Higgs bundles and abelianization

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The main characters

Consider $G \leq G^{\mathbb{C}}$ a real form of a reductive connected algebraic group ($\mathrm{SL}(n,\mathbb{R}) < \mathrm{SL}(n,\mathbb{C})$, $(G^{\mathbb{C}})_{\mathbb{R}} < G^{\mathbb{C}} \times G^{\mathbb{C}}$...)

A G-**Higgs bundle** over a smooth projective complex curve X is a pair (E,ϕ) where
- $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle.
- $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ is the **Higgs field**.

Here $K = T^*X$, $H < G$ is maximally compact and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the associated Cartan decomposition.

e.g.• $\mathrm{SL}(n,\mathbb{R})$-Higgs bundles are pairs of $\mathrm{SO}(n,\mathbb{C})$-bundles and symmetric Higgs fields.
• $(G^{\mathbb{C}})_{\mathbb{R}}$-Higgs bundles are $G^{\mathbb{C}}$-Higgs bundles.

The stack of Higgs bundles

A **stack** over X is a **sheaf of categories**. We may define the stack of G-Higgs bundles by:

$$\begin{array}{rcl} \mathcal{H}_G : (Sch/X)_{an} & \to & \textbf{Cat} \\ Y \xrightarrow{f} X & \mapsto & \left\{ \begin{array}{c} f^*(E,\phi) \\ (E,\phi) \to X \ G\text{-Higgs} \end{array} \right\} \end{array}$$

This is the **quotient stack** $[\mathfrak{m}^{\mathbb{C}} \otimes K/H^{\mathbb{C}}]$. Thus, a Higgs bundle over X is a section $(E,\phi) \in \mathcal{H}_G(X)$.

The Hitchin map

Using the isomorphism

$$\mathfrak{m}^{\mathbb{C}}//H^{\mathbb{C}} \cong \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a}^{\mathbb{C}}),$$

where $\mathfrak{a} \subset \mathfrak{m}$ is a maximal abelian and $W(\mathfrak{a}^{\mathbb{C}})$ is the restricted Weyl group, we obtain $\chi : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}//W(\mathfrak{a})$, which induces the **Hitchin map**:

$$\begin{array}{rcl} [h_G] : [\mathfrak{m}^{\mathbb{C}} \otimes K/H^{\mathbb{C}}] & \to & \mathcal{A}_G := \mathfrak{a}^{\mathbb{C}} \otimes K//W(\mathfrak{a}^{\mathbb{C}}) \\ (E,\phi) & \mapsto & \chi(\phi) \end{array}$$

In the case of complex algebraic groups, **abelianization** holds. Given $D^{\mathbb{C}} < G^{\mathbb{C}}$ a maximal torus, we have correspondences

$$\left\{ \begin{array}{c} (E,\phi) \to X \\ G^{\mathbb{C}}\text{-Higgs bundle} \\ [h_{G^{\mathbb{C}}}](E,\phi) = a \end{array} \right\} \overset{\cong}{\Leftrightarrow} \left\{ \begin{array}{c} L \in BD^{\mathbb{C}}(\widehat{X}_a) \\ +\text{conditions} \\ \widehat{X}_a \text{ cameral cover [2]} \end{array} \right\}$$

$$\xrightarrow{\text{mod isom.}} \mathrm{Prym}(\overline{X}_a) \ : \ \overline{X}_a/X \text{ spectral cover [5].}$$

Global structure: Gerbes

A **gerbe** over X is a stack which is locally isomorphic to $B\mathcal{G}$ for some sheaf of groups $\mathcal{G}$. In a sense, it is a twisted torsor.

To get a gerbe we need to control automorphisms: a G-Higgs bundle $(E,G) \to X$) is **regular** if $\phi(x)$ has minimal dimensional centraliser

$$\mathfrak{c}_{\mathfrak{m}^{\mathbb{C}}}(\phi(x)) = \{y \in \mathfrak{m}^{\mathbb{C}} \ : \ [y,\phi(x)] = 0\}$$

for all $x \in X$.

Theorem 1. *([3, 6])*
• *The stack of everywhere regular G-Higgs bundles $[\mathfrak{m}^{\mathbb{C}} \otimes K/H^{\mathbb{C}}]^{reg}$ is a gerbe over $\mathcal{A}_G$.*
• *If $Ad(H) < Ad(G)$ is maximally compact, there exists a global section (Hitchin–Kostant–Rallis (HKR) section [4, 6]):*

$$s : (Sch/\mathcal{A}_G)_{an} \to [\mathfrak{m}^{\mathbb{C}} \otimes K/H^{\mathbb{C}}]^{reg}.$$

• *The gerbe is abelian if and only if the form is quasi-split. In this case, the gerbe is banded by a sheaf of tori $\mathcal{D}$ on $\mathcal{A}_G$.*

Moral: **abelianization is only possible for quasi-split real forms**. Simple quasi-split real forms are: split real forms, $\mathfrak{su}(p,p)$, $\mathfrak{su}(p,p+1)$, $\mathfrak{so}(p,p+2)$ and $\mathfrak{e}_{6(2)}$.

