomology has a clear geometrical meaning when we look at it in terms of Čech cohomology

<u>Examples</u>

1) $0 \to \mathbb{Z}_2 \to \mathcal{O}_X^\times \xrightarrow{\wedge 2} \mathcal{O}_X^\times \to 0$

The LES

$$0 \to H^0(X,\mathbb{Z}_2) \to H^0(X,\mathcal{O}_X^\times) \to H^0(X,\mathcal{O}_X^\times) \to$$
$$\to H^1(X,\mathbb{Z}_2) \to H^1(X,\mathcal{O}_X^\times) \to H^1(X,\mathcal{O}_X^\times) \to$$
$$\to H^2(X,\mathbb{Z}_2) \to \cdots$$

tells us that the obstruction for a never vanishing holo function on X to be $f=g^2$ for some other function is due to some line bundle

$L \to X$ of order 2.

Likewise, the obstruction for a line bundle to be $L=(L')^2 := L'\otimes L'$ is a cohomology class on $H^2(X,\mathbb{Z}_2)$. This group classifies $\mathbb{Z}_2$-gerbes.

2) X R.S. $0 \to \mathcal{O}_X^\times \hookrightarrow \mathcal{M}_X^\times \to \mathcal{D}_X \to 0$ induces

$$0 \to H^0(X,\mathcal{O}_X^\times) \to H^0(X,\mathcal{M}_X^\times) \to H^0(X,\mathcal{D}_X) \xrightarrow{\delta}$$
$$\to H^1(X,\mathcal{O}_X^\times) \to H^1(X,\mathcal{M}_X^\times) \to \cdots$$

Now, the obstruction to the existence of a meromorphic function with prescribed zeroes p_i & poles q_j of orders a_i, b_j is also codified in terms of bundles: indeed, we have that $D=\sum_i a_i p_i - \sum_j b_j q_j \in \mathcal{D}_X(X) \underset{\text{by def.}}{=} H^0(X,\mathcal{D}_X)$

So $\exists f\in \mathcal{M}_X^\times(X)$ with the given poles & zeroes $\Leftrightarrow$ $\delta(D)=0$, that is, if the corresponding line bundle is trivial.