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# The non-local symplectic vortex equations and gauged Gromov-Witten invariants

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— *to my parents*

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## Abstract

The goal of this thesis is to define gauged Gromov-Witten invariants for closed monotone symplectic manifolds equipped with a Hamiltonian action of a compact Lie group. Gauged Gromov-Witten invariants were first studied by Cieliebak, Gaio, and Salamon for actions of arbitrary compact Lie groups on symplectically aspherical manifolds, and by Mundet i Riera for semi-free circle actions on monotone manifolds. Our definition of the invariants follows the pseudocycle approach to the definition of Gromov-Witten invariants for semipositive symplectic manifolds due to McDuff and Salamon. Gauged Gromov-Witten invariants are defined by counting solutions of the symplectic vortex equations. Solutions of these equations are called vortices and may be regarded as gauge-theoretical deformations of pseudoholomorphic curves. Traditionally, the symplectic vortex equations are formulated in terms of invariant almost complex structures on the target manifold. As it turns out, this invariance condition makes it impossible to obtain transversality for simple bubbles appearing in the compactification of the moduli space. Hence the pseudocycle approach to the definition of the invariants is bound to fail in this case. To overcome this problem we introduce a new non-local perturbation scheme for the symplectic vortex equations. The basic idea is to drop the invariance condition on the almost complex structure at the cost of introducing an additional dependence of the almost complex structure on the solution itself. This dependence is implemented by means of a holonomy perturbation satisfying certain axioms. We give an explicit construction of this perturbation. In this way, we are led to the definition of the non-local symplectic vortex equations. Solutions of these equations are called non-local vortices. We prove that the moduli space of gauge equivalence classes of marked non-local vortices admits a Gromov compactification by polystable non-local vortices. Our proof combines techniques from gauge theory and symplectic topology. We prove a non-local mean value inequality for subharmonic functions on the disk and establish an a priori estimate for non-local vortices. We prove a removable singularity theorem for vortices. This allows us to apply weak Uhlenbeck compactness in order to reduce the compactification problem to Gromov compactness for pseudoholomorphic curves. We construct an evaluation map from the moduli space of marked non-local vortices to the Borel construction and prove that it is a pseudocycle. This enables us to define the gauged Gromov-Witten invariants as an intersection number of pseudocycles in the Borel construction. We conclude by showing that the invariants do not depend on the data used to define them.

# Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Definition von geeichten Gromov-Witten-Invarianten für solche abgeschlossenen monotonen symplektischen Mannigfaltigkeiten, auf welchen eine kompakte Liegruppe auf Hamiltonsche Art und Weise wirkt. Geeichte Gromov-Witten-Invarianten wurden erstmalig von Cieliebak, Gaio und Salamon, sowie von Mundet i Riera studiert. Dieser betrachtet halb-freie Wirkungen des Einheitskreises auf monotonen Mannigfaltigkeiten, während jene den Fall von Wirkungen beliebiger kompakter Liegruppen auf symplektisch-asphärischen Mannigfaltigkeiten studieren. Unsere Definition der Invarianten folgt einem von McDuff und Salamon zur Definition der Gromov-Witten-Invarianten für semipositive symplektische Mannigfaltigkeiten entwickelten Zugang, welcher sich Pseudozykel bedient. Geeichte Gromov-Witten-Invarianten werden definiert, indem man Lösungen der symplektischen Vortexgleichungen zählt. Die Lösungen dieser Gleichungen heissen Vortices und können als eichtheoretische Deformationen pseudoholomorpher Kurven angesehen werden. Gewöhnlich werden die Vortexgleichungen unter Verwendung invarianter fast-komplexer Strukturen auf der Zielmannigfaltigkeit formuliert. Es zeigt sich jedoch, dass diese Invarianzbedingung es unmöglich macht, Transversalität für die einfachen Blasen, welche bei der Kompaktifizierung des Modulraumes auftreten, zu erreichen. Dies bedeutet, dass der auf Pseudozykeln beruhende Zugang zur Definition der Invarianten in diesem Falle scheitern muss. Um diese Schwierigkeiten zu überwinden, führen wir ein neues Störungsschema für die symplektischen Vortexgleichungen ein. Dem liegt folgende Idee zugrunde: Wir lassen beliebige fast-komplexe Strukturen zu, müssen dafür aber eine zusätzliche Abhängigkeit der fast-komplexen Struktur von den Lösungen einführen. Diese Abhängigkeit wird bewerkstelligt durch eine Holonomiestörung, die gewissen Axiomen genügt. Wir geben eine explizite Konstruktion einer solchen Störung an. Auf diese Weise gelangen wir zur Definition der nicht-lokalen symplektischen Vortexgleichungen. Deren Lösungen werden als nicht-lokale Vortices bezeichnet. Wir zeigen, dass der Modulraum von Eichäquivalenzklassen markierter nicht-lokaler Vortices durch polystabile nicht-lokale Vortices Gromov-kompaktifizierbar ist. Unser Beweis vereint Techniken aus der Eichtheorie sowie aus der symplektischen Topologie. Wir beweisen eine nicht-lokale Mittelwertungleichung für subharmonische Funktionen auf der Scheibe und zeigen eine A-priori-Abschätzung für nicht-lokale Vortices. Ferner beweisen wir einen Hebbarkeitssatz für Vortices. Dies erlaubt die Anwendung von schwacher Uhlenbeck-Kompaktheit, um das Kompaktifizierungsproblem für Vortices auf Gromov-Kompaktheit für pseudoholomorphe Kurven zu reduzieren. Wir konstruieren eine Auswertungsabbildung vom Modulraum markierter nicht-lokaler Vortices in die Borel-Konstruktion und zeigen, dass diese ein Pseudozykel ist. Dies versetzt uns sodann in die Lage, die geeichten Gromov-Witten-Invarianten als Schnittzahlen von Pseudozykeln in der Borel-Konstruktion zu definieren. Schliesslich zeigen wir, dass die Invarianten nicht von den zu ihrer Definition verwendeten Daten abhängen.

# Contents

CHAPTER 1

# Introduction

Gauged Gromov-Witten invariants for Hamiltonian actions of compact Lie groups on symplectic manifolds were first studied by Cieliebak, Gaio, and Salamon [**4**] and Mundet i Riera [**26**]. Cieliebak et. al. defined these invariants for actions of arbitrary compact Lie groups on symplectically aspherical manifolds satisfying certain natural technical conditions; the definition given by Mundet i Riera applies to semi-free circle actions on closed monotone manifolds also satisfying certain further technical conditions. Recently, González and Woodward [**13**] and Frenkel, Teleman, and Tolland [**10**] defined similar invariants for smooth projective varieties equipped with an action of a reductive algebraic group.

The goal of this thesis is to remove the asphericity assumption in Cieliebak et. al. [**4**] and to define gauged Gromov-Witten invariants for arbitrary closed monotone symplectic manifolds endowed with a Hamiltonian action of a compact Lie group under the assumption that the group acts freely on the zero level set of the moment map. This is part of a joint project with E. González, C. Woodward, and F. Ziltener [**12**] whose aim is to define gauged Gromov-Witten invariants with holonomy conditions at the marked points.

We begin with a review of the basic ideas underlying the definition of the gauged Gromov-Witten invariants. Let $(M, \omega, \mu)$ be a Hamiltonian $G$-manifold with moment map $\mu\colon M \to \mathfrak{g}$, where $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$. The invariants are then defined by counting solutions $(A, u)$ of the *symplectic vortex equations*

$$\overline{\partial}_{J,A}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0, \tag{1.1}$$

where $A$ is a connection on some fixed principal $G$-bundle $P$ over a closed Riemann surface $\Sigma$ with fixed complex structure, $F_A$ denotes its curvature, $u$ is a $G$-equivariant map $P \to M$, $\mathrm{dvol}_\Sigma$ is a fixed area form on $\Sigma$, and $J\colon \Sigma \to \mathcal{J}(M, \omega)^G$ is a family of $G$-invariant $\omega$-compatible almost complex structures on $M$. The symplectic vortex equations were introduced by Cieliebak et. al. [**4**] and, independently, Mundet i Riera [**25**]. They are invariant under the action of the group of gauge transformations of the bundle $P$. Note that for this invariance to hold the almost complex structure $J$ necessarily has to be $G$-invariant. Solutions of equations (1.1) are called *vortices* and may be regarded as a gauge-theoretical deformation of pseudoholomorphic curves. Over the past several years, the symplectic vortex equations have been studied from different perspectives [**3, 5, 8, 9, 11, 12, 13, 14, 27, 28, 29, 30, 33, 37, 39, 40, 41**]. In mathematical physics they are known as *gauged sigma models* [**18, 36**].

In more technical terms, gauged Gromov-Witten invariants are defined by means of intersection theory on a suitable compactification of the moduli space of gauge equivalence classes of vortices of fixed degree. Under certain technical assumptions, this moduli space is compact whenever the manifold $(M, \omega)$ is *symplectically aspherical* [**4**]. The latter condition means that the symplectic form $\omega$ vanishes on all spherical homology classes, which in particular implies that $M$ does not contain any nontrivial pseudoholomorphic spheres. Once this asphericity assumption is removed, the moduli space will in general no longer be compact and *spherical fiber bubbles* in the fibers of the associated bundle $P \times_G M \to \Sigma$ may bubble off. However, in this case the moduli space admits a Gromov compactification by polystable vortices [**31**]. The definition of the invariants then requires transversality for the spherical fiber bubbles so as to ensure that certain boundary strata of this compactification are actually smooth manifolds. While it is a well-known fact that for monotone $(M, \omega)$ transversality for spherical fiber bubbles holds for generic almost complex structure in the fiber, we cannot expect this to be true in the present situation since the almost complex structure $J$ was assumed to be $G$-invariant. In order to define the invariants in the non-aspherical case we thus need a device which enables us to avoid $G$-invariance of the almost complex structure $J$ on the one hand, but still allows for a compactification of the moduli space on the other hand. This is where the non-local vortex equations come into play.

The non-local vortex equations arise naturally from a *non-local* perturbation scheme for the standard vortex equations (1.1). The main idea behind this construction is as follows. We drop the $G$-invariance condition on the almost complex structure by replacing $J$ with a *$G$-equivariant* family $P \to \mathcal{J}(M, \omega)$ of arbitrary $\omega$-compatible almost complex structures. The vortex equations (1.1) would then in general no longer be gauge-invariant. To compensate for this, we let the family $J$ also depend on the pair $(A, u)$ in a way that is equivariant with respect to the action of the group of gauge transformations. This dependence is accomplished by means of a carefully chosen holonomy perturbation, which may be regarded as a classifying map for the action of the group of gauge transformations on the space of pairs $(A, u)$.

Technically, this perturbation scheme is realized as follows (see Chapter 2). Assume that $G$ acts freely on $\mu^{-1}(0)$. Fix a real constant $E > 0$, and choose an area form $\mathrm{dvol}_\Sigma$ on the Riemann surface $\Sigma$ that is *$E$-admissible* (Definition 2.1.1). Roughly speaking, this means that the area is evenly distributed over $\Sigma$ and the total area is sufficiently large with respect to $E$. We then consider the Banach manifold $\mathcal{B}$ of pairs $(A, u)$ consisting of a connection $A$ on $P$ and a $G$-equivariant map $u \colon P \to M$, both of class $W^{1,p}$ for some fixed real number $p > 2$, satisfying the taming condition

$$\int_\Sigma |\mu(u)|^2_{\mathfrak{g}} \, \mathrm{dvol}_\Sigma < E.$$

This condition ensures that the canonical projection $\mathcal{B} \to \mathcal{B}/\mathcal{G}^{2,p}$ by the action of the group $\mathcal{G}^{2,p}$ of gauge transformations of $P$ of class $W^{2,p}$ is a principal $\mathcal{G}^{2,p}$-bundle (see Section 2.1) and hence admits a *regular* $\mathcal{G}^{2,p}$-equivariant classifying map

$$\Theta\colon \mathcal{B} \to W^{1,p}\big(P, EG^N\big)^G$$

(Theorem 2.1.5) satisfying a number of axioms which will later be indispensable for the study of the analytical properties of the non-local vortex equations (Definition 2.1.3). The classifying map $\Theta$ assigns to every pair $(A,u) \in \mathcal{B}$ a $G$-equivariant map

$$\Theta_{(A,u)}\colon P \to EG^N$$

of class $W^{1,p}$ taking values in a fixed finite-dimensional model $EG^N \to BG^N$ of the universal $G$-bundle $EG \to BG$ (for technical reasons we have to work with $EG^N$ instead of $EG$). We next fix a smooth $G$-equivariant family

$$J\colon EG^N \to \mathcal{J}(M,\omega), \qquad e \mapsto J_e$$

of $\omega$-compatible almost complex structures on $M$, which in turn gives rise to a map that assigns to every pair $(A,u) \in \mathcal{B}$ a $G$-equivariant family

$$J_{\Theta(A,u)}\colon P \to \mathcal{J}(M,\omega), \qquad p \mapsto J_{\Theta(A,u)(p)}$$

of almost complex structures on $M$. We may then define the *non-local* Cauchy-Riemann operator

$$\overline{\partial}_{J,A,\Theta}(u) := \frac{1}{2}\Big(\mathrm{d}_A u + J_{\Theta(A,u)}(u) \circ \mathrm{d}_A u \circ j_\Sigma\Big),$$

where $\mathrm{d}_A u := \mathrm{d}u + X_A(u)$ is the twisted derivative of $u$ and $j_\Sigma$ denotes the fixed complex structure on $\Sigma$. The *non-local symplectic vortex equations* now take the form

$$\overline{\partial}_{J,A,\Theta}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0 \tag{1.2}$$

for pairs $(A,u) \in \mathcal{B}$ (see Section 2.2).

Our strategy for the definition of the gauged Gromov-Witten invariants is to adapt the pseudocycle approach to the definition of the Gromov-Witten invariants for semipositive symplectic manifolds due to McDuff and Salamon [**22**] to the present situation.

The first step is to construct a compactification of the moduli space

$$\mathcal{M}_n(P,M;B)$$

of $n$-marked non-local vortices of fixed degree $B$ (see Chapter 3), where $B \in H_2^G(M;\mathbb{Z})$ is an integral equivariant homology class in $M$. The elements of this space are gauge equivalence classes of solutions of equations (1.2) such that the equivariant degree of the map $u$ equals the class $B$ (see Section 2.2), where the $G$-bundle $P \to \Sigma$ is determined by the characteristic class coming from $B$ under the canonical projection $H_2^G(M;\mathbb{Z}) \to H_2(BG;\mathbb{Z})$. As it turns out, the moduli space $\mathcal{M}_n(P,M;B)$ admits a Gromov compactification by *polystable non-local vortices* in a way similar to the Gromov compactification of the moduli space of solutions of the standard vortex equations (1.1) constructed in [**31**] (Theorem 3.1.7). The strategy for constructing this compactification is to reformulate the problem in such a way that it may be solved by means of standard techniques from gauge theory (weak Uhlenbeck compactness for connections [**32**]) and symplectic topology (Gromov compactness for pseudoholomorphic curves [**15**]). This was first carried out by Mundet i Riera [**26**] for solutions of the standard vortex equations (1.1) under the assumption that $G$ is the circle. Our approach relies on techniques developed by Cieliebak et. al. [**3**] and by McDuff and Salamon [**22**]. The key step is to prove a removable singularity theorem for vortices (Theorem 3.3.2) that fits into the non-local framework. It relies on an a priori estimate for non-local vortices (Theorem 3.2.1), which in turn follows from a *non-local* version of the mean value inequality for subharmonic functions on the disk (Proposition 3.2.2).

The next step in the definition of the invariants is to establish an evaluation map

$$\mathrm{ev}\colon \mathcal{M}_n(P, M; B) \to (EG \times_G M)^n$$

on the moduli space of $n$-marked vortices taking values in the $n$-fold product of the Borel construction (see Section 5.1). Under the assumption that $(M, \omega)$ is monotone and after incorporating an additional Hamiltonian perturbation $H$ into the non-local vortex equations (1.2) (see Section 2.2), this evaluation map turns out to be a *pseudocycle* of (real) dimension

$$\frac{1}{2}\big(\dim M - 2\,\dim G\big)\cdot\chi(\Sigma) + 2\,\big\langle c_1^G(TM), B\big\rangle + 2n$$

for generic almost complex structure $J$ and Hamiltonian perturbation $H$ (Proposition 5.2.1). Recall that $(M, \omega)$ is called *monotone* if there exists a number $\tau > 0$ such that

$$\langle[\omega], A\rangle = \tau\cdot\langle c_1(TM), A\rangle$$

for every spherical homology class $A \in H_2(M; \mathbb{Z})$.

The proof of the pseudocycle property of the evaluation map relies on a transversality result for polystable non-local vortices (see Chapter 4). More precisely, we prove that the moduli space of *simple polystable non-local vortices* of fixed combinatorial type and prescribed degrees (see Section 4.3) is a manifold of the expected dimension (Theorem 4.3.5). This in particular requires transversality for the spherical fiber bubbles in the associated bundle $P \times_G M \to \Sigma$, which can now be achieved since the almost complex structure $J$ is no longer assumed to be $G$-invariant. It is useful to think of the fiber bubbles in $P \times_G M$ as fiber bubbles in the bundle $EG \times_G M \to BG$ associated to the universal bundle. Transversality for the fiber bubbles in $P \times_G M$ then appears as an instance of parametric transversality for fiber bubbles in $EG \times_G M$ (see Section A.3).

We may now define the gauged Gromov-Witten invariants

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,J,H}\colon H^*_G(M)^{\otimes n} \to \mathbb{Z} \tag{1.3}$$

as an intersection number of pseudocycles by

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,J,H}(a_1, \ldots, a_n) := f\cdot\mathrm{ev},$$

where $f\colon V \to \big(EG \times_G M\big)^n$ is a pseudocycle Poincaré dual to the product $a_1 \otimes \cdots \otimes a_n$ (see Section 5.3). Here we denote by $H^*_G(M)$ the torsion-free part of the integral equivariant cohomology of $M$. A priori, the invariants (1.3) depend on the almost complex structure $J$ and the Hamiltonian perturbation $H$. However, the Main Theorem below asserts that they are in fact invariants in the sense that they do not depend on the choice of generic $J$ and $H$.

We may now state the main result of this work (Theorem 5.3.2).

Main Theorem. *Let $G$ be a compact connected Lie group, and let $(M, \omega, \mu)$ be a closed Hamiltonian $G$-manifold. Assume that $(M, \omega)$ is monotone and that $G$ acts freely on $\mu^{-1}(0)$.*

*Fix an equivariant homology class $B \in H_2^G(M; \mathbb{Z})$ and a positive integer $n$. Then the homomorphism*

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,J,H}\colon H^*_G(M)^{\otimes n} \to \mathbb{Z}$$

*defined by*

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,J,H}(a_1, \dots, a_n) := f \cdot \mathrm{ev}$$

*is independent of the generic almost complex structure $J$, the generic Hamiltonian perturbation $H$, and the pseudocycle $f$ used to define it.*

The homomorphisms (1.3) are invariants of the Hamiltonian $G$-manifold $(M, \omega, \mu)$, depending only on the degree $B$, the area form $\mathrm{dvol}_\Sigma$ and the classifying map $\Theta$. We expect that they are in fact independent of the *admissible* area form $\mathrm{dvol}_\Sigma$ and the *regular* classifying map $\Theta$, although this has yet to be proven. They will be called *gauged Gromov-Witten invariants.*

CHAPTER 2

# The non-local symplectic vortex equations

The goal of this chapter is to introduce the non-local symplectic vortex equations. They arise naturally from a certain perturbation scheme for the standard vortex equations (1.1). This perturbation scheme is reminiscent of a construction of a holonomy perturbation due to Floer [**7**]. We will give an axiomatic characterization of this perturbation scheme in Section 2.1, deferring the details of its construction to Section 2.3 at the end of this chapter. The non-local vortex equations will then be introduced in Section 2.2.

Throughout this thesis, we fix the following notation. Let $G$ be a compact connected Lie group, with Lie algebra denoted by $\mathfrak{g}$, and let $(M, \omega, \mu)$ be a closed Hamiltonian $G$-manifold. Explicitly, this means that $M$ is a $G$-manifold equipped with a $G$-invariant symplectic form $\omega$ and a $G$-equivariant moment map $\mu\colon M \to \mathfrak{g}^* \cong \mathfrak{g}$ such that the identity

$$\iota(X_\xi)\,\omega = \mathrm{d}\langle \mu, \xi \rangle$$

holds for every $\xi \in \mathfrak{g}$, where $X_\xi$ denotes the infinitesimal action of $\xi$ on $M$. Here we identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by means of some fixed invariant inner product on $\mathfrak{g}$. We will always assume that $G$ acts freely on $\mu^{-1}(0)$. Note that this implies that 0 is a regular value of $\mu$. Moreover, it follows from this assumption that there exists a real constant $\delta > 0$ such that

$$M_\delta := \big\{ x \in M \,\big|\, |\mu(u)| \leq \delta \big\}$$

is a smooth compact submanifold with boundary of $M$ upon which $G$ acts freely.

Let $\Sigma$ be a closed Riemann surface with fixed complex structure $j_\Sigma$. Let $\pi\colon P \to \Sigma$ be a principal $G$-bundle over $\Sigma$. We denote by $P(\mathfrak{g}) := P \times_G \mathfrak{g}$ the adjoint bundle.

## 2.1. Regular classifying maps

The goal of this section is to give an axiomatic characterization of the perturbation scheme that will later be used to define the non-local vortex equations. We begin by introducing some notation.

**2.1.1. Admissible area forms and the configuration space of pairs.** In this subsection we introduce a configuration space consisting of pairs $(A, u)$, where $A$ is a connection on $P$ and $u\colon P \to M$ is a $G$-equivariant map, satisfying a certain taming condition. This configuration space will later serve as the domain of definition for the non-local vortex equations.

We begin by defining a certain class of area forms on the Riemann surface $\Sigma$. Given an area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, we denote by $\langle\cdot\,,\cdot\rangle_\Sigma := \mathrm{dvol}_\Sigma(\cdot\,, j_\Sigma\,\cdot)$ the Kähler metric on $\Sigma$ determined by $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$.

Definition 2.1.1. (Admissible area forms) Let $E$ be a positive real constant. An area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ is called $E$**-admissible** if the corresponding Kähler metric $\langle\cdot\,,\cdot\rangle_\Sigma$ has the following property. There exist a positive real number $R$, smaller than half the injectivity radius of $\Sigma$, and finitely many points $z_1,\ldots,z_a$ on $\Sigma$ such that the closed geodesic disks $B_{R/3}(z_1),\ldots,B_{R/3}(z_a)$ form a covering of $\Sigma$ and the area of every annulus $B_R(z_i)\setminus B_{R/2}(z_i)$ satisfies

$$\mathrm{Vol}\big(B_R(z_i)\setminus B_{R/2}(z_i)\big) > \frac{E}{\delta^2}$$

for $i=1,\ldots,a$.

Fix a real number $p>2$. We denote by $\mathcal{A}^{1,p}(P)$ the space of connections on $P$ of class $W^{1,p}$, and by $W^{1,p}(P,M)^G$ the space of $G$-equivariant maps $u\colon P\to M$ of class $W^{1,p}$ (see [**34**], Appendices A and B for details on these spaces). By Rellich's theorem ([**34**], Theorem B.2), any pair $(A,u)\in\mathcal{A}^{1,p}(P)\times W^{1,p}(P,M)^G$ is of class $C^0$. Fix a positive constant $E$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, and consider the configuration space of all pairs $(A,u)\in\mathcal{A}^{1,p}(P)\times W^{1,p}(P,M)^G$ such that the map $u$ satisfies the taming condition

$$\int_\Sigma |\mu(u)|_{\mathfrak{g}}^2\,\mathrm{dvol}_\Sigma < E, \tag{2.1}$$

where the norm is understood with respect to the invariant inner product on the Lie algebra $\mathfrak{g}$. We will denote this configuration space by

$$\mathcal{B}^{1,p} := \mathcal{B}^{1,p}(P,M;E,\mathrm{dvol}_\Sigma) := \big\{(A,u)\in\mathcal{A}^{1,p}(P)\times W^{1,p}(P,M)^G\,\big|\,u \text{ satisfies (2.1)}\big\}.$$

The group $\mathcal{G}^{2,p}:=\mathcal{G}^{2,p}(P):=W^{2,p}(P,G)^G$ of gauge transformations of class $W^{2,p}$ of the bundle $P$ acts on the space $\mathcal{A}^{1,p}(P)\times W^{1,p}(P,M)^G$ from the right by

$$g^*(A,u) := \big(g^{-1}Ag+g^{-1}\,\mathrm{d}g,\, g^{-1}u\big) \tag{2.2}$$

(see [**34**], Lemmata A.6 and B.3 for details). This action leaves the taming condition (2.1) invariant and hence induces an action of $\mathcal{G}^{2,p}$ on the configuration space $\mathcal{B}^{1,p}$.

Lemma 2.1.2. *The action (2.2) of the group of gauge transformations $\mathcal{G}^{2,p}$ on the configuration space $\mathcal{B}^{1,p}$ is free. The projection $\mathcal{B}^{1,p}\to\mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ is a principal fiber bundle with structure group $\mathcal{G}^{2,p}$.*

Proof. We first prove that the action is free. Consider a pair $(A,u)\in\mathcal{B}^{1,p}$ and a gauge transformation $g\in\mathcal{G}^{2,p}$, and suppose that

$$g^{-1}Ag+g^{-1}\,\mathrm{d}g = A \quad\text{and}\quad g^{-1}u=u. \tag{2.3}$$

Since $u$ satisfies the taming condition (2.1), admissibility of the area form $\mathrm{dvol}_\Sigma$ implies that there exists a base point $p_0 \in P$ such that $u(p_0) \in M_\delta$, that is, the action of $G$ on $M$ is free at $u(p_0)$. In fact, if there were no such point then Definition 2.1.1 would imply existence of a point $z_{i_0} \in \Sigma$ such that $|\mu(u(p))| > \delta$ for all points $p \in P$ such that $\pi(p)$ is contained in the annulus $B_R(z_{i_0}) \setminus B_{R/2}(z_{i_0})$. We would then further conclude from Definition 2.1.1 that

$$\int_\Sigma |\mu(u)|^2 \,\mathrm{dvol}_\Sigma \geq \int_{B_R(z_{i_0})\setminus B_{R/2}(z_{i_0})} |\mu(u)|^2 \,\mathrm{dvol}_\Sigma > \delta^2 \cdot \frac{E}{\delta^2} = E,$$

contradicting (2.1). Hence it follows from assumption (2.3) that $g$ is a solution of the ordinary differential equation

$$g^{-1}Ag + g^{-1}\,\mathrm{d}g = A, \qquad g(p_0) = 1, \tag{2.4}$$

where $1 \in G$ denotes the unit element. By the Cauchy-Lipschitz theorem ([**17**], Ch. V, Thm. 3.1), $g$ is uniquely determined by the initial condition $g(p_0) = 1$. Whence $g(p) = 1$ for all $p \in P$, that is, $g$ is the identity gauge transformation.

In order to see that $\mathcal{B}^{1,p} \to \mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ is a principal $\mathcal{G}^{2,p}$-bundle, we combine the local slice theorem ([**34**], Theorem F and [**3**], Theorem B.1) for the action of the group

$$\mathcal{G}_0^{2,p} := \left\{ g \in \mathcal{G}^{2,p} \,\middle|\, g(p_0) = 1 \right\}$$

of based gauge transformations on the space of connections $\mathcal{A}^{1,p}(P)$ with the local slice theorem (see [**6**], Section 2.4) for the residual action of $G$ on the fiber of $P$ over the base point $p_0$. □

**2.1.2. The classifying space for the group of gauge transformations.** Fix a classifying space $BG$ of the group $G$ and a universal $G$-bundle $EG \to BG$. Let $\mathcal{G}^0(P)$ be the group of continuous gauge transformations of $P$. Denote by $C^0(P, EG)^G$ the space of continuous $G$-equivariant maps $\theta\colon P \to EG$, and by $C^0_P(\Sigma, BG)$ the space of continuous maps $\bar{\theta}\colon \Sigma \to BG$ such that $P \cong \bar{\theta}^* EG$. The latter space is a classifying space for the group of gauge transformations $\mathcal{G}^0(P)$, and the natural map

$$C^0(P, EG)^G \to C^0_P(\Sigma, BG)$$

which sends a $G$-equivariant map $\theta\colon P \to EG$ to the corresponding map $\bar{\theta}\colon \Sigma \to BG$ between quotients defines a principal $\mathcal{G}^0(P)$-bundle which is universal for $\mathcal{G}^0(P)$ (see Husemoller [**19**], Sections 7.2 and 7.3 for details). Here $\mathcal{G}^0(P)$ acts on $C^0(P, EG)^G$ from the right by

$$g^*\theta := \theta\, g. \tag{2.5}$$

By Lemma 2.1.2, the canonical projection $\mathcal{B}^{1,p} \to \mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ defines a principal $\mathcal{G}^{2,p}$-bundle. Any continuous $\mathcal{G}^{2,p}$-equivariant map

$$\mathcal{B}^{1,p} \to C^0(P, EG)^G \tag{2.6}$$

descends to a classifying map

$$\mathcal{B}^{1,p}/\mathcal{G}^{2,p} \to C^0_P(\Sigma, BG)$$

for this bundle ([**19**], Sec. 4.4, Thm. 4.2). By abuse of language we shall therefore call any map (2.6) a **classifying map** for the configuration space $\mathcal{B}^{1,p}$. Note that this classifying map is unique up to equivariant homotopy.

**2.1.3. Regular classifying maps.** In order to carry out the analysis in the subsequent chapters we have to restrict to a certain class of classifying maps (2.6). The next definition gives an axiomatic characterization of such classifying maps.

For every positive integer $N$ we fix a finite dimensional approximation $EG^N \subset EG$ of the universal $G$-bundle $EG$ (see Husemoller [**19**], Sec. 4.11). Note that $EG^N$ is a finite dimensional smooth manifold upon which $G$ acts freely.

Definition 2.1.3 (Regular classifying maps). Fix a real number $p > 2$, a real constant $E > 0$, and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. A classifying map

$$\Theta\colon \mathcal{B}^{1,p} = \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to C^0(P, EG)^G, \quad (A, u) \mapsto \Theta_{(A,u)} \tag{2.7}$$

in the sense of Section 2.1.2 is called **regular** if it satisfies the following axioms (see Remark 2.1.4 below for an explanation of the notation).

**(Finiteness)** There exists a finite dimensional approximation $EG^N$ of $EG$ such that for every pair $(A, u) \in \mathcal{B}^{1,p}$ the map $\Theta_{(A,u)}\colon P \to EG$ takes values in $EG^N$.

**(Regularity)** (i) For every pair $(A, u) \in \mathcal{B}^{1,p}$, the map $\Theta_{(A,u)}\colon P \to EG^N$ is of class $W^{1,p}$.

(ii) If $A$ is of class $C^k$ for some $k \geq 1$, then $\Theta_{(A,u)}$ is of class $C^k$ as well.

(iii) The map

$$\Theta\colon \mathcal{B}^{1,p} \to W^{1,p}\big(P, EG^N\big)^G$$

is Fréchet differentiable.

**(Continuity)** Let $Z$ be a finite subset of $\Sigma$, and suppose that $(A_\nu, u_\nu)$ is a sequence of pairs in $\mathcal{B}^{1,p}$ that converges to a pair $(A, u) \in \mathcal{B}^{1,p}$ in the following sense.

(i) $A_\nu$ converges to $A$ in the $C^0$-topology on $\Sigma$;

(ii) $u_\nu$ converges to $u$ in the $C^0$-topology on compact subsets of $\Sigma \setminus Z$.

Then the sequence $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_{(A,u)}$ in the $C^0$-topology.

**(Estimates)** Fix a $G$-invariant metric on $EG^N$.

(i) There exists a constant $C > 0$ such that for all $(A, u) \in \mathcal{B}^{1,p}$ we have

$$\left\|\Theta_{(A,u)}\right\|_{L^p(\Sigma)} + \left\|\mathrm{d}_A\Theta_{(A,u)}\right\|_{L^p(\Sigma)} \leq C \cdot \Big(1 + \left\|F_A\right\|_{L^p(\Sigma)}\Big).$$

(ii) There exists a constant $C' > 0$ such that for all $(A, u) \in \mathcal{B}^{1,p}$ of class $C^1$ we have

$$\left\|\mathrm{d}_A\Theta_{(A,u)}\right\|_{L^\infty(\Sigma)} \leq C' \cdot \Big(1 + \left\|F_A\right\|_{L^\infty(\Sigma)}\Big).$$

(iii) There exists a constant $C'' > 0$ such that for all $(A, u) \in \mathcal{B}^{1,p}$ of class $C^2$ we have

$$\left|\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(z)\right| \leq C'' \cdot \Bigg(1 + \left\|F_A\right\|^2_{L^\infty(\Sigma)} + \int_{B_{\iota(\Sigma)}(z)} \frac{|\nabla_A F_A(z')|}{\mathrm{d}_\Sigma(z, z')}\, \mathrm{dvol}_\Sigma(z')\Bigg)$$

for all points $z \in \Sigma$. Here, we write $B_{\iota(\Sigma)}(z)$ for the geodesic disk on $\Sigma$ around $z$ whose radius is the injectivity radius $\iota(\Sigma)$ of $\Sigma$, and we denote by $\mathrm{d}_\Sigma(\cdot, \cdot)$ the Riemannian distance function on $\Sigma$.

We will denote by $\mathcal{C}^p_{\mathrm{reg}} := \mathcal{C}^p_{\mathrm{reg}}(P, M; E, \mathrm{dvol}_\Sigma; EG)$ the space of all regular classifying maps (2.7).

Remark 2.1.4. We explain the notation appearing in Definition 2.1.3.

In the (Regularity) axiom, the Sobolev space of classifying maps of class $W^{1,p}$ is to be understood as follows (see [**34**], App. B for more details).

Recall that $G$ acts freely on the finite dimensional approximation $EG^N$ of the universal bundle $EG$. Denoting the corresponding quotient by $BG^N$ we thus obtain a principal $G$-bundle $EG^N \to BG^N$. There is a one-to-one correspondence between $G$-equivariant maps $\theta\colon P \to EG^N$ and sections $\bar{\theta}\colon \Sigma \to P \times_G EG^N$ of the fiber bundle $P \times_G EG^N \to \Sigma$ associated to the $G$-bundle $\pi\colon P \to \Sigma$ by the $G$-action on $EG^N$. More specifically, the section $\bar{\theta}$ is given by $\bar{\theta}(z) := [p, \theta(p)]$ for all $z \in \Sigma$, where $p \in P$ is such that $\pi(p) = z$. We will usually not distinguish in the notation between a $G$-equivariant map $\theta\colon P \to EG^N$ and its corresponding section $\bar{\theta}\colon \Sigma \to P \times_G EG^N$.

Next, let us fix a $G$-invariant metric on $EG^N$. By the equivariant Nash embedding theorem [**23**] there exists a Euclidean vector space $V$ of sufficiently large dimension that is equipped with a right action of $G$ by isometries, together with an isometric $G$-equivariant embedding of the manifold $EG^N$ into the vector space $V$. This embedding makes the fiber bundle $P \times_G EG^N \to \Sigma$ into a subbundle of the associated vector bundle $E := P \times_G V \to \Sigma$. In particular, we may think of a $G$-equivariant map $P \to EG^N$ as a section of the vector bundle $E$.

Fix a smooth reference connection $A_0 \in \mathcal{A}(P)$ on $P$. It defines a covariant derivative

$$\nabla_{A_0} \colon \Gamma(\Sigma, E) \to \Gamma(\Sigma, T^*\Sigma \otimes E), \qquad \nabla_{A_0}\alpha := \mathrm{d}\alpha + X_{A_0}(\alpha)$$

on smooth sections of the vector bundle $E$. Here, $X_\eta$ denotes the infinitesimal action of $\eta \in \mathfrak{g}$ on $V$. Note that in this definition we think of $\alpha$ as a $G$-equivariant map $P \to V$, and that the covariant derivative $\nabla_{A_0}\alpha$ is $G$-equivariant and horizontal and hence descends to a 1-form on $\Sigma$.

The Kähler metric on $\Sigma$ determined by the area form $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$, together with the $G$-invariant metric on $V$ give rise to a fiberwise metric on the tensor bundle $\otimes^k T^*\Sigma \otimes E \to \Sigma$, $k = 0, 1$. We may define the $L^p$-norm of a section $\alpha \in \Gamma(\Sigma, \otimes^k T^*\Sigma \otimes E)$, $k = 0, 1$, by

$$\left\|\alpha\right\|_{L^p} := \left(\int_\Sigma |\alpha|^p \,\mathrm{dvol}_\Sigma\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Here $|\alpha|$ denotes the pointwise norm of $\alpha$ understood with respect to the fiberwise metric on the bundle $\otimes^k T^*\Sigma \otimes E$. The $W^{1,p}$-Sobolev norm of a section $\alpha \in C^1(\Sigma, E)$ of class $C^1$ is then given by

$$\left\|\alpha\right\|_{W^{1,p}} := \left(\left\|\alpha\right\|_{L^p}^p + \left\|\nabla_{A_0}\alpha\right\|_{L^p}^p\right)^{\frac{1}{p}}. \tag{2.8}$$

The Sobolev space $W^{1,p}(\Sigma, E)$ of $W^{1,p}$-sections of the vector bundle $E$ is then defined as the completion of the space of smooth sections of $E$ with respect to the $W^{1,p}$-norm (2.8). The Sobolev norm extends to the whole Sobolev space, making $W^{1,p}(\Sigma, E)$ into a Banach space. Note that all Sobolev norms obtained in this way are equivalent. The topology on $W^{1,p}(\Sigma, E)$ defined by these norms will be called the $W^{1,p}$-topology.

We may now define the Sobolev space $W^{1,p}(P, EG^N)^G$ of all $G$-equivariant maps $P \to EG^N$ of class $W^{1,p}$ as the space of all continuous $G$-equivariant maps $P \to EG^N$ that, when considered as a section of the vector bundle $E \to \Sigma$ as described above, have finite $W^{1,p}$-norm. In this way we obtain an inclusion

$$W^{1,p}(P, EG^N)^G \hookrightarrow W^{1,p}(\Sigma, E)$$

which endows the space $W^{1,p}(P, EG^N)^G$ with the structure of a smooth Banach manifold. Note in particular that this Banach manifold structure does not depend on the choice of $G$-invariant metric on $EG^N$.

In the (Estimates) axiom, the covariant derivatives of the map $\Theta_{(A,u)}$ and of the curvature $F_A$ are defined as follows.

Let us first consider the map $\Theta_{(A,u)} \colon P \to EG^N$. Its twisted derivative

$$\mathrm{d}_A\Theta_{(A,u)} \colon TP \to T_{\Theta_{(A,u)}} EG^N$$

is given by

$$\mathrm{d}_A\Theta_{(A,u)} := \mathrm{d}\Theta_{(A,u)} + X_A\big(\Theta_{(A,u)}\big),$$

where $X_\eta$ denotes the infinitesimal action of $\eta \in \mathfrak{g}$ on $EG^N$. We may equivalently think of this map as a 1-form on $\Sigma$, that is, as a section of the tensor bundle $T^*\Sigma \otimes F \to \Sigma$, where $F$ denotes the vector bundle

$$F := \Theta^*_{(A,u)} TEG^N / G \to \Sigma. \tag{2.9}$$

Fix a $G$-invariant metric on $EG^N$, and denote by $\nabla^{EG}$ the corresponding Levi-Civita connection on $EG^N$. The connection $A$ then gives rise to a covariant derivative

$$\nabla^F_A \colon C^1(\Sigma, F) \to C^0(\Sigma, T^*\Sigma \otimes F)$$

on the vector bundle $F$ defined by

$$\nabla^F_A \xi = \nabla^{EG}\xi + \nabla^{EG}_\xi X_A(\xi),$$

where $X_\eta$ denotes the infinitesimal action of $\eta \in \mathfrak{g}$ on $EG^N$.

Let $\nabla^\Sigma$ be the Levi-Civita connection on $\Sigma$, understood with respect to the Kähler metric determined by the area form $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$. The connections $\nabla^\Sigma$ and $\nabla^F_A$ then define a covariant derivative

$$\nabla_A \colon C^1\big(\Sigma, T^*\Sigma \otimes F\big) \to C^0\big(\Sigma, \otimes^2 T^*\Sigma \otimes F\big)$$

acting on the 1-form $\mathrm{d}_A\Theta_{(A,u)}$ by the formula

$$\begin{aligned}\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(w,v) = \nabla^{EG}_{A,w}\big(\mathrm{d}_A\Theta_{(A,u)}(v)\big) + \nabla^{EG}_{\mathrm{d}_A\Theta_{(A,u)}(v)} X_{A(w)}\big(\mathrm{d}_A\Theta_{(A,u)}\big)\\ - \mathrm{d}_A\Theta_{(A,u)}\big(\nabla^\Sigma_w v\big)\end{aligned} \tag{2.10}$$

for all smooth vector fields $w, v$ on $\Sigma$.

Lastly, we consider the covariant derivative of the curvature $F_A$. As above, let $\nabla^\Sigma$ be the Levi-Civita connection on $\Sigma$. Recall that $P(\mathfrak{g}) = P \times_G \mathfrak{g}$ denotes the adjoint bundle of $P$. The connection $A$ defines a twisted derivative

$$\nabla_A \colon C^1\big(\Sigma, P(\mathfrak{g})\big) \to C^0\big(\Sigma, T^*\Sigma \otimes P(\mathfrak{g})\big)$$

by the formula

$$\nabla_A \eta := \mathrm{d}\eta + [A, \eta].$$

The operators $\nabla^\Sigma$ and $\nabla_A$ then give rise to a covariant derivative

$$\nabla_A \colon C^1\big(\Sigma, \otimes^2 T^*\Sigma \otimes P(\mathfrak{g})\big) \to C^0\big(\Sigma, \otimes^3 T^*\Sigma \otimes P(\mathfrak{g})\big)$$

acting on the curvature $F_A$ by the formula

$$\nabla_A F_A\,(v_0, v_1, v_2) := \nabla_{A,v_0}\big(F_A(v_1, v_2)\big) - F_A\big(\nabla^\Sigma_{v_0} v_1, v_2\big) - F_A\big(v_1, \nabla^\Sigma_{v_0} v_2\big)$$

for all smooth vector fields $v_0, v_1, v_2$ on $\Sigma$.

In the (Estimates) axiom, the norms $\|F_A\|^2_{L^p(\Sigma)}$ and $\|F_A\|^2_{L^\infty(\Sigma)}$ are understood with respect to the fiberwise metric on the tensor bundle $\otimes^2 T^*\Sigma \otimes P(\mathfrak{g})$ induced by the Kähler metric on $\Sigma$ and the $G$-invariant inner product on $\mathfrak{g}$. Likewise, for any point $z \in \Sigma$ the expression $|\nabla_A F_A(z)|$ denotes the norm of the section $\nabla_A F_A$ at the point $z$, taken with respect to the fiber metric on the tensor bundle $\otimes^3 T^*\Sigma \otimes P(\mathfrak{g})$. Furthermore, the norms $\|\mathrm{d}_A\Theta_{(A,u)}\|_{L^p(\Sigma)}$ and $\|\mathrm{d}_A\Theta_{(A,u)}\|_{L^\infty(\Sigma)}$ are understood with respect to the fiberwise metric on the tensor bundle $T^*\Sigma \otimes F$ induced by the Kähler metric on $\Sigma$ and the $G$-invariant metric on $EG^N$, where the vector bundle $F$ is as in (2.9) above. Likewise, for any point $z \in \Sigma$ the expression $|\nabla_A(\mathrm{d}_A\Theta_{(A,u)})(z)|$ denotes the norm of the section $\nabla_A(\mathrm{d}_A\Theta_{(A,u)})$ at the point $z$, taken with respect to the fiber metric on the tensor bundle $\otimes^2 T^*\Sigma \otimes F$.

**2.1.4. Existence of regular classifying maps.** The next theorem, which is the main result of this section, ensures existence of regular classifying maps.

THEOREM 2.1.5. *Fix a real constant $E > 0$ and an $E$-admissible area form* $\mathrm{dvol}_\Sigma$ *on $\Sigma$. Let $p > 2$ be a real number. Then there exists a classifying map*

$$\Theta\colon \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to C^0(P, EG)^G$$

*that is regular in the sense of Definition 2.1.3.*

The proof of Theorem 2.1.5 is deferred to Section 2.3.

## 2.2. Non-local symplectic vortices

In this section, we introduce the non-local symplectic vortex equations. We define Hamiltonian perturbations of these equations, and prove an energy identity and a regularity result for solutions of the corresponding equations.

**2.2.1. The non-local symplectic vortex equations.** Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number. By Theorem 2.1.5 we may choose a finite dimensional approximation $EG^N$ of $EG$ and a regular classifying map

$$\Theta\colon \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to W^{1,p}\big(P, EG^N\big)^G, \quad (A, u) \mapsto \Theta_{(A,u)} := \Theta(A, u) \qquad (2.11)$$

for the configuration space

$$\mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) := \mathcal{A}^{1,p}(P) \times \left\{ u \in W^{1,p}(P, M)^G \,\middle|\, \int_\Sigma |\mu(u)|^2 \,\mathrm{dvol}_\Sigma < E \right\}$$

(see Section 2.1.3 for the definition of regular classifying maps).

Let $\mathcal{J}(M, \omega)$ denote the space of smooth $\omega$-compatible almost complex structures on $M$. Fix a smooth $G$-equivariant family

$$J\colon EG^N \to \mathcal{J}(M, \omega), \qquad e \mapsto J_e \qquad (2.12)$$

of $\omega$-compatible almost complex structures on $M$. It gives rise to a map

$$J_\Theta\colon \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to W^{1,p}\big(P, \mathcal{J}(M, \omega)\big)^G, \qquad (A, u) \mapsto J_{\Theta(A,u)}$$

which assigns to a pair $(A, u)$ a $G$-equivariant family

$$J_{\Theta(A,u)}\colon P \to \mathcal{J}(M, \omega), \qquad p \mapsto J_{\Theta(A,u)(p)} \tag{2.13}$$

of almost complex structures on $M$.

Remark 2.2.1. We may think of the family (2.12) as a vertical almost complex structure on the fiber bundle $EG^N \times_G M \to BG^N$. It pulls back along the map $\Theta_{(A,u)}$ to the family (2.13) which is a vertical almost complex structure on the fiber bundle $P(M) = P \times_G M \to \Sigma$. This family further pulls back along the section $u\colon \Sigma \to P(M)$ to a complex structure $J_{\Theta(A,u)}(u)$ on the vector bundle $u^*TM/G \to \Sigma$. In fact, the family (2.13) induces a $G$-equivariant family

$$J_{\Theta(A,u)}(u)\colon P \to \mathrm{End}(u^*TM), \qquad p \mapsto J_{\Theta(A,u)(p)}(u(p))$$

of almost complex structures on the vector bundle $u^*TM \to P$. To verify the equivariance property of this family we compute

$$\begin{aligned} J_{\Theta(A,u)}(u)(ph) &= J_{\Theta(A,u)(ph)}\big(u(ph)\big) \\ &= J_{h^{-1}\Theta(A,u)(p)}\big(h^{-1}u(p)\big) \\ &= \big(h^* J_{\Theta(A,u)(p)}\big)\big(h^{-1}u(p)\big) \\ &= h^*\big(J_{\Theta(A,u)}(u)(p)\big) \end{aligned}$$

for $p \in P$ and $h \in G$. Hence $J_{\Theta(A,u)}(u)$ descends to a complex structure on the vector bundle $u^*TM/G \to \Sigma$.

We will denote by $\mathcal{J} := \mathcal{J}(EG, M, \omega)$ the stratified space of all smooth $G$-equivariant families (2.12) of $\omega$-compatible almost complex structures on $M$, where $N$ runs through all positive integers.

For any pair $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ we denote by

$$\mathrm{d}_A u := \mathrm{d}u + X_A(u)$$

the twisted derivative of $u$. Recall at this point that $X_\xi$ denotes the infinitesimal action of $\xi$ on $M$, for every $\xi \in \mathfrak{g}$. The complex antilinear part

$$\overline{\partial}_{J,A,\Theta}(u) := \frac{1}{2}\Big(\mathrm{d}_A u + J_{\Theta(A,u)}(u) \circ \mathrm{d}_A u \circ j_\Sigma\Big)$$

will be called the **non-local Cauchy-Riemann operator**. The **non-local symplectic vortex equations** are the system of first order non-linear partial differential equations

$$\overline{\partial}_{J,A,\Theta}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0 \tag{2.14}$$

for pairs $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$, where $F_A$ denotes the curvature form of the connection $A$.

The non-local vortex equations are invariant under the action of the group of gauge transformations $\mathcal{G}^{2,p}(P)$. In fact, let $(A,u)$ be a solution of (2.14) and let $g \in \mathcal{G}^{2,p}$. Then

$$\begin{aligned}
\overline{\partial}_{J,g^*A,\Theta}\big(g^{-1}u\big) &= \frac{1}{2}\Big(\mathrm{d}_{g^*A}\big(g^{-1}u\big) + J_{\Theta(g^*A,g^{-1}u)}\big(g^{-1}u\big)\circ \mathrm{d}_{g^*A}\big(g^{-1}u\big)\circ j_\Sigma\Big)\\
&= \frac{1}{2}\Big(g^*\mathrm{d}_A u + J_{\Theta(A,u)\,g}\big(g^{-1}u\big)\circ g^*\mathrm{d}_A u\circ j_\Sigma\Big)\\
&= \frac{1}{2}\Big(g^*\mathrm{d}_A u + \big(g^* J_{\Theta(A,u)}\big)\big(g^{-1}u\big)\circ g^*\mathrm{d}_A u\circ j_\Sigma\Big)\\
&= \frac{1}{2}\, g^*\Big(\mathrm{d}_A u + J_{\Theta(A,u)}(u)\circ \mathrm{d}_A u\circ j_\Sigma\Big)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
F_{g^*A} + \mu\big(g^{-1}u\big)\,\mathrm{dvol}_\Sigma &= g^*F_A + g^{-1}\mu(u)\\
&= g\big(F_A + \mu(u)\big)g^{-1}\\
&= 0.
\end{aligned}$$

In the first calculation we used $\mathcal{G}^{2,p}(P)$-equivariance of the classifying map $\Theta$ and $G$-equivariance of the family $J$. The solutions of equations (2.14) are called **non-local vortices**.

REMARK 2.2.2. The non-local vortex equations (2.14) reduce to the standard vortex equations (1.1) whenever the family

$$J\colon EG^N \to \mathcal{J}(M,\omega)$$

of almost complex structures is constant. In fact, in this case it follows that $J$ takes values in the space $\mathcal{J}(M,\omega)^G$ of $G$-invariant almost complex structures, which is the set of fixed points for the $G$-action on $\mathcal{J}(M,\omega)$.

**2.2.2. Hamiltonian perturbations.** In order to obtain transversality we will later need to consider Hamiltonian perturbations of the non-local vortex equations (2.14). Our exposition follows Cieliebak et. al. [**3**].

Let $C^\infty(M)^G$ denote the space of smooth $G$-invariant functions on $M$, and denote by

$$\mathcal{H} := \mathcal{H}(\Sigma, M, \omega, \mu) := \Omega^1\big(\Sigma, C^\infty(M)^G\big)$$

the space of smooth Hamiltonian perturbations. For $H \in \mathcal{H}$ and $v_z \in T_z\Sigma$ we write $H_{v_z} := H_z(v_z) \in C^\infty(M)^G$, and denote by $X_{H_{v_z}} \in \mathrm{Vect}(M,\omega)$ the $G$-invariant Hamiltonian vector field of the function $H_{v_z}$ determined by the relation

$$\iota(X_{H_{v_z}})\omega = \mathrm{d}H_{v_z}. \tag{2.15}$$

For every section $u \in C^\infty(\Sigma, P(M))$, a Hamiltonian perturbation $H \in \mathcal{H}$ determines a 1-form $X_H(u) \in \Omega^1(P, u^*TM)$ by

$$\big(X_H(u)\big)_p(\tilde{v}_p) := X_{H_{\mathrm{d}_p\pi(\tilde{v}_p)}}(u(p))$$

for all $p \in P$ and $\tilde{v}_p \in T_pP$. This 1-form is $G$-equivariant and horizontal and thus descends to a 1-form

$$X_H(u) \in \Omega^1\big(\Sigma, u^*TM/G\big).$$

For any pair $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ we denote by

$$\mathrm{d}_{A,H}u := \mathrm{d}_A u + X_H(u)$$

the perturbed twisted derivative of $u$. Its complex antilinear part

$$\overline{\partial}_{J,A,H,\Theta}(u) := \frac{1}{2}\Big(\mathrm{d}_{A,H}u + J_{\Theta(A,u)}(u) \circ \mathrm{d}_{A,H}u \circ j_\Sigma\Big)$$

will be called the perturbed non-local Cauchy-Riemann operator. The perturbed non-local symplectic vortex equations are the system

$$\overline{\partial}_{J,A,H,\Theta}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0 \tag{2.16}$$

for pairs $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$. Note that these equations are invariant under the action of the group of gauge transformations $\mathcal{G}^{2,p}(P)$. A pair $(A, u)$ that solves these equations will be called a perturbed non-local vortex. Note further that the perturbed equations (2.16) reduce to equations (2.14) in the case $H = 0$.

REMARK 2.2.3. To simplify terminology, we will usually refer to equations (2.14) and (2.16) as the *vortex equations*, and call their solutions *vortices*.

**2.2.3. Perturbation data.** The perturbed non-local vortex equations (2.16) depend on the choice of a regular classifying map $\Theta$, a family of almost complex structures $J$, and a Hamiltonian perturbation $H$. We now formalize this dependence. Recall from Sections 2.1.3, 2.2.1 and 2.2.2 the definition of the spaces $\mathcal{C}^p_{\mathrm{reg}}$, $\mathcal{J}$ and $\mathcal{H}$.

DEFINITION 2.2.4. A triple $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$ is called a **perturbation datum** if the composition

$$J_\Theta \colon \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to \mathrm{Map}(P, \mathcal{J}(M, \omega))^G$$

given by

$$J_{\Theta(A,u)} \colon P \xrightarrow{\Theta(A,u)} EG^N \xrightarrow{J} \mathcal{J}(M, \omega)$$

is well-defined in the set-theoretical sense.

The perturbed non-local vortex equations (2.16) are thus defined for any choice of perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

**2.2.4. Energy identity.** The energy identity for the standard vortex equations (1.1) (see [**3**], Proposition 2.2 for details) generalizes to the non-local case in a straightforward way, as we shall now explain.

We define the **Yang-Mills-Higgs energy** of a pair $(A,u) \in \mathcal{B}^{1,p}(P,M;E,\mathrm{dvol}_\Sigma)$ on an open subset $U \subset \Sigma$ by

$$E(A,u;U) := \frac{1}{2}\int_U \Big(\big|F_A\big|_{\mathfrak{g}}^2 + \big|\mathrm{d}_{A,H}u\big|_{J_\Theta}^2 + \big|\mu(u)\big|_{\mathfrak{g}}^2\Big)\,\mathrm{dvol}_\Sigma,$$

and write $E(A,u)$ for the Yang-Mills-Higgs energy of $(A,u)$ on $\Sigma$. The norms appearing in this definition are understood as follows. First of all, for every pair $(A,u)$ the $G$-equivariant family (2.13) of $\omega$-compatible almost complex structures on $M$ gives rise to a fiber metric

$$\langle\cdot,\cdot\rangle_{J_{\Theta(A,u)}} := \omega(\cdot, J_{\Theta(A,u)}\cdot) \tag{2.17}$$

on the vector bundle $u^*TM/G \to \Sigma$. The norm of the 1-form $\mathrm{d}_{A,H}u$ is then pointwise defined to be the operator norm of the linear map

$$\mathrm{d}_{A,H}u(z)\colon T_z\Sigma \to u^*TM/G$$

for $z \in \Sigma$, understood with respect to the Kähler metric $\langle\cdot,\cdot\rangle_\Sigma$ on $\Sigma$ determined by the area form $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$, and the fiber metric (2.17) on $u^*TM/G$. The norm of $\mu(u)$ is poinwise understood with respect to the $G$-invariant inner product on $\mathfrak{g}$. Note that the Yang-Mills-Higgs energy is invariant under the action of the group of gauge transformations.

Remark 2.2.5. For later reference we remark that the Yang-Mills-Higgs energy of a vortex $(A,u)$ on an open subset $U \subset \Sigma$ may be expressed in the form

$$E(A,u;U) = \int_U \left(\frac{1}{2}\big|\mathrm{d}_{A,H}u\big|_{J_\Theta}^2 + \big|\mu(u)\big|_{\mathfrak{g}}^2\right)\mathrm{dvol}_\Sigma\,.$$

This is an immediate consequence of the second vortex equation in (2.16).

For a $G$-equivariant map $u\colon P \to M$, its equivariant **degree** is defined to be the class $[u]^G \in H_2^G(M;\mathbb{Z})$ in the integral equivariant homology of $M$ that is obtained by equivariant push-forward of the fundamental class of $\Sigma$ along $u$ (see [**3**], Section 2.3 for details).

Every pair $(A,u) \in \mathcal{B}^{1,p}(P,M;E,\mathrm{dvol}_\Sigma)$ satisfies the following energy identity which relates the Yang-Mills-Higgs energy of $(A,u)$ with the equivariant symplectic area of the map $u$. It is adapted from Proposition 2.2 in Cieliebak et. al. [**3**]. Recall that the curvature of a Hamiltonian connection $H$ is

$$\Omega_H\,\mathrm{dvol}_\Sigma := \mathrm{d}H + \frac{1}{2}\{H \wedge H\} \in \Omega^2\big(\Sigma, C^\infty(M)^G\big),$$

where $\{\cdot\,,\,\cdot\}$ denotes the Poisson bracket on the space of smooth functions on $M$. Its Hofer norm is then given by

$$\|\Omega_H\| := \int_\Sigma \left( \sup_{x\in M} \Omega_H(\cdot\,,x) - \inf_{x\in M} \Omega_H(\cdot\,,x) \right) \mathrm{dvol}_\Sigma\,.$$

Note that this norm does not depend on the area form $\mathrm{dvol}_\Sigma$.

PROPOSITION 2.2.6 (Energy identity). *Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.*

*Let $(A,u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$. The Yang-Mills-Higgs energy of the pair $(A,u)$ and the equivariant symplectic area of the map $u$ are related by*

$$E(A,u) = \int_\Sigma \left( \left|\overline{\partial}_{J,A,H,\Theta}(u)\right|^2_{J_\Theta} + \frac{1}{2}\left|F_A + \mu(u)\,\mathrm{dvol}_\Sigma\right|^2_{\mathfrak{g}} \right) \mathrm{dvol}_\Sigma \\ + \left\langle [\omega-\mu]_G, [u]^G \right\rangle + \int_\Sigma \Omega_H(u)\,\mathrm{dvol}_\Sigma\,.$$

*Here the equivariant symplectic area of the map $u$ is given by*

$$\left\langle [\omega-\mu]_G, [u]^G \right\rangle = \int_\Sigma u^*\omega - d\langle \mu(u), A\rangle,$$

*independently of $A$, and $\Omega_H \in C^\infty(\Sigma, C^\infty(M)^G)$ denotes the curvature of the Hamiltonian connection $H$. In particular, the Yang-Mills-Higgs energy of a vortex $(A,u)$ solving equations (2.16) satisfies an estimate*

$$E(A,u) \le \left\langle [\omega-\mu]_G, [u]^G \right\rangle + \left\|\Omega_H(u)\right\|,$$

*where $\|\Omega_H\|$ is the Hofer norm of the curvature $\Omega_H$.*

PROOF. The proof of Proposition 2.2 in Cieliebak et. al. [**3**] carries over word by word to our situation. In fact, the family of almost complex structures $J_{\Theta(A,u)}(u)$ plays the role of the family of almost complex structures $J_u$ in that proof. □

**2.2.5. Local coordinates.** We will frequently need to consider the vortex equations (2.16) in local coordinates. Let $D \subset \mathbb{C}$ be an open subset of the complex plane $\mathbb{C}$ with complex coordinate $s + \mathrm{i}\,t$. Let $\varphi\colon \mathbb{C} \supset D \to \Sigma$ be a holomorphic chart map which trivializes the bundle $P$, and choose a lift $\tilde{\varphi}\colon D \to P$ of this map.

A vortex $(A,u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ solving equations (2.16) determines a triple $(\Phi, \Psi, u^{\mathrm{loc}})$ consisting of functions $\Phi, \Psi\colon D \to \mathfrak{g}$ and a map $u^{\mathrm{loc}}\colon D \to M$, both of class $W^{1,p}$, by the relations

$$\tilde{\varphi}^*A = \Phi\,\mathrm{d}s + \Psi\,\mathrm{d}t \quad \text{and} \quad u^{\mathrm{loc}} = u \circ \tilde{\varphi}.$$

Moreover, the area form $\mathrm{dvol}_\Sigma$ gives rise to a smooth function $\lambda\colon D \to (0,\infty)$ by

$$\varphi^* \,\mathrm{dvol}_\Sigma = \lambda^2 \,\mathrm{d}s \wedge \mathrm{d}t,$$

and the Hamiltonian perturbation determines smooth functions $F, G\colon D \times M \to \mathbb{R}$ by

$$\varphi^* H = F \,\mathrm{d}s + G \,\mathrm{d}t.$$

A short calculation ([**3**], Proof of Prop. 2.2) now shows that the triple $(\Phi, \Psi, u^{\mathrm{loc}})$ satisfies the equations

$$\begin{aligned}\partial_s u^{\mathrm{loc}} + X_\Phi\big(u^{\mathrm{loc}}\big) + X_F\big(u^{\mathrm{loc}}\big)& \\ + \big(J_{\Theta(A,u)} \circ \varphi\big)\big(u^{\mathrm{loc}}\big)\big(\partial_t u^{\mathrm{loc}} + X_\Psi\big(u^{\mathrm{loc}}\big) + X_G\big(u^{\mathrm{loc}}\big)\big) &= 0 \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \lambda^2 \mu\big(u^{\mathrm{loc}}\big) &= 0.\end{aligned} \tag{2.18}$$

For fixed $(A, u)$, these equations may be regarded as the standard vortex equations on the trivial $G$-bundle over $D$, and the triple $(\Phi, \Psi, u^{\mathrm{loc}})$ becomes a vortex in the usual sense.

Remark 2.2.7. We emphasize that the first equation in (2.18) depends on the vortex $(A, u)$. In fact, $(J_{\Theta(A,u)} \circ \varphi)(u^{\mathrm{loc}})$ is the complex structure on the vector bundle $(u^{\mathrm{loc}})^* TM \cong \varphi^*(u^* TM/G) \to \Sigma$ induced by the $G$-equivariant family of almost complex structures $J_{\Theta(A,u)}\colon P \to \mathcal{J}(M, \omega)$ (see Remark 2.2.1) and thus depends on $(A, u)$ via the classifying map $\Theta$. It is in this sense that the vortex equations (2.14) and (2.16) are *non-local.*

We define the Yang-Mills-Higgs energy density of the vortex $(\Phi, \Psi, u^{\mathrm{loc}})$ by

$$e\big(\Phi, \Psi, u^{\mathrm{loc}}\big) := \big|\partial_s u^{\mathrm{loc}} + X_\Phi\big(u^{\mathrm{loc}}\big) + X_F\big(u^{\mathrm{loc}}\big)\big|^2_{J_\Theta} + \lambda^2 \big|\mu\big(u^{\mathrm{loc}}\big)\big|^2_{\mathfrak{g}},$$

and its Yang-Mills-Higgs energy by

$$E\big(\Phi, \Psi, u^{\mathrm{loc}}\big) := \int_D e\big(\Phi, \Psi, u^{\mathrm{loc}}\big) \,\mathrm{d}s \wedge \mathrm{d}t.$$

We further define the Yang-Mills-Higgs energy density of the vortex $(A, u)$ on $D$ by

$$e_\varphi(A, u) := \left(\frac{1}{2}\big|\mathrm{d}_{A,H} u \circ \tilde{\varphi}\big|^2_{J_\Theta \circ \varphi} + \big|\mu(u \circ \tilde{\varphi})\big|^2_{\mathfrak{g}}\right) \cdot \lambda^2.$$

Note that the Yang-Mills-Higgs energy densities $e_\varphi(A, u)$ and $e(\Phi, \Psi, u^{\mathrm{loc}})$ and the Yang-Mills-Higgs energy $E(\Phi, \Psi, u^{\mathrm{loc}})$ are invariant under the action of the group of gauge transformations and only depend on the chart map $\varphi$ but not on the choice of lift $\tilde{\varphi}$. The two energy densities are then related by

$$e\big(\Phi, \Psi, u^{\mathrm{loc}}\big) = e_\varphi(A, u), \tag{2.19}$$

and the Yang-Mills-Higgs energy of the vortex $(\Phi, \Psi, u^{\mathrm{loc}})$ agrees with the Yang-Mills-Higgs energy of the vortex $(A, u)$ on $D$ in the sense that

$$E\big(\Phi, \Psi, u^{\mathrm{loc}}\big) = E\big(A, u; \varphi(D)\big). \tag{2.20}$$

**2.2.6. Elliptic regularity.** Elliptic regularity continues to hold for non-local vortices. We prove the following proposition, which generalizes Theorem 3.1 in Cieliebak et. al. [**3**]. For any positive integer $\ell$ we denote by $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(M)^G)$ the space of Hamiltonian perturbations of class $C^\ell$.

Proposition 2.2.8 (Elliptic regularity). *Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some positive integer $\ell$.*

*If the pair $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ solves the non-local vortex equations (2.16), then there exists a gauge transformation $g \in \mathcal{G}^{2,p}(P)$ such that the pair $(g^*A, g^{-1}u)$ is of class $W^{\ell,p}$ and of class $C^{\ell-1}$.*

The proof of Proposition 2.2.8 is based on the following lemma. It is adapted from the proof of Theorem 3.1 in [**3**].

Lemma 2.2.9. *Assume that the hypotheses of Proposition 2.2.8 are satisfied. Suppose in addition that there exists a smooth connection $A_0 \in \mathcal{A}(P)$ such that $A$ is in Coulomb gauge relative to $A_0$, that is,*

$$\mathrm{d}^*_{A_0}(A - A_0) = 0. \tag{2.21}$$

*Then $(A, u)$ is of class $W^{\ell,p}$ and of class $C^{\ell-1}$.*

Proof. The proof is by standard elliptic bootstrapping. It is modeled on the proof of Theorem 3.1 in [**3**]. Let $\ell \geq 1$. We will prove by induction that $A$ and $u$ are of class $W^{k+1,p}$ and of class $C^k$, for $1 \leq k \leq \ell - 1$.

For $k = 1$, this is true since $(A, u)$ is of class $W^{1,p}$ by assumption.

Now suppose that the claim is true for some $1 \leq k \leq \ell - 1$.

Consider the connection $A$. Set $\alpha := A - A_0 \in W^{k,p}(\Sigma, T^*\Sigma \otimes P(\mathfrak{g}))$. Combining the Coulomb gauge condition (2.21) with the second vortex equation in (2.16) we obtain

$$\begin{aligned}
\mathrm{d}_{A_0}\alpha &= \mathrm{d}\alpha + [A_0 \wedge \alpha] \\
&= \mathrm{d}A - \mathrm{d}A_0 + [A_0 \wedge A] - [A_0 \wedge A_0] \\
&= -\left(\mathrm{d}A_0 + \frac{1}{2}[A_0 \wedge A_0]\right) - \frac{1}{2}[A_0 \wedge A_0] + [A_0 \wedge A] + \mathrm{d}A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] + \frac{1}{2}[A \wedge A] + \mathrm{d}A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] + F_A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] - \mu(u)\,\mathrm{dvol}_\Sigma\,.
\end{aligned} \tag{2.22}$$

Since $A$ and $u$ are of class $W^{k,p}$ by assumption, it follows from formula (2.22) that $\mathrm{d}_{A_0}\alpha$ is of class $W^{k,p}$. Furthermore, it is obvious from formula (2.21) that $\mathrm{d}_{A_0}^*\alpha$ is of class $W^{k,p}$. Hence it follows from elliptic regularity for the Hodge-Laplace operator $\mathrm{d}_{A_0}\mathrm{d}_{A_0}^* + \mathrm{d}_{A_0}^*\mathrm{d}_{A_0}$ that $\alpha$ is of class $W^{k+1,p}$. Thus $A = A_0 + \alpha$ is of class $W^{k+1,p}$. By Rellich's theorem ([**34**], Thm. B.2) it follows that $A$ is of class $C^k$.

Next we prove that $u$ is of class $W^{k+1,p}$ and of class $C^k$. We shall work in local coordinates. Choose a holomorphic coordinate chart $\mathbb{C} \supset D \to \Sigma$ and a Darboux chart on $M$. We may thus assume without loss of generality that $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$, where $\omega_0$ is the standard symplectic form on $\mathbb{R}^{2n}$. By (2.18), with respect to these local coordinates, the first of the vortex equations (2.16) takes the form

$$\partial_s u + X_\Phi(u) + X_F(u) + J_{\Theta(A,u)}(u)\big(\partial_t u + X_\Psi(u) + X_G(u)\big) = 0 \tag{2.23}$$

where we express the connection $A$ as

$$A = \Phi\,\mathrm{d}s + \Psi\,\mathrm{d}t, \qquad \Phi, \Psi \colon D \to \mathfrak{g},$$

the Hamiltonian perturbation $H$ as

$$H = F\,\mathrm{d}s + G\,\mathrm{d}t, \qquad F, G \colon D \times M \to \mathbb{R},$$

and consider $J_{\Theta(A,u)}(u)$ as a matrix-valued map

$$J_{\Theta(A,u)}(u) \colon D \to M_{2n,2n}(\mathbb{R}). \tag{2.24}$$

We may then rewrite equation (2.23) as

$$\partial_s u + J_{\Theta(A,u)}(u)\,\partial_t u = -X_\Phi(u) - X_F(u) - J_{\Theta(A,u)}(u)\big(X_\Psi(u) + X_G(u)\big). \tag{2.25}$$

Let us investigate the regularity of the right-hand side of this equation. Firstly, we have already proved that $A$ and hence $\Phi$ and $\Psi$ are of class $C^k$. It follows that the families of smooth vector fields $X_\Phi$ and $X_\Psi$ on $M$ are of class $C^k$ on $D$. Likewise, $H$ and hence $F$ and $G$ are of class $C^\ell$ by assumption. Thus we see from (2.15) that the families of smooth vector fields $X_F$ and $X_G$ on $M$ are of class $C^{\ell-1}$, whence of class $C^k$, on $D$ (since $k \leq \ell - 1$). Since $u$ is of class $W^{k,p}$ by assumption, it follows that each of the sections $X_\Phi(u)$, $X_\Psi(u)$, $X_F(u)$ and $X_G(u)$ is of class $C^k$ on $D$ ([**34**], Lemma B.8). Secondly, it follows from part (ii) of the (Regularity) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) that the map $\Theta_{(A,u)}$ is of class $C^k$. Hence, since the family of almost complex structures $J$ is smooth and $u$ is of class $W^{k,p}$ by assumption, it follows that the map (2.24) is of class $W^{k,p}_{\mathrm{loc}}$ on $D$ ([**34**], Lemma B.8). We therefore conclude that the right-hand side of equation (2.25) is of class $W^{k,p}_{\mathrm{loc}}$ ([**34**], Lemma B.3).

Applying Proposition B.4.9 (i) in [**22**] (see also Lemma 3.3 in [**3**]) it follows that $u$ is of class $W^{k+1,p}$ on $D$. By Rellich's theorem we infer that $u$ is of class $C^k$ on $D$. $\square$

We are now ready for the proof of Proposition 2.2.8.

PROOF OF PROPOSITION 2.2.8. The proof is modeled on the proof of Theorem 3.1 in [**3**]. Suppose that $(A,u) \in \mathcal{B}^{1,p}(P,M;E,\mathrm{dvol}_\Sigma)$ solves the non-local vortex equations (2.16). We apply the local slice theorem ([**34**], Thm. F and [**3**], Thm. B.1). To this end, we take $A$ as reference connection and choose a smooth connection $A_0 \in \mathcal{A}(P)$ such that $\|A - A_0\|_{W^{1,p}(\Sigma)}$ (and hence also $\|A - A_0\|_{L^p(\Sigma)}$) is sufficiently small. Then the local slice theorem (taking $q = p$) asserts the existence of a gauge transformation $g \in \mathcal{G}^{2,p}(P)$ such that

$$\mathrm{d}_A^*\big(g_*A_0 - A\big) = 0.$$

This implies ([**34**], Lemma 8.4 (ii)) that

$$\mathrm{d}_{g_*A_0}^*\big(A - g_*A_0\big) = 0,$$

whence

$$\mathrm{d}_{A_0}^*\big(g^*A - A_0\big) = g^*\,\mathrm{d}_{g_*A_0}^*\big(A - g_*A_0\big) = 0.$$

That is, $g^*A$ is in Coulomb gauge relative to $A_0$. By gauge invariance of the vortex equations (2.16), we may apply Lemma 2.2.9 to the pair $(g^*A, g^{-1}u)$. □

## 2.3. Existence of regular classifying maps

This section is devoted to the proof of Theorem 2.1.5 which claims that regular classifying maps do exist. We give an explicit construction of a regular classifying map and prove that it satisfies the regularity axioms of Definition 2.1.3. Our construction is reminiscent of a construction of a holonomy perturbation due to Floer [**7**].

**2.3.1. Systems of holonomy maps.** Recall that $\Sigma$ carries a Kähler metric determined by the complex structure $j_\Sigma$ and the area form $\mathrm{dvol}_\Sigma$. Let $z_0 \in \Sigma$ be an arbitrary point, and fix a positive real number $R$ that is smaller than half the injectivity radius of $\Sigma$. Denote by

$$B := B(z_0;R) \quad \text{and} \quad D := B(z_0;R/2)$$

the closed geodesic disks of radius $R$ and $R/2$ around $z_0$. Define a smooth family of geodesics

$$\gamma\colon D\times(B\setminus D) \to C^1\big([0,1],B\big), \qquad (z,z') \mapsto \gamma_{(z,z')} \tag{2.26}$$

by assigning to every pair of points $(z,z') \in D\times(B\setminus D)$ the geodesic

$$\gamma_{(z,z')}\colon [0,1] \to B, \qquad \tau \mapsto \gamma_{(z,z')}(\tau)$$

joining the point $z$ in the disk $D$ with the point $z'$ in the annulus $B\setminus D$, understood with respect to the Kähler metric on $\Sigma$.

Fix a connection $A$ on $P$ of class $C^1$. We may then lift the family of paths (2.26) to a family of $A$-horizontal paths in $P$, obtaining a family

$$h^A\gamma\colon P|_D \times (B\backslash D) \to C^1\big([0,1], P\big) \tag{2.27}$$

that assigns to every pair of points $(p, z') \in P|_D \times (B\backslash D)$ the $A$-horizontal lift

$$h^A\gamma_{(\pi(p),z')}\colon [0,1] \to P, \qquad h_A\gamma_{(\pi(p),z')}(0) = p$$

of the geodesic $\gamma_{(\pi(p),z')}\colon [0,1] \to \Sigma$. Note that the path $h^A\gamma_{(\pi(p),z')}$ is the unique solution of the ordinary differential equation

$$A\bigg(\frac{\mathrm{d}}{\mathrm{d}\tau} h^A\gamma_{(\pi(p),z')}\bigg) = 0, \qquad h^A\gamma_{(\pi(p),z')}(0) = p. \tag{2.28}$$

By construction, it has the property that

$$\pi\big(h^A\gamma_{(\pi(p),z')}(\tau)\big) = \gamma_{(\pi(p),z')}(\tau)$$

for all $\tau \in [0,1]$, where $\pi\colon P \to \Sigma$ denotes the bundle projection. Note moreover that the path $h^A\gamma_{(\pi(p),z')}$ is of class $C^1$ since $A$ is of class $C^1$ by assumption, by the Cauchy-Lipschitz theorem ([**17**], Ch. V, Thm. 3.1).

Fix a smooth trivializing section $\sigma\colon B \to P|_B$ of the bundle $P$ over the disk $B$. We associate to the triple $(\gamma, \sigma, A)$ a system of **holonomy maps**

$$\mathrm{hol}^A := \mathrm{hol}^A(\gamma,\sigma)\colon P|_D \times (B\backslash D) \to G, \qquad (p,z') \mapsto \mathrm{hol}^A(p,z') \tag{2.29}$$

defined by the relation

$$h^A\gamma_{(\pi(p),z')}(1) = \sigma(z') \cdot \mathrm{hol}^A(p,z') \tag{2.30}$$

in the fiber of $P$ over $z'$. Fixing the endpoint $z'$ in the annulus $B\backslash D$, the system (2.29) gives rise to a **reduced holonomy map**

$$\mathrm{hol}^A_{z'}\colon P|_D \to G, \qquad p \mapsto \mathrm{hol}^A_{z'}(p) := \mathrm{hol}^A(p,z'). \tag{2.31}$$

It has the following transformation property.

LEMMA 2.3.1. *Let $g \in C^2(P,G)^G$ be a gauge transformation of $P$ class $C^2$, and let $h \in G$. Then*

$$\mathrm{hol}^{g^*A}_{z'}\big(p.g(p)^{-1}h\big) = g(\sigma(z'))^{-1} \cdot \mathrm{hol}^A_{z'}(p) \cdot h.$$

PROOF. Let us first consider the family of $A$-horizontal paths (2.27). For a gauge transformation $g \in C^2(P,G)^G$ and $h \in G$ we have, by definition of the family (2.27),

$$h^{g^*A}\gamma_{(\pi(pg(p)^{-1}h),z')}(0) = h^A\gamma_{(\pi(p),z')}(0) \cdot g\big(h^A\gamma_{(\pi(p),z')}(0)\big)^{-1} \cdot h$$

and

$$\begin{aligned}
&A\bigg(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(h^A\gamma_{(\pi(p),z')}\cdot g\big(h^A\gamma_{(\pi(p),z')}\big)^{-1}\cdot h\Big)\bigg)\\
&=A\bigg(\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}h^A\gamma_{(\pi(p),z')}\Big)\cdot g\big(h^A\gamma_{(\pi(p),z')}\big)^{-1}\cdot h\bigg)\\
&=g\big(h^A\gamma_{(\pi(p),z')}\big)\cdot h^{-1}\cdot A\bigg(\frac{\mathrm{d}}{\mathrm{d}\tau}h^A\gamma_{(\pi(p),z')}\bigg)\cdot g\big(h^A\gamma_{(\pi(p),z')}\big)^{-1}\cdot h=0.
\end{aligned}$$

The Cauchy-Lipschitz theorem on uniqueness of solutions of ordinary differential equations ([**17**], Ch. V, Thm. 3.1) then implies that

$$h^{g^*A}\gamma_{(\pi(pg(p)^{-1}h),z')}(\tau)=h^A\gamma_{(\pi(p),z')}(\tau)\cdot g\big(h^A\gamma_{(\pi(p),z')}(\tau)\big)^{-1}\cdot h$$

for all $\tau\in[0,1]$. Let us abbreviate $\gamma:=\gamma_{(\pi(p),z')}$ and $\rho^A:=\mathrm{hol}^A_{z'}(p)$. By relation (2.30) we further get

$$\begin{aligned}
&\sigma(z')\cdot\mathrm{hol}^{g^*A}_{z'}\big(p.g(p)^{-1}h\big)\\
=\;&h^{g^*A}\gamma_{(\pi(p.g(p)^{-1}h),z')}(1)\\
=\;&h^A\gamma_{(\pi(p),z')}(1)\cdot g\big(h^A\gamma_{(\pi(p),z')}(1)\big)^{-1}\cdot h\\
=\;&\sigma(z')\cdot\mathrm{hol}^A_{z'}(p)\cdot g\big(\sigma(z')\cdot\mathrm{hol}^A_{z'}(p)\big)^{-1}\cdot h\\
=\;&\sigma(z')\cdot\mathrm{hol}^A_{z'}(p)\cdot\mathrm{hol}^A_{z'}(p)^{-1}\cdot g(\sigma(z'))^{-1}\cdot\mathrm{hol}^A_{z'}(p)\cdot h\\
=\;&\sigma(z')\cdot g(\sigma(z'))^{-1}\cdot\mathrm{hol}^A_{z'}(p)\cdot h,
\end{aligned}$$

whence

$$\mathrm{hol}^{g^*A}_{z'}\big(p.g(p)^{-1}h\big)=g(\sigma(z'))^{-1}\cdot\mathrm{hol}^A_{z'}(p)\cdot h$$

as claimed. □

The lemma shows that the reduced holonomy map (2.31) is $G$-equivariant with respect to the standard right action of $G$ on itself. We will frequently be interested in the dependence of this map on the connection $A$. This leads us to consider the **relative reduced holonomy map**

$$\mathrm{hol}_{z'}\colon\mathcal{A}^1(P)\to C^1(P|_D,G)^G,\qquad A\mapsto\mathrm{hol}^A_{z'}\tag{2.32}$$

for every point $z'$ in the annulus $B\setminus D$. Here and throughout this section, $\mathcal{A}^1(P)$ denotes the space of connections on $P$ that are of class $C^1$.

Next we collect some basic properties of the holonomy maps (2.31) and (2.32) that will later be needed in proving regularity of the classifying map that we shall construct. Of particular importance to us will be explicit formulas for the derivatives of the reduced holonomy maps (2.31). In order to formulate our results it will be convenient to have the following notation.

Recall from (2.26) above the definition of the geodesics $\gamma_{(z,z')}$ joining $z \in D$ with $z' \in B \backslash D$. We now consider infinitesimal variations of these paths under the assumption that the endpoint $z'$ is kept fixed. More precisely, we consider the derivative of the map

$$\gamma_{z'} \colon D \to C^1\big([0,1], B\big), \qquad z \mapsto \gamma_{z'}(z) := \gamma_{(z,z')}. \tag{2.33}$$

We shall denote its derivative at a point $z \in D$ by

$$\delta_z\gamma_{z'} \colon T_zD \to C^1\big([0,1], T_{\gamma_{(z,z')}}B\big), \qquad v_z \mapsto \delta_z\gamma_{z'}(v_z). \tag{2.34}$$

It assigns to every tangent vector $v_z$ at $z$ the Jacobi field along the path $\gamma_{(z,z')}$ obtained by infinitesimal variation of the family (2.33) in the direction of $v_z$.

Fix a smooth vector field $v$ on $B$ and consider the geodesic disk $B_{2R}(z')$ on $\Sigma$ around the point $z'$. For every $\tau \in [0,1)$ the geodesics $\gamma_{z'}$ and their infinitesimal variations $\delta\gamma_{z'}$ give rise to smooth vector fields $X^\tau$ and $Y_v^\tau$ on the punctured disk $B_{2R\cdot(1-\tau)}(z') \setminus \{z'\}$ in the following way. Denote by

$$\gamma_\tau := \gamma_{z',\tau} \colon B_{2R}(z') \setminus \{z'\} \to B_{2R\cdot(1-\tau)}(z') \setminus \{z'\}, \qquad z \mapsto \gamma_{(z,z')}(\tau) \tag{2.35}$$

the map that assigns to a point $z \in B_{2R}(z')$ the point at time $\tau$ on the geodesic $\gamma_{(z,z')} \colon [0,1] \to B$ joining $z'$ with $z$. This map is injective, and we denote its inverse by

$$\gamma^\tau \colon B_{2R\cdot(1-\tau)}(z') \setminus \{z'\} \to B_{2R}(z') \setminus \{z'\}. \tag{2.36}$$

We then define the vector fields $X^\tau$ and $Y_v^\tau$ by

$$X^\tau(z) := \partial_\tau\gamma_{(\gamma^\tau(z),z')}(\tau) \in T_z\Sigma \tag{2.37}$$

and

$$Y_v^\tau(z) := \big(\delta_{\gamma^\tau(z)}\gamma_{z'}(v_{\gamma^\tau(z)})\big)(\tau) \in T_z\Sigma \tag{2.38}$$

for all points $z \in B_{2R\cdot(1-\tau)}(z') \setminus \{z'\}$. Here, $\gamma_{(\gamma^\tau(z),z')} \colon [0,1] \to B$ is the geodesic joining the point $\gamma^\tau(z)$ with $z'$, and $\delta_{\gamma^\tau(z)}\gamma_{z'}(v_{\gamma^\tau(z)})$ is the derivative of the map (2.33) at the point $\gamma^\tau(z)$ in the direction of the tangent vector $v_{\gamma^\tau(z)} \in T_{\gamma^\tau(z)}\Sigma$ as defined in (2.34).

Denote by $\omega_{\mathrm{MC}} \in \Omega^1(G,\mathfrak{g})$ the Maurer-Cartan form on $G$ (see [**20**], Sec. I.4), and recall that we have fixed a trivializing section $\sigma \colon B \to P|_B$ of the bundle $P$ over the disk $B$. The following two propositions collect basic properties of the holonomy maps (2.31) and (2.32).

PROPOSITION 2.3.2. *Fix a point $z' \in B \setminus D$ and a connection $A$ on $P$ of class $C^1$. Then the reduced holonomy map (2.31) has the following properties.*

(i) *Let $k \geq 1$, and assume that $A$ is of class $C^k$. Then the system of holonomy maps (2.29) is of class $C^k$.*

(ii) *By (i) the reduced holonomy map (2.31) is of class $C^1$. Its covariant derivative satisfies the following identities in the Lie algebra $\mathfrak{g}$, where $\omega_{\mathrm{MC}} \in \Omega^1(G, \mathfrak{g})$ denotes the Maurer-Cartan form and $\sigma\colon B \to P|_B$ is a fixed trivializing section of the bundle $P$ over $B$.*

(a) *Fix a point $z \in D$ and a tangent vector $v_z \in T_z\Sigma$. Then we have*

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big) = \int_0^1 (\sigma^* F_A)\Big(\partial_\tau \gamma_{(z,z')}(\tau), \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\Big)\, \mathrm{d}\tau.$$

(b) *Fix a $G$-invariant metric on $G$, and assume that $A$ is of class $C^2$. Then by (i) the reduced holonomy map (2.31) is of class $C^2$. Fix a point $z \in D$ and tangent vectors $v_z, w_z \in T_z\Sigma$, and let $v$ and $w$ be smooth vector fields on $D$ that restrict to $v_z$ and $w_z$ at the point $z$. Then we have*

$$\begin{aligned}
&\omega_{\mathrm{MC}}\big(\nabla_A\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big)(w_z, v_z)\big)\\
&= \int_0^1 \Big(\big(\nabla_A(\sigma^* F_A)\big)\big(\mathrm{d}_z\gamma_\tau(w_z), \partial_\tau \gamma_{(z,z')}(\tau), \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\big)\\
&\qquad\qquad + (\sigma^* F_A)\big(\nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)} X^\tau, \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\big)\\
&\qquad\qquad + (\sigma^* F_A)\big(\partial_\tau \gamma_{(z,z')}(\tau), \nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_v - Y^\tau_{\nabla^\Sigma_w v}(\gamma_\tau(z))\big)\Big)\, \mathrm{d}\tau\\
&\qquad\qquad\qquad - \big(\nabla^G_{\sigma^*(\mathrm{d}_A \mathrm{hol}^A_{z'})(w_z)} \omega_{\mathrm{MC}}\big)\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big).
\end{aligned}$$

*Here we use the notation introduced in (2.33), (2.34), (2.35), (2.37) and (2.38) above (see Remark 2.3.3 below for an explanation of the covariant derivatives appearing in the formulas).*

REMARK 2.3.3. We explain the notation appearing in Proposition 2.3.2. First of all, the covariant derivative of the map

$$\mathrm{hol}^A_{z'}\colon P|_D \to G$$

is given by

$$\mathrm{d}_A \mathrm{hol}^A_{z'} := \mathrm{d}\, \mathrm{hol}^A_{z'} + X_A\big(\mathrm{hol}^A_{z'}\big),$$

where $X_\eta$ denotes the infinitesimal action of $\eta \in \mathfrak{g}$ on $G$. Locally on $D$ this induces a map

$$\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\colon TD \to T_{\mathrm{hol}^A_{z'} \circ \sigma} G$$

We may equivalently think of this map as a 1-form on $D$, that is, as a section of the tensor bundle $T^*D \otimes F \to D$, where $F$ denotes the vector bundle

$$F := \big(\mathrm{hol}^A_{z'} \circ \sigma\big)^* TG \to D.$$

Fix a $G$-invariant metric on $G$, and denote by $\nabla^G$ the corresponding Levi-Civita connection on $G$. The connection $A$ and the section $\sigma$ then give rise to a covariant derivative

$$\nabla_A^F\colon C^1(D,F)\to C^0(D,T^*\Sigma\otimes F)$$

on the vector bundle $F$ defined by

$$\nabla_A^F\xi:=\nabla^G\xi+\nabla^G_\xi X_{\sigma^*A}(\xi),$$

where $X_\eta$ denotes the infinitesimal right action of $\eta\in\mathfrak{g}$ on $G$.

Let $\nabla^\Sigma$ be the Levi-Civita connection on $\Sigma$, understood with respect to the Kähler metric determined by the area form $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$. The connections $\nabla^\Sigma$ and $\nabla_A^F$ then define a covariant derivative

$$\nabla_A\colon C^1\big(D,T^*D\otimes F\big)\to C^0\big(D,\otimes^2T^*D\otimes F\big)$$

acting on the 1-form $\sigma^*(\mathrm{d}_A\mathrm{hol}^A_{z'})$ by the formula

$$\begin{aligned}\nabla_A\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big)(w,v)&=\nabla^G_{A,w}\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(v)\big)\\&\quad+\nabla^G_{\sigma^*(\mathrm{d}_A\mathrm{hol}^A_{z'})(v)}X_{\sigma^*A(w)}\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big)-\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big(\nabla^\Sigma_w(v)\big)\end{aligned}\tag{2.39}$$

for all smooth vector fields $w,v$ on $D$.

Next we consider the covariant derivative of the curvature $\sigma^*F_A$. The connection $A$ defines a covariant derivative

$$\nabla_A\colon C^1(D,\mathfrak{g})\to C^0(D,\mathfrak{g})$$

by

$$\nabla_A\eta:=\mathrm{d}\eta+[\sigma^*A,\eta].$$

As above, let $\nabla^\Sigma$ be the Levi-Civita connection on $\Sigma$. The connections $\nabla_A$ and $\nabla^\Sigma$ then give rise to a covariant derivative

$$\nabla_A\colon C^1\big(D,\otimes^2T^*D\otimes\mathfrak{g}\big)\to C^0\big(D,\otimes^3T^*D\otimes\mathfrak{g}\big)$$

acting on the curvature $\sigma^*F_A$ by the formula

$$\begin{aligned}\big(\nabla_A(\sigma^*F_A)\big)(v_0,v_1,v_2):=\nabla_{A,v_0}\big((\sigma^*F_A)(v_1,v_2)\big)-(\sigma^*F_A)\big(\nabla^\Sigma_{v_0}v_1,v_2\big)\\-(\sigma^*F_A)\big(v_1,\nabla^\Sigma_{v_0}v_2\big)\end{aligned}\tag{2.40}$$

for all smooth vector fields $v_0,v_1,v_2$ on $D$.

Lastly, we denote by $\nabla^G\omega_{\mathrm{MC}}$ the 1-form on $G$ obtained by covariant differentiation of the Maurer-Cartan form $\omega_{\mathrm{MC}}$ with respect to the Levi-Civita connection $\nabla^G$ on $G$.

PROPOSITION 2.3.4. *Fix a $G$-invariant metric on $G$. Recall that $\mathcal{A}^1(P)$ denotes the space of connections on $P$ of class $C^1$, and that $\sigma\colon B \to P|_B$ is a fixed trivializing section of the bundle $P$ over $B$. Then the holonomy maps (2.31) and (2.32) have the following properties (see Remarks 2.1.4 and 2.3.3 above for an explanation of the notation).*

(i) *For every point $z' \in B \setminus D$ the map (2.32) is continuous with respect to the $C^0$-topologies on $\mathcal{A}^1(P)$ and on $C^1(P|_D, G)^G$.*

(ii) *Fix a real number $p > 2$. There exists a constant $C_1 > 0$ such that for all points $z' \in B \setminus D$ and all connections $A \in \mathcal{A}^1(P)$ we have*

$$\big\|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big\|_{L^p(D)} \leq C_1 \cdot \big\|F_A\big\|_{L^p(\Sigma)}.$$

(iii) *There exists a constant $C_2 > 0$ such that for all points $z' \in B \setminus D$ and for all connections $A \in \mathcal{A}^1(P)$ we have*

$$\big\|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big\|_{L^\infty(D)} \leq C_2 \cdot \big\|F_A\big\|_{L^\infty(\Sigma)}.$$

(iv) *There exists a constant $C_3 > 0$ such that for all points $z' \in B \setminus D$ and all connections $A \in \mathcal{A}^1(P)$ of class $C^2$ we have*

$$\begin{aligned}\big|\big(\nabla_A\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big)\big)(z)\big| \;\leq\; C_3 \cdot \Bigg(1 &+ \big\|F_A\big\|^2_{L^\infty(\Sigma)} \\ &+ \int_0^1 \big|\big(\nabla_A(\sigma^* F_A)\big)\big(\gamma_{(z,z')}(\tau)\big)\big|\,\mathrm{d}\tau\Bigg)\end{aligned}$$

*for all points $z \in D$. Here $\gamma_{(z,z')}$ denotes the geodesic introduced in (2.26) above, joining the points $z$ and $z'$.*

We prove Propsition 2.3.2 first; the proof of Proposition 2.3.4 starts on page 33.

PROOF OF PROPOSITION 2.3.2. Proof of (i): The system of holonomy maps (2.29) is defined in terms of the horizontal lifts (2.27) which are solutions of the ordinary differential equation (2.28). The coefficients of this equation are given in terms of the connection $A$. Hence the claim is a consequence of the Cauchy-Lipschitz theorem ([**17**], Ch. V, Thm. 4.1), which states that the solution of an ordinary differential equation with $C^k$-coefficients is a function of class $C^k$.

Proof of (ii): Fix a point $z' \in B\setminus D$, and let $A$ be a connection on $P$ of class $C^1$. Consider the map

$$h_{z'}\colon P|_D \to P_{z'}, \qquad p \mapsto h_{z'}(p) := h^A\gamma_{(\pi(p),z')}(1) \tag{2.41}$$

which assigns to every point $p \in P|_D$ the endpoint of the $A$-horizontal lift of the ray $\gamma_{(\pi(p),z')}$ starting at $p$. As we have seen in the proof of (i) above, this map is of class $C^1$ since $A$ is of class $C^1$. Moreover, the proof of Lemma 2.3.1 shows that it is $G$-equivariant. In order to prove assertion (ii, a) of the proposition we first derive a formula for the twisted derivative of the map (2.41).

Recall from (2.33) the notation $\gamma_{z'}(z) = \gamma_{(z,z')}$, and recall moreover that $\sigma\colon B \to P|_B$ denotes a trivializing section of $P$ over $B$.

Claim. *For every point $p \in P|_D$ and every tangent vector $\tilde{v}_p \in T_pP$, the covariant derivative of the map (2.41) satisfies the identity*

$$A_{h_{z'}(p)}\big(\mathrm{d}_A h_{z'}(\tilde{v}_p)\big) = \int_0^1 (\sigma^* F_A)\Big(\partial_\tau \gamma_{(\pi(p),z')}(\tau), \big(\delta_{\pi(p)}\gamma_{z'}(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big)\,\mathrm{d}\tau. \tag{2.42}$$

Proof of Claim. First, we claim that the covariant derivative of the map (2.41) may be expressed in terms of the infinitesimal variation of the family of horizontal paths (2.27) in the direction of the tangent vector $\tilde{v}_p$ by

$$\mathrm{d}_A h(\tilde{v}_p) = \big(\mathrm{d}_{\pi(p)}(h^A\gamma_{(\pi(\cdot),z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(1). \tag{2.43}$$

In fact, we compute

$$\mathrm{d}_A h(\tilde{v}_p) = \mathrm{d}_p h\big(\tilde{v}_p^{\mathrm{hor}}\big) = \mathrm{d}_p\big(h^A\gamma_{(\pi(\cdot),z')}(1)\big)\big(\tilde{v}_p^{\mathrm{hor}}\big) = \big(\mathrm{d}_{\pi(p)}(h^A\gamma_{(\pi(\cdot),z')})(\mathrm{d}\pi(\tilde{v}_p^{\mathrm{hor}}))\big)(1),$$

where $\tilde{v}_p^{\mathrm{hor}}$ denotes the $A$-horizontal component of $\tilde{v}_p$. Since $\mathrm{d}\pi(\tilde{v}_p^{\mathrm{hor}}) = \mathrm{d}\pi(\tilde{v}_p)$, formula (2.43) follows.

Next, for $\epsilon > 0$ sufficiently small we choose a smooth path $s\colon (-\epsilon,\epsilon) \to P$, $\lambda \mapsto s(\lambda)$ in $P$ such that

$$s(0) = p, \qquad \left.\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right|_{\lambda=0} = \tilde{v}_p.$$

It gives rise to a 1-parameter family of geodesics in $B$

$$(-\epsilon,\epsilon) \to C^1\big([0,1], B\big), \qquad \lambda \mapsto \gamma_\lambda := \gamma_{(\pi(s(\lambda)),z')}$$

which associates to $\lambda$ the geodesic joining the point $\pi(s(\lambda))$ with the point $z'$. This family then lifts to a 1-parameter family of $A$-horizontal paths in $P$

$$(-\epsilon,\epsilon) \to C^1\big([0,1], P\big), \qquad \lambda \mapsto h_\lambda := h^A\gamma_\lambda.$$

Thus, for every $\lambda \in (-\epsilon,\epsilon)$, the path $h_\lambda$ is the unique solution of the ordinary differential equation

$$A(\partial_\tau h_\lambda) = 0, \qquad h_\lambda(0) = s(\lambda). \tag{2.44}$$

Since $A$ is of class $C^1$ we may, by the Cauchy-Lipschitz theorem ([**17**], Ch. V, Thm. 3.1), write the family $\lambda \mapsto h_\lambda$ as a map of class $C^1$

$$h\colon (-\epsilon,\epsilon) \times [0,1] \to P, \qquad (\lambda,\tau) \mapsto h_\lambda(\tau).$$

Then $[\partial_\lambda h, \partial_\tau h] = 0$, so we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} A(\partial_\lambda h) &= \frac{\mathrm{d}}{\mathrm{d}\tau} A(\partial_\lambda h) - \frac{\mathrm{d}}{\mathrm{d}\lambda} A(\partial_\tau h) \\
&= \mathcal{L}_{\partial_\tau h} A(\partial_\lambda h) - \mathcal{L}_{\partial_\lambda h} A(\partial_\tau h) \\
&= \mathrm{d}A(\partial_\tau h, \partial_\lambda h) + A\big([\partial_\tau h, \partial_\lambda h]\big) \\
&= F_A(\partial_\tau h, \partial_\lambda h) - \big[A(\partial_\tau h), A(\partial_\lambda h)\big] \\
&= F_A(\partial_\tau h, \partial_\lambda h). \qquad (2.45)
\end{aligned}$$

Let us explain this computation. In the first equality, we used equation (2.44). The second equality is just a reformulation in terms of Lie derivatives on $P$. The third equality uses a basic identity for Lie derivatives ([**20**], Prop. I.3.11), while the fourth equality uses the identity $[\partial_\lambda h, \partial_\tau h] = 0$ and the definition of the curvature. The fifth equality is again a consequence of equation (2.44). We further have

$$\partial_\lambda h(\tau) = \big(\mathrm{d}_{\pi(s(\lambda))}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau),$$

whence

$$\partial_\lambda h(\tau)\Big|_{\lambda=0} = \big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)$$

for all $\tau \in [0,1]$. Setting $\lambda = 0$ in (2.45) we therefore obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau} A\Big(\big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big) \\
&= F_A\Big(\partial_\tau h^A \gamma_{(\pi(p), z')}(\tau), \big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big).
\end{aligned}$$

Since $F_A$ is horizontal and equivariant, we may now pull the right-hand side of this relation back to the disk $B$ along the section $\sigma\colon B \to P$. In fact, since $\pi \circ \sigma = \mathrm{id}$ we have

$$\sigma^* \, \partial_\tau h^A \gamma_{(\pi(p), z')}(\tau) = \partial_\tau \gamma_{(\pi(p), z')}(\tau)$$

and

$$\sigma^* \, \mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau) = \big(\delta_{\pi(p)} \gamma_{z'}(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau).$$

Hence we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau} A\Big(\big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big) \\
&\qquad = (\sigma^* F_A)\Big(\partial_\tau \gamma_{(\pi(p), z')}(\tau), \big(\delta_{\pi(p)} \gamma_{z'}(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big).
\end{aligned}$$

Since $\big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(0) = 0$, integrating this differential equation yields

$$\begin{aligned}
&A\Big(\big(\mathrm{d}_{\pi(p)}(h^A \gamma_{(\pi(\cdot), z')})(\mathrm{d}\pi(\tilde{v}_p))\big)(1)\Big) \\
&\qquad = \int_0^1 (\sigma^* F_A)\Big(\partial_\tau \gamma_{(\pi(p), z')}(\tau), \big(\delta_{\pi(p)} \gamma_{z'}(\mathrm{d}\pi(\tilde{v}_p))\big)(\tau)\Big)\, \mathrm{d}\tau.
\end{aligned}$$

Combining this with identity (2.43), the Claim follows. $\square$

We now use the result of the Claim above to deduce the formula of assertion (ii, a). Fix a point $z \in D$ and a tangent vector $v_z \in T_z\Sigma$. Using the section $\sigma\colon D \to P|_D$ we may define lifts $p := \sigma(z) \in P|_D$ and $\tilde{v}_p := \mathrm{d}\sigma(v_z) \in T_pP$. Then

$$\pi(p) = z \quad \text{and} \quad \mathrm{d}\pi(\tilde{v}_p) = v_z. \tag{2.46}$$

Recall from formula (2.30) that the reduced holonomy map was defined by

$$h^A\gamma_{(\pi(p),z')}(1) = \sigma(z') \cdot \mathrm{hol}^A_{z'}(p).$$

Differentiating this identity covariantly at the point $p$ in the direction of $\tilde{v}_p$, and using the notation $h_{z'}(p) = h^A\gamma_{(\pi(p),z')}(1)$ introduced in (2.41) above, we obtain

$$\mathrm{d}_A h_{z'}(\tilde{v}_p) = h_{z'}(p).\, \omega_{\mathrm{MC}}\big(\mathrm{d}_A \mathrm{hol}^A_{z'}(\tilde{v}_p)\big),$$

where $\omega_{\mathrm{MC}} \in \Omega^1(G, \mathfrak{g})$ denotes the Maurer-Cartan form on $G$. Hence

$$A_{h_{z'}(p)}\big(\mathrm{d}_A h_{z'}(\tilde{v}_p)\big) = \omega_{\mathrm{MC}}\big(\mathrm{d}_A \mathrm{hol}^A_{z'}(\tilde{v}_p)\big). \tag{2.47}$$

Since $p = \sigma(z)$ and $\tilde{v}_p = \mathrm{d}\sigma(v_z)$ we have

$$\mathrm{d}_A \mathrm{hol}^A_{z'}(\tilde{v}_p) = \sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z). \tag{2.48}$$

Combining formulas (2.46), (2.47) and (2.48) with formula (2.42) in the Claim above, we finally obtain

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big) = \int_0^1 (\sigma^* F_A)\Big(\partial_\tau \gamma_{(z,z')}(\tau), \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\Big)\,\mathrm{d}\tau. \tag{2.49}$$

This proves (ii, a).

Let us now prove assertion (ii, b). Fix a $G$-invariant metric on $G$ and let $\nabla^G$ denote the corresponding Levi-Civita connection on $G$. Let $v$ be a smooth vector field on $\Sigma$, fix a point $z \in D$, and let $w_z \in T_z\Sigma$ be a tangent vector. First of all, by formula (2.39) we have

$$\nabla_A\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big)(w_z, v_z) = \nabla_{A,w_z}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v)\big) - \sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big(\nabla^\Sigma_{w_z} v\big), \tag{2.50}$$

where $\nabla^\Sigma$ is the Levi-Civita connection on $\Sigma$, understood with respect to the Kähler metric.

Let us consider the first term on the right-hand side of equation (2.50). By the Leibniz rule we have

$$\begin{aligned} \omega_{\mathrm{MC}}\Big(\nabla_{A,w_z}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v)\big)\Big) &= \nabla_{A,w_z}\Big(\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v)\big)\Big) \\ &\quad - \big(\nabla^G_{\sigma^*(\mathrm{d}_A \mathrm{hol}^A_{z'})(w_z)} \omega_{\mathrm{MC}}\big)\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big). \end{aligned} \tag{2.51}$$

For an explanation of the covariant derivatives appearing in this computation we refer the reader to Remark 2.3.3 above. By formula (2.49) above, using the notation introduced in (2.35), (2.37) and (2.38) we have

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big) = \int_0^1 (\sigma^* F_A)\big(X^\tau(\gamma_\tau(z)), Y^\tau_v(\gamma_\tau(z))\big)\,\mathrm{d}\tau.$$

We may thus compute the first term on the right-hand side of formula (2.51) by differentiating the map

$$D \ni z'' \mapsto \omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(v_{z''})\big) = \int_0^1 (\sigma^* F_A)\big(X^\tau(\gamma_\tau(z'')), Y^\tau_v(\gamma_\tau(z''))\big)\,\mathrm{d}\tau$$

at the point $z$, obtaining

$$\begin{aligned}\nabla_{A,w_z}\Big(\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(v)\big)\Big) &= \int_0^1 \nabla_{A,w_z}\Big((\sigma^* F_A)\big(X^\tau(\gamma_\tau(\,\cdot\,)), Y^\tau_v(\gamma_\tau(\,\cdot\,))\big)\Big)\,\mathrm{d}\tau \\ &= \int_0^1 \nabla_{A,\gamma_\tau w_z}\big((\sigma^* F_A)(X^\tau, Y^\tau_v)\big)\,\mathrm{d}\tau. \qquad (2.52)\end{aligned}$$

Here we use the abbreviation $\gamma_\tau w_z := \mathrm{d}_z\gamma_\tau(w_z)$ in the tangent space $T_{\gamma_\tau(z)}\Sigma$. By formula (2.40) we further have

$$\begin{aligned}\nabla_{A,\gamma_\tau w_z}\big((\sigma^* F_A)(X^\tau, Y^\tau_v)\big) &= \big(\nabla_A(\sigma^* F_A)\big)\big(\gamma_\tau w_z, X^\tau(\gamma_\tau(z)), Y^\tau_v(\gamma_\tau(z))\big) \\ &\quad + (\sigma^* F_A)\big(\nabla^\Sigma_{\gamma_\tau w_z} X^\tau, Y^\tau_v(\gamma_\tau(z))\big) + (\sigma^* F_A)\big(X^\tau(\gamma_\tau(z)), \nabla^\Sigma_{\gamma_\tau w_z} Y^\tau_v\big). \qquad (2.53)\end{aligned}$$

Likewise, for the second term on the right-hand side of equation (2.50) we obtain from (2.49) the expression

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big(\nabla^\Sigma_{w_z} v\big)\big) = \int_0^1 (\sigma^* F_A)\big(X^\tau(\gamma_\tau(z)), Y^\tau_{\nabla^\Sigma_{w_z} v}(\gamma_\tau(z))\big)\,\mathrm{d}\tau. \qquad (2.54)$$

Combining (2.54), (2.53), (2.52) and (2.51) with equation (2.50) above, we finally obtain

$$\begin{aligned}\nabla_A\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big)(w_z, v_z) &= \int_0^1 \Big(\big(\nabla_A(\sigma^* F_A)\big)\big(\gamma_\tau w_z, X^\tau(\gamma_\tau(z)), Y^\tau_v(\gamma_\tau(z))\big) \\ &\quad + (\sigma^* F_A)\big(\nabla^\Sigma_{\gamma_\tau w_z} X^\tau, Y^\tau_v(\gamma_\tau(z))\big) + (\sigma^* F_A)\big(X^\tau(\gamma_\tau(z)), \nabla^\Sigma_{\gamma_\tau w_z} Y^\tau_v - Y^\tau_{\nabla_{w_z} v}(\gamma_\tau(z))\big)\Big)\,\mathrm{d}\tau \\ &\quad - \big(\nabla^G_{\sigma^*(\mathrm{d}_A\mathrm{hol}^A_{z'})(w_z)}\omega_{\mathrm{MC}}\big)\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(v_z)\big).\end{aligned}$$

Using the definition of the vector fields $X^\tau$ and $Y^\tau_v$ in (2.37) and (2.38), and using $\gamma_\tau w_z = \mathrm{d}_z\gamma_\tau(w_z)$, the formula claimed in (ii, b) follows. This completes the proof of Proposition 2.3.2. □

Proof of Proposition 2.3.4. Fix a $G$-invariant metric on $G$.

Proof of (i): The system of holonomy maps (2.29) is defined in terms of the horizontal lifts (2.27) which are solutions of the ordinary differential equation (2.28). The coefficients of this equation are given in terms of the connection $A$. Hence the claim is a consequence of the Cauchy-Lipschitz theorem ([**17**], Ch. V, Thm. 3.1), which implies that the solutions of an ordinary differential equation depend continuously on its coefficients.

Proof of (ii): Let $z' \in B \setminus D$ and $A \in \mathcal{A}^1(P)$, and consider the reduced holonomy map

$$\mathrm{hol}^A_{z'}\colon P|_D \to G.$$

It will be most convenient to work in local polar geodesic coordinates around the point $z'$ on $\Sigma$. More precisely, we denote by

$$\exp_{z'}\colon T_{z'}\Sigma \supset B_{2R}(0) \to B_{2R}(z'), \qquad \exp_{z'}(0) = z' \tag{2.55}$$

the Riemannian exponential map on the tangent space of $\Sigma$ at $z'$. It maps the Euclidean disk $B_{2R}(0)$ in the tangent space $T_{z'}\Sigma \cong \mathbb{R}^2$ diffeomorphically onto the geodesic disk $B_{2R}(z')$ on $\Sigma$. Recall at this point that the constant $R$ was chosen to be smaller than half the injectivity radius of $\Sigma$. Note that the disk $B$ is contained in the chart $B_{2R}(z')$. We endow the punctured disk $B_{2R}(0) \setminus \{0\}$ with polar coordinates $(r, \varphi)$, thinking of the point $(r, \varphi)$ as the lift of the point $z$. For every angle $\varphi \in [0, 2\pi)$, let us denote by

$$I(\varphi) := \big\{ r \in (0, 2R] \,\big|\, \exp_{z'}(r, \varphi) \in D \big\} \tag{2.56}$$

the set of all numbers $r$ such that the point $(r, \varphi) \in B_{2R}(0)$ gets mapped into the disk $D$ under the exponential map (2.55). We further write the lift of the area form on $\Sigma$ under the exponential map in the form

$$\exp^*_{z'} \mathrm{dvol}_\Sigma(z) = \lambda(r, \varphi)^2 \cdot r \, \mathrm{d}r \wedge \mathrm{d}\varphi \tag{2.57}$$

for some bounded smooth function $\lambda\colon B_{2R}(0) \setminus \{0\} \to (0, \infty)$.

Fix a point $z \in \Sigma$ and a tangent vector $v_z \in T_z\Sigma$. Applying Proposition 2.3.2 (ii, a) and using the polar geodesic coordinates introduced above, we obtain for the covariant derivative of the reduced holonomy map at $z$ in the direction of $v_z$ the relation

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big) = \int_0^1 \widetilde{F}_A\Big(\partial_\tau \tilde{\gamma}_{((r,\varphi),0)}(\tau), \big(\delta_0 \tilde{\gamma}_{(r,\varphi)}(\tilde{v})\big)(\tau)\Big)\, \mathrm{d}\tau, \tag{2.58}$$

where we abbreviate

$$\widetilde{F}_A := (\sigma \circ \exp_{z'})^* F_A.$$

Here, $\sigma\colon B \to P|_B$ denotes a fixed section. We further denote by

$$\tilde{\gamma}_{((r,\varphi),0)}\colon [0,1] \to B_{2R}(0), \qquad \tau \mapsto ((1-\tau)\cdot r, \varphi) \tag{2.59}$$

the line segment joining the point $(r, \varphi)$ with the origin in $\mathbb{R}^2$. Then $\tilde{\gamma}_{((r,\varphi),0)}$ is the lift of the geodesic $\gamma_{(z,z')}$. As before, we use the abbreviation $\tilde{\gamma}_{(r,\varphi)} := \tilde{\gamma}_{((r,\varphi),0)}$. The vector $\tilde{v} \in \mathbb{R}^2$ denotes the lift of the tangent vector $v_z$, and we denote by $\delta_0 \tilde{\gamma}_{(r,\varphi)}(\tilde{v})$ the Jacobi field along $\tilde{\gamma}_{((r,\varphi),0)}$ obtained by differentiation of the map $(r, \varphi) \mapsto \tilde{\gamma}_{((r,\varphi),0)}$ in the direction of $\tilde{v}$. Using the reparametrization $\tau' := r \cdot (1-\tau)$, the integral on the right-hand side of (2.58) may then be written as

$$-\int_0^r \widetilde{F}_A\Big(\partial_{\tau'} \tilde{\gamma}_{((r,\varphi),0)}(1-\tau'/r), \big(\delta_0 \tilde{\gamma}_{(r,\varphi)}(\tilde{v})\big)(1-\tau'/r)\Big)\, \mathrm{d}\tau'.$$

Note that there exists a constant $c_1 > 0$ (not depending on $z$ and $z'$) such that

$$\left|\big(\delta_0\tilde{\gamma}_{(r,\varphi)}(\tilde{v})\big)(1-\tau'/r)\right| \leq c_1 \cdot |\tilde{v}| \cdot \frac{\tau'}{r}$$

for all $\tau' \in [0, r]$; and note moreover that we have an identity

$$\tilde{\gamma}_{((r,\varphi),0)}(1-\tau'/r) = (\tau', \varphi)$$

in $\mathbb{R}^2$, by (2.59). Passing to operator norms, we thus obtain from (2.58) an estimate

$$\left|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(z)\right| \leq c_2 \cdot \int_0^r \left|\widetilde{F}_A((\tau',\varphi))\right| \cdot \frac{\tau'}{r}\,\mathrm{d}\tau' \tag{2.60}$$

for some constant $c_2 > 0$ (not depending on $z$, $z'$ and $A$).

Fix a real number $p > 2$. Applying Hölder's inequality we obtain from (2.60) the relation

$$\left|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(z)\right|^p \leq c_3 \cdot \int_0^r \left|\widetilde{F}_A((\tau',\varphi))\right|^p \cdot \frac{\tau'}{r}\,\mathrm{d}\tau'$$

for some constant $c_3 > 0$ (not depending on $z$, $z'$ and $A$). Here we used that $0 \leq \tau'/r \leq 1$. Using the notation introduced in (2.56) and (2.57) above, we may now integrate this inequality over the disk $D$, obtaining

$$\begin{aligned}
&\int_D \left|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(z)\right|^p \mathrm{dvol}_\Sigma(z) \\
\leq\ & c_3 \cdot \int_0^{2\pi}\int_{I(\varphi)} \left(\int_0^r \left|\widetilde{F}_A((\tau',\varphi))\right|^p \cdot \frac{\tau'}{r}\,\mathrm{d}\tau'\right) \cdot \lambda(r,\varphi)^2 \cdot r\,\mathrm{d}r\,\mathrm{d}\varphi \\
\leq\ & c_3 \cdot \int_0^{2R} \left(\int_0^{2\pi}\int_0^r \left|\widetilde{F}_A((\tau',\varphi))\right|^p \cdot \tau'\,\mathrm{d}\tau'\,\mathrm{d}\varphi\right) \cdot \lambda(r,\varphi)^2\,\mathrm{d}r. 
\end{aligned} \tag{2.61}$$

Here, for the second estimate we used Fubini's theorem and the fact that $I(\varphi) \subset (0, 2R]$. Now

$$\begin{aligned}
\int_0^{2\pi}\int_0^r \left|\widetilde{F}_A((\tau',\varphi))\right|^p \cdot \tau'\,\mathrm{d}\tau'\,\mathrm{d}\varphi &\leq \int_{B_{2R}(0)} \left|\widetilde{F}_A((\tau',\varphi))\right|^p \cdot \tau'\,\mathrm{d}\tau'\,\mathrm{d}\varphi \\
&\leq c_4 \cdot \int_{B_{2R}(z')} |F_A(z)|^p\,\mathrm{dvol}_\Sigma(z) \\
&\leq c_4 \cdot \int_\Sigma |F_A(z)|^p\,\mathrm{dvol}_\Sigma(z)
\end{aligned}$$

for some constant $c_4 > 0$ (not depending on $z'$ and $A$). Here, we think of $(\tau', \varphi)$ as polar coordinates on the disk $B_{2R}(0)$. Furthermore, for the second estimate we used relation (2.57). Hence we obtain from (2.61) an estimate

$$\int_D \left|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(z)\right|^p \mathrm{dvol}_\Sigma(z) \leq c_5 \cdot \int_\Sigma |F_A(z)|^p\,\mathrm{dvol}_\Sigma(z)$$

for some constant $c_5 > 0$ (not depending on $z'$ and $A$). Setting $C_1 := (c_5)^{1/p}$ it follows that

$$\left\|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\right\|_{L^p(D)} \leq C_1 \cdot \left\|F_A\right\|_{L^p(\Sigma)}.$$

This proves assertion (ii).

Proof of (iii): Let $z' \in B \setminus D$ and $A \in \mathcal{A}^1(P)$, and consider the reduced holonomy map

$$\mathrm{hol}^A_{z'} \colon P|_D \to G.$$

Fix a point $z \in \Sigma$ and a tangent vector $v_z \in T_z\Sigma$. Applying Proposition 2.3.2 (ii, a), we obtain for the covariant derivative of the reduced holonomy map at $z$ in the direction of $v_z$ the relation

$$\omega_{\mathrm{MC}}\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big) = \int_0^1 (\sigma^* F_A)\Big(\partial_\tau \gamma_{(z,z')}(\tau), \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\Big)\, \mathrm{d}\tau.$$

Passing to operator norms, it follows that there exists a constant $c_6 > 0$ (not depending on $z'$ and $A$) such that

$$\left|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(z)\right| \leq c_6 \cdot \int_0^1 \left|F_A\big(\gamma_{(z,z')}(\tau)\big)\right| \mathrm{d}\tau$$

for all points $z \in D$. We conclude from this that

$$\left\|\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\right\|_{L^\infty(D)} \leq C_2 \cdot \left\|F_A\right\|_{L^\infty(\Sigma)}$$

for some constant $C_2 > 0$ (not depending on $z'$ and $A$). This proves assertion (iii).

Proof of (iv): Let $z' \in B \setminus D$ and $A \in \mathcal{A}^1(P)$ be of class $C^2$, and consider the pull-back

$$\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big) \colon TD \to TG \tag{2.62}$$

of the covariant derivative of the reduced holonomy map. Fix a point $z \in D$ and tangent vectors $v_z, w_z \in T_z D$. Choose smooth vector fields $v$ and $w$ on $D$ that restrict to $v_z$ and $w_z$ at the point $z$. Applying Proposition 2.3.2 (ii, b), we obtain for the covariant derivative of the map (2.62) at $z$ in the direction of $w_z$ and $v_z$ the relation

$$\begin{aligned}
&\omega_{\mathrm{MC}}\big(\nabla_A\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big)(w_z, v_z)\big)\\
&\quad = \int_0^1 \Big(\big(\nabla_A(\sigma^* F_A)\big)\big(\mathrm{d}_z \gamma_\tau(w_z), \partial_\tau \gamma_{(z,z')}(\tau), \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\big)\\
&\qquad\qquad + (\sigma^* F_A)\big(\nabla^\Sigma_{\mathrm{d}_z \gamma_\tau(w_z)} X^\tau, \big(\delta_z \gamma_{z'}(v_z)\big)(\tau)\big)\\
&\qquad + (\sigma^* F_A)\big(\partial_\tau \gamma_{(z,z')}(\tau), \nabla^\Sigma_{\mathrm{d}_z \gamma_\tau(w_z)} Y^\tau_v - Y^\tau_{\nabla_w v}(\gamma_\tau(z))\big)\Big)\, \mathrm{d}\tau\\
&\qquad\qquad - \big(\nabla^G_{\sigma^*(\mathrm{d}_A \mathrm{hol}^A_{z'})(w_z)} \omega_{\mathrm{MC}}\big)\big(\sigma^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)(v_z)\big).
\end{aligned} \tag{2.63}$$

Here we use the notation introduced in (2.33), (2.34), (2.35), (2.37), (2.38) and Remark 2.3.3. We may then estimate the first two terms appearing on the right-hand side of this formula by

$$\begin{aligned} &\Big|\big(\nabla_A(\sigma^*F_A)\big)\big(\mathrm{d}_z\gamma_\tau(w_z),\partial_\tau\gamma_{(z,z')}(\tau),\big(\delta_z\gamma_{z'}(v_z)\big)(\tau)\big)\Big| \\ &\qquad\qquad \leq c_7\cdot|w_z|\cdot|v_z|\cdot\Big|\big(\nabla_A(\sigma^*F_A)\big)\big(\gamma_{(z,z')}(\tau)\big)\Big| \end{aligned} \tag{2.64}$$

and

$$\Big|(\sigma^*F_A)\big(\nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)}X^\tau,\big(\delta_z\gamma_{z'}(v_z)\big)(\tau)\big)\Big| \leq c_7\cdot|w_z|\cdot|v_z|\cdot\big\|F_A\big\|_{L^\infty(\Sigma)} \tag{2.65}$$

for some constant $c_7>0$ (not depending on $z$, $z'$ and $A$). Using assertion (iii) above, the last term may be estimated by

$$\begin{aligned} &\Big|\big(\nabla^G_{\sigma^*(\mathrm{d}_A\mathrm{hol}^A_{z'})(w_z)}\omega_{\mathrm{MC}}\big)\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(v_z)\big)\Big| \\ \leq\;& c_8\cdot|w_z|\cdot|v_z|\cdot\big\|\nabla^G\omega_{\mathrm{MC}}\big\|_{L^\infty(G)}\cdot\big|\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)(z)\big|^2 \\ \leq\;& c_8\cdot C_2^2\cdot|w_z|\cdot|v_z|\cdot\big\|\nabla^G\omega_{\mathrm{MC}}\big\|_{L^\infty(G)}\cdot\big\|F_A\big\|^2_{L^\infty(\Sigma)} \end{aligned} \tag{2.66}$$

for some constant $c_8>0$ (not depending on $z$, $z'$ and $A$). For the third term we have an inequality

$$\begin{aligned} &\Big|(\sigma^*F_A)\big(\partial_\tau\gamma_{(z,z')}(\tau),\nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)}Y^\tau_v-Y^\tau_{\nabla^\Sigma_w v}(\gamma_\tau(z))\big)\Big| \\ &\qquad\qquad \leq c_9\cdot\Big|\nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)}Y^\tau_v-Y^\tau_{\nabla^\Sigma_w v}(\gamma_\tau(z))\Big|\cdot\big\|F_A\big\|_{L^\infty(\Sigma)} \end{aligned} \tag{2.67}$$

for some constant $c_9>0$ (not depending on $z$, $z'$ and $A$). In order to estimate this further, we need the following claim, which states that the map

$$\alpha\colon \mathrm{Vect}(D)\times\mathrm{Vect}(D)\to\mathrm{Vect}(D),\qquad \alpha(w,v):=\nabla^\Sigma_{\mathrm{d}\gamma_\tau(w)}Y^\tau_v-Y^\tau_{\nabla^\Sigma_w v}(\gamma_\tau) \tag{2.68}$$

given by the first term on the right-hand side of inequality (2.67) is tensorial. Here, $\mathrm{Vect}(D)$ denotes smooth vector fields on $D$.

Claim. *The map (2.68) is linear over the smooth functions on $D$.*

Proof of Claim. Let us abbreviate $\nabla:=\nabla^\Sigma$. Linearity in the first argument is clear. Let now $f$ be a smooth function on $D$ and consider $\alpha(w,f\cdot v)$. Let $z\in D$. Then

$$\alpha(w,f\cdot v)(z) \;=\; \nabla_{\mathrm{d}_z\gamma_\tau(w_z)}Y^\tau_{f\cdot v}-Y^\tau_{\nabla_w(f\cdot v)}(\gamma_\tau(z)). \tag{2.69}$$

Recall from (2.35) and (2.38) that the vector field $Y^\tau_{f\cdot v}$ is defined in terms of the Jacobi field by

$$Y^\tau_{f\cdot v}(z):=\big(\delta_{\gamma^\tau(z)}\gamma_{z'}\big(f(\gamma^\tau(z))\cdot v_{\gamma^\tau(z)}\big)\big)(\tau),$$

where $\gamma^\tau$ denotes the inverse of the map $\gamma_\tau$ given by $\gamma_\tau(z'')=\gamma_{(z'',z')}(\tau)$. It follows from this that

$$Y^\tau_{f\cdot v}=f(\gamma^\tau)\cdot Y^\tau_v.$$

Hence we have

$$\begin{aligned}\nabla_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_{f\cdot v} &= \nabla_{\mathrm{d}_z\gamma_\tau(w_z)}\big(f(\gamma^\tau)\cdot Y^\tau_v\big)\\ &= \big(\mathrm{d}_{(\gamma^\tau\circ\gamma_\tau)(z)} f\circ \mathrm{d}_{\gamma_\tau(z)}\gamma^\tau\big)(\mathrm{d}_z\gamma_\tau(w_z))\cdot Y^\tau_v(\gamma_\tau(z))\\ &\qquad\qquad + f((\gamma^\tau\circ\gamma_\tau)(z))\cdot \nabla_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_v\\ &= \mathrm{d}_z f(w_z)\cdot Y^\tau_v(\gamma_\tau(z)) + f(z)\cdot \nabla_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_v. \end{aligned} \tag{2.70}$$

Likewise, using the Leibniz rule

$$\nabla_w(f\cdot v) = \mathrm{d}f(w)\cdot v + f\cdot\nabla_w v$$

we obtain

$$\begin{aligned} Y^\tau_{\nabla_w(f\cdot v)}(\gamma_\tau(z)) &= Y^\tau_{\mathrm{d}f(w)\cdot v}(\gamma_\tau(z)) + Y^\tau_{f\cdot\nabla_w v}(\gamma_\tau(z))\\ &= \mathrm{d}f(w)((\gamma^\tau\circ\gamma_\tau))(z)\cdot Y^\tau_v(\gamma_\tau(z))\\ &\qquad + f((\gamma^\tau\circ\gamma_\tau)(z))\cdot Y^\tau_{\nabla_w v}(\gamma_\tau(z))\\ &= \mathrm{d}_z f(w_z)\cdot Y^\tau_v(\gamma_\tau(z)) + f(z)\cdot Y^\tau_{\nabla_w v}(\gamma_\tau(z)). \end{aligned} \tag{2.71}$$

Combining (2.70) and (2.71) with (2.69) then yields

$$\alpha(w, f\cdot v)(z) = f(z)\cdot\nabla_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_v - f(z)\cdot Y^\tau_{\nabla_w v}(\gamma_\tau(z)) = f(z)\cdot\alpha(w,v)(z).$$

This proves the claim. □

By the Claim there exists a constant $c_{10} > 0$ (not depending on $z$, $z'$ and $A$) such that

$$\Big|\nabla^\Sigma_{\mathrm{d}_z\gamma_\tau(w_z)} Y^\tau_v - Y^\tau_{\nabla^\Sigma_w v}(\gamma_\tau(z))\Big| \le c_{10}\cdot|v_z|\cdot|w_z|. \tag{2.72}$$

Whence combining inequalities (2.64), (2.65), (2.67) and (2.66) with (2.63) and passing to operator norms we obtain an estimate

$$\begin{aligned}\big|\big(\nabla_A\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big)\big)(z)\big| \le c_{11}\cdot\int_0^1 \Big(\big|\big(\nabla_A(\sigma^* F_A)\big)(\gamma_{(z,z')}(\tau))\big|\\ + \big\|F_A\big\|^2_{L^\infty(\Sigma)} + \big\|F_A\big\|_{L^\infty(\Sigma)} + 1\Big)\,\mathrm{d}\tau + \big\|\nabla^G\omega_{\mathrm{MC}}\big\|_{L^\infty(G)}\end{aligned}$$

for some constant $c_{11} > 0$ (not depending on $z$, $z'$ and $A$). This finally implies that

$$\begin{aligned}\big|\big(\nabla_A\big(\sigma^*\big(\mathrm{d}_A\mathrm{hol}^A_{z'}\big)\big)\big)(z)\big| \le C_3\cdot\Bigg(1 + \big\|F_A\big\|^2_{L^\infty(\Sigma)}\\ + \int_0^1\Big(\big|\big(\nabla_A(\sigma^* F_A)\big)(\gamma_{(z,z')}(\tau))\big|\,\mathrm{d}\tau\Bigg)\end{aligned}$$

for some constant $C_3 > 0$ (not depending on $z$, $z'$ and $A$). This proves assertion (iv) and finishes the proof of Proposition 2.3.4. □

**2.3.2. Construction of a classifying map.** Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ (see Definition 2.1.1). Let $p > 2$ be a real number. Our goal in this subsection is to give an explicit construction of a regular classifying map

$$\Theta\colon \mathcal{B}^{1,p} := \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma) \to C^0(P, EG)^G$$

in the sense of Section 2.1.2 and Definition 2.1.3. More specifically, we shall construct a $\mathcal{G}^{2,p}$-equivariant map

$$\Theta\colon \mathcal{B}^{1,p} \to W^{1,p}\big(P, EG^N\big)^G \tag{2.73}$$

taking values in $G$-equivariant maps $P \to EG^N$ of class $W^{1,p}$ (see Remark 2.1.4). We remind the reader that $W^{1,p}(P, EG^N)^G$ is contained in $C^0(P, EG)^G$ by Rellich's theorem.

We will proceed as follows. Recall from Section 2.1.1 that the configuration space $\mathcal{B}^{1,p}$ consists of all pairs $(A, u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P, M)^G$ such that the map $u$ satisfies the taming condition

$$\int_\Sigma |\mu(u)|_{\mathfrak{g}}^2 \,\mathrm{dvol}_\Sigma < E. \tag{2.74}$$

Let us consider the subspace

$$\mathcal{B} := \big\{(A, u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P, M)^G \,\big|\, (A, u) \text{ is of class } C^1 \text{ and } u \text{ satisfies (2.74)}\big\}$$

consisting of pairs of class $C^1$. Denote by $\mathcal{G}$ the group of gauge transformations of $P$ that are of class $C^2$. The free action (2.2) of the group of gauge transformations $\mathcal{G}^{2,p}$ on the configuration space $\mathcal{B}^{1,p}$ then restricts to a well-defined action of $\mathcal{G}$ on the subspace $\mathcal{B}$. Our strategy for the construction of the classifying map (2.73) now is to first construct a $\mathcal{G}$-equivariant classifying map

$$\Theta\colon \mathcal{B} \to C^1\big(P, EG^N\big)^G \tag{2.75}$$

for the space $\mathcal{B}$ and then to extend it to all of $\mathcal{B}^{1,p}$ by continuity.

The first step in the construction of the classifying map (2.75) is to exhibit sufficiently many systems of holonomy maps on $\Sigma$ (see Section 2.3.1). By admissiblity of the area form $\mathrm{dvol}_\Sigma$ (see Definition 2.1.1) we may pick a positive real number $R$, smaller than half the injectivity radius of $\Sigma$, and finitely many points $z_1, \ldots, z_a$ on $\Sigma$ such that the closed geodesic disks $B_{R/3}(z_1), \ldots, B_{R/3}(z_a)$ form a covering of $\Sigma$ and the area of every annulus $B_R(z_i) \setminus B_{R/2}(z_i)$ satisfies

$$\mathrm{Vol}\big(B_R(z_i) \setminus B_{R/2}(z_i)\big) > \frac{E}{\delta^2} \tag{2.76}$$

for $i = 1, \ldots, a$. Let us denote

$$B_i := B_R(z_i) \quad \text{and} \quad D_i := B_{R/2}(z_i)$$

for $i = 1, \ldots, a$. We then have the following basic lemma.

LEMMA 2.3.5. *For every $1 \leq i \leq a$ and for every continuous $G$-equivariant map $u\colon P \to M$ satisfying the taming condition (2.74) there exists an open subset $V_u^i \subset B_i \backslash D_i$ such that*

$$|\mu(u)(z)| \leq \delta/2$$

*for all $z \in V_u^i$.*

PROOF. Let $1 \leq i \leq a$ and suppose that $u \in W_G^{1,p}(P, M; \mathrm{dvol}_\Sigma)$. Assume for contradiction that $|\mu(u)(z)| > \delta/2$ for all $z \in B_i \backslash D_i$. Combining inequalities (2.74) and (2.76) we then obtain

$$\int_\Sigma |\mu(u)|^2 \,\mathrm{dvol}_\Sigma \geq \int_{B_i \backslash D_i} |\mu(u)|^2 \,\mathrm{dvol}_\Sigma > \frac{\delta^2}{4} \cdot \mathrm{Vol}(B_i \backslash D_i) > \frac{\delta^2}{4} \cdot \frac{8}{\delta^2} = 2,$$

contradicting inequality (2.74). Hence there exists $z_0 \in \Sigma$ such that $|\mu(u)(z_0)| \leq \delta/2$. Since $u$ is continuous the lemma follows. □

For the actual definition of the systems of holonomy maps on the disks $B_i$ let us now consider, for $i = 1, \dots, a$, the smooth family of geodesics

$$\gamma^i\colon D_i \times (B_i \setminus D_i) \to C^\infty\big([0,1], B_i\big), \qquad (z, z') \mapsto \gamma^i_{(z,z')} \tag{2.77}$$

that assigns to every pair of points $(z, z') \in D_i \times (B_i \backslash D_i)$ the geodesic

$$\gamma^i_{(z,z')}\colon [0,1] \to B_i, \qquad \tau \mapsto \gamma^i_{(z,z')}(\tau)$$

joining the point $z$ in the disk $D_i$ with the point $z'$ in the annulus $B_i \backslash D_i$. Fix moreover smooth sections

$$\sigma_i\colon B_i \to P|_{B_i} \tag{2.78}$$

that trivialize the bundle $P$ over each disk $B_i$. As we have seen in Section 2.3.1, for any connection $A$ on $P$ of class $C^1$ the triple $(\gamma^i, \sigma_i, A)$ gives rise to a system of holonomy maps

$$\mathrm{hol}_i^A := \mathrm{hol}^A(\gamma^i, \sigma_i)\colon P|_{D_i} \times (B_i \backslash D_i) \to G, \qquad (p, z') \mapsto \mathrm{hol}_i^A(p, z') \tag{2.79}$$

on the disk $B_i$. We will frequently think of these systems as families of reduced holonomy maps

$$\mathrm{hol}^A_{i,z'}\colon P|_{D_i} \to G, \qquad p \mapsto \mathrm{hol}^A_{i,z'}(p) := \mathrm{hol}_i^A(p, z') \tag{2.80}$$

parametrized by the points $z'$ in the annuli $B_i \backslash D_i$.

Continuing with the construction of the classifying map (2.75) we first note that the compact connected Lie group $G$ embeds into the unitary group $U(\ell)$, for some positive integer $\ell$ (see [**1**], Sec. III.3). For the purposes of this subsection we may therefore assume without loss of generality that the group $G$ equals the unitary group $U(\ell)$. We will, however, not indicate this in the notation. Recall that $G$ acts freely on the submanifold

$M_\delta = \{x \in M \mid |\mu(x)| \le \delta\}$. Hence there exists a positive integer $n$ and a smooth classifying map

$$\theta\colon M_\delta \to V(\ell, n) \subset M_{n,\ell}(\mathbb{C}) \tag{2.81}$$

for the principal $G$-bundle $M_\delta \to M_\delta/G$ taking values in the compact Stiefel manifold $V(\ell, n)$ of unitary $\ell$-frames in $\mathbb{C}^n$. Here we think of unitary $\ell$-frames in $\mathbb{C}^n$ as $n$-by-$\ell$ matrices whose column vectors form an orthonormal set in $\mathbb{C}^n$. In this way $V(\ell, n)$ becomes a submanifold of $M_{n,\ell}(\mathbb{C})$ which can be expressed by

$$V(\ell, n) = \big\{U \in M_{n,\ell}(\mathbb{C}) \,\big|\, U^*U = \mathrm{Id}_\ell \,\big\}.$$

Choose moreover a smooth bump function $\varrho\colon M \to [0, 1]$ such that

$$\varrho(x) = \begin{cases} 0 & \text{if } |\mu(x)| > \delta, \\ 1 & \text{if } |\mu(x)| \le \delta/2. \end{cases}$$

We then define pre-classifying maps

$$\Phi^i\colon \mathcal{B} \times P|_{D_i} \times (B_i \backslash D_i) \to \mathbb{R}_{\ge 0} \cdot V(\ell, n) \subset M_{n,\ell}(\mathbb{C}), \quad i = 1, \dots, a, \tag{2.82}$$

by

$$\Phi^i(A, u; p, z') := \varrho(u(\sigma_i(z'))) \cdot \theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z'), \tag{2.83}$$

where $\sigma_i\colon B_i \to P|_{B_i}$ are the trivializing sections (2.78). Here $\mathbb{R}_{\ge 0} \cdot V(\ell, n)$ is the cone in $M_{n,\ell}(\mathbb{C})$ over the Stiefel manifold $V(\ell, n)$.

Claim. *The maps (2.82) are well-defined.*

Proof of Claim. The expression $\theta(u(\sigma_i(z')))$ is well-defined if $u(u(\sigma_i(z')))$ lies in the submanifold $M_\delta = \{x \in M \mid |\mu(x)| \le \delta\}$. This in turn is the case whenever $\varrho(u(\sigma_i(z'))) > 0$. □

The product space $\mathcal{B} \times P$ comes with two natural group actions. The group of gauge transformations $\mathcal{G}$ acts from the right by

$$g^*(A, u; p) := \big(g^*A, g^{-1}u; p.g(p)^{-1}\big) \tag{2.84a}$$

(see (2.2)), and $G$ acts from the right on the second factor by

$$(A, u; p) \cdot h := \big(A, u; p.h\big). \tag{2.84b}$$

These two group actions commute because

$$(p.h).(g(p.h))^{-1} = p.\big(hh^{-1}g(p)^{-1}h\big) = \big(p.g(p)^{-1}\big).h$$

for $p \in P$, $g \in \mathcal{G}$ and $h \in G$. The next lemma shows that the pre-classifying maps (2.82) are invariant under the action (2.84a) and equivariant with respect to the action (2.84b) and the standard right action of $G$ on $M_{n,\ell}(\mathbb{C})$.

LEMMA 2.3.6. *Let $(A,u) \in \mathcal{B}$, $p \in P|_{D_i}$, $g \in \mathcal{G}$ and $h \in G$. Then the pre-classifying maps (2.82) transform by the rule*

$$\Phi^i\big(g^*A, g^{-1}u; p.g(p)^{-1}h, z'\big) = \Phi^i\big(A, u; p, z'\big) \cdot h.$$

PROOF. We see from formula (2.83) that, since the bump function $\varrho$ is $G$-invariant, it will be enough to verify the claimed relation for the product $\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z')$. Recall from Lemma 2.3.1 that

$$\mathrm{hol}_i^{g^*A}(p.g(p)^{-1}h, z') = g(\sigma(z'))^{-1} \cdot \mathrm{hol}_i^A(p, z') \cdot h.$$

Hence we obtain

$$\begin{aligned} &\theta\big(g^{-1}u(\sigma_i(z'))\big) \cdot \mathrm{hol}_i^{g^*A}\big(p.g(p)^{-1}h, z'\big) \\ &= \theta\big(u(\sigma_i(z'))\big) \cdot g(\sigma(z')) \cdot g(\sigma(z'))^{-1} \cdot \mathrm{hol}_i^A(p, z') \cdot h = \theta\big(u(\sigma_i(z'))\big) \cdot \mathrm{hol}_i^A(p, z') \cdot h, \end{aligned}$$

which proves the lemma. $\square$

We now average the pre-classifying maps (2.82) over the annuli $B_i \setminus D_i$, as follows. Denote by $F(\ell, n)$ the non-compact Stiefel manifold of $\ell$-frames in $\mathbb{C}^n$, thinking of such frames as $n$-by-$\ell$ matrices whose column vectors form a linearly independent set in $\mathbb{C}^n$. In this way $F(\ell, n)$ becomes an open submanifold of $M_{n,\ell}(\mathbb{C})$ which can be expressed by

$$F(\ell, n) = \big\{F \in M_{n,\ell}(\mathbb{C}) \,\big|\, F^*F \in U(\ell)\big\}$$

and which contains the compact Stiefel manifold $V(\ell, n)$ as a proper submanifold. We fix finitely many open convex cones $C_1, \ldots, C_b$ in $F(\ell, n)$ such that the sets $C_j \cap V(\ell, n)$ form an open covering of $V(\ell, n)$. We further fix a smooth partition of unity

$$\varrho_j \colon V(\ell, n) \to [0, 1], \qquad j = 1, \ldots, b$$

subordinate to this covering. Then we define averaged pre-classifying maps

$$F_j^i \colon \mathcal{B} \times P|_{D_i} \to \{0\} \cup F(\ell, n), \quad i = 1, \ldots, a,\ j = 1, \ldots, b \tag{2.85}$$

by

$$F_j^i(A, u; p) := \int_{B_i \setminus D_i} \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z')\big) \cdot \Phi^i(A, u; p, z') \,\mathrm{dvol}_\Sigma(z'). \tag{2.86}$$

CLAIM. *The maps (2.85) are well-defined.*

PROOF OF CLAIM. Fix $(A, u; p) \in \mathcal{B} \times P|_{D_i}$ and consider the map

$$B_i \setminus D_i \ni z \mapsto \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z')\big) \cdot \Phi^i(A, u; p, z') \in \overline{C_j}, \tag{2.87}$$

where $\overline{C_j}$ denotes the closure of the cone $C_j$. By Proposition 2.3.2 (i) the map (2.87) is continuous. This ensures that the integral on the right-hand-side in the definition of $F_j^i(A, u; p)$ converges. This integral yields an $n$-by-$\ell$ matrix $F_j^i(A, u; p) \in M_{n,\ell}(\mathbb{C})$. It remains to check that this matrix is in fact zero or a frame. But this is immediate: since the map (2.87) takes values in the closure of the convex cone $C_j$ we conclude that $F_j^i(A, u; p)$ must also be contained in the closure of $C_j$, that is, it is zero or a frame. $\square$

Next, patching together the averaged pre-classifying maps (2.85) we define maps

$$F^i\colon \mathcal{B}\times P|_{D_i}\to F(\ell,bn), \qquad i=1,\dots,a \tag{2.88}$$

by

$$F^i(A,u;p) := \begin{pmatrix} F^i_1(A,u;p)\\ \vdots \\ F^i_j(A,u;p)\\ \vdots \\ F^i_b(A,u;p)\end{pmatrix}. \tag{2.89}$$

Claim. *The maps (2.88) are well-defined.*

Proof of Claim. We have to check that the matrix $F^i(A,u;p)$ is a frame in $\mathbb{C}^{bn}$. Since each matrix $F^i_j(A,u;p)$ is either zero or a frame in $\mathbb{C}^n$, it will be enough to check that $F^i_{j_0}(A,u;p)\neq 0$ for some $j_0$. To this end, we recall from Lemma 2.3.5 that there exists an open subset $V^i_u\subset B_i\setminus D_i$ such that $|\mu(u)(z')|\le \delta/2$ for all $z'\in V^i_u$. It then follows from the definition of the pre-clssifying map (2.82) that

$$\sum_{j=1}^b \varrho_j\big(\theta(u(\sigma_i(z')))\cdot \mathrm{hol}^A_i(p,z')\big)\cdot \Phi^i(A,u;p,z') = \theta(u(\sigma_i(z')))\cdot \mathrm{hol}^A_i(p,z')\in V(\ell,n).$$

This identity shows that there exists $j_0$ and an open subset $V^{ij_0}_u\subset V^i_u$ such that for all $z'\in V^{ij_0}_u$ the matrix

$$\varrho_{j_0}\big(\theta(u(\sigma_i(z')))\cdot \mathrm{hol}^A_i(p,z')\big)\cdot \Phi^i(A,u;p,z')$$

is an element of the cone $C_{j_0}$. By convexity of $C_{j_0}$ it follows that

$$F^i_{j_0}(A,u;p) = \int_{B_i\setminus D_i} \varrho_{j_0}\big(\theta(u(\sigma_i(z')))\cdot \mathrm{hol}^A_i(p,z')\big)\cdot \Phi^i(A,u;p,z')\ \mathrm{dvol}_\Sigma(z')\in C_{j_0},$$

whence $F^i_{j_0}(A,u;p)\neq 0$. $\square$

We now apply the Gram-Schmidt process in order to modify the maps (2.88) so as to take values in the compact $G$-space $V(\ell,bn)$. Recall that the Gram-Schmidt process determines a $G$-equivariant smooth retraction map

$$\Pi_{\mathrm{GS}}\colon F(\ell,bn)\to V(\ell,bn).$$

We may then consider maps

$$\Pi_{\mathrm{GS}}\circ F^i\colon \mathcal{B}\times P|_{D_i}\to V(\ell,bn), \qquad i=1,\dots,a. \tag{2.90}$$

Next we patch together these maps in order to obtain a map that is defined on $\mathcal{B}\times P$. To this end we choose smooth bump functions

$$\lambda_i\colon \Sigma\to[0,1], \qquad i=1,\dots,a$$

subordinate to the covering $\{D_1, \dots, D_a\}$ and such that

$$\overline{\mathrm{supp}(\lambda_i)} \subset D_i \quad \text{and} \quad \sum_{i=1}^{a} \lambda_i^2 = 1. \tag{2.91}$$

Then we define a map

$$\Theta \colon \mathcal{B} \times P \to V(\ell, abn) \tag{2.92}$$

by setting

$$\Theta(A, u; p) := \begin{pmatrix} \lambda_1(\pi(p)) \cdot \Pi_{\mathrm{GS}}\big(F^1(A, u; p)\big) \\ \vdots \\ \lambda_i(\pi(p)) \cdot \Pi_{\mathrm{GS}}\big(F^i(A, u; p)\big) \\ \vdots \\ \lambda_a(\pi(p)) \cdot \Pi_{\mathrm{GS}}\big(F^a(A, u; p)\big) \end{pmatrix}. \tag{2.93}$$

Claim. *The map (2.92) is well-defined.*

Proof of Claim. We have to check that the matrix $\Theta(A, u; p)$ is a unitary $\ell$-frame in $\mathbb{C}^{abn}$. Since each matrix $\Pi_{\mathrm{GS}}\big(F^i(A, u; p)\big)$ is a unitary $\ell$-frame in $\mathbb{C}^{bn}$, this is an immediate consequence of the normalization property (2.91) of the patching functions $\lambda_i$. $\square$

Recall that we are assuming $G = U(\ell)$. In this case we may choose the universal $G$-bundle $EG \to BG$ to be the $U(\ell)$-bundle

$$V(\ell, \infty) \to Gr(\ell, \infty),$$

where $V(\ell, \infty)$ denotes the infinite Stiefel manifold of unitary $\ell$-frames and $Gr(\ell, \infty)$ is the infinite Grassmannian. If we set $N := abn - 1$, the Stiefel manifold $V(\ell, abn)$ then appears as the finite dimensional approximation $EU(\ell)^N$ to $EU(\ell)$ (see Husemoller [**19**], Ch. 4, Sec. 11). In more invariant terms we may hence write the map (2.92) in the form

$$\Theta \colon \mathcal{B} \times P \to EG^N. \tag{2.94}$$

Lemma 2.3.7. *The map (2.94) is equivariant with respect to the $G$-action (2.84b) and the standard right action of $G$ on $EG^N$.*

Proof. This follows from the definition of the maps (2.92), (2.90), (2.85), (2.82) together with Lemma 2.3.6. $\square$

By Lemma 2.3.7 we may write the map (2.94) as a map

$$\Theta \colon \mathcal{B} \to \mathrm{Map}\big(P, EG^N\big)^G, \qquad (A, u) \mapsto \Theta_{(A,u)} \tag{2.95}$$

which assigns to every pair $(A, u) \in \mathcal{B}$ a $G$-equivariant map $\Theta_{(A,u)} \colon P \to EG^N$. It has the following basic properties.

LEMMA 2.3.8. *The map (2.95) has the following properties.*

(i) *It is equivariant with respect to the actions (2.2) and (2.5) of the group of gauge transformations $\mathcal{G}$.*
(ii) *For every pair $(A, u) \in \mathcal{B}$ the map $\Theta_{(A,u)} \colon P \to EG^N$ is of class $C^1$.*

PROOF. Proof of (i): By Lemma 2.3.6 the map (2.94) is invariant under the action (2.84a) of the group of gauge transformations $\mathcal{G}$. Since the $\mathcal{G}$-action (2.84a) and the $G$-action (2.84b) commute, this implies that the map

$$\Theta \colon \mathcal{B} \to \mathrm{Map}\big(P, EG^N\big)^G$$

is $\mathcal{G}$-equivariant with respect to the actions (2.2) and (2.5). In fact, for $g \in \mathcal{G}$ we have

$$\begin{aligned}
\Theta_{(g^*A, g^{-1}u)}(p) &= \Theta\big(g^*A, g^{-1}u; p\big)\\
&= \Theta\big(g^*A, g^{-1}u; pg(p)\, g(pg(p))^{-1}\big)\\
&= \Theta(A, u; pg(p))\\
&= \Theta_{(A,u)}(pg(p))\\
&= \big(\Theta_{(A,u)}\, g\big)(p).
\end{aligned}$$

Proof of (ii): Recalling the definitions of the maps (2.92), (2.90), (2.85) and (2.82) we see that assertion (ii) follows from Proposition 2.3.2 (i) since $(A, u) \in \mathcal{B}$ is of class $C^1$. □

By Lemma 2.3.8 we may consider the map (2.94) as a $\mathcal{G}$-equivariant map

$$\Theta \colon \mathcal{B} \to C^1\big(P, EG^N\big)^G. \tag{2.96}$$

It remains to prove that this map extends to a $\mathcal{G}^{2,p}$-equivariant map

$$\Theta \colon \mathcal{B}^{1,p} \to W^{1,p}\big(P, EG^N\big)^G$$

on the configuration space $\mathcal{B}^{1,p}$, which will then be a classifying map in the sense of Section 2.1.2. But this follows immediately once we have proved that the map (2.96) is continuous with respect to the $W^{1,p}$-topologies on $\mathcal{B}$ and $C^1(P, EG^N)^G$. This is the content of the next proposition.

PROPOSITION 2.3.9. *The map (2.96) is continuous with respect to the $W^{1,p}$-topologies on $\mathcal{B}$ and on $C^1(P, EG^N)^G$.*

PROOF. First of all, we briefly recall the definition of the $W^{1,p}$-topologies on the spaces $\mathcal{B}$ and $C^1(P, EG^N)^G$.

Since the taming condition (2.74) is an open condition, the space $\mathcal{B}$ inherits its $W^{1,p}$-topology from the $W^{1,p}$-topology on the product space $\mathcal{A}(P) \times C^1(P, M)^G$. The $W^{1,p}$-topology on $\mathcal{A}(P)$ is defined in terms of the $W^{1,p}$-norm on the space $C^1(\Sigma, T^*\Sigma \otimes P(\mathfrak{g}))$, where $P(\mathfrak{g}) = P \times_G \mathfrak{g}$ is the adjoint bundle, by means of the decomposition

$$\mathcal{A}(P) = A_0 + C^1(\Sigma, T^*\Sigma \otimes P(\mathfrak{g}))$$

for a fixed smooth reference connection $A_0$ on $P$. The $W^{1,p}$-topology on $C^1(P,M)^G$ is defined in terms of a fixed $G$-invariant metric on $P$. However, in the present situation it will be useful to to make this more concrete by passing to a linear target space for the maps $P \to M$. More precisely, by the equivariant Nash embedding theorem [**23**] there exists a Euclidean vector space $V$ of sufficiently large dimension that is equipped with an action of $G$ by isometries, together with an isometric $G$-equivariant embedding of $P$ into $V$. The $W^{1,p}$-topology on $C^1(P,M)^G$ is then induced from the $W^{1,p}$-topology on $C^1(P,V)^G \cong C^1(\Sigma, P \times_G V)$ via the inclusion

$$C^1(P,M)^G \hookrightarrow C^1(P,V)^G. \tag{2.97}$$

The $W^{1,p}$-topology on the space $C^1(P,EG^N)^G$ is defined as follows. Recall that we are assuming $G = U(\ell)$, whence the finite dimensional approximation $EG^N$ of $EG$ is given by the Stiefel manifold $V(\ell, N+1)$ of unitary $\ell$-frames in $C^{N+1}$, for $N$ sufficiently large. Thus $EG^N$ is $G$-equivariantly embedded into the linear space $M_{N+1,\ell}(\mathbb{C})$ of complex $N+1$ by $\ell$ matrices, hence we have an inclusion

$$C^1\big(P, EG^N\big)^G \hookrightarrow C^1\big(P, M_{N+1,\ell}(\mathbb{C})\big)^G. \tag{2.98}$$

The $W^{1,p}$-topology on the space $C^1(P,EG^N)^G$ is now induced from the $W^{1,p}$-topology on $C^1(P,M_{N+1,\ell}(\mathbb{C}))^G \cong C^1(\Sigma, P\times_G M_{N+1,\ell}(\mathbb{C}))$ via this inclusion. In particular, we may consider the classifying map (2.97) as a map

$$\Theta\colon \mathcal{B} \to C^1\big(P, M_{N+1,\ell}(\mathbb{C})\big)^G$$

which assigns to a pair $(A,u)$ of class $C^1$ a $G$-equivariant map $\Theta_{(A,u)}\colon P \to M_{N+1,\ell}(\mathbb{C})$ of class $C^1$. Note again that, as in the case of $C^1(P,M)^G$, we are passing to a linear target space for the maps $\Theta_{(A,u)}$ since the estimates we shall be dealing with later on will most conveniently be formulated in terms of a linear structure on the space $C^1(P,EG^N)^G$.

Let us now consider a sequence of pairs $(A_\nu, u_\nu) \in \mathcal{B}$ that converges to $(A,u) \in \mathcal{B}$ in the $W^{1,p}$-topology. We have to prove that the corresponding sequence of classifying maps $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_{(A,u)}$ in the $W^{1,p}$-topology on $C^1(P,EG^N)^G$.

Using the inclusion (2.98) we may consider the sequence of norms

$$\big\|\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big\|_{W^{1,p}} = \big\|\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big\|_{L^p} + \big\|\nabla_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big)\big\|_{L^p} \tag{2.99}$$

for some fixed smooth reference connection $A_0 \in \mathcal{A}(P)$ on $P$ (see Remark 2.1.4). We prove that this sequence tends to zero as $\nu \to \infty$. In fact, this will be a consequence of the following two claims, as we shall now explain. The proofs of these claims will be deferred to the end of the current proof of Proposition 2.3.9.

CLAIM 1. *The map (2.96) is continuous with respect to the $C^0$-topologies on $\mathcal{B}$ and on $C^1\big(P,EG^N\big)^G$.*

CLAIM 2. *There exists a constant $C_1 > 0$ such that for all $\nu$ we have*

$$\big\|\mathrm{d}_A\Theta_{(A,u)} - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}\big\|_{L^p} \leq C_1 \cdot \big\|F_A - F_{A_\nu}\big\|_{L^p}.$$

To see how these claims imply that sequence (2.99) converges to zero first note that, by Rellich's theorem ([**34**], Thm. B.2 (iii)), the sequence $(A_\nu, u_\nu)$ converges to $(A, u)$ in the $C^0$-topology. Hence by Claim 1 the sequence $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_{(A,u)}$ in the $C^0$-topology as well. But this implies that

$$\left\|\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\right\|_{L^p} \to 0 \tag{2.100}$$

as $\nu \to \infty$. In order to estimate the second term on the right hand side of (2.99) we first recall that (see Remark 2.1.4)

$$\begin{aligned}\nabla_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big) &= \mathrm{d}_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big)\\ &= \mathrm{d}_A\Theta_{(A,u)} - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)} + X_{A_0-A}(u) - X_{A_0-A_\nu}(u_\nu).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\left\|\nabla_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big)\right\|_{L^p} \leq \left\|\mathrm{d}_A\Theta_{(A,u)} - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}\right\|_{L^p}&\\ + \left\|X_{A_0-A}(u) - X_{A_0-A_\nu}(u_\nu)\right\|_{L^p}&.\end{aligned}$$

Since $u$ and $u'$ evaluate into the linear space $V$ by assumption, there exists a constant $C_2 > 0$ such that for all $\nu$ we have an estimate

$$\left\|X_{A_0-A}(u) - X_{A_0-A_\nu}(u_\nu)\right\|_{L^p} \leq C_2 \cdot \left\|A - A_\nu\right\|_{L^p}.$$

Hence by Claim 2 we conclude

$$\left\|\nabla_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big)\right\|_{L^p} \leq C_1 \cdot \left\|F_A - F_{A_\nu}\right\|_{L^p} + C_2 \cdot \left\|A - A_\nu\right\|_{L^p}.$$

Since $A_\nu$ converges to $A$ in the $W^{1,p}$-topology, it follows that $F_{A_\nu}$ converges to $F_A$ in the $L^p$-topology (here we use that $p > 2$), whence

$$\left\|\nabla_{A_0}\big(\Theta_{(A,u)} - \Theta_{(A_\nu,u_\nu)}\big)\right\|_{L^p} \to 0$$

as $\nu \to \infty$. Combining this with (2.100) we finally conclude that sequence (2.99) converges to zero.

Proof of Claim 1. A careful inspection of formulas (2.83), (2.86), (2.89) and (2.93) appearing in the construction of the classifying map (2.96) at the beginning of this subsection shows that the classifying map is continuous with respect to the $C^0$-topology. We emphasize that this argument crucially relies on Proposition 2.3.4 (i), which ensures that the relative reduced holonomy map (2.32) is continuous with respect to the $C^0$-topology. $\square$

Proof of Claim 2. It follows from formulas (2.89) and (2.93) that there exists a constant $c_1 > 0$ (not depending on $\nu$) such that for all points $z \in \Sigma$ and all tangent vectors $v_z \in T_z\Sigma$ we have

$$\begin{aligned} &\left|\mathrm{d}_A\Theta_{(A,u)}(v_z) - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}(v_z)\right| \\ &\qquad \le c_1 \cdot \sum_{\substack{i,j \\ z\in D_i}} \left|\sigma_i^*\big(\mathrm{d}_A F_j^i(A,u;\cdot)\big)(v_z) - \sigma_i^*\big(\mathrm{d}_{A_\nu} F_j^i(A_\nu,u_\nu;\cdot)\big)(v_z)\right|, \end{aligned} \tag{2.101}$$

where the sum runs over all pairs $(i,j)$ such that the point $z$ is contained in the disk $D_i$ and $\sigma\colon B_i \to P|_{B_i}$ denotes a fixed section for each $i$. Recall from (2.83) and (2.86) that

$$\begin{aligned} F_j^i(A,u;p) = \int_{B_i\setminus D_i} &\varrho\big(u(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p,z')\big) \\ &\cdot \theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p,z')\ \mathrm{dvol}_\Sigma(z') \end{aligned}$$

and

$$\begin{aligned} F_j^i(A_\nu,u_\nu;p) = \int_{B_i\setminus D_i} &\varrho\big(u_\nu(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u_\nu(\sigma_i(z'))) \cdot \mathrm{hol}_i^{A_\nu}(p,z')\big) \\ &\cdot \theta(u_\nu(\sigma_i(z'))) \cdot \mathrm{hol}_i^{A_\nu}(p,z')\ \mathrm{dvol}_\Sigma(z') \end{aligned}$$

for $p \in P|_{D_i}$. We conclude from these formulas that there exist constants $c_2^{ij} > 0$ (not depending on $\nu$) such that

$$\begin{aligned} &\left|\sigma_i^*\big(\mathrm{d}_A F_j^i(A,u;\cdot)\big)(v_z) - \sigma_i^*\big(\mathrm{d}_{A_\nu} F_j^i(A_\nu,u_\nu;\cdot)\big)(v_z)\right| \\ &\qquad \le c_2^{ij} \cdot \int_{B_i\setminus D_i} \left|\omega_{\mathrm{MC}}\Big(\sigma_i^*\big(\mathrm{d}_A \mathrm{hol}_{i,z'}^A\big)(v_z) - \sigma_i^*\big(\mathrm{d}_{A_\nu}\mathrm{hol}_{i,z'}^{A_\nu}\big)(v_z)\Big)\right|_{\mathfrak{g}} \mathrm{dvol}_\Sigma(z') \end{aligned}$$

for all points $z \in D_i$ and all $v_z \in T_z\Sigma$, where $\omega_{\mathrm{MC}} \in \Omega^1(G;\mathfrak{g})$ denotes the Maurer-Cartan form on $G$. Here we used that the reduced holonomy maps (2.80) have compact target $G$. Combining this with estimate (2.101) and passing to operator norms, it follows that there exists a constant $c_3 > 0$ (not depending on $\nu$) such that

$$\begin{aligned} &\left|\mathrm{d}_A\Theta_{(A,u)}(z) - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}(z)\right| \\ &\qquad \le c_3 \cdot \sum_{\substack{i \\ z\in D_i}} \int_{B_i\setminus D_i} \left|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}_{i,z'}^A\big)(z) - \sigma_i^*\big(\mathrm{d}_{A_\nu}\mathrm{hol}_{i,z'}^{A_\nu}\big)(z)\right| \mathrm{dvol}_\Sigma(z') \end{aligned} \tag{2.102}$$

for all points $z \in \Sigma$, where the sum runs over all $i$ such that $z$ is contained in the disk $D_i$.

Fix a real number $p > 2$. Applying Hölder's inequality in combination with Fubini's theorem, it follows from (2.102) that

$$\begin{aligned}\big\|\mathrm{d}_A\Theta_{(A,u)} - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}\big\|_{L^p(\Sigma)} \\ \leq c_3 \cdot \sum_{i=1}^{a} \int_{B_i\setminus D_i} \big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big) - \sigma_i^*\big(\mathrm{d}_{A_\nu}\mathrm{hol}^{A_\nu}_{i,z'}\big)\big\|_{L^p(D_i)}\,\mathrm{dvol}_\Sigma(z').\end{aligned} \tag{2.103}$$

Here we used that $\Sigma$ is covered by the disks $D_i$ for $i = 1, \dots, a$. Now in complete analogy to statement (ii) of Proposition 2.3.4 we have an estimate

$$\big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big) - \sigma_i^*\big(\mathrm{d}_{A_\nu}\mathrm{hol}^{A_\nu}_{i,z'}\big)\big\|_{L^p(D_i)} \leq c_4 \cdot \big\|F_A - F_{A_\nu}\big\|_{L^p(\Sigma)}$$

for some constant $c_4 > 0$ (not depending on $\nu$). Hence we obtain from (2.103) an estimate

$$\big\|\mathrm{d}_A\Theta_{(A,u)} - \mathrm{d}_{A_\nu}\Theta_{(A_\nu,u_\nu)}\big\|_{L^p(\Sigma)} \leq C_1 \cdot \big\|F_A - F_{A_\nu}\big\|_{L^p(\Sigma)}$$

for some constant $C_1 > 0$ (not depending on $\nu$). This proves Claim 2. □

The proof of Proposition 2.3.9 is now complete. □

**2.3.3. Proof of Theorem 2.1.5.** We prove that the classifying map

$$\Theta\colon \mathcal{B}^{1,p} \to W^{1,p}\big(P, EG^N\big)^G \tag{2.104}$$

defined in the previous subsection satisfies the regularity axioms of Definition 2.1.3. Recall that this map was obtained by continuous extension from an explicitly constructed classifying map

$$\Theta\colon \mathcal{B} \to C^1\big(P, EG^N\big)^G \tag{2.105}$$

that is defined on the dense subspace $\mathcal{B} \subset \mathcal{B}^{1,p}$ of pairs of class $C^1$.

Proof of **(Finiteness)**: This axiom is satisfied by construction of the map (2.104) in Section 2.3.2.

Proof of **(Regularity)**: Proof of (i): This property is satisfied by construction of the map (2.104) in Section 2.3.2.

Proof of (ii): A careful inspection of formulas (2.83), (2.86), (2.89) and (2.93) appearing in the construction of the classifying map (2.104) in Section 2.3.2 shows that this property is a consequence of the corresponding regularity result for the systems of holonomy maps (2.79), which holds by Proposition 2.3.2 (i).

Proof of (iii): Again, formulas (2.83), (2.86), (2.89) and (2.93) show that the classifying map (2.104) is in fact Fréchet differentiable. Here we use Proposition B.1.20 in [**22**].

This proves (Regularity).

Proof of **(Continuity)**: The configuration space $\mathcal{B}$ of pairs of class $C^1$ is dense in $\mathcal{B}^{1,p}$, whence it suffices to verify that (Continuity) holds for the classifying map (2.105).

Let $Z$ be a finite subset of $\Sigma$ and consider a sequence $(A_\nu, u_\nu)$ in $\mathcal{B}$ that converges to $(A, u) \in \mathcal{B}$ in the following sense.

(a) $A_\nu$ converges to $A$ in the $C^0$-topology on $\Sigma$;
(b) $u_\nu$ converges to $u$ in the $C^0$-topology on compact subsets of $\Sigma \setminus Z$.

We shall prove that the sequence $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_{(A,u)}$ in the $C^0$-topology on $\Sigma$.

First of all, it follows from formulas (2.83), (2.86), (2.89) and (2.93) appearing in the construction of the classifying map (2.105) that it will be enough to prove that, for every pair $(i, j)$, the sequence of maps

$$P|_{D_i} \ni p \mapsto F^i_j(A_\nu, u_\nu; p)$$

converges to the map

$$P|_{D_i} \ni p \mapsto F^i_j(A, u; p)$$

in the $C^0$-topology. This in turn will follow by compactness of the disks $D_i$ once we have proved that

$$F^i_j(A_\nu, u_\nu; p) \to F^i_j(A, u; p)$$

in the space of matrices $M_{n,\ell}(\mathbb{C})$ (with respect to the standard Euclidean norm thereon) for every point $p \in P|_{D_i}$.

In order to prove this we fix a point $p \in P|_{D_i}$ and define functions

$$f^\nu_p, f_p \colon B_i \backslash D_i \to M_{n,\ell}(\mathbb{C})$$

by

$$f^\nu_p(z') := \varrho\big(u_\nu(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u_\nu(\sigma_i(z'))) \cdot \mathrm{hol}^{A_\nu}_i(p, z')\big) \cdot \theta(u_\nu(\sigma_i(z'))) \cdot \mathrm{hol}^{A_\nu}_i(p, z')$$

and

$$f_p(z') := \varrho\big(u(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}^{A}_i(p, z')\big) \cdot \theta(u(\sigma_i(z'))) \cdot \mathrm{hol}^{A}_i(p, z').$$

The definition of these functions is motivated by formulas (2.83) and (2.86). Since the classifying map $\theta$ introduced in (2.81) takes values in the compact Stiefel manifold, the functions $f^\nu_p$ are uniformly bounded in $M_{n,\ell}(\mathbb{C})$. Moreover, by Proposition 2.3.4 (i) and assumption (a) above,

$$\mathrm{hol}^{A_\nu}_i(p, z') \to \mathrm{hol}^{A}_i(p, z')$$

in $U(\ell)$ for every $z' \in B_i \backslash D_i$; by assumption (b),

$$u_\nu(\sigma_i(z')) \to u(\sigma_i(z'))$$

in $M$ for almost every $z' \in B_i \backslash D_i$. Hence we obtain

$$f^\nu_p(z') \to f_p(z')$$

for almost every $z' \in B_i \backslash D_i$. Then Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{\nu\to\infty} F_j^i(A_\nu, u_\nu; p) = \lim_{\nu\to\infty} \int_{B_i\backslash D_i} f_p^\nu \ \mathrm{dvol}_\Sigma = \int_{B_i\backslash D_i} f_p \ \mathrm{dvol}_\Sigma = F_j^i(A, u; p).$$

This proves (Continuity).

Proof of **(Estimates)**: First of all, note that since $\mathcal{B}$ is dense in $\mathcal{B}^{1,p}$, it will be sufficient to prove estimates (i)–(iii) for the classifying map (2.105).

Let $(A, u) \in \mathcal{B}$. Then the map $\Theta_{(A,u)}$ is of class $C^1$ by (Regularity, ii) above.

Proof of (i): Since $\Theta_{(A,u)}$ has compact target $EG^N$ it will be enough to prove that

$$\big\|\mathrm{d}_A\Theta_{(A,u)}\big\|_{L^p} \leq c_1 \cdot \Big(1 + \big\|F_A\big\|_{L^p}\Big) \tag{2.106}$$

for some constant $c_1 > 0$ (not depending on $(A, u)$).

Recall from Section 2.3.2 the construction of the classifying map (2.105). It follows from formulas (2.89) and (2.93) that there exists a constant $c_2 > 0$ (not depending on $(A, u)$) such that for all points $z \in \Sigma$ and all tangent vectors $v_z \in T_z\Sigma$ we have

$$\big|\mathrm{d}_A\Theta_{(A,u)}(v_z)\big| \leq c_2 \cdot \Big(|v_z| + \sum_{\substack{i,j\\ z\in D_i}} \big|\sigma_i^*\big(\mathrm{d}_A F_j^i(A, u; \cdot)\big)(v_z)\big|\Big), \tag{2.107}$$

where the sum runs over all pairs $(i, j)$ such that the point $z$ is contained in the disk $D_i$. Here the $\sigma_i \colon D_i \to P|_{D_i}$ denote fixed sections. Recall from (2.83) and (2.86) that

$$F_j^i(A, u; p) = \int_{B_i\backslash D_i} \varrho\big(u(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z')\big) \\ \cdot \theta(u(\sigma_i(z'))) \cdot \mathrm{hol}_i^A(p, z') \ \mathrm{dvol}_\Sigma(z')$$

for $p \in P|_{D_i}$. We conclude from this formula that there exist constants $c_3^{ij} > 0$ (not depending on $(A, u)$) such that

$$\big|\sigma_i^*\big(\mathrm{d}_A F_j^i(A, u; \cdot)\big)(v_z)\big| \leq c_3^{ij} \cdot \int_{B_i\backslash D_i} \big|\omega_{\mathrm{MC}}\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}_{i,z'}^A\big)(v_z)\big)\big|_{\mathfrak{g}} \,\mathrm{dvol}_\Sigma(z')$$

for all points $z \in D_i$ and all tangent vectors $v_z \in T_z\Sigma$, where $\omega_{\mathrm{MC}} \in \Omega^1(G; \mathfrak{g})$ denotes the Maurer-Cartan form on $G$. Here we used that the reduced holonomy maps (2.80) have compact target $G$. Combining this with estimate (2.107) and passing to operator norms, it follows that there exists a constant $c_4 > 0$ (not depending on $(A, u)$) such that

$$\big|\mathrm{d}_A\Theta_{(A,u)}(z)\big| \leq c_4 \cdot \Bigg(1 + \sum_{\substack{i\\ z\in D_i}} \int_{B_i\backslash D_i} \big|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}_{i,z'}^A\big)(z)\big| \,\mathrm{dvol}_\Sigma(z')\Bigg) \tag{2.108}$$

for all points $z \in \Sigma$, where the sum runs over all $i$ such that $z$ is contained in the disk $D_i$.

Fix a real number $p > 2$. Applying Hölder's inequality in combination with Fubini's theorem, it follows from (2.108) that

$$\begin{aligned}\big\|\mathrm{d}_A\Theta_{(A,u)}\big\|_{L^p(\Sigma)} \leq c_4 \cdot \Bigg(&\big\|1\big\|_{L^p(\Sigma)} \\ &+ \sum_{i=1}^{a} \int_{B_i\setminus D_i} \big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big\|_{L^p(D_i)}\,\mathrm{dvol}_\Sigma(z')\Bigg). \quad (2.109)\end{aligned}$$

Here we used that $\Sigma$ is covered by the disks $D_i$ for $i = 1, \ldots, a$. By Proposition 2.3.4 (ii) we have

$$\big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big\|_{L^p(D_i)} \leq c_5 \cdot \big\|F_A\big\|_{L^p(\Sigma)}$$

for some constant $c_5 > 0$ (not depending on $(A,u)$). Hence we obtain from (2.109) the desired estimate (2.106). This proves (i).

Proof of (ii): By inequality (2.108) in the proof of (i) above there exists a constant $c_4 > 0$ (not depending on $(A,u)$) such that

$$\big|\mathrm{d}_A\Theta_{(A,u)}(z)\big| \leq c_4 \cdot \Bigg(1 + \sum_{\substack{i\\ z\in D_i}} \int_{B_i\setminus D_i} \big|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(z)\big|\,\mathrm{dvol}_\Sigma(z')\Bigg)$$

for all points $z \in \Sigma$, where the sum runs over all $i$ such that $z$ is contained in the disk $D_i$. It follows from this that

$$\big\|\mathrm{d}_A\Theta_{(A,u)}\big\|_{L^\infty(\Sigma)} \leq c_4 \cdot \Bigg(1 + \sum_{i=1}^{a} \int_{B_i\setminus D_i} \big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big\|_{L^\infty(D_i)}\,\mathrm{dvol}_\Sigma(z')\Bigg). \quad (2.110)$$

Here we used that $\Sigma$ is covered by the disks $D_i$ for $i = 1, \ldots, a$. By Proposition 2.3.4 (iii) we have

$$\big\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big\|_{L^\infty(D_i)} \leq c_6 \cdot \big\|F_A\big\|_{L^\infty(\Sigma)}$$

for some constant $c_6 > 0$ (not depending on $(A,u)$). Hence we obtain from (2.110) that

$$\big\|\mathrm{d}_A\Theta_{(A,u)}\big\|_{L^\infty(\Sigma)} \leq C' \cdot \Big(1 + \big\|F_A\big\|_{L^\infty(\Sigma)}\Big).$$

for some constant $C' > 0$ (not depending on $(A,u)$). This proves (ii).

Proof of (iii): Assume that $A$ is of class $C^2$. Then $\Theta_{(A,u)}$ is of class $C^2$ by (Regularity, ii). Recall from Section 2.3.2 the construction of the classifying map (2.105). It follows from formulas (2.89) and (2.93) that there exists a constant $c_7 > 0$ (not depending on $(A,u)$) such that for all points $z \in \Sigma$ and all tangent vectors $w_z, v_z \in T_z\Sigma$ we have

$$\begin{aligned}\big|\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(w_z,v_z)\big| \leq c_7 \cdot \Bigg(&|w_z|\cdot|v_z| + |w_z|\cdot\big|\mathrm{d}_A\Theta_{(A,u)}(v_z)\big| \\ &+ \sum_{\substack{i,j\\ z\in D_i}} \big|\big(\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A F^i_j(A,u;\cdot)\big)\big)\big)(w_z,v_z)\big|\Bigg), \quad (2.111)\end{aligned}$$

where the sum runs over all pairs $(i,j)$ such that the point $z$ is contained in the disk $D_i$. In order to estimate this further recall that by (2.108) above we have

$$\left|\mathrm{d}_A\Theta_{(A,u)}(v_z)\right| \leq c_4 \cdot |v_z| \cdot \left(1 + \sum_{\substack{i\\ z\in D_i}} \int_{B_i\setminus D_i} \left|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(z)\right| \mathrm{dvol}_\Sigma(z')\right). \tag{2.112}$$

Recall moreover from (2.83) and (2.86) that

$$F^i_j(A,u;p) = \int_{B_i\setminus D_i} \varrho\big(u(\sigma_i(z'))\big) \cdot \varrho_j\big(\theta(u(\sigma_i(z'))) \cdot \mathrm{hol}^A_i(p,z')\big) \\ \cdot \theta(u(\sigma_i(z'))) \cdot \mathrm{hol}^A_i(p,z') \; \mathrm{dvol}_\Sigma(z')$$

for $p \in P|_{D_i}$. We conclude from this formula that there exist constants $c_8^{ij} > 0$ (not depending on $(A,u)$) such that

$$\left|\big(\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A F^i_j(A,u;\cdot)\big)\big)\big)(w_z,v_z)\right| \\ \leq c_8^{ij} \cdot \int_{B_i\setminus D_i} \Bigg( \left|\omega_{\mathrm{MC}}\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(w_z)\big)\right|_{\mathfrak{g}} \cdot \left|\omega_{\mathrm{MC}}\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(v_z)\big)\right|_{\mathfrak{g}} \\ + \left|\omega_{\mathrm{MC}}\Big(\big(\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big)\big)(w_z,v_z)\Big)\right|_{\mathfrak{g}}\Bigg) \mathrm{dvol}_\Sigma(z') \tag{2.113}$$

for all $z \in D_i$ and all $w_z, v_z \in T_z\Sigma$, where $\omega_{\mathrm{MC}} \in \Omega^1(G;\mathfrak{g})$ denotes the Maurer-Cartan form on $G$. Here we used that the reduced holonomy maps (2.80) have compact target $G$. Combining (2.112) and (2.113) with (2.111) and passing to operator norms, it follows that there exists a constant $c_9 > 0$ (not depending on $(A,u)$) such that

$$\left|\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(z)\right| \leq c_9 \cdot \Bigg(1 + \sum_{\substack{i\\ z\in D_i}} \int_{B_i\setminus D_i} \Big(\left|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(z)\right| \\ + \left|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)(z)\right|^2 + \left|\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big)(z)\right|\Big) \mathrm{dvol}_\Sigma(z')\Bigg)$$

for all points $z \in \Sigma$, where the sum runs over all $i$ such that $z$ is contained in the disk $D_i$. We may further estimate this by

$$\left|\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(z)\right| \leq c_9 \cdot \Bigg(1 + \sum_{i=1}^{a} \int_{B_i\setminus D_i} \Big(\left\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\right\|_{L^\infty(D_i)} \\ + \left\|\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\right\|^2_{L^\infty(D_i)}\Big) \mathrm{dvol}_\Sigma(z') \\ + \sum_{\substack{i\\ z\in D_i}} \int_{B_i\setminus D_i} \left|\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A\mathrm{hol}^A_{i,z'}\big)\big)(z)\right| \mathrm{dvol}_\Sigma(z')\Bigg). \tag{2.114}$$

Here we used that $\Sigma$ is covered by the disks $D_i$ for $i = 1, \ldots, a$. Recall from Proposition 2.3.4 (iii, iv) that

$$\left\|\sigma_i^*\big(\mathrm{d}_A \mathrm{hol}^A_{i,z'}\big)\right\|_{L^\infty(D_i)} \leq c_{10} \cdot \left\|F_A\right\|_{L^\infty(\Sigma)}$$

and

$$\begin{aligned}\left|\big(\nabla_A\big(\sigma_i^*\big(\mathrm{d}_A \mathrm{hol}^A_{z'}\big)\big)\big)(z)\right| \ \leq \ c_{10} \cdot \Bigg(1 + \left\|F_A\right\|^2_{L^\infty(\Sigma)} \\ + \int_0^1 \left|\big(\nabla_A(\sigma^* F_A)\big)(\gamma_{(z,z')}(\tau))\right| \mathrm{d}\tau\Bigg)\end{aligned}$$

for some constant $c_{10} > 0$ (not depending on $z$ and $(A,u)$). Here $\gamma_{(z,z')}$ denotes the geodesic introduced in (2.26) above, joining the points $z$ and $z'$. Plugging this into inequality (2.114) above we finally conclude that there exists a constant $c_{11} > 0$ (not depending $(A,u)$) such that

$$\begin{aligned}\left|\nabla_A\big(\mathrm{d}_A \Theta_{(A,u)}\big)(z)\right| \leq c_{11} \cdot \Bigg(1 + \left\|F_A\right\|^2_{L^\infty(\Sigma)} \\ + \sum_{\substack{i \\ z \in D_i}} \int_{B_i \setminus D_i} \left(\int_0^1 \left|\big(\nabla_A(\sigma^* F_A)\big)(\gamma_{(z,z')}(\tau))\right| \mathrm{d}\tau\right) \mathrm{dvol}_\Sigma(z')\Bigg) \quad (2.115)\end{aligned}$$

for all points $z \in \Sigma$.

In order to estimate the integral on the right-hand side of inequality (2.115) it will be most convenient to work in local polar geodesic coordinates around the point $z$ on $\Sigma$. More precisely, we denote by

$$\exp_z \colon T_z\Sigma \supset B_{2R}(0) \to B_{2R}(z), \qquad \exp_z(0) = z \tag{2.116}$$

the Riemannian exponential map on the tangent space of $\Sigma$ at $z$. It maps the Euclidean disk $B_{2R}(0)$ in the tangent space $T_z\Sigma \cong \mathbb{R}^2$ diffeomorphically onto the geodesic disk $B_{2R}(z)$ on $\Sigma$. Recall at this point that the constant $R$ was chosen to be smaller than half the injectivity radius of $\Sigma$; and that $B_i = B_R(z_i)$, $D_i = B_{R/2}(z_i)$ and $z \in D_i$. Hence the disk $B_i$ is contained in the chart $B_{2R}(z)$. We endow the punctured disk $B_{2R}(0) \setminus \{0\}$ with polar coordinates $(r, \varphi)$. For every angle $\varphi \in [0, 2\pi)$, let us denote by

$$I(\varphi) := \big\{r \in (0, 2R] \,\big|\, \exp_z(r,\varphi) \in B_i \setminus D_i\big\}$$

the set of all numbers $r$ such that the point $(r,\varphi) \in B_{2R}(0)$ gets mapped into the annulus $B_i \setminus D_i$ under the exponential map (2.116). We write the lift of the area form on $\Sigma$ under the exponential map in the form

$$\exp_z^* \mathrm{dvol}_\Sigma(z') = \lambda(r,\varphi)^2 \cdot r \,\mathrm{d}r \wedge \mathrm{d}\varphi \tag{2.117}$$

for some bounded smooth function $\lambda \colon B_{2R}(0) \setminus \{0\} \to (0, \infty)$, and denote

$$\widetilde{F}_A := (\sigma \circ \exp_z)^* F_A.$$

We may then express the integral on the right-hand side of inequality (2.115) in the form

$$\int_0^{2\pi}\int_{I(\varphi)}\left(\int_0^1\big|\nabla_A\widetilde{F}_A\big(\tilde{\gamma}_{(0,(r,\varphi))}(\tau)\big)\big|\,\mathrm{d}\tau\right)\cdot\lambda^2(r,\varphi)\cdot r\,\mathrm{d}r\,\mathrm{d}\varphi.$$

Here we think of $(r,\varphi)$ as the lift of the point $z'$ under the exponential map (2.116), and we further denote by

$$\tilde{\gamma}_{(0,(r,\varphi))}\colon [0,1]\to B_{2R}(0),\qquad \tau\mapsto(\tau\cdot r,\varphi)$$

the line segment joining the origin with the point $(r,\varphi)$ in $\mathbb{R}^2$. Then $\tilde{\gamma}_{(0,(r,\varphi))}$ is the lift of the geodesic $\gamma_{(z,z')}$. Using the reparametrization $\tau':=r\cdot\tau$ and the fact that $I(\varphi)\subset(0,2R]$, we may estimate this integral further from the above by

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2R}\left(\int_0^r\big|\nabla_A\widetilde{F}_A\big(\tilde{\gamma}_{(0,(r,\varphi))}(\tau'/r)\big)\big|\cdot\frac{1}{r}\,\mathrm{d}\tau'\right)\cdot\lambda^2(r,\varphi)\cdot r\,\mathrm{d}r\,\mathrm{d}\varphi\\
=\;&\int_0^r\left(\int_0^{2\pi}\left(\int_0^{2R}\frac{1}{\tau'}\cdot\big|\nabla_A\widetilde{F}_A\big(\tilde{\gamma}_{(0,(r,\varphi))}(\tau'/r)\big)\big|\cdot\tau'\,\mathrm{d}\tau'\right)\right.\\
&\qquad\qquad\left.\cdot\lambda^2(\tau',\varphi)\cdot\frac{\lambda^2(r,\varphi)}{\lambda^2(\tau',\varphi)}\,\mathrm{d}\varphi\right)\mathrm{d}r\\
\le\;&c_{12}\cdot\int_0^{2R}\left(\int_0^{2\pi}\int_0^{2R}\frac{1}{\tau'}\cdot\big|\nabla_A\widetilde{F}_A\big(\tilde{\gamma}_{(0,(r,\varphi))}(\tau'/r)\big)\big|\cdot\lambda^2(\tau',\varphi)\cdot\tau'\,\mathrm{d}\tau'\,\mathrm{d}\varphi\right)\mathrm{d}r\\
=\;&2R\cdot c_{12}\cdot\int_{B_{2R}(0)}\frac{\big|\nabla_A\widetilde{F}_A((\tau',\varphi))\big|}{\tau'}\cdot\lambda^2(\tau',\varphi)\cdot\tau'\,\mathrm{d}\tau'\,\mathrm{d}\varphi\\
\le\;&c_{13}\cdot\int_{B_{\iota(\Sigma)}(z)}\frac{|\nabla_A F_A(z')|}{\mathrm{d}_\Sigma(z,z')}\,\mathrm{dvol}_\Sigma(z')
\end{aligned}\tag{2.118}$$

for constants $c_{12},c_{13}>0$ (not depending on $z$ and $(A,u)$). Here $\iota(\Sigma)$ denotes the injectivity radius of $\Sigma$, and $\mathrm{d}_\Sigma(\cdot,\cdot)$ is the Riemannian distance function on $\Sigma$. In the first equality we used Fubini's theorem; in the first inequality we used that the quotient

$$\frac{\lambda(r,\varphi)}{\lambda(\tau',\varphi)}$$

is uniformly bounded and that $r\le R$; in the second equality we used the identity

$$\tilde{\gamma}_{(0,(r,\varphi))}(\tau')=(\tau',\varphi)$$

in $\mathbb{R}^2$; for the last inequality recall that $R$ was chosen such that $2R\le\iota(\Sigma)$. Combining estimate (2.118) with inequality (2.115) we finally conclude that

$$\big|\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(z)\big|\le C''\cdot\left(1+\big\|F_A\big\|^2_{L^\infty(\Sigma)}+\int_{B_{\iota(\Sigma)}(z)}\frac{|\nabla_A F_A(z')|}{\mathrm{d}_\Sigma(z,z')}\,\mathrm{dvol}_\Sigma(z')\right)$$

for some constant $C''>0$ (not depending on $(A,u)$ and $z$). This proves (iii).

The proof of Theorem 2.1.5 is now complete.

CHAPTER 3

# Gromov compactness

In this chapter, we prove that the moduli space of gauge equivalence classes of solutions of the perturbed non-local vortex equations (2.16) admits a Gromov compactification by polystable non-local vortices (Theorem 3.1.7).

We begin by adapting the concept of Gromov convergence for vortices due to Mundet i Riera [**26**] and González and Woodward [**13**] to the non-local case, in Section 3.1. Our approach is very much inspired by the definition of Gromov convergence for pseudoholomorphic curves in McDuff and Salamon [**22**]. The section closes with the statement of Theorem 3.1.7. The other sections are then devoted to the proof of this result.

The basic strategy of our proof is to reformulate the problem in such a way that it may be solved by combining weak Uhlenbeck compactness for connections with Gromov compactness for pseudoholomorphic curves. This was first carried out by Mundet i Riera [**26**] in the case where the Lie group $G$ is the circle. Our goal in this chapter is to give a proof that works for arbitrary compact connected Lie groups $G$ and that relies on the techniques developed by McDuff and Salamon [**22**] in their proof of Gromov compactness for pseudoholomorphic curves.

Compactness for solutions of the standard vortex equations (1.1) was proved by Cieliebak et. al. [**3**] under the assumption that the manifold $M$ is symplectically aspherical. As a first step towards the proof of Theorem 3.1.7 we will in Section 3.4 adapt their proof to solutions of the perturbed non-local vortex equations (2.16) for not necessarily aspherical target $M$. This yields compactness for non-local vortices up to possible bubbling phenomena which we ignore for the moment. The proof of this result is based on a removable singularity theorem for non-local vortices, which will be established in Section 3.3. This theorem in turn requires an a priori estimate for non-local vortices. We will derive this estimate in Section 3.2.

In order to complete the proof of Theorem 3.1.7 it remains to deal with the bubbling phenomena. In Section 3.5 we will apply Gromov's graph construction in order to transform vortices into pseudoholomorphic curves. This will finally allow us in Section 3.6 to construct the Gromov compactification for non-local vortices by reducing the problem to Gromov compactness for ordinary pseudoholomorphic curves. Again, the analytical methods entering into these arguments crucially rely on the a priori estimate for non-local vortices obtained in Section 3.2.

## 3.1. Gromov convergence

The goal of this section is to introduce the concept of Gromov convergence for non-local vortices, and to state Theorem 3.1.7 on Gromov compactness for non-local vortices.

**3.1.1. Trees and nodal curves.** We begin by recalling some basic facts about trees and nodal curves from McDuff and Salamon [**22**], Section D.2, modifying the notation and terminology a little. A **tree** is a directed connected graph without cycles. We denote it by $(V, E)$, where $V$ is a finite set of vertices and $E \subset V \times V$ is the edge relation. A rooted tree is a tree $(V, E)$ which has a distinguished **root vertex** $0 \in V$. We will indicate this in the notation by writing the set of vertices $V$ as a disjoint union

$$V = \{0\} \sqcup V_S.$$

The elements of $V_S$ are called **spherical vertices**. Note that $V$ always contains the root vertex 0 whereas $V_S$ may be empty. Let $n$ be a nonnegative integer. An **n-labeled tree** is a triple $T = (V, E, \Lambda)$ consisting of a rooted tree $(V = \{0\} \sqcup V_S, E)$ and a labeling

$$\Lambda\colon \{1, \dots, n\} \to V, \qquad i \mapsto \alpha_i.$$

Given an $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$, we mean by a normalized nodal curve of combinatorial type $T$ a tuple

$$(\mathbf{\Sigma}, \mathbf{z}) := \big(\{\Sigma_\alpha\}_{\alpha \in V}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big),$$

often just written as

$$\mathbf{z} = \big(\{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big),$$

consisting of a closed Riemann surface $\Sigma_0$, called the **principal component**, associated to the root vertex 0; rational curves $\Sigma_\alpha := \mathbb{P}^1$, called **spherical components**, for every $\alpha \in V_S$; nodal points $z_{\alpha\beta} \in \Sigma_\alpha$ labeled by the directed edges $\alpha E \beta$ of $T$; and $n$ distinct marked points $z_i \in \Sigma_{\alpha_i}$, $i = 1, \dots, n$, such that for every $\alpha \in V$ the points $z_{\alpha\beta}$ for $\alpha E \beta$ and $z_i$ for $\alpha_i = \alpha$ are pairwise distinct. For $\alpha \in V$ we denote the set of nodal points on the component $\Sigma_\alpha$ by

$$Z_\alpha := \big\{z_{\alpha\beta} \,\big|\, \alpha E \beta\big\},$$

and we define the set of special points on $\Sigma_\alpha$ by

$$Y_\alpha := Z_\alpha \cup \big\{z_i \,\big|\, \alpha_i = \alpha\big\}.$$

If two vertices $\alpha, \beta \in V$ are not connected by an edge, denote by $z_{\alpha\beta}$ the unique nodal point on $\Sigma_\alpha$ which corresponds to the first edge on the chain running from $\alpha$ to $\beta$. As a special case of this, we define the point $z_{0i}$ on the principal component $\Sigma_0$ to be

$$z_{0i} := \begin{cases} z_i & \text{if } \alpha_i = 0, \\ z_{0\alpha_i} & \text{if } \alpha_i \in V_S \end{cases}$$

In other words, if $z_i$ lies on a spherical component, then $z_{0i}$ is the unique nodal point on the principal component at which the bubble tree containing $z_i$ is attached. Otherwise, $z_{0i}$ coincides with $z_i$.

**3.1.2. Polystable non-local vortices.** Polystable vortices were first introduced by Mundet i Riera [**26**] and by González and Woodward [**13**]. We adapt their definition here to the case of non-local vortices. Denote by $P(M) := P \times_G M \to \Sigma$ the symplectic fiber bundle associated to the $G$-bundle $P \to \Sigma$ and the $G$-manifold $M$.

REMARK 3.1.1. We may equivalently think of a $G$-equivariant map $u\colon P \to M$ as a section $u\colon \Sigma \to P(M)$. In fact, this section is defined by

$$\Sigma \ni z \mapsto \big[p, u(p)\big] \in P(M), \quad \pi(p) = z,$$

where $\pi\colon P \to \Sigma$ denotes the bundle projection. We will usually not distinguish between these two viewpoints in the notation and switch freely from one to the other, depending on the situation.

DEFINITION 3.1.2 (Polystable non-local vortices). Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

Let $n$ be a nonnegative integer, and let $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ be an $n$-labeled tree. A **polystable non-local vortex of combinatorial type** $T$ is a tuple

$$(A, \mathbf{u}, \mathbf{z}) := \big((A, u_0), \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big)$$

consisting of

- a normalized nodal curve $\big(\{\Sigma_\alpha\}_{\alpha \in V}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big)$ of combinatorial type $T$ with principal component $\Sigma_0 = \Sigma$;
- a perturbed non-local vortex $(A, u_0) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ on the principal component $\Sigma_0$ satisfying equations (2.16);
- a $J_{\Theta(A,u_0)}(z_{0\alpha})$-holomorphic curve $u_\alpha\colon \Sigma_\alpha \to P(M)_{z_{0\alpha}}$ in the fiber of $P(M)$ over the nodal point $z_{0\alpha} \in \Sigma_0$ for every $\alpha \in V_S$

such that the following conditions are satisfied.

**(Connectedness)** $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$ for all $\alpha, \beta \in V$ such that $\alpha E \beta$.

**(Polystability)** $|Y_\alpha| \ge 3$ for all $\alpha \in V_S$ such that $u_\alpha$ is constant.

To understand the meaning of the (Connectedness) condition in the case $\alpha = 0$, we think of the $G$-equivariant map $u_0\colon P \to M$ as a section $u_0\colon \Sigma \to P(M)$ as explained in Remark 3.1.1. The (Connectedness) condition then means that $u_0(z_{0\beta}) = u_\beta(z_{\beta 0})$ in the fiber of $P(M)$ over the nodal point $z_{0\beta} \in \Sigma_0$.

REMARK 3.1.3. Recall from Remark 2.2.1 that we may think of the almost complex structure $J_{\Theta(A,u_0)}$ as a vertical almost complex structure on the fiber bundle $P(M)$. The $J_{\Theta(A,u_0)}(z_{0\alpha})$-holomorphic curves $u_\alpha\colon \Sigma_\alpha \to P(M)_{z_{0\alpha}}$ are also called **spherical fiber bubbles**. Fixing an identification $P(M)_{z_{0\alpha}} \cong M$, they may be regarded as ordinary pseudoholomorphic curves $\Sigma_\alpha \to M$. To make this more precise, for each point $p \in P$ we define a trivialization of the fiber of the bundle $P(M)$ over the point $\pi(p)$ by

$$\iota_p\colon M \xrightarrow{\cong} P(M)_{\pi(p)} = P_{\pi(p)} \times_G M, \quad x \mapsto [p, x],$$

where $\pi\colon P \to \Sigma$ denotes the bundle projection. In this way, the map $u_\alpha\colon \Sigma_\alpha \to P(M)_{z_{0\alpha}}$ is seen to be equivalent to the family of pairs

$$\big(p_\alpha, v_\alpha := \iota_{p_\alpha}^{-1} \circ u_\alpha\colon \Sigma_\alpha \to M\big),$$

where $p_\alpha$ ranges over the fiber $P_{z_{0\alpha}}$ and the map $v_\alpha\colon \Sigma_\alpha \to M$ is an ordinary $J_{\Theta(A,u_0)(p_\alpha)}$-holomorphic curve.

Given a polystable non-local vortex $(A, \mathbf{u}, \mathbf{z})$ of combinatorial type $T = (V = \{0\} \sqcup V_S, E, \Lambda)$, we define its **energy** to be

$$E(A, \mathbf{u}) := E(A, u_0) + \sum_{\alpha \in V_S} E(u_\alpha),$$

where $E(A, u_0)$ is the Yang-Mills-Higgs energy of the non-local vortex $(A, u_0)$, and $E(u_\alpha)$ denotes the energy of the $J_{\Theta(A,u_0)}(z_{0\alpha})$-holomorphic curve $u_\alpha\colon \Sigma_\alpha \to P(M)_{z_{0\alpha}}$ (see [**22**], Section 2.2).

The energy identity of Proposition 2.2.6 extends to polystable non-local vortices in the following way. Define the **degree** of the polystable non-local vortex $(A, \mathbf{u}, \mathbf{z})$ to be the class

$$[\mathbf{u}]^G := [u]^G + \sum_{\alpha \in V_S} [u_\alpha]^G \in H_2^G(M; \mathbb{Z})$$

in equivariant cohomology, where $[u]^G \in H_2^G(M; \mathbb{Z})$ denotes the degree of the non-local vortex $(A, u_0)$ (see Section 2.2), and the class $[u_\alpha]^G$ is defined as follows. Let $\theta\colon P \to EG$ be a classifying map for the bundle $P \to \Sigma$. The composition of maps

$$\Sigma_\alpha \xrightarrow{u_\alpha} (P_{z_{0\alpha}} \times M)/G \hookrightarrow (P \times M)/G \xrightarrow{\theta \times \mathrm{id}} (EG \times M)/G = M_G$$

then defines a class $[u_\alpha]^G \in H_2^G(M; \mathbb{Z})$ which is independent of the choice of $\theta$.

Proposition 3.1.4 (Energy identity). *Let $(A, \boldsymbol{u}, \boldsymbol{z})$ be a polystable non-local vortex. Its energy and degree are related by*

$$E(A, \boldsymbol{u}) = \big\langle [\omega - \mu]_G, [u]^G \big\rangle + \int_\Sigma \Omega_H(u)\, \mathrm{dvol}_\Sigma + \sum_{\alpha \in V_S} \big\langle [\omega], [u_\alpha] \big\rangle,$$

*where the Poincaré pairing of the classes $[\omega]$ and $[u_\alpha]$ is given by*

$$\big\langle [\omega], [u_\alpha] \big\rangle = \int_{\Sigma_\alpha} u_\alpha^* \omega.$$

Proof. The energy identity for pseudoholomorphic curves of Lemma 2.2.1 in [**22**] yields

$$E(u_\alpha) = \big\langle [\omega], [u_\alpha] \big\rangle = \int_{\Sigma_\alpha} u_\alpha^* \omega$$

for every $\alpha \in V_S$. Hence the claimed identity follows from the energy identity for non-local vortices of Proposition 2.2.6. □

**3.1.3. Gromov convergence.** As a special case of Definition 3.1.2, we shall mean by an $n$**-marked non-local vortex** a tuple

$$(A, u, \mathbf{z}) = (A, u, z_1, \dots, z_n)$$

consisting of a perturbed non-local vortex $(A, u)$ satisfying equations (2.16) and a sequence $z_1, \dots, z_n$ of $n$ distinct marked points on $\Sigma$. For $z_0 \in \Sigma$ and $r > 0$, denote by

$$B_r(z_0) := \big\{ z \in \Sigma \,\big|\, |z - z_0| \leq r \big\}$$

the closed geodesic disk in $\Sigma$ of radius $r$ centered at the point $z_0$, understood with respect to the Kähler metric determined by the area form $\mathrm{dvol}_{\Sigma_0}$ and the complex structure $j_{\Sigma_0}$. Let $B \subset \mathbb{C}$ be the closed unit disk.

Definition 3.1.5 (Gromov convergence). Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

Let $n$ be a nonnegative integer, and let $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ be an $n$-labeled tree. A sequence of $n$-marked non-local vortices

$$(A_\nu, u_\nu, \mathbf{z}_\nu) = (A_\nu, u_\nu, z_1^\nu, \dots, z_n^\nu)$$

is said to **Gromov converge** to a polystable non-local vortex of combinatorial type $T$

$$(A, \mathbf{u}, \mathbf{z}) = \big((A, u_0), \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \leq i \leq n}\big)$$

if there exist

(a) a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ of the bundle $P$;
(b) a positive real number $r$ such that $B_r(z_{0\alpha}) \cap Z_0 = \{z_{0\alpha}\}$ for every nodal point $z_{0\alpha} \in Z_0$;
(c) a holomorphic disk $\varphi_\alpha \colon B \to B_r(z_{0\alpha})$ such that $\varphi_\alpha(0) = z_{0\alpha}$, for each $\alpha \in V_S$;
(d) a sequence of biholomorphic maps $\phi_\alpha^\nu \in \mathrm{Aut}(\Sigma_\alpha)$, for each $\alpha \in V$, where we define $\phi_0^\nu := \mathrm{id}_{\Sigma_0}$ for all $\nu$

such that the following holds.

**(Map)** The sequence

$$\Big(g_\nu^* A_\nu, g_\nu^{-1} u_\nu, \big\{(g_\nu^{-1} u_\nu) \circ \varphi_{z_{0\alpha}} \circ \phi_\alpha^\nu\big\}_{\alpha \in V_S}\Big)$$

converges to

$$\big(A, u_0, \{u_\alpha\}_{\alpha \in V_S}\big)$$

in the following sense.

(i) The sequence $g_\nu^* A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on $\Sigma$;
(ii) the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ converges to $(A, u_0)$ weakly in the $W^{2,p}$-topology and strongly in the $C^1$-topology on every compact subset of $\Sigma_0 \setminus Z_0$;

(iii) for every $\alpha \in V_S$ the sequence

$$u_\alpha^\nu := \left(g_\nu^{-1} u_\nu\right) \circ \varphi_{z_{0\alpha}} \circ \phi_\alpha^\nu \colon \Omega_\alpha^\nu \to P(M),$$

where

$$\Omega_\alpha^\nu := (\phi_\alpha^\nu)^{-1} B \subset \mathbb{C}$$

is an open subset of the bubble $\Sigma_\alpha$, converges to

$$u_\alpha \colon \Sigma_\alpha \to P(M)_{z_{0\alpha}}$$

strongly in the $C^\infty$-topology on every compact subset of $\Sigma_\alpha \setminus Z_\alpha$.

**(Energy)** The sequence $E(A_\nu, u_\nu)$ converges to $E(A, \mathbf{u})$.

**(Rescaling)** If $\alpha, \beta \in V_S$ are such that $\alpha E \beta$ then the sequence

$$\phi_{\alpha\beta}^\nu := (\phi_\alpha^\nu)^{-1} \circ \phi_\beta^\nu$$

converges to $z_{\alpha\beta}$ in the $C^\infty$-topology on every compact subset of $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$.

**(Marked point)** For each $i = 1, \ldots, n$, the sequence of marked points $z_i^\nu$ converges in the following sense:

(i) If $\alpha_i = 0$, then the sequence $z_i^\nu$ converges to $z_i$ in $\Sigma_0$.

(ii) If $\alpha_i \in V_S$, then the sequence $(\varphi_{z_{0\alpha_i}} \circ \phi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$ converges to $z_i$ in $\Sigma_{\alpha_i}$.

**3.1.4. Conservation of the degree.** As in the case of pseudoholomorphic curves, the degrees of a Gromov converging sequence of non-local vortices stabilize. We have the following result.

Proposition 3.1.6 (Conservation of the degree). *Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. Let $n$ be a nonnegative integer.*

*Let $(A_\nu, u_\nu, \boldsymbol{z}_\nu)$ be a sequence of $n$-marked non-local vortices that Gromov converges to a non-local vortex $(A, \boldsymbol{u}, \boldsymbol{z})$. Then the degree is preserved in the sense that*

$$[u_\nu]^G = [\boldsymbol{u}]^G$$

*in $H_2^G(M; \mathbb{Z})$ for $\nu$ sufficiently large.*

Proof. The proof is similar to the proof of conservation of the degree for a Gromov convergent sequence of pseudoholomorphic curves ([**22**], Thm. 5.2.2 (ii)). $\square$

**3.1.5. Main result.** The main result of this chapter is the following theorem.

THEOREM 3.1.7 (Gromov compactness). *Fix a real constant $E > 0$ and an $E$-admissible area form* $\mathrm{dvol}_\Sigma$ *on* $\Sigma$. *Let $p > 2$ be a real number, and fix a perturbation datum* $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. *Let $n$ be a nonnegative integer.*

*Let $(A_\nu, u_\nu, \boldsymbol{z}_\nu)$ be a sequence of $n$-marked non-local vortices solving equations (2.16) such that the Yang-Mills-Higgs energy satisfies a uniform bound*

$$\sup_\nu E(A_\nu, u_\nu) < \infty.$$

*Then the sequence $(A_\nu, u_\nu, \boldsymbol{z}_\nu)$ has a Gromov convergent subsequence.*

We will prove Theorem 3.1.7 in Section 3.6. The strategy of our proof is to combine techniques from gauge theory and symplectic topology, namely, Uhlenbeck compactness for connections and Gromov compactness for pseudoholomorphic curves. More precisely, we reduce the construction of the limit polystable vortex to the construction of the limit stable curve of a suitably chosen sequence of pseudoholomorphic curves with uniformly bounded energy as in McDuff and Salamon [**22**]. This requires some preliminary results, which we provide in the subsequent sections.

Sections 3.2 and 3.3 are of preparatory nature. In Section 3.2 we prove an a priori estimate for non-local vortices. This estimate is the main analytical tool for the analysis of this chapter. In Section 3.3 we establish removal of singularities for non-local vortices.

In Section 3.4 we begin with the actual proof of Gromov compactness. We prove a compactness result for non-local vortices, ignoring any bubbling. In Section 3.5 we apply Gromov's graph construction in order to transform non-local vortices into pseudoholomorphic curves. The actual proof of Theorem 3.1.7 is contained in Section 3.6, where we combine the results of the previous Sections 3.4 and 3.5.

## 3.2. A priori estimate

The aim of this section is to prove Theorem 3.2.1 below which provides an a priori estimate for solutions of the perturbed non-local vortex equations (2.16). A priori estimates for solutions of the standard vortex equations (1.1) were proved before by Gaio and Salamon [**11**] and Ziltener [**40**] (see also Frauenfelder [**8**]). The a priori estimate of Theorem 3.2.1 will be the main analytical tool for all the analysis of the remaining sections of this chapter.

**3.2.1. Main result.** Given an area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, we denote by $\langle\cdot,\cdot\rangle_\Sigma$ the Kähler metric determined by $j_\Sigma$ and $\mathrm{dvol}_\Sigma$; and, moreover, for $z_0 \in \Sigma$ and $r > 0$ $B_r(z_0)$ denotes the closed geodesic disk in $\Sigma$ of radius $r$ centered at the point $z_0$. Recall that we denote by $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(M)^G)$ the space of Hamiltonian perturbations of class $C^\ell$, for any positive integer $\ell$.

The main result of this section is the following theorem.

THEOREM 3.2.1 (A priori estimate). *Fix real constants $E_0 > E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$.*

*Then there exist constants $\hbar > 0$, $C > 0$ and $R > 0$, depending on $E_0$, such that for all $z_0 \in \Sigma$ and all $0 < r \leq R$ the following holds. If a solution $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ of the perturbed non-local vortex equations (2.16) has Yang-Mills-Higgs energy bounded by*

$$E(A, u) \leq E_0,$$

*then*

$$E\big(A, u; B_r(z_0)\big) < \hbar \quad \Longrightarrow \quad \frac{1}{2}\big|\mathrm{d}_{A,H} u(z_0)\big|^2_{J_\Theta} + \big|\mu\big(u(z_0)\big)\big|^2_{\mathfrak{g}} \leq \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big).$$

The proof of Theorem 3.2.1 will be given in Section 3.2.2 below. It is based on the following two propositions.

Let $B_r(0) \subset \mathbb{C}$ denote the closed unit disk of radius $r$ centered at the origin, equipped with the standard Euclidean metric.

PROPOSITION 3.2.2 (Mean value inequality). *There exist constants $c_1, c_2 > 0$ such that the following holds for all $0 < r \leq 1$. Let $w \colon B_r(0) \to [0, \infty)$ be a nonnegative function of class $C^2$ and assume that it satisfies for some constant $a \geq 0$ the partial differential inequality*

$$\Delta w(x) \geq -a \left( w^2(x) + \frac{4\, w(x)}{r^2} + w(x) \cdot \int_{B_r(0)} \frac{1}{|y - x|} \sqrt{w(y)}\, \mathrm{d}y \right)$$

*for all $x \in B_{r/2}(0)$. Then*

$$\int_{B_r(0)} w \leq \frac{c_1}{a} \qquad \Longrightarrow \qquad w(0) \leq \frac{(1 + a + c_1)\, c_2}{r^2} \cdot \int_{B_r(0)} w.$$

The proof of Proposition 3.2.2 will be given after the proof of Theorem 3.2.1 in Section 3.2.3 below.

Before we state the next proposition we recall from Section 2.2.5 the definition of the Yang-Mills-Higgs energy density of a vortex in terms of local coordinates. Let $B \subset \mathbb{C}$ denote the closed unit disk with complex coordinate $s + \mathrm{i}t$. Let $\varphi \colon B \to \Sigma$ be a holomorphic disk in $\Sigma$ which trivializes the bundle $P$, and choose a lift $\tilde{\varphi} \colon B \to P$ of this map. The area form $\mathrm{dvol}_\Sigma$ gives rise to a smooth function $\lambda \colon B \to (0, \infty)$ by

$$\varphi^* \,\mathrm{dvol}_\Sigma = \lambda^2 \;\mathrm{d}s \wedge \mathrm{d}t.$$

Then the Yang-Mills-Higgs energy density of a vortex $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ in the chart $B$ is the function

$$e_\varphi(A, u) := \left( \frac{1}{2}\big|\mathrm{d}_{A,H} u \circ \tilde{\varphi}\big|^2_{J_\Theta \circ \varphi} + \big|\mu(u \circ \tilde{\varphi})\big|^2_{\mathfrak{g}} \right) \cdot \lambda^2$$

on the disk $B$.

Proposition 3.2.3. *Let us fix real constants $E_0 > E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$. Let $\varphi \colon B \to \Sigma$ be a holomorphic disk in $\Sigma$.*

*Then there exists a constant $c \geq 0$, depending on $E_0$, such that for every vortex $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ whose Yang-Mills-Higgs energy is bounded by*

$$E(A, u) \leq E_0$$

*the following holds. For all $x_0 \in B$ and all $r > 0$ such that $B_r(x_0) \subset B$ the Yang-Mills-Higgs energy density $e := e_\varphi(A, u)$ satisfies the partial differential inequality*

$$\Delta e(x) \geq -c \left( e^2(x) + \frac{4\, e(x)}{r^2} + e(x) \cdot \int_{B_r(x_0)} \frac{1}{|y - x|} \sqrt{e(y)}\, \mathrm{d}y \right) \tag{3.1}$$

*for all $x \in B_{r/2}(x_0)$.*

The proof of Proposition 3.2.3 is deferred to Section 3.2.4. We are now ready for the proof of Theorem 3.2.1.

**3.2.2. Proof of Theorem 3.2.1.** We prove Theorem 3.2.1 assuming the above Propositions 3.2.2 and 3.2.3. Let us fix real constants $E_0 > E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$. Let $B \subset \mathbb{C}$ denote the closed unit disk in $\mathbb{C}$.

The proof is in two steps. We first establish an a priori estimate in local coordinates, and then globalize this estimate.

Claim. *Let $\varphi \colon B \to \Sigma$ be a holomorphic disk in $\Sigma$. Then there exist constants $\delta > 0$ and $C > 0$ such that for every vortex $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ whose Yang-Mills-Higgs energy is bounded by*

$$E(A, u) \leq E_0$$

*the following holds. For all $x_0 \in B$ and all $r > 0$ such that $B_r(x_0) \subset B$ the Yang-Mills-Higgs energy density $e := e_\varphi(A, u)$ satisfies an a priori estimate*

$$E\big(A, u; B_r(x_0)\big) < \delta \quad \Longrightarrow \quad e_\varphi(A, u)(x_0) \leq \frac{C}{r^2} \cdot E\big(A, u; B_r(x_0)\big). \tag{3.2}$$

Proof of Claim. Let $\varphi \colon B \to \Sigma$ be a holomorphic disk in $\Sigma$. Let $c$ be the constant from Proposition 3.2.3 (depending on $E_0$), and let $c_1$, $c_2$ be the constants from Proposition 3.2.2. Define constants

$$\delta := \frac{c_1}{c} \quad \text{and} \quad C := (1 + c + c_1)\, c_2. \tag{3.3}$$

Let $x_0 \in B$ and $r > 0$ be such that $B_r(x_0) \subset B$. Let $(A, u)$ be a vortex such that $E(A, u) \leq E_0$, and denote by $e := e_\varphi(A, u)$ its Yang-Mills-Higgs energy density on the disk $B$. Define a function $w\colon B_r(0) \to \mathbb{R}$ by translation

$$w(x) := e(x + x_0).$$

Then Proposition 3.2.3 implies that $w \geq 0$ and

$$\Delta w(x) \geq -c\left(w^2(x) + \frac{4\,w(x)}{r^2} + w(x) \cdot \int_{B_r(0)} \frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y\right)$$

for all $x \in B_{r/2}(0)$. Hence it follows from Proposition 3.2.2 that

$$\int_{B_r(0)} w \leq \frac{c_1}{c} \qquad \Longrightarrow \qquad w(0) \leq \frac{(1+c+c_1)\,c_2}{r^2} \cdot \int_{B_r(0)} w. \tag{3.4}$$

Since

$$E\big(A, u; B_r(x_0)\big) = \int_{B_r(x_0)} e = \int_{B_r(0)} w,$$

the a priori estimate (3.2) follows from (3.3) and (3.4). □

We now globalize the a priori estimate of the Claim above. To this end, we choose a finite collection of holomorphic disks

$$\varphi_j\colon \mathbb{C} \supset B \xrightarrow{\simeq} \varphi_j(B) \subset \Sigma, \qquad j = 1, \ldots, N$$

in $\Sigma$ in such a way that the open subsets

$$U_j := \varphi_j(B \backslash \partial B)$$

form a covering of $\Sigma$. By the Lebesgue Number Lemma, there exists a constant $R > 0$ such that for every $z_0 \in \Sigma$ and every $0 < r < R$ there exists $j_0 \in \{1, \ldots, N\}$ such that $B_r(z_0) \subset U_{j_0}$.

For every $j$ the area form $\mathrm{dvol}_\Sigma$ defines a smooth function $\lambda_j\colon B \to (0, \infty)$ by the relation

$$\varphi_j^*\,\mathrm{dvol}_\Sigma = \lambda_j^2\;\mathrm{d}s \wedge \mathrm{d}t.$$

Denote by $\mathrm{d}_\Sigma\colon \Sigma \times \Sigma \to [0, \infty)$ the distance function on $\Sigma$ determined by the Kähler metric $\langle\cdot,\cdot\rangle_\Sigma$, and by $\mathrm{d}_B\colon B \times B \to [0, \infty)$ the distance function on $B$ determined by the Euclidean metric. By compactness of $B$ there exist constants $c_j > 0$ such that

$$\mathrm{d}_\Sigma\big(\varphi_j(x_1), \varphi_j(x_2)\big) \leq c_j \cdot \mathrm{d}_B(x_1, x_2) \tag{3.5}$$

for all $x_1, x_2 \in B$. Finally we denote, for every $j$, by $\delta_j$ and $C_j$ the constants of the Claim above associated to the coordinate chart $\varphi_j$.

We then define constants

$$\hbar := \min_{1 \leq j \leq N}\big\{\delta_j\big\} \quad \text{and} \quad C := \max_{1 \leq j \leq N}\big\{C_j \cdot c_j^2 \cdot \|\lambda_j\|_{C^0(B)}^{-2}\big\}. \tag{3.6}$$

Let now $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ be a vortex whose Yang-Mills-Higgs energy is bounded by $E(A, u) \leq E_0$. Let $z_0 \in \Sigma$ and $0 < r < R$, and assume that

$$E\big(A, u; B_r(z_0)\big) < \hbar. \tag{3.7}$$

As we have seen above, there exists $j_0 \in \{1, \ldots, N\}$ such that $B_r(z_0) \subset U_{j_0}$. Define

$$x_0 := \varphi_{j_0}^{-1}(z_0) \quad \text{and} \quad \rho_0 := c_{j_0}^{-1} \cdot r, \tag{3.8}$$

where the constant $c_{j_0}$ was defined by relation (3.5). We consider the Yang-Mills-Higgs energy of the vortex $(A, u)$ on the disk $B_{\rho_0}(\zeta_0)$. It follows from estimate (3.5) that

$$B_{\rho_0}(x_0) \subset \varphi_{j_0}^{-1}\big(B_r(z_0)\big) \subset B.$$

Hence it follows from formula (2.20) that

$$E\big(A, u; B_{\rho_0}(x_0)\big) \leq E\big(A, u; \varphi_{j_0}^{-1}(B_r(z_0)\big) = E\big(A, u; B_r(z_0)\big). \tag{3.9}$$

By assumption (3.7) and the definition of the constant $\hbar$ in (3.6) we conclude

$$E\big(A, u; B_{\rho_0}(x_0)\big) < \hbar \leq \delta_{j_0}.$$

Hence the a priori estimate (3.2) of the Claim above yields

$$e_{\varphi_{j_0}}(A, u)(x_0) \leq \frac{C_{j_0}}{\rho_0^2} \cdot E\big(A, u; B_{\rho_0}(x_0)\big).$$

Using inequality (3.9) and the definition of $\rho_0$ in (3.8), we obtain from this

$$e_{\varphi_{j_0}}(A, u)(x_0) \leq \frac{C_{j_0} \cdot c_{j_0}^2}{r^2} \cdot E\big(A, u; B_r(z_0)\big).$$

The claimed a priori estimate will now follow from the identity

$$\frac{1}{2}\big|\mathrm{d}_{A,H} u(z_0)\big|_{J_\Theta}^2 + \big|\mu(u(z_0))\big|^2 = e_{\varphi_{j_0}}(A, u)(x_0) \cdot \lambda_{j_0}^{-2}(x_0),$$

which holds by definition of the energy density $e_{\varphi_{j_0}}(A, u)$. In fact, by definition of the constant $C$ in (3.6) we then obtain from the previous inequality the estimate

$$\begin{aligned}\frac{1}{2}\big|\mathrm{d}_{A,H} u(z_0)\big|_{J_\Theta}^2 + \big|\mu(u(z_0))\big|^2 &\leq \frac{C_{j_0} \cdot c_{j_0}^2 \|\lambda_{j_0}\|_{C^0(B)}^{-2}}{r^2} \cdot E\big(A, u; B_r(z_0)\big) \\ &= \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big).\end{aligned}$$

This completes the proof of Theorem 3.2.1.

**3.2.3. Proof of Proposition 3.2.2.** The proof is adapted from the proof of the mean value inequality of Theorem 1.1 in Wehrheim [**35**]. It relies on the following mean value inequality for subharmonic functions due to Morrey [**24**] (see [**35**] for an elementary proof of this inequality).

Lemma 3.2.4 (Wehrheim [**35**], Lemma 3.2). *There exists a constant $c \geq 1$ such that the following holds for all $0 < r \leq 1$. Let $w\colon B_r(0) \to [0, \infty)$ be a nonnegative function of class $C^2$ such that $\Delta w \geq 0$. Then*

$$w(0) \leq \frac{c}{r^2} \cdot \int_{B_r(0)} w.$$

We now begin with the proof of Proposition 3.2.2. Let $c$ be the positive constant of Lemma 3.2.4 above. Define constants

$$c_1 := \frac{1}{1024c^2} \quad \text{and} \quad c_2 := 8192c^2.$$

Let $0 < r \leq 1$ and let $w\colon B_r(0) \to [0, \infty)$ be a nonnegative function of class $C^2$. Assume that it satisfies for some constant $a \geq 0$ the inequalities

$$\Delta w(x) \geq -a \left( w^2(x) + \frac{4\,w(x)}{r^2} + w(x) \cdot \int_{B_r(0)} \frac{1}{|y - x|} \sqrt{w(y)}\, \mathrm{d}y \right) \tag{3.10}$$

for all $x \in B_{r/2}(0)$, and that

$$\int_{B_r(0)} w \leq \frac{c_1}{a}. \tag{3.11}$$

Consider the function $f\colon [0,1] \to [0, \infty)$ defined by

$$f(\rho) := (1 - \rho)^2 \cdot \sup_{B_{\rho r/2}(0)} w$$

for $0 \leq \rho \leq 1$. This function is continuous with $f(1) = 0$, so there exists a number $0 \leq \bar\rho < 1$ such that

$$f(\bar\rho) = \max_{0 \leq \rho \leq 1} f(\rho).$$

Then there exists a point $\bar x \in B_{\bar\rho r/2}(0)$ such that

$$\bar c := \sup_{B_{\bar\rho r/2}(0)} w = w(\bar x). \tag{3.12}$$

Define

$$\varepsilon := \frac{1}{2}(1 - \bar\rho) \leq \frac{1}{2}. \tag{3.13}$$

Claim 1. *The function $w$ satisfies*

$$w(0) \leq 4\varepsilon^2\, \bar c \qquad \textit{and} \qquad \sup_{B_{(\bar\rho+\varepsilon)r/2}(0)} w \leq 4\,\bar c. \tag{3.14}$$

Proof of Claim 1. The claimed estimates follow right from the definition of the function $f$ and the number $\bar\rho$. As for the first estimate we obtain, using (3.13),

$$w(0) = f(0) \leq f(\bar\rho) = (1 - \bar\rho)^2 \sup_{B_{\bar\rho r/2}(0)} w = 4\varepsilon^2\, \bar c.$$

To verify the second estimate, first note that

$$\bar\rho + \varepsilon = \bar\rho + \frac{1}{2}(1 - \bar\rho) = \frac{\bar\rho + 1}{2} < 1,$$

whence $B_{(\bar\rho+\varepsilon)r/2}(0) \subset B_{r/2}(0)$. Here we used (3.13) and that $\bar\rho < 1$ by definition. Hence we obtain

$$\sup_{B_{(\bar\rho+\varepsilon)r/2}(0)} w = (1-\bar\rho-\varepsilon)^{-2} f(\bar\rho+\varepsilon) \le 4\,(1-\bar\rho)^{-2} f(\bar\rho) = 4\cdot \sup_{B_{\bar\rho r/2}(0)} w = 4\,\bar c.$$

This proves Claim 1. □

We next prove that the function $w$ is subharmonic. This will then allow us to apply Lemma 3.2.4 above.

Claim 2. *For $0<\delta\le\varepsilon r/4$ the function $w$ satisfies an inequality*

$$\Delta w \ge -144a\,\bar c^2 - 32a\,\bar c\left(\frac{4}{r^2}+1+\delta^2+\frac{1}{\delta^2}\cdot\int_{B_r(0)} w\right) \tag{3.15}$$

*on the disk $B_{\varepsilon r/4}(\bar x)$.*

Proof of Claim 2. Let $0<\delta\le\varepsilon r/4$ and $x\in B_{\varepsilon r/4}(\bar x)$. By assumption (3.10) we have

$$\Delta w(x) \;\ge\; -a\left(w(x)^2+\frac{4\,w(x)}{r^2}+w(x)\cdot\int_{B_r(0)}\frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y\right). \tag{3.16}$$

We may write the third term on the right-hand side of this inequality as

$$\begin{aligned} w(x)\cdot\int_{B_r(0)}\frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y &= w(x)\cdot\int_{B_\delta(x)}\frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y \\ &\quad + w(x)\cdot\int_{B_r(0)\setminus B_\delta(x)}\frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y. \end{aligned} \tag{3.17}$$

Note that $\delta\le\varepsilon r/4$ and hence $\bar\rho r/2+\varepsilon r/4+\delta\le(\bar\rho+\varepsilon)r/2$, so we have an inclusion

$$B_\delta(x)\subset B_{(\bar\rho+\varepsilon)r/2}(0).$$

Using Young's inequality and the second inequality in (3.14) we may therefore estimate the first term on the right-hand side of (3.17) by

$$\begin{aligned} w(x)\cdot\int_{B_\delta(x)}\frac{1}{|y-x|}\sqrt{w(y)}\,\mathrm{d}y &\le w(x)\cdot\sup_{B_\delta(x)}\sqrt{w}\cdot\int_{B_\delta(x)}\frac{1}{|y-x|}\,\mathrm{d}y \\ &= w(x)\cdot\sup_{B_\delta(x)}\sqrt{w}\cdot 2\pi\cdot\delta \\ &\le 8\,w(x)^2+8\,\delta^2\cdot\sup_{B_\delta(x)} w \\ &\le 128\,\bar c^2+32\,\delta^2\,\bar c. \end{aligned}$$

Likewise, using Hölder's inequality and the fact that $r \leq 1$ by assumption, we obtain for the second term on the right-hand side of (3.17)

$$\begin{aligned} w(x) \cdot \int_{B_r(0)\setminus B_\delta(x)} \frac{1}{|y-x|} \sqrt{w(y)}\, \mathrm{d}y &\leq w(x) \cdot \frac{1}{\delta} \cdot \int_{B_r(0)} \sqrt{w} \\ &\leq w(x) + w(x) \cdot \frac{1}{\delta^2} \cdot \left( \int_{B_r(0)} \sqrt{w} \right)^2 \\ &\leq w(x) + r^2 \pi\, w(x) \cdot \frac{1}{\delta^2} \cdot \int_{B_r(0)} w \\ &\leq 4\,\bar{c} + 16\,\bar{c} \cdot \frac{1}{\delta^2} \cdot \int_{B_r(0)} w. \end{aligned}$$

Inserting these estimates into (3.16) and using the second estimate in (3.14) again, inequality (3.15) follows. $\square$

Now let $0 < \delta \leq \varepsilon r/4$. It follows from (3.15) that the function

$$v(x) := 8a\,\bar{c} \left( 8\bar{c} + \frac{4}{r^2} + 1 + \delta^2 + \frac{1}{\delta^2} \cdot \int_{B_r(0)} w \right) \cdot |x - \bar{x}|^2 + w(x)$$

is subharmonic on the disk $B_{\varepsilon r/4}(\bar{x})$. Here we use that $\Delta\, |x-\bar{x}|^2 = 4$. Hence, using (3.12) and applying Lemma 3.2.4, we obtain for $0 < \rho \leq \varepsilon$

$$\begin{aligned} \bar{c} &= v(\bar{x}) \leq \frac{16c}{(\rho r)^2} \cdot \int_{B_{\rho r/4}(\bar{x})} v \\ &= 128ac\,\bar{c} \left( 8\bar{c} + \frac{4}{r^2} + 1 + \delta^2 + \frac{1}{\delta^2} \cdot \int_{B_r(0)} w \right) \cdot \frac{1}{(\rho r)^2} \cdot \int_{B_{\rho r/4}(\bar{x})} |x-\bar{x}|^2 \\ &\qquad + \frac{16c}{(\rho r)^2} \cdot \int_{B_{\rho r/4}(\bar{x})} w \\ &\leq ac\,\bar{c} \left( 8\bar{c} + \frac{4}{r^2} + 1 + \delta^2 + \frac{1}{\delta^2} \cdot \int_{B_r(0)} w \right) \cdot (\rho r)^2 + \frac{16c}{(\rho r)^2} \cdot \int_{B_r(0)} w. \end{aligned} \tag{3.18}$$

Here we used that

$$\int_{B_{\rho r/4}(\bar{x})} |x-\bar{x}|^2 = 2\pi \cdot \int_0^{\rho r/4} t^3 dt = \frac{\pi}{512} (\rho r)^4 \leq \frac{1}{128} (\rho r)^4.$$

Let us abbreviate

$$A := ac \left( 8\bar{c} + \frac{4}{r^2} + 1 + \frac{(\varepsilon r)^2}{16} + \frac{16}{(\varepsilon r)^2} \cdot \int_{B_r(0)} w \right) \cdot (\varepsilon r)^2.$$

We distinguish two cases.

CASE 1. $A \le \frac{1}{2}$. Setting $\delta = \varepsilon r/4$ and $\rho = \varepsilon$, estimate (3.18) implies that

$$\bar{c} \le \frac{32c}{(\varepsilon r)^2} \cdot \int_{B_r(0)} w.$$

Hence we obtain from the first estimate in (3.14) that

$$w(0) \le 4\varepsilon^2\,\bar{c} \le \frac{128c}{r^2} \cdot \int_{B_r(0)} w \le \frac{c_2}{r^2} \cdot \int_{B_r(0)} w.$$

Here we used that $c > 1$. This proves the mean value inequality in Case 1.

CASE 2. $A > \frac{1}{2}$. In this case, we may choose $0 < \rho < \varepsilon$ such that

$$\frac{1}{2} \cdot \frac{1}{(\rho r)^2} = ac\left(8\bar{c} + \frac{4}{r^2} + 1 + \frac{(\varepsilon r)^2}{16} + \frac{16}{(\varepsilon r)^2} \cdot \int_{B_r(0)} w\right). \tag{3.19}$$

Then, setting $\delta = \varepsilon r/4$, estimate (3.18) implies that

$$\bar{c} \le \frac{32c}{(\rho r)^2} \cdot \int_{B_r(0)} w.$$

Plugging this into the first estimate in (3.14) and using (3.19) again, we obtain

$$\begin{aligned} w(0) \le 4\varepsilon^2\,\bar{c} &\le 256ac^2\left(8\varepsilon^2\,\bar{c} + \varepsilon^2\left(\frac{4}{r^2} + 1\right) + \frac{\varepsilon^4 r^2}{16} + \frac{16}{r^2} \cdot \int_{B_r(0)} w\right) \cdot \int_{B_r(0)} w \\ &\le 256ac^2\left(8\varepsilon^2\,\bar{c} + \frac{2}{r^2} + \frac{1}{2} + \frac{16}{r^2} \cdot \int_{B_r(0)} w\right) \cdot \int_{B_r(0)} w. \end{aligned} \tag{3.20}$$

In the last inequality we used that $\varepsilon \le 1/2$ by (3.13) and that $r \le 1$ by assumption. We distinguish two subcases.

SUBCASE 2A. $8\varepsilon^2\,\bar{c} \le \frac{2}{r^2} + \frac{1}{2} + \frac{16}{r^2}\int_{B_r(0)} w$. Then estimate (3.20) and inequality (3.11) yield

$$\begin{aligned} w(0) &\le 512ac^2\left(\frac{2}{r^2} + \frac{1}{2} + \frac{16}{r^2} \cdot \int_{B_r(0)} w\right) \cdot \int_{B_r(0)} w \\ &\le 8192c^2 \cdot \frac{a + c_1}{r^2} \cdot \int_{B_r(0)} w \\ &\le \frac{(1 + a + c_1)\,c_2}{r^2} \cdot \int_{B_r(0)} w. \end{aligned}$$

Here we used that $r \le 1$ by assumption. This proves the mean value inequality in Subcase 2a.

Subcase 2b. $8\varepsilon^2\,\bar{c} > \frac{2}{r^2} + \frac{1}{2} + \frac{16}{r^2}\int_{B_r(0)} w$. In this case, it follows from estimate (3.20) that

$$4\,\varepsilon^2\,\bar{c} < 4096ac^2\,\varepsilon^2\,\bar{c}\cdot\int_{B_r(0)} w$$

whence

$$\int_{B_r(0)} w > \frac{1}{1024ac^2} = \frac{c_1}{a},$$

contradicting inequality (3.11).

This proves the mean value inequality in Case 2. The proof of Proposition 3.2.2 is now complete.

**3.2.4. Proof of Proposition 3.2.3.** Fix real constants $E_0 > E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$.

Let $(A, u) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ be a solution of the vortex equations (2.16) such that the Yang-Mills-Higgs energy is bounded by

$$E(A, u) \leq E_0. \tag{3.21}$$

Let $B \subset \mathbb{C}$ denote the closed unit disk with complex coordinate $x = s + \mathrm{i}\,t$. Let further $\varphi\colon B \to \Sigma$ be a holomorphic disk in $\Sigma$ which trivializes the bundle $P$, and choose a lift $\tilde{\varphi}\colon B \to P$ of this map. Recall from Section 2.2.5 that, locally in the chart $B$, the vortex $(A, u)$ determines a triple $(\Phi, \Psi, u^{\mathrm{loc}})$ consisting of functions $\Phi, \Psi\colon D \to \mathfrak{g}$ and a map $u^{\mathrm{loc}}\colon D \to M$, both of class $W^{1,p}$, by the relations

$$A^{\mathrm{loc}} := \tilde{\varphi}^* A = \Phi\,\mathrm{d}s + \Psi\,\mathrm{d}t \quad \text{and} \quad u^{\mathrm{loc}} := u \circ \tilde{\varphi}.$$

The area form $\mathrm{dvol}_\Sigma$ gives rise to a smooth function $\lambda\colon D \to (0, \infty)$ by

$$\varphi^*\,\mathrm{dvol}_\Sigma = \lambda^2\ \mathrm{d}s \wedge \mathrm{d}t,$$

and the Hamiltonian perturbation determines smooth functions $F, G\colon D \times M \to \mathbb{R}$ by

$$\varphi^* H = F\,\mathrm{d}s + G\,\mathrm{d}t.$$

Moreover, the $G$-equivariant map

$$\Theta_{(A,u)}\colon P \to EG^N$$

gives rise to a map

$$\Theta^{\mathrm{loc}}_{(A,u)} := \Theta_{(A,u)} \circ \tilde{\varphi}\colon B \to EG^N,$$

of class $W^{1,p}$, and there is a corresponding family of almost complex structures

$$I := J_{\Theta^{\mathrm{loc}}(A,u)}\colon B \to \mathcal{J}(M, \omega),$$

also of class $W^{1,p}$. We may think of this family as a complex structure on the vector bundle $(u^{\mathrm{loc}})^* TM \to B$ (see Remark 2.2.1).

The triple $(\Phi, \Psi, u^{\mathrm{loc}})$ satisfies the vortex equations (2.18), which in the present notation take the form

$$\begin{aligned} \partial_s u^{\mathrm{loc}} + X_\Phi\big(u^{\mathrm{loc}}\big) + X_F\big(u^{\mathrm{loc}}\big) + I\big(u^{\mathrm{loc}}\big)\big(\partial_t u^{\mathrm{loc}} + X_\Psi\big(u^{\mathrm{loc}}\big) + X_G\big(u^{\mathrm{loc}}\big)\big) &= 0 \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \lambda^2\, \mu\big(u^{\mathrm{loc}}\big) &= 0. \end{aligned} \tag{3.22}$$

Moreover, by formula (2.19) the Yang-Mills-Higgs energy density of the vortex $(A, u)$ in the chart $B$ can be expressed in terms of $(\Phi, \Psi, u^{\mathrm{loc}})$ by

$$e := e_\varphi(A, u) = \big|\partial_s u^{\mathrm{loc}} + X_\Phi\big(u^{\mathrm{loc}}\big) + X_F\big(u^{\mathrm{loc}}\big)\big|_I^2 + \lambda^2 \,\big|\mu\big(u^{\mathrm{loc}}\big)\big|^2. \tag{3.23}$$

In the following we shall drop the superscripts from the notation and write $A$, $u$ and $\Theta$ for $A^{\mathrm{loc}}$, $u^{\mathrm{loc}}$ and $\Theta^{\mathrm{loc}}$, respectively.

Note that the Laplacian $\Delta e$ is well-defined because the energy density $e$ is actually of class $C^2$. In fact, by elliptic regularity for vortices (see Proposition 2.2.8; here we use that $\ell \geq 4$ by assumption) and gauge invariance of the energy density $e$ (see Section 2.2.4) we may without loss of generality assume that the vortex $(A, u)$ is of class $C^3$. Then formula (3.23) above shows that $e$ is of class $C^2$.

The proof of Proposition 3.2.3 will be in two stages. We first compute an explicit formula for the Laplacian $\Delta e$ of the energy density (Claim 1 on p. 75), and then obtain the desired inequality (3.1) from a straightforward estimate (Claim 2 on p. 82).

Before we may begin with the computations we need to introduce some more notation. First of all, it will be convenient to use the abbreviations

$$\begin{aligned} v_s &:= \mathrm{d}_{A,H} u(\partial_s) = \partial_s u + X_\Phi(u) + X_F(u), \\ v_t &:= \mathrm{d}_{A,H} u(\partial_t) = \partial_t u + X_\Psi(u) + X_G(u), \\ \kappa &:= F_A(\partial_s, \partial_t) = \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]. \end{aligned} \tag{3.24}$$

Note that $v_s$ and $v_t$ are smooth sections of the vector bundle $u^*TM \to B$ and that $\kappa$ is a smooth function $B \to \mathfrak{g}$. With this notation, the vortex equations (3.22) take the form

$$v_s + I(u)\, v_t = 0, \qquad \kappa + \lambda^2\, \mu(u) = 0, \tag{3.25}$$

and the Yang-Mills-Higgs energy density (3.23) of the vortex $(A, u)$ is given by

$$e := e_\varphi(A, u) := |v_s|_I^2 + \lambda^2\, |\mu(u)|^2\,. \tag{3.26}$$

Furthermore, we recall from Appendix A.1 the definition of certain operators. The family $I\colon B \to \mathcal{J}(M, \omega)$ gives rise to a family

$$\langle \cdot, \cdot \rangle_I \colon B \ni x \mapsto \langle \cdot, \cdot \rangle_{I_x}$$

of Riemannian metrics on $M$, which in turn determines a family

$$\nabla \colon B \ni x \mapsto \nabla_{I_x}$$

of associated Levi-Civita connections on $M$ and a family of corresponding Riemann curvature tensors

$$R\colon B \ni z \mapsto R_z \in \Omega^2(M, \mathrm{End}(TM)).$$

We will also denote the infinitesimal action of $G$ on $M$ at a point $x \in M$ by

$$\mathrm{L}_x\colon \mathfrak{g} \to T_xM, \qquad \xi \mapsto \mathrm{L}_x\xi = X_\xi(x),$$

and its adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and the family of metrics $\langle\cdot,\cdot\rangle_I$ on $M$ by

$$\mathrm{L}_x^*\colon T_xM \to \mathfrak{g}, \qquad v \mapsto \mathrm{L}_x^* v.$$

We have a twisted covariant derivative

$$\nabla_A\colon \Omega^0(B,\mathfrak{g}) \to \Omega^1(B,\mathfrak{g}), \qquad \nabla_A\eta = \mathrm{d}\eta + [A,\eta],$$

acting on smooth $\mathfrak{g}$-valued functions on $B$. Evaluating in the directions of the coordinate vector fields $\partial_s$ and $\partial_t$, we obtain operators

$$\nabla_{A,s}, \nabla_{A,t}\colon \Omega^0(D,\mathfrak{g}) \to \Omega^0(D,\mathfrak{g})$$

given by the formulas

$$\nabla_{A,s}\eta = \partial_s\eta + [\Phi,\eta], \qquad \nabla_{A,t}\eta = \partial_t\eta + [\Psi,\eta].$$

In addition, we have a twisted covariant derivative

$$\nabla_A\colon \Omega^0(B,u^*TM) \to \Omega^1(B,u^*TM), \qquad \nabla_A\xi = \nabla\xi + \nabla_\xi X_A(u),$$

acting on sections of the vector bundle $u^*TM \to B$. Evaluating in the directions of the coordinate vector fields $\partial_s$ and $\partial_t$, we obtain operators

$$\nabla_{A,s}, \nabla_{A,t}\colon \Omega^0(B,u^*TM) \to \Omega^0(B,u^*TM)$$

given by the formulas

$$\nabla_{A,s}\xi = \nabla_s\xi + \nabla_\xi X_\Phi(u), \qquad \nabla_{A,t}\xi = \nabla_t\xi + \nabla_\xi X_\Psi(u).$$

Moreover, we have a twisted covariant derivative

$$\delta_A\colon \Omega^0\big(B,\Omega^0(M,\mathrm{End}(TM))\big) \to \Omega^1\big(B,\Omega^0(M,\mathrm{End}(TM))\big)$$

acting on functions $B \to \Omega^0(M,\mathrm{End}(TM))$ given by

$$\delta_A I' = \mathrm{d}I' - \mathcal{L}_{X_A} I',$$

where $\mathcal{L}_{X_A}I'$ is the Lie derivative of $I'$ along the vector field $X_A$. Evaluating in the directions of the coordinate vector fields $\partial_s$ and $\partial_t$, we obtain operators

$$\delta_{A,s}, \delta_{A,t}\colon \Omega^0\big(B,\Omega^0(M,\mathrm{End}(TM)\big) \to \Omega^0\big(B,\Omega^0(M,\mathrm{End}(TM)\big),$$

given by

$$\delta_{A,s}I' = \partial_s I' - \mathcal{L}_{X_\Phi}I', \qquad \delta_{A,t}I' = \partial_t I' - \mathcal{L}_{X_\Psi}I'.$$

Lastly, recall that the twisted derivative of the family $\nabla\colon B \ni z \mapsto \nabla_{J_z}$ of connections on $M$ is given by the map

$$\mathrm{D}_A\nabla = \mathrm{d}\nabla^{\mathrm{LC}} \circ \delta_A I\colon TB \to \Omega^1(M,\mathrm{End}(TM)), \tag{3.27}$$

where $\mathrm{d}\nabla^{\mathrm{LC}}$ is the derivative of the map $\nabla^{\mathrm{LC}}\colon \mathcal{J}(M,\omega) \ni I' \mapsto \nabla_{I'}$, which assigns to every $\omega$-compatible almost complex structure $I'$ on $M$ the Levi-Civita connection determined by the corresponding Riemannian metric $\langle\cdot,\cdot\rangle_{I'} = \omega(\cdot, I'\cdot)$. Evaluating in the directions of the coordinate vector fields $\partial_s$ and $\partial_t$, we obtain functions

$$\mathrm{D}_{A,s}\nabla, \mathrm{D}_{A,t}\nabla \colon B \to \Omega^1(M, \mathrm{End}(TM))$$

given by

$$\mathrm{D}_{A,s}\nabla = \mathrm{d}\nabla^{\mathrm{LC}}\big(\delta_{A,s} I\big), \qquad \mathrm{D}_{A,t}\nabla = \mathrm{d}\nabla^{\mathrm{LC}}\big(\delta_{A,t} I\big).$$

We close this review of notation by recalling from Appendix A.1 that there is a $\mathfrak{g}$-valued bilinear form $\rho\colon u^*TM \otimes u^*TM \to \mathfrak{g}$ satisfying

$$\langle \eta, \rho(\xi_1, \xi_2)\rangle_{\mathfrak{g}} = \langle \nabla_{\xi_1} X_\eta(u), \xi_2\rangle_I$$

for $\xi_1, \xi_2 \in \Omega^0(B, u^*TM)$ and $\eta \in \Omega^0(B, \mathfrak{g})$.

We are now ready to begin with the actual proof of Proposition 3.2.3. We start with the following identity.

Claim 1. *The Laplacian of the Yang-Mills-Higgs energy density is given by the formula*

$$\begin{aligned}
\Delta e = {}& 2\,|\nabla_{A,s} v_s|^2 + 2\,|\nabla_{A,t} v_s|^2 - 2\,|(\delta_{A,s} I) v_s|^2 - 2\,\langle v_s, R(v_s, v_t) v_t\rangle \\
& - 2\,\langle v_s, (\mathrm{D}_{A,s}\nabla)(v_t)\, v_t\rangle + 2\,\langle v_s, (\mathrm{D}_{A,t}\nabla)(v_s)\, v_t\rangle - 6\,\partial_s(\lambda^2)\langle \mu(u), \mathrm{L}_u^* v_t\rangle \\
& + 6\,\partial_t(\lambda^2)\langle \mu(u), \mathrm{L}_u^* v_s\rangle - 2\lambda^2\langle (\nabla_{v_s} I) v_s, \mathrm{L}_u \mu(u)\rangle + 2\lambda^2\langle \mu(u), 3\rho(v_t, v_s) \\
& - 2\rho(v_s, v_t)\rangle + 4\lambda^2 |\mathrm{L}_u^* v_t|^2 + 4\lambda^2 |\mathrm{L}_u^* v_s|^2 + \Delta(\lambda^2)|\mu(u)|^2 + 2\lambda^4 |\mathrm{L}_u \mu(u)|^2 \\
& - 2\lambda^2\langle (\delta_{A,t} I) v_t, \mathrm{L}_u \mu(u)\rangle + 2\,\langle (\delta_{A,s} I) v_t, \nabla_{A,s} v_s\rangle + 2\,\langle (\delta_{A,t} I) v_s, \nabla_{A,s} v_s\rangle \\
& + 4\,\langle (\delta_{A,t} I) v_t, \nabla_{A,t} v_s\rangle - 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,s} I)) v_t\rangle + 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,t} I)) v_s\rangle \\
& + \langle v_t, (\nabla_{v_t}(\delta_{A,t} I)) v_s\rangle + \langle v_t, (\nabla_{v_s}(\delta_{A,s} I)) v_s\rangle - \langle (\delta_{A,s} I) v_s, (\nabla_{v_s} I) v_s\rangle \\
& + \langle (\delta_{A,t} I) v_s, (\nabla_{v_t} I) v_s\rangle - 2\,\langle v_s, (\delta_{A,s}(\delta_{A,s} I)) v_t\rangle + 2\,\langle v_s, (\delta_{A,s}(\delta_{A,t} I)) v_s\rangle \\
& + \langle v_t, (\delta_{A,s}(\delta_{A,s} I)) v_s\rangle + \langle v_t, (\delta_{A,t}(\delta_{A,t} I)) v_s\rangle. \qquad (3.28)
\end{aligned}$$

Our proof builds on the computations carried out by Gaio and Salamon [**11**], Section 9. Recall from (3.26) that

$$e = |v_s|^2 + \lambda^2\, |\mu(u)|^2,$$

whence

$$\Delta e = \Delta\big(|v_s|^2\big) + \Delta\big(\lambda^2\, |\mu(u)|^2\big). \qquad (3.29)$$

We will compute each term separately. Let us first consider

$$\Delta\big(|v_s|^2\big) = \partial_s^2\big(|v_s|^2\big) + \partial_t^2\big(|v_s|^2\big).$$

By Proposition A.1.6, we have

$$\begin{aligned}\partial_s\big(|v_s|^2\big) &= \partial_s\langle v_s, v_s\rangle \\ &= \langle \nabla_{A,s}v_s, v_s\rangle + \langle v_s, \nabla_{A,s}v_s\rangle + \langle Iv_s, (\delta_{A,s}I)v_s\rangle \\ &= 2\,\langle v_s, \nabla_{A,s}v_s\rangle + \langle Iv_s, (\delta_{A,s}I)v_s\rangle\end{aligned}$$

and hence

$$\begin{aligned}\partial_s^2(|v_s|^2) \;&=\; 2\;\partial_s\langle v_s, \nabla_{A,s}v_s\rangle + \partial_s\langle Iv_s, (\delta_{A,s}I)v_s\rangle \\ &=\; 2\,\langle v_s, \nabla_{A,s}\nabla_{A,s}v_s\rangle + 2\,\langle \nabla_{A,s}v_s, \nabla_{A,s}v_s\rangle + 2\,\langle Iv_s, (\delta_{A,s}I)\nabla_{A,s}v_s\rangle \\ &\quad\; + \langle \nabla_{A,s}(Iv_s), (\delta_{A,s}I)v_s\rangle + \langle Iv_s, \nabla_{A,s}((\delta_{A,s}I)v_s)\rangle \\ &\quad\; + \langle (\delta_{A,s}I)Iv_s, I(\delta_{A,s}I)v_s)\rangle \\ &=\; 2\,\langle v_s, \nabla_{A,s}\nabla_{A,s}v_s\rangle + 2\;|\nabla_{A,s}v_s|^2 + 2\,\langle Iv_s, (\delta_{A,s}I)\nabla_{A,s}v_s\rangle \\ &\quad\; + \langle (\delta_{A,s}I)v_s, \nabla_{A,s}(Iv_s)\rangle + \langle Iv_s, \nabla_{A,s}((\delta_{A,s}I)v_s)\rangle - |I(\delta_{A,s}I)v_s|^2\,,\end{aligned}$$

where the last equality holds by Lemma A.1.4 (i). By Proposition A.1.5,

$$\begin{aligned}&\nabla_{A,s}(Iv_s) = I\nabla_{A,s}v_s + (\nabla_{v_s}I)v_s + (\delta_{A,s}I)v_s, \\ &\nabla_{A,s}((\delta_{A,s}I)v_s) = (\delta_{A,s}I)\nabla_{A,s}v_s + (\nabla_{v_s}(\delta_{A,s}I))v_s + (\delta_{A,s}(\delta_{A,s}I))v_s.\end{aligned}$$

Plugging this into the previous expression, we get

$$\begin{aligned}\partial_s^2\big(|v_s|^2\big) = 2\,&\langle v_s, \nabla_{A,s}\nabla_{A,s}v_s\rangle + 2\;|\nabla_{A,s}v_s|^2 + 2\,\langle Iv_s, (\delta_{A,s}I)\nabla_{A,s}v_s\rangle \\ &+ \langle (\delta_{A,s}I)v_s, I\nabla_{A,s}v_s\rangle + \langle (\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle + \langle (\delta_{A,s}I)v_s, (\delta_{A,s}I)v_s\rangle \\ &+ \langle Iv_s, (\delta_{A,s}I)\nabla_{A,s}v_s\rangle + \langle Iv_s, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle Iv_s, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle \\ &- |I(\delta_{A,s}I)v_s|^2\,.\end{aligned}$$

Using Lemma A.1.4 (i), (iii) this becomes

$$\begin{aligned}\partial_s^2\big(|v_s|^2\big) \;&=\; 2\,\langle v_s, \nabla_{A,s}\nabla_{A,s}v_s\rangle + 2\;|\nabla_{A,s}v_s|^2 + 2\,\langle (\delta_{A,s}I)Iv_s, \nabla_{A,s}v_s\rangle \\ &\quad\; + \langle (\delta_{A,s}I)Iv_s, \nabla_{A,s}v_s\rangle + \langle (\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle + |(\delta_{A,s}I)v_s|^2 \\ &\quad\; + \langle (\delta_{A,s}I)Iv_s, \nabla_{A,s}v_s\rangle + \langle Iv_s, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle Iv_s, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle \\ &\quad\; - \;|(\delta_{A,s}I)v_s|^2 \\ &=\; 2\;|\nabla_{A,s}v_s|^2 + 2\,\langle v_s, \nabla_{A,s}\nabla_{A,s}v_s\rangle + 4\,\langle (\delta_{A,s}I)Iv_s, \nabla_{A,s}v_s\rangle \\ &\quad\; + \langle (\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle + \langle Iv_s, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle Iv_s, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle.\end{aligned}$$

Likewise, we compute

$$\begin{aligned}\partial_t^2\big(|v_s|^2\big) = 2\;&|\nabla_{A,t}v_s|^2 + 2\,\langle v_s, \nabla_{A,t}\nabla_{A,t}v_s\rangle + 4\,\langle (\delta_{A,t}I)Iv_s, \nabla_{A,t}v_s\rangle \\ &+ \langle (\delta_{A,t}I)v_s, (\nabla_{v_t}I)v_s\rangle + \langle Iv_s, (\nabla_{v_t}(\delta_{A,t}I))v_s\rangle + \langle Iv_s, (\delta_{A,t}(\delta_{A,t}I))v_s\rangle.\end{aligned}$$

Combining the last two formulas and using the first vortex equation (3.25) we finally obtain

$$\begin{aligned}\Delta\big(|v_s|^2\big) &= 2\,|\nabla_{A,s}v_s|^2 + 2\,|\nabla_{A,t}v_s|^2 + 2\,\langle v_s, (\nabla_{A,s}\nabla_{A,s} + \nabla_{A,t}\nabla_{A,t})v_s\rangle\\
&\quad + 4\,\langle(\delta_{A,s}I)v_t, \nabla_{A,s}v_s\rangle + 4\,\langle(\delta_{A,t}I)v_t, \nabla_{A,t}v_s\rangle + \langle(\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle\\
&\quad + \langle(\delta_{A,t}I)v_s, (\nabla_{v_t}I)v_s\rangle + \langle v_t, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\nabla_{v_t}(\delta_{A,t}I))v_s\rangle\\
&\quad + \langle v_t, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\delta_{A,t}(\delta_{A,t}I))v_s\rangle. \end{aligned}\tag{3.30}$$

We can compute this further using the following two lemmata.

LEMMA 3.2.5. *The sections $v_s$, $v_t$ and the function $\kappa$ satisfy the following identities.*

(i) $\nabla_{A,s}v_t - \nabla_{A,t}v_s = \mathrm{L}_u\kappa$

(ii) $\nabla_{A,s}v_s + \nabla_{A,t}v_t = -I\,\mathrm{L}_u\kappa - (\delta_{A,s}I)v_t + (\delta_{A,t}I)v_s$

PROOF. Using the formulas

$$\nabla_s X_\Psi(u) = X_{\partial_s\Psi}(u) + \nabla_{\partial_s u}X_\Psi(u), \qquad \nabla_t X_\Phi(u) = X_{\partial_t\Phi}(u) + \nabla_{\partial_t u}X_\Phi(u)$$

and rearranging terms, we obtain

$$\begin{aligned}&\nabla_{A,s}v_t - \nabla_{A,t}v_s\\
&= \nabla_s\,\partial_t u + \nabla_s X_\Psi(u) + \nabla_{\partial_t u}X_\Phi(u) + \nabla_{X_\Psi(u)}X_\Phi(u) - \nabla_t\,\partial_s u - \nabla_t X_\Phi(u)\\
&\qquad - \nabla_{\partial_s u}X_\Psi(u) - \nabla_{X_\Phi(u)}X_\Psi(u)\\
&= \nabla_s\,\partial_t u - \nabla_t\,\partial_s u + X_{\partial_s\Phi}(u) - X_{\partial_t\Phi}(u) + \nabla_{X_\Psi(u)}X_\Phi(u) - \nabla_{X_\Phi(u)}X_\Psi(u) + \nabla_{\partial_s u}X_\Psi(u)\\
&\qquad - \nabla_{\partial_s u}X_\Psi(u) - \nabla_{\partial_t u}X_\Phi(u) + \nabla_{\partial_t u}X_\Phi(u)\\
&= \nabla_{\partial_s u}\,\partial_t u - \nabla_{\partial_t u}\,\partial_s u + X_{\partial_s\Phi}(u) - X_{\partial_t\Phi}(u) + \nabla_{X_\Psi(u)}X_\Phi(u) - \nabla_{X_\Phi(u)}X_\Psi(u).\end{aligned}$$

Since the Levi-Civita connection $\nabla_z$ is torsion free for $z \in D$,

$$\nabla_{\partial_s u}\,\partial_t u - \nabla_{\partial_t u}\,\partial_s u = [\partial_t u, \partial_s u] = 0$$

and

$$\nabla_{X_\Psi(u)}X_\Phi(u) - \nabla_{X_\Phi(u)}X_\Psi(u) = [X_\Phi(u), X_\Psi(u)] = X_{[\Phi,\Psi]}(u).$$

Note that, to be consistent with [**11**], we use the sign convention for the Lie bracket defined in [**21**], Remark 3.3. That is, for vector fields $X, Y$ on $M$ we have $L_{[X,Y]} = [L_Y, L_X]$. Whence

$$\begin{aligned}\nabla_{A,s}v_t - \nabla_{A,t}v_s &= X_{\partial_s\Psi}(u) - X_{\partial_t\Phi}(u) + X_{[\Phi,\Psi]}(u)\\
&= \mathrm{L}_u\big(\,\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]\big)\\
&= \mathrm{L}_u\kappa.\end{aligned}$$

For the proof of (ii) we will need the following formula:

$$(\nabla_{v_s}I)v_t - (\nabla_{v_t}I)v_s = 0. \tag{3.31}$$

To see this, just note that by the first vortex equation (3.25),

$$(\nabla_{v_s} I)v_t = -(\nabla_{Iv_t} I)(Iv_s) = I^2(\nabla_{Iv_t} I)(Iv_s),$$

and using Lemma A.1.1 (ii), (i), we therefore get

$$(\nabla_{v_s} I)v_t = I(\nabla_{v_t} I)(Iv_s) = -I^2(\nabla_{v_t} I)v_s = (\nabla_{v_t} I)v_s.$$

Now by the first vortex equation (3.25) and Proposition A.1.5,

$$\begin{aligned}
&\nabla_{A,s}v_s + \nabla_{A,t}v_t \\
&= -\nabla_{A,s}(Iv_t) + \nabla_{A,t}(Iv_s) \\
&= -I\nabla_{A,s}v_t - (\nabla_{v_s} I)v_t - (\delta_{A,s}I)v_t + I\nabla_{A,t}v_s + (\nabla_{v_t} I)v_s + (\delta_{A,t}I)v_s \\
&= -I(\nabla_{A,s}v_t - \nabla_{A,t}v_s) - (\nabla_{v_s} I)v_t + (\nabla_{v_t} I)v_s - (\delta_{A,s}I)v_t + (\delta_{A,t}I)v_s.
\end{aligned}$$

By Assertion (i) and Formula (3.31) we thus obtain

$$\nabla_{A,s}v_s + \nabla_{A,t}v_t = -I\,\mathrm{L}_u\kappa - (\delta_{A,s}I)v_t + (\delta_{A,t}I)v_s.$$

□

LEMMA 3.2.6. *The twisted Laplacian of the section $v_s$ is given by the formula*

$$\begin{aligned}
(\nabla_{A,s}\nabla_{A,s} + \nabla_{A,t}\nabla_{A,t})v_s = &-R(v_s, v_t)v_t - (\mathrm{D}_{A,s}\nabla)(v_t)\,v_t + (\mathrm{D}_{A,t}\nabla)(v_s)\,v_t \\
&+ \partial_s(\lambda^2)I\,\mathrm{L}_u\mu(u) + \partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) + \lambda^2(\nabla_{v_s} I)\,\mathrm{L}_u\mu(u) + \lambda^2(\delta_{A,s}I)\,\mathrm{L}_u\mu(u) \\
&+ \lambda^2 I\nabla_{v_s}X_{\mu(u)}(u) + 2\lambda^2\nabla_{v_t}X_{\mu(u)}(u) - \lambda^2 I\,\mathrm{L}_u\mathrm{L}_u^* Iv_s + \lambda^2\,\mathrm{L}_u\mathrm{L}_u^* v_s \\
&+ I(\delta_{A,s}I)\nabla_{A,s}v_s + (\delta_{A,t}I)\nabla_{A,s}v_s - (\delta_{A,s}I)(\nabla_{v_s} I)v_s - (\delta_{A,s}I)(\delta_{A,s}I)v_s \\
&- (\nabla_{v_s}(\delta_{A,s}I))v_t + (\nabla_{v_s}(\delta_{A,t}I))v_s - (\delta_{A,s}(\delta_{A,s}I))v_t + (\delta_{A,s}(\delta_{A,t}I))v_s.
\end{aligned}$$

PROOF. By Lemma A.1.10, the second vortex equation (3.25) and Lemma 3.2.5,

$$\begin{aligned}
&(\nabla_{A,s}\nabla_{A,s} + \nabla_{A,t}\nabla_{A,t})v_s \\
&= (\nabla_{A,t}\nabla_{A,s} - \nabla_{A,s}\nabla_{A,t})v_t + \nabla_{A,s}(\nabla_{A,s}v_s + \nabla_{A,t}v_t) - \nabla_{A,t}(\nabla_{A,s}v_t - \nabla_{A,t}v_s) \\
&= -R(v_s, v_t)v_t + \lambda^2\,\nabla_{v_t}X_{\mu(u)}(u) - \big((\mathrm{D}_{A,s}\nabla)(v_t) - (\mathrm{D}_{A,t}\nabla)(v_s)\big)v_t - \nabla_{A,s}(I\,\mathrm{L}_u\kappa) \\
&\qquad - \nabla_{A,s}((\delta_{A,s}I)Iv_s) + \nabla_{A,s}((\delta_{A,t}I)v_s) - \nabla_{A,t}(\mathrm{L}_u\kappa).
\end{aligned} \tag{3.32}$$

We will inspect the last four terms on the right-hand side of this equality separately. By the second vortex equation (3.25) and Proposition A.1.5, we get

$$\begin{aligned}
&\nabla_{A,s}(I\,\mathrm{L}_u\kappa) \\
&= -\nabla_{A,s}(\lambda^2 I\,\mathrm{L}_u\mu(u)) \\
&= -\,\partial_s(\lambda^2)I\,\mathrm{L}_u\mu(u) - \lambda^2\,\nabla_{A,s}(I\,\mathrm{L}_u\mu(u)) \\
&= -\,\partial_s(\lambda^2)I\,\mathrm{L}_u\mu(u) - \lambda^2 I\nabla_{A,s}\,\mathrm{L}_u\mu(u) - \lambda^2(\nabla_{v_s} I)\,\mathrm{L}_u\mu(u) - \lambda^2(\delta_{A,s}I)\,\mathrm{L}_u\mu(u).
\end{aligned}$$

Moreover, by Lemmas A.1.9 and A.1.3,

$$\nabla_{A,s}(\mathrm{L}_u\mu(u)) = \mathrm{L}_u(\nabla_{A,s}\mu(u)) + \nabla_{v_s}X_{\mu(u)}(u) = -\,\mathrm{L}_u\mathrm{L}_u^*Iv_s + \nabla_{v_s}X_{\mu(u)}(u).$$

Whence

$$\begin{aligned}\nabla_{A,s}(I\,\mathrm{L}_u\kappa) = -\,\partial_s(\lambda^2)I\,\mathrm{L}_u\mu(u) - \lambda^2(\nabla_{v_s}I)\,\mathrm{L}_u\mu(u) + \lambda^2 I\,\mathrm{L}_u\mathrm{L}_u^*Iv_s\\ -\,\lambda^2 I\nabla_{v_s}X_{\mu(u)}(u) - \lambda^2(\delta_{A,s}I)\,\mathrm{L}_u\mu(u).\end{aligned} \tag{3.33}$$

Similarly, using the vortex equations (3.25) and Lemmata A.1.9 and A.1.3, we further obtain

$$\begin{aligned}\nabla_{A,t}(\mathrm{L}_u\kappa) &= -\nabla_{A,t}(\lambda^2\,\mathrm{L}_u\mu(u))\\ &= -\,\partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) - \lambda^2\,\nabla_{A,t}(\mathrm{L}_u\mu(u))\\ &= -\,\partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) - \lambda^2\,\mathrm{L}_u\nabla_{A,t}\mu(u) - \lambda^2\,\nabla_{v_t}X_{\mu(u)}(u)\\ &= -\,\partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) + \lambda^2\,\mathrm{L}_u\mathrm{L}_u^*Iv_t - \lambda^2\,\nabla_{v_t}X_{\mu(u)}(u)\\ &= -\,\partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) - \lambda^2\,\mathrm{L}_u\mathrm{L}_u^*\,v_s - \lambda^2\,\nabla_{v_t}X_{\mu(u)}(u).\end{aligned} \tag{3.34}$$

Furthermore, by Proposition A.1.5 we have

$$\nabla_{A,s}((\delta_{A,t}I)v_s) = (\delta_{A,t}I)\nabla_{A,s}v_s + (\nabla_{v_s}(\delta_{A,t}I))v_s + (\delta_{A,s}(\delta_{A,t}I))v_s \tag{3.35}$$

and

$$\begin{aligned}\nabla_{A,s}((\delta_{A,s}I)Iv_s) &= (\delta_{A,s}I)\nabla_{A,s}(Iv_s) + (\nabla_{v_s}(\delta_{A,s}I))Iv_s + (\delta_{A,s}(\delta_{A,s}I))Iv_s\\ &= -\,I(\delta_{A,s}I)\nabla_{A,s}v_s + (\delta_{A,s}I)(\nabla_{v_s}I)v_s + (\delta_{A,s}I)(\delta_{A,s}I)v_s\\ &\quad +(\nabla_{v_s}(\delta_{A,s}I))v_t + (\delta_{A,s}(\delta_{A,s}I))v_t,\end{aligned} \tag{3.36}$$

where in the last equality we used Lemma A.1.4 (i) and the first vortex equation (3.25). Plugging (3.33), (3.34), (3.35) and (3.36) into Formula (3.32) above, we finally get

$$\begin{aligned}(\nabla_{A,s}\nabla_{A,s} + \nabla_{A,t}\nabla_{A,t})v_s = -R(v_s,v_t)v_t - (\mathrm{D}_{A,s}\nabla)(v_t)\,v_t + (\mathrm{D}_{A,t}\nabla)(v_s)\,v_t\\ +\,\partial_s(\lambda^2)I\,\mathrm{L}_u\mu(u) + \partial_t(\lambda^2)\,\mathrm{L}_u\mu(u) + \lambda^2(\nabla_{v_s}I)\,\mathrm{L}_u\mu(u) + \lambda^2(\delta_{A,s}I)\,\mathrm{L}_u\mu(u)\\ +\,\lambda^2 I\nabla_{v_s}X_{\mu(u)}(u) + 2\lambda^2\nabla_{v_t}X_{\mu(u)}(u) - \lambda^2 I\,\mathrm{L}_u\mathrm{L}_u^*Iv_s + \lambda^2\,\mathrm{L}_u\mathrm{L}_u^*v_s\\ +\,I(\delta_{A,s}I)\nabla_{A,s}v_s + (\delta_{A,t}I)\nabla_{A,s}v_s - (\delta_{A,s}I)(\nabla_{v_s}I)v_s - (\delta_{A,s}I)(\delta_{A,s}I)v_s\\ -\,(\nabla_{v_s}(\delta_{A,s}I))v_t + (\nabla_{v_s}(\delta_{A,t}I))v_s - (\delta_{A,s}(\delta_{A,s}I))v_t + (\delta_{A,s}(\delta_{A,t}I))v_s.\end{aligned}$$

This proves the lemma. $\square$

We may now proceed with the computation of Formula (3.30). Using Lemma 3.2.6 we obtain

$$\begin{aligned}
\Delta\big(|v_s|^2\big) &= 2\,|\nabla_{A,s}v_s|^2 + 2\,|\nabla_{A,t}v_s|^2 - 2\,\langle v_s, R(v_s,v_t)v_t\rangle - 2\,\langle v_s, (\mathrm{D}_{A,s}\nabla)(v_t)\,v_t\rangle \\
&\quad + 2\,\langle v_s, (\mathrm{D}_{A,t}\nabla)(v_s)\,v_t\rangle + 2\,\partial_s(\lambda^2)\langle v_s, I\,\mathrm{L}_u\mu(u)\rangle + 2\,\partial_t(\lambda^2)\langle v_s, \mathrm{L}_u\mu(u)\rangle \\
&\quad + 2\lambda^2\langle v_s, (\nabla_{v_s}I)\,\mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle v_s, (\delta_{A,s}I)\,\mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle v_s, I\nabla_{v_s}X_{\mu(u)}(u)\rangle \\
&\quad + 4\lambda^2\langle v_s, \nabla_{v_t}X_{\mu(u)}(u)\rangle - 2\lambda^2\langle v_s, I\,\mathrm{L}_u\mathrm{L}_u^*Iv_s\rangle + 2\lambda^2\langle v_s, \mathrm{L}_u\mathrm{L}_u^*v_s\rangle \\
&\quad + 2\,\langle v_s, I(\delta_{A,s}I)\nabla_{A,s}v_s\rangle + 2\,\langle v_s, (\delta_{A,t}I)\nabla_{A,s}v_s\rangle - 2\,\langle v_s, (\delta_{A,s}I)(\nabla_{v_s}I)v_s\rangle \\
&\quad - 2\,\langle v_s, (\delta_{A,s}I)(\delta_{A,s}I)v_s\rangle - 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,s}I))v_t\rangle + 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,t}I))\,v_s\rangle \\
&\quad - 2\,\langle v_s, (\delta_{A,s}(\delta_{A,s}I))v_t\rangle + 2\,\langle v_s, (\delta_{A,s}(\delta_{A,t}I))v_s\rangle + 4\,\langle(\delta_{A,s}I)v_t, \nabla_{A,s}v_s\rangle \\
&\quad + 4\,\langle(\delta_{A,t}I)v_t, \nabla_{A,t}v_s\rangle + \langle(\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle + \langle(\delta_{A,t}I)v_s, (\nabla_{v_t}I)v_s\rangle \\
&\quad + \langle v_t, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\nabla_{v_t}(\delta_{A,t}I))v_s\rangle + \langle v_t, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\delta_{A,t}(\delta_{A,t}I))v_s\rangle.
\end{aligned}$$

By the first vortex equation (3.25) and the definition of the bilinear form $\rho$ above, we have

$$\langle v_s, I\nabla_{v_s}X_{\mu(u)}(u)\rangle = -\langle Iv_s, \nabla_{v_s}X_{\mu(u)}(u)\rangle = -\langle v_t, \nabla_{v_s}X_{\mu(u)}(u)\rangle = \langle \mu(u), -\rho(v_s,v_t)\rangle$$

and

$$\langle v_s, \nabla_{v_t}X_{\mu(u)}(u)\rangle = \langle \mu(u), \rho(v_t,v_s)\rangle.$$

Plugging this into the previous expression and using Lemma A.1.1 (iv), Lemma A.1.4 (iii) and the first vortex equation (3.25) we further obtain

$$\begin{aligned}
\Delta\big(|v_s|^2\big) &= 2\,|\nabla_{A,s}v_s|^2 + 2\,|\nabla_{A,t}v_s|^2 - 2\,\langle v_s, R(v_s,v_t)v_t\rangle - 2\,\langle v_s, (\mathrm{D}_{A,s}\nabla)(v_t)\,v_t\rangle \\
&\quad + 2\,\langle v_s, (\mathrm{D}_{A,t}\nabla)(v_s)\,v_t\rangle - 2\,\partial_s(\lambda^2)\langle \mu(u), \mathrm{L}_u^*v_t\rangle + 2\,\partial_t(\lambda^2)\langle \mu(u), \mathrm{L}_u^*v_s\rangle \\
&\quad - 2\lambda^2\langle(\nabla_{v_s}I)v_s, \mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle(\delta_{A,s}I)v_s, \mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle\mu(u), 2\rho(v_t,v_s) - \rho(v_s,v_t)\rangle \\
&\quad + 2\lambda^2|\mathrm{L}_u^*v_t|^2 + 2\lambda^2|\mathrm{L}_u^*v_s|^2 - 2\,\langle(\delta_{A,s}I)v_t, \nabla_{A,s}v_s\rangle + 2\,\langle(\delta_{A,t}I)v_s, \nabla_{A,s}v_s\rangle \\
&\quad - 2\,\langle(\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle - 2\,\langle(\delta_{A,s}I)v_s, (\delta_{A,s}I)v_s\rangle - 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,s}I))v_t\rangle \\
&\quad + 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,t}I))\,v_s\rangle - 2\,\langle v_s, (\delta_{A,s}(\delta_{A,s}I))v_t\rangle + 2\,\langle v_s, (\delta_{A,s}(\delta_{A,t}I))v_s\rangle \\
&\quad + 4\,\langle(\delta_{A,s}I)v_t, \nabla_{A,s}v_s\rangle + 4\,\langle(\delta_{A,t}I)v_t, \nabla_{A,t}v_s\rangle + \langle(\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle \\
&\quad + \langle(\delta_{A,t}I)v_s, (\nabla_{v_t}I)v_s\rangle + \langle v_t, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\nabla_{v_t}(\delta_{A,t}I))v_s\rangle \\
&\quad + \langle v_t, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\delta_{A,t}(\delta_{A,t}I))v_s\rangle.
\end{aligned}$$

Cancelling and rearranging terms we finally get

$$\begin{aligned}\Delta\big(|v_s|^2\big) &= 2\,|\nabla_{A,s}v_s|^2 + 2\,|\nabla_{A,t}v_s|^2 - 2\,\,|(\delta_{A,s}I)v_s|^2 - 2\,\langle v_s, R(v_s,v_t)v_t\rangle \\ &\quad - 2\,\langle v_s, (\mathrm{D}_{A,s}\nabla)(v_t)\,v_t\rangle + 2\,\langle v_s, (\mathrm{D}_{A,t}\nabla)(v_s)\,v_t\rangle - 2\,\partial_s(\lambda^2)\langle\mu(u), \mathrm{L}_u^*v_t\rangle \\ &\quad + 2\,\partial_t(\lambda^2)\langle\mu(u), \mathrm{L}_u^*v_s\rangle - 2\lambda^2\langle(\nabla_{v_s}I)v_s, \mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle(\delta_{A,s}I)v_s, \mathrm{L}_u\mu(u)\rangle \\ &\quad + 2\lambda^2\langle\mu(u), 2\rho(v_t,v_s) - \rho(v_s,v_t)\rangle + 2\lambda^2|\mathrm{L}_u^*v_t|^2 + 2\lambda^2|\mathrm{L}_u^*v_s|^2 \\ &\quad + 2\,\langle(\delta_{A,s}I)v_t, \nabla_{A,s}v_s\rangle + 2\,\langle(\delta_{A,t}I)v_s, \nabla_{A,s}v_s\rangle + 4\,\langle(\delta_{A,t}I)v_t, \nabla_{A,t}v_s\rangle \\ &\quad - 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,s}I))v_t\rangle + 2\,\langle v_s, (\nabla_{v_s}(\delta_{A,t}I))v_s\rangle + \langle v_t, (\nabla_{v_t}(\delta_{A,t}I))v_s\rangle \\ &\quad + \langle v_t, (\nabla_{v_s}(\delta_{A,s}I))v_s\rangle - \langle(\delta_{A,s}I)v_s, (\nabla_{v_s}I)v_s\rangle + \langle(\delta_{A,t}I)v_s, (\nabla_{v_t}I)v_s\rangle \\ &\quad - 2\,\langle v_s, (\delta_{A,s}(\delta_{A,s}I))v_t\rangle + 2\,\langle v_s, (\delta_{A,s}(\delta_{A,t}I))v_s\rangle \\ &\quad + \langle v_t, (\delta_{A,s}(\delta_{A,s}I))v_s\rangle + \langle v_t, (\delta_{A,t}(\delta_{A,t}I))v_s\rangle. \qquad (3.37)\end{aligned}$$

Next we consider the second term $\Delta(\lambda^2|\mu(u)|^2)$ in (3.29). This term was computed by Gaio and Salamon [**11**], Section 9, in the case where $I_z$ does not depend on the point $z$. Generalizing their formula to our situation, we obtain

$$\begin{aligned}\Delta(\lambda^2|\mu(u)|^2) &= \Delta(\lambda^2)|\mu(u)|^2 + 4\,\partial_t(\lambda^2)\langle\mu(u), \mathrm{L}_u^*v_s\rangle - 4\,\partial_s(\lambda^2)\langle\mu(u), \mathrm{L}_u^*Iv_s\rangle \\ &\quad + 2\lambda^2|\mathrm{L}_u^*v_s|^2 + 2\lambda^2|\mathrm{L}_u^*v_t|^2 + 2\lambda^4|\mathrm{L}_u\mu(u)|^2 - 2\lambda^2\langle\mu(u), \rho(v_s,v_t) - \rho(v_t,v_s)\rangle \\ &\quad - 2\lambda^2\langle\mu(u), \mathrm{L}_u^*I(\delta_{A,s}I)v_t\rangle + 2\lambda^2\langle\mu(u), \mathrm{L}_u^*I(\delta_{A,t}I)v_s\rangle. \qquad (3.38)\end{aligned}$$

Here we use the relation

$$\begin{aligned}&\nabla_{A,t}\,\mathrm{L}_u^*v_s - \nabla_{A,s}\,\mathrm{L}_u^*v_t \\ &= \mathrm{L}_u^*\big(\nabla_{A,t}v_s - \nabla_{A,s}v_t\big) + \rho(v_t,v_s) - \rho(v_s,v_t) - \mathrm{L}_u^*I(\delta_{A,t}I)v_s + \mathrm{L}_u^*I(\delta_{A,s}I)v_t \\ &= \lambda^2\,\mathrm{L}_u^*\mathrm{L}_u\mu(u) + \rho(v_t,v_s) - \rho(v_s,v_t) - \mathrm{L}_u^*I(\delta_{A,t}I)v_s + \mathrm{L}_u^*I(\delta_{A,s}I)v_t,\end{aligned}$$

which is a direct consequence of Lemma A.1.9, Lemma 3.2.5 (i) and the second vortex equation (3.25). Now by Lemma A.1.4 (ii),(i) and the first vortex equation (3.25), we have

$$\begin{aligned}&- 2\lambda^2\langle\mu(u), \mathrm{L}_u^*I(\delta_{A,s}I)v_t\rangle + 2\lambda^2\langle\mu(u), \mathrm{L}_u^*I(\delta_{A,t}I)v_s\rangle \\ &= -2\lambda^2\langle I(\delta_{A,s}I)v_t, \mathrm{L}_u\mu(u)\rangle + 2\lambda^2\langle I(\delta_{A,t}I)v_s, \mathrm{L}_u\mu(u)\rangle \\ &= -2\lambda^2\langle(\delta_{A,s}I)v_s, \mathrm{L}_u\mu(u)\rangle - 2\lambda^2\langle(\delta_{A,t}I)v_t, \mathrm{L}_u\mu(u)\rangle.\end{aligned}$$

Plugging this into (3.38), we have

$$\begin{aligned}\Delta(\lambda^2|\mu(u)|^2) &= 2\lambda^2|\mathrm{L}_u^*v_t|^2 + 2\lambda^2|\mathrm{L}_u^*v_s|^2 - 4\,\partial_s(\lambda^2)\langle\mu(u), \mathrm{L}_u^*v_t\rangle + 4\,\partial_t(\lambda^2)\langle\mu(u), \mathrm{L}_u^*v_s\rangle \\ &\quad + \Delta(\lambda^2)|\mu(u)|^2 + 2\lambda^4|\mathrm{L}_u\mu(u)|^2 - 2\lambda^2\langle\mu(u), \rho(v_s,v_t) - \rho(v_t,v_s)\rangle \\ &\quad - 2\lambda^2\langle(\delta_{A,s}I)v_s, \mathrm{L}_u\mu(u)\rangle - 2\lambda^2\langle(\delta_{A,t}I)v_t, \mathrm{L}_u\mu(u)\rangle. \qquad (3.39)\end{aligned}$$

Inserting (3.37) and (3.39) into (3.29) and rearranging terms, we obtain the claimed formula (3.28) for the Laplacian of the Yang-Mills-Higgs energy density. This completes the proof of Claim 1.

The second part of the proof of Proposition 3.2.3 is to prove the following claim.

CLAIM 2. *There exists a constant $c > 0$ (not depending on $e$) such that the following holds. For all $x_0 \in B$ and all $r > 0$ such that $B_r(x_0) \subset B$ the Yang-Mills-Higgs energy density satisfies the partial differential inequality*

$$\Delta e(x) \geq -c\left(e^2(x) + \frac{e(x)}{r^2} + e(x) \cdot \int_{B_r(x_0)} \frac{1}{|y-x|}\sqrt{e(y)}\,\mathrm{d}y\right) \tag{3.40}$$

*for all $x \in B_{r/2}(x_0)$.*

We begin by noting that, from the first vortex equation (3.25), we have

$$|v_s| = |-Iv_t| = |v_t|\,. \tag{3.41}$$

Using the Cauchy-Schwarz inequality and formula (3.28), we immediately obtain the straightforward estimate

$$\begin{aligned}
\Delta e \geq\ & 2\,|\nabla_{A,s}v_s|^2 + 2\,|\nabla_{A,t}v_s|^2 - 2\,|(\delta_{A,s}I)|^2\,|v_s|^2 - 2\,\|R\|\,|v_s|^4 \\
& - 2\,\|\mathrm{D}_{A,s}\nabla\|\,|v_s|^4 - 2\,\|\mathrm{D}_{A,t}\nabla\|\,|v_s|^4 - 6\,\partial_s(\lambda^2)\,|\mu(u)|\,|\mathrm{L}_u^*v_t| \\
& - 6\,\partial_t(\lambda^2)\,|\mu(u)|\,|\mathrm{L}_u^*v_s| - 2\lambda^2\|\nabla I\|\,|\mathrm{L}_u\mu(u)|\,|v_s|^2 - 2\lambda^2\|\delta_{A,t}I\|\,|\mathrm{L}_u\mu(u)|\,|v_s| \\
& + 2\lambda^4\,|\mathrm{L}_u\mu(u)|^2 - 10\lambda^2\,|\mu(u)|\,\|\rho\|\,|v_s|^2 + \Delta(\lambda^2)\,|\mu(u)|^2 + 4\lambda^2\,|\mathrm{L}_u^*v_t|^2 + 4\lambda^2\,|\mathrm{L}_u^*v_s|^2 \\
& - 2\,\|\delta_{A,s}I\|\,|\nabla_{A,s}v_s|\,|v_s| - 2\,\|\delta_{A,t}I\|\,|\nabla_{A,s}v_s|\,|v_s| - 4\,\|\delta_{A,t}I\|\,|\nabla_{A,t}v_s|\,|v_s| \\
& - 3\,\|\nabla(\delta_{A,s}I)\|\,|v_s|^3 - 3\,\|\nabla(\delta_{A,t}I)\|\,|v_s|^3 - \|\delta_{A,s}I\|\|\nabla I\|\,|v_s|^3 - \|\delta_{A,t}I\|\|\nabla I\|\,|v_s|^3 \\
& - 3\,\|\delta_{A,s}(\delta_{A,s}I)\|\,|v_s|^2 - 2\,\|\delta_{A,s}(\delta_{A,t}I)\|\,|v_s|^2 - \|\delta_{A,t}(\delta_{A,t}I)\|\,|v_s|^2\,.
\end{aligned}$$

Here we denote pointwise operator norms by $\|\cdot\|$. We proceed by estimating this further using Young's inequality. We begin with the 5th and 6th term. Recall from (3.27) that

$$\mathrm{D}_{A,v}\nabla = \mathrm{d}\nabla^{\mathrm{LC}} \circ \delta_{A,v}I$$

for $v \in TB$, where $\mathrm{d}\nabla^{\mathrm{LC}}$ is the derivative of the map $\nabla^{\mathrm{LC}}\colon \mathcal{J}(M,\omega) \ni I \mapsto \nabla_I$. Hence

$$\|\mathrm{D}_{A,v}\nabla\| \leq \left\|\mathrm{d}\nabla^{\mathrm{LC}}\right\|\|\delta_{A,v}I\|$$

for $v \in TB$, which implies that

$$\|\mathrm{D}_{A,s}\nabla\| \leq \left\|\mathrm{d}\nabla^{\mathrm{LC}}\right\|^2 + \|\delta_{A,s}I\|^2$$

and

$$\|\mathrm{D}_{A,t}\nabla\| \leq \left\|\mathrm{d}\nabla^{\mathrm{LC}}\right\|^2 + \|\delta_{A,t}I\|^2.$$

Next for the 7th and 8th term we have

$$\begin{aligned}6\,\partial_s(\lambda^2)\,|\mu(u)|\;|\mathrm{L}_u^* v_t| &\leq \frac{9\,(\partial_s(\lambda^2)\,|\mu(u)|)^2}{4\lambda^2} + 4\lambda^2\,|\mathrm{L}_u^* v_t|^2 \\ &= 9\,(\partial_s\lambda)^2\,|\mu(u)|^2 + 4\lambda^2\,|\mathrm{L}_u^* v_t|^2\end{aligned}$$

and similarly

$$6\,\partial_t(\lambda^2)\,|\mu(u)|\;|\mathrm{L}_u^* v_s| \leq 9\,(\partial_t\lambda)^2\,|\mu(u)|^2 + 4\lambda^2\,|\mathrm{L}_u^* v_s|^2\,.$$

For the 9th and 10th term we get, respectively,

$$2\lambda^2\|\nabla I\|\,|\mathrm{L}_u\mu(u)|\,|v_s|^2 \leq \|\nabla I\|^2\,|v_s|^4 + \lambda^4\,|\mathrm{L}_u\mu(u)|^2$$

and

$$2\lambda^2\|\delta_{A,t}I\|\;|\mathrm{L}_u\mu(u)|\,|v_s| \leq \|\delta_{A,t}I\|^2\,|v_s|^2 + \lambda^4\,|\mathrm{L}_u\mu(u)|^2\,.$$

For the 12th term,

$$10\lambda^2\,|\mu(u)|\,\|\rho\|\,|v_s|^2 \leq \lambda^4\,|\mu(u)|^2 + 25\,\|\rho\|^2\,|v_s|^4\,.$$

For the 16th, 17th and 18th term,

$$\begin{aligned}2\,\|\delta_{A,s}I\|\,|\nabla_{A,s}v_s|\,|v_s| &\leq \|\delta_{A,s}I\|^2\,|v_s|^2 + |\nabla_{A,s}v_s|^2\,,\\ 2\,\|\delta_{A,t}I\|\,|\nabla_{A,s}v_s|\,|v_s| &\leq \|\delta_{A,t}I\|^2\,|v_s|^2 + |\nabla_{A,s}v_s|^2\end{aligned}$$

and

$$4\,\|\delta_{A,t}I\|\,|\nabla_{A,t}v_s|\,|v_s| \leq 2\,\|\delta_{A,t}I\|^2\,|v_s|^2 + 2\;|\nabla_{A,t}v_s|^2\,.$$

For the 19th and 20th term,

$$\|\nabla(\delta_{A,s}I)\|\,|v_s| \leq \|\nabla(\delta_{A,s}I)\|^2 + |v_s|^2$$

and

$$\|\nabla(\delta_{A,t}I)\|\,|v_s| \leq \|\nabla(\delta_{A,t}I)\|^2 + |v_s|^2\,.$$

Finally, for the 21st and 22nd term, we have

$$\|\delta_{A,s}I\|\|\nabla I\|\,|v_s| \leq \|\delta_{A,s}I\|^2 + \|\nabla I\|^2\,|v_s|^2$$

and

$$\|\delta_{A,t}I\|\|\nabla I\|\,|v_s| \leq \|\delta_{A,t}I\|^2 + \|\nabla I\|^2\,|v_s|^2\,.$$

Applying all these inequalities and using the identity $|\mathrm{d}\lambda|^2 = (\partial_s\lambda)^2 + (\partial_t\lambda)^2$, we may estimate $\Delta e$ further by

$$\begin{aligned}\Delta e \geq &-25\Big(\|R\| + \|\rho\|^2 + \|\nabla I\|^2 + \left\|\mathrm{d}\nabla^{\mathrm{LC}}\right\|^2 + \|\delta_{A,s}I\|^2 + \|\delta_{A,t}I\|^2 + 1\Big)\,|v_s|^4 \\ &- \Big(\lambda^{-2}\big(9\,|\mathrm{d}\lambda|^2 - \Delta(\lambda^2)\big) + \lambda^2\Big)\lambda^2\,|\mu(u)|^2 - 5\,\Big(\|\delta_{A,s}I\|^2 + \|\delta_{A,t}I\|^2 + \|\nabla(\delta_{A,s}I)\|^2 \\ &\qquad + \|\nabla(\delta_{A,t}I)\|^2 + \|\delta_{A,s}(\delta_{A,s}I)\| + \|\delta_{A,s}(\delta_{A,t}I)\| + \|\delta_{A,t}(\delta_{A,t}I)\|\Big)\,|v_s|^2\,.\end{aligned}$$

Since

$$e = |v_s|^2 + \lambda^2 |\mu(u)|^2$$

by (3.26) we conclude that

$$\begin{aligned}\Delta e \geq -25 \Big( \|R\| + \|\rho\|^2 + \|\nabla I\|^2 + \big\|\mathrm{d}\nabla^{\mathrm{LC}}\big\|^2 + \|\delta_{A,s} I\|^2 + \|\delta_{A,t} I\|^2 + 1 \Big) e^2 \\ - \Big( \lambda^{-2}\big(9\,|\mathrm{d}\lambda|^2 - \Delta(\lambda^2)\big) + \lambda^2 \Big) e - 5 \Big( \|\delta_{A,s} I\|^2 + \|\delta_{A,t} I\|^2 + \|\nabla(\delta_{A,s} I)\|^2 \\ + \|\nabla(\delta_{A,t} I)\|^2 + \|\delta_{A,s}(\delta_{A,s} I)\| + \|\delta_{A,s}(\delta_{A,t} I)\| + \|\delta_{A,t}(\delta_{A,t} I)\| \Big) e. \quad (3.42)\end{aligned}$$

The right-hand side of this inequality can be estimated further in the following way (see p. 73 for definitions). Because the domain of the smooth function $\lambda\colon B \to (0,\infty)$ is compact, it follows that there exists an upper bound $c_1 > 0$ (not depending on $(A,u)$) such that

$$\lambda^{-2}\big(9\,|\mathrm{d}\lambda|^2 - \Delta(\lambda^2)\big) + \lambda^2 \leq c_1 \quad (3.43)$$

pointwise on $B$. Likewise, the smooth family $I\colon B \to \mathcal{J}(M,\omega)$ has compact parameter space, so there exists a constant $c_2 > 0$ (not depending on $(A,u)$) such that

$$\|R_z\| + \|\rho_z\|^2 + \|\nabla I_z\|^2 \leq c_2 \quad (3.44)$$

for all $z \in B$.

Note further that $\|\mathrm{d}\nabla^{\mathrm{LC}}\|$ is a constant that does not depend on $(A,u)$.

It remains to estimate those terms in inequality (3.42) that involve $A$-twisted derivatives. It will be convenient to write the family of almost complex structures $I$ in the form

$$I = J_{\Theta(A,u)} = J \circ \Theta_{(A,u)}. \quad (3.45)$$

We begin by considering the first order twisted derivatives of $I$. By formula (3.45), we obtain

$$\delta_{A,v} I = \mathrm{d}J\big(\mathrm{d}_A \Theta_{(A,u)}(v)\big) \quad (3.46)$$

for every smooth vector field $v$ on $B$. Hence we get a pointwise estimate

$$\|\delta_{A,v} I\| \leq \|\mathrm{d}J\| \cdot \left|\mathrm{d}_A \Theta_{(A,u)}(v)\right|,$$

where the norm $\|\mathrm{d}J\|$ does not depend on $(A,u)$. By part (ii) of the (Estimates) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3), we have

$$\left|\mathrm{d}_A \Theta_{(A,u)}(v)\right| \leq c_3 \cdot |v| \cdot \Big(1 + \|F_A\|_{L^\infty(\Sigma)}\Big) \quad (3.47)$$

for some constant $c_3 > 0$ (not depending on $(A,u)$). By compactness of $M$, it follows from the second vortex equation

$$F_A = -\mu(u)\ \mathrm{dvol}_\Sigma$$

that

$$\|F_A\|_{L^\infty(\Sigma)} \leq c_4 \tag{3.48}$$

for some constant $c_4 > 0$ (not depending on $(A, u)$). Hence we conclude that there exists a constant $c_5 > 0$ (not depending on $(A, u)$) such that

$$\|\delta_{A,s} I\|, \|\delta_{A,t} I\| \leq c_5 \tag{3.49}$$

pointwise on $B$.

In a similar fashion we obtain estimates for the second order covariant derivatives of $I$, as follows. Recall from (3.46) that

$$\delta_{A,v} I = \mathrm{d}J\big(\mathrm{d}_A \Theta_{(A,u)}(v)\big).$$

Hence

$$\nabla(\delta_{A,v} I) = \big(\nabla(\mathrm{d}J)\big)\big(\mathrm{d}_A \Theta_{(A,u)}(v)\big).$$

Now it follows from estimates (3.47) and (3.48) that the map

$$\mathrm{d}_A \Theta_{(A,u)}(v) \colon P \to TEG^N$$

takes values in a certain compact subset $K = K(v)$ of the tangent bundle $TEG^N$ which does not depend on $(A, u)$. Thus we conclude that there exists a constant $c_6 > 0$ (not depending on $(A, u)$) such that

$$\|\nabla(\delta_{A,s} I)\|, \|\nabla(\delta_{A,t} I)\| \leq c_6 \tag{3.50}$$

pointwise on $B$.

Let us next estimate the second order twisted derivatives of $I$. Recall that $B$ denotes the closed unit disk in $\mathbb{C}$. Fix a point $x_0 \in B$ and a constant $r > 0$ such that $B_r(x_0) \subset B$. In particular, this means that $r \leq 1$. Let $x \in B_{r/2}(x_0)$.

Let $v$ be a smooth vector field on $B$. Differentiating formula (3.46) we obtain

$$\delta_{A,w}(\delta_{A,v} I) = \mathrm{d}(\mathrm{d}J)\Big(\nabla_{A,w}\big(\mathrm{d}_A \Theta_{(A,u)}(v)\big)\Big),$$

whence

$$\|\delta_{A,w}(\delta_{A,v} I)\| \leq \|\mathrm{d}(\mathrm{d}J)\| \cdot \big|\nabla_{A,w}\big(\mathrm{d}_A \Theta_{(A,u)}(v)\big)\big| \tag{3.51}$$

for every tangent vector $w \in T_x B$. Here the norm $\|\mathrm{d}(\mathrm{d}J)\|$ does not depend on $(A, u)$.

By part (iii) of the (Estimates) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3), we have

$$\big|\nabla_A\big(\mathrm{d}_A \Theta_{(A,u)}\big)(\varphi(x))\big| \leq C'' \cdot \left(1 + \big\|F_A\big\|^2_{L^\infty(\Sigma)} + \int_{B_{\iota(\Sigma)}(\varphi(x))} \frac{|\nabla_A F_A(z')|}{\mathrm{d}_\Sigma(\varphi(x), z')} \,\mathrm{dvol}_\Sigma(z')\right)$$

for all points $x \in B$. Here the point $\varphi(x)$ is the image in $\Sigma$ of the point $x \in B$ under the chart map $\varphi \colon B \to \Sigma$, the integral is over the geodesic disk $B_{\iota(\Sigma)}(\varphi(x)) \subset \Sigma$ around the point $\varphi(x)$ with radius the injectivity radius $\iota(\Sigma)$ of $\Sigma$, and $\mathrm{d}_\Sigma(\cdot, \cdot)$ denotes the Riemannian distance function on $\Sigma$.

Recall moreover from formula (2.10) that

$$\nabla_A\big(\mathrm{d}_A\Theta_{(A,u)}\big)(w,v) = \nabla_{A,w}\big(\mathrm{d}_A\Theta_{(A,u)}(v)\big) + \nabla_{\mathrm{d}_A\Theta_{(A,u)}(v)}X_{A(w)}\big(\mathrm{d}_A\Theta_{(A,u)}\big) - \mathrm{d}_A\Theta_{(A,u)}\big(\nabla_w v\big).$$

This shows that there exists a constant $c_7(v) > 0$ (not depending on $(A,u)$) such that

$$\big|\nabla_{A,w}\big(\mathrm{d}_A\Theta_{(A,u)}(v)\big)\big| \leq c_7(v)\cdot|w|\cdot\left(1+\big\|F_A\big\|^2_{L^\infty(\Sigma)} + \int_{B_{\iota(\Sigma)}(\varphi(x))}\frac{\big|\nabla_A F_A(z')\big|}{\mathrm{d}_\Sigma(\varphi(x),z')}\,\mathrm{dvol}_\Sigma(z')\right). \tag{3.52}$$

We now estimate the integral on the right-hand side of (3.52). By the Leibniz rule it follows from the second vortex equation

$$F_A = -\mu(u)\,\mathrm{dvol}_\Sigma$$

that

$$\begin{aligned}\nabla_A F_A &= -\big(\nabla_A(\mu(u))\big)\,\mathrm{dvol}_\Sigma - \mu(u)\,\nabla\mathrm{dvol}_\Sigma\\ &= -\big(\mathrm{d}\mu\circ\mathrm{d}_A u\big)\,\mathrm{dvol}_\Sigma - \mu(u)\,\nabla\mathrm{dvol}_\Sigma.\end{aligned}$$

We obtain from this a pointwise estimate

$$|\nabla_A F_A| \leq c_8\cdot\Big(\|\mathrm{d}\mu\|_{L^\infty(M)}\cdot|\mathrm{d}_A u|\cdot|\mathrm{dvol}_\Sigma| + \|\mu\|_{L^\infty(M)}\cdot|\nabla\mathrm{dvol}_\Sigma|\Big) \tag{3.53}$$

for some constant $c_8 > 0$ (not depending on $(A,u)$). Note that the norms $\|\mu\|_{L^\infty(M)}$, $\|\mathrm{d}\mu\|_{L^\infty(M)}$, $|\mathrm{dvol}_\Sigma|$ and $|\nabla\mathrm{dvol}_\Sigma|$ do not depend on $(A,u)$, whence there exists a constant $c_9 > 0$ (not depending on $(A,u)$) such that

$$|\nabla_A F_A| \leq c_9\cdot\big(1+|\mathrm{d}_A u|\big). \tag{3.54}$$

Recall that

$$e = |v_s|^2 + \lambda^2\,|\mu(u)|^2$$

by (3.26), that

$$|v_s| = |v_t|$$

by (3.41), and that

$$v_s = \mathrm{d}_A u(\partial_s) + X_F(u), \qquad v_t = \mathrm{d}_A u(\partial_t) + X_G(u).$$

by (3.24). Thus we obtain an estimate

$$\begin{aligned}|\mathrm{d}_A u| &\leq c_{10}\cdot|\mathrm{d}_A u(\partial_s+\partial_t)|\\ &\leq c_{10}\cdot\big(|\mathrm{d}_A u(\partial_s)| + |\mathrm{d}_A u(\partial_t)|\big)\\ &\leq c_{10}\cdot\big(|v_s| + |X_F(u)| + |v_t| + |X_G(u)|\big)\\ &\leq c_{11}\cdot\big(|v_s|+1\big)\\ &\leq c_{11}\cdot\big(\sqrt{e}+1\big)\end{aligned}$$

for constants $c_{10}, c_{11} > 0$ (not depending on $(A, u)$). Combining this with (3.54) we get, locally on the chart $B$, an estimate

$$|\nabla_A F_A| \leq c_{12} \cdot \big(1 + \sqrt{e}\big) \tag{3.55}$$

for some constant $c_{12} > 0$ (not depending on $(A, u)$). Moreover, by Young's inequality it follows from (3.54) that

$$|\nabla_A F_A| \leq c_9 \cdot \Big(2 + |\mathrm{d}_A u|^2\Big). \tag{3.56}$$

Using (3.55) and (3.56), we may now estimate the integral on the right-hand side of inequality (3.52) by

$$\begin{aligned}\int_{B_{\iota(\Sigma)}(\varphi(x))} \frac{\big|\nabla_A F_A(z')\big|}{\mathrm{d}_\Sigma(\varphi(x), z')} \,\mathrm{dvol}_\Sigma(z') \leq c_{13} \cdot \Bigg( &\int_{B_r(x_0)} \frac{1}{|y-x|} \Big(1 + \sqrt{e(y)}\Big) \,\mathrm{d}y \\ &+ \frac{1}{r^2} \cdot \int_{B_{\iota(\Sigma)}(\varphi(x)) \setminus \varphi(B_r(x_0))} \Big(2 + |\mathrm{d}_A u(z')|^2\Big) \,\mathrm{dvol}_\Sigma(z')\Bigg)\end{aligned} \tag{3.57}$$

for some constant $c_{13} > 0$ (not depending on $(A, u)$). Here we used that the chart map $\varphi$ and its inverse are both continuously differentiable, and that $r \leq 1$ by assumption. Since

$$\int_{B_r(x_0)} \frac{1}{|y-x|} \,\mathrm{d}y \leq \int_{B_{2r}(x)} \frac{1}{|y|} \,\mathrm{d}y = 4\pi\, r$$

we may estimate the integral in the first term on the right-hand side of inequality (3.57) by

$$4\pi\, r + \int_{B_r(x_0)} \frac{1}{|y-x|} \sqrt{e(y)} \,\mathrm{d}y.$$

Likewise, by assumption (3.21) we may estimate the integral in the second term on the right-hand side of inequality (3.57) in terms of the area of $\Sigma$ and the Yang-Mills-Higgs energy of $(A, u)$ by

$$\int_\Sigma \big(2 + |\mathrm{d}_A u|^2\big) \,\mathrm{dvol}_\Sigma \leq 2\,\big(\mathrm{Vol}(\Sigma) + E(A, u)\big) \leq 2\,\big(\mathrm{Vol}(\Sigma) + E_0\big).$$

Hence we conclude that there exists a constant $c_{14} > 0$ (not depending on $(A, u)$) such that

$$\begin{aligned}&\int_{B_{\iota(\Sigma)}(\varphi(x))} \frac{\big|\nabla_A F_A(z')\big|}{\mathrm{d}_\Sigma(\varphi(x), z')} \,\mathrm{dvol}_\Sigma(z') \\ &\qquad \leq c_{14} \cdot \Bigg(1 + \frac{1}{r^2} + \int_{B_r(x_0)} \frac{1}{|y-x|} \sqrt{e(y)} \,\mathrm{d}y\Bigg).\end{aligned} \tag{3.58}$$

Combining (3.51), (3.52), (3.48) and (3.58), we finally get estimates

$$\|\delta_{A,s}(\delta_{A,s}J)\|, \|\delta_{A,s}(\delta_{A,t}J)\|, \|\delta_{A,t}(\delta_{A,t}J)\| \\ \leq c_{15} \cdot \left(1 + \frac{1}{r^2} + \int_{B_r(x_0)} \frac{1}{|y-x|} \sqrt{e(y)}\,\mathrm{d}y\right) \tag{3.59}$$

pointwise on $B$ for some constant $c_{15} > 0$ (not depending on $(A, u)$).

Applying estimates (3.43), (3.44), (3.49), (3.50) and (3.59) to inequality (3.42), we conclude that there exists a constant $c_{16} > 0$ (not depending on $(A, u)$) such that

$$\Delta e(x) \geq -c_{16} \left(e^2(x) + e(x) + \frac{e(x)}{r^2} + e(x) \cdot \int_{B_r(x_0)} \frac{1}{|y-x|} \sqrt{e(y)}\,\mathrm{d}y\right)$$

for all $x \in B_r(x_0)$. Since $r \leq 1$ by assumption, the partial differential inequality (3.40) follows, which proves Claim 2.

This completes the proof of Proposition 3.2.3.

## 3.3. Removal of singularities

The goal of this section is to prove a Removable Singularity Theorem for vortices on the punctured unit disk (Theorem 3.3.2). We use Gromov's graph trick in order to reduce this problem to removal of singularities for certain punctured