In the remaining cases, generically, we know the fibers. Let $C_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}}) = \{h \in H^{\mathbb{C}} \ : \ Ad(h)|_{\mathfrak{a}^{\mathbb{C}}} \equiv id\}$.

Proposition. *For a generic $a \in H^0(X, \mathcal{A}_G)$, there exists $X_{reg} \subset X$ open and a $W(\mathfrak{a})$-cover $\widehat{X}_{reg} \to X_{reg}$ such that $[h_G]^{-1}(a)|_{X_{reg}} \cong BC_{H^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})(\widehat{X}_{reg})$.*

Description of the fibers: the quasi-split form case

From now on, let $G < G^{\mathbb{C}}$ be quasi-split. We have a morphism

$$\kappa : [\mathfrak{m}^{\mathbb{C}} \otimes K/H^{\mathbb{C}}]^{reg} \to [\mathfrak{g}^{\mathbb{C}} \otimes K/G^{\mathbb{C}}]^{reg}$$

which commutes to the respective Hitchin maps $[h_G]$ and $[h_{G^{\mathbb{C}}}]$. Using [2] we are able to further describe the fibers of $[h_G]$.

Cameral data

We have that

$$\mathfrak{d}^{\mathbb{C}} := \{x \in \mathfrak{g}^{\mathbb{C}} \ : \ [x, \mathfrak{a}^{\mathbb{C}}] = 0\} = \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$. Let $D^{\mathbb{C}} = T^{\mathbb{C}}A^{\mathbb{C}}$ be the corresponding maximal torus and W its Weyl group.

The **cameral cover** associated to $a \in H^0(X, \mathcal{A}_G)$ is the pullback:

$$\begin{array}{ccccc} \widehat{X}_a & & \longrightarrow & & \mathfrak{d}^{\mathbb{C}} \otimes K \\ \downarrow & & & & \downarrow \\ X & \longrightarrow & \mathfrak{a}^{\mathbb{C}} \otimes K//W(\mathfrak{a}) & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \otimes K//W. \end{array}$$

The Cartan involution on $\mathfrak{g}$ induces one on cameral covers. We consider the quotient cameral cover $\widetilde{X}_a := \widehat{X}_a/\theta$.

Theorem 2. *([6, 3]) Let $a \in H^0(X, \mathcal{A}_G)$. There is an equivalence of categories:*

$$\left\{ \begin{array}{c} (E,\phi) \to X \in Im(\kappa) \\ [h_G](E,\phi) = a \end{array} \right\} \overset{\cong}{\Leftrightarrow} \left\{ \begin{array}{c} L \in BT^{\mathbb{C}}(\widetilde{X}_a) \\ +\text{conditions} \end{array} \right\}$$

*We call the latter the **category of cameral data**.*

SU$(p+1,p)$-Higgs bundles

An $\mathrm{SU}(p+1,p)$-Higgs bundle is a pair (E,ϕ) where:

- $E = V \oplus W$ with V a rank $p+1$ vector bundle, W a rank p vector bundle, $\det(V \oplus W) = \mathcal{O}_X$.
- $\phi = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, $C : V \to W \otimes K$, $B : W \to V \otimes K$.

The Hitchin map assigns to each pair the **characteristic coefficients** of the Higgs field:

$$h : (E,\phi) \mapsto a = (\mathrm{tr} \wedge^2 \phi, \ldots, \mathrm{tr} \wedge^{2p} \phi).$$

Thus, $\mathcal{A}_{\mathrm{SU}(p+1,p)} = \oplus_{i=1}^p K^{2i}$. On the other hand, $\mathbb{C}^p \cong \mathfrak{a}^{\mathbb{C}} \subset \mathfrak{d}^{\mathbb{C}} \cong \mathbb{C}^{2p}$, so the cameral cover is

$$\begin{array}{ccc} \widehat{X}_a & \hookrightarrow & K^{\oplus 2p} \\ \downarrow & & \downarrow \\ X & \xrightarrow{(a_1,\ldots,a_p)} & \oplus_{i=1}^p K^{2i} \end{array} \qquad \begin{array}{c} (k_1, \ldots, k_{2p}, -\sum_i k_i) \\ \downarrow \\ (\sigma_2(k_j), \ldots, \sigma_{2p}(k_j)) \end{array}$$

with σ_{2i} the $2i$-th symmetric polynomial.

Generically, a suitable $W_P \leq W$ satisfies

$$\widehat{X}_a/W_P \cong \overline{X}_a = SpecSym^{\bullet}(K^*/\lambda(\lambda^{2p} \sum_i a_i \lambda^{2(p-i)}).$$

Theorem 2 automatically yields the spectral data, relating our methods to those in [8, 9].

Theorem 3. *([7, 6]) For generic $a \in \mathcal{A}_G$ we have that $\pi_0(\kappa(h^{-1}(a))) \to Pic(Y)$ is a $(\mathbb{C}^\times)^{2p(g-1)-1}$ torsor. Here $Y \subset \overline{X}_a/\overline{\theta}$ is an irreducible component.*

As an application, the Milnor-Wood inequality, the gerby structure and the HKR section recover [1]:

Theorem 4. *([7]) There are $(2p-1)(g-1)$ connected components in $\mathcal{M}(SU(p+1,p))^{reg}$, the moduli space of regular polystable Higgs bundles.*

References

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