Higgs bundles and abelianization

Second GEAR Junior Retreat
Ann Arbor, May 2014

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Remark

$(G^{\mathbb{C}})_{\mathbb{R}}$-Higgs bundles are the usual $G^{\mathbb{C}}$-Higgs bundles.

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$$h_G^{-1}(a) \cap \mathcal{M}(G)^{reg} \overset{1-1}{\leftrightarrow} \left\{ P \to \widehat{X}_a \quad \begin{array}{l} \text{principal } D^{\mathbb{C}} - \text{bundle} \\ P|_{\widehat{X}_{G,a}} \cong \theta^* P|_{\widehat{X}_{G,a}} \\ +\text{conditions.} \end{array} \right\}$$

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There are $2(p-1)(g-1)+1$ connected components in the moduli space $\mathcal{M}(SU(p+1,p))^{reg}$.

THANKS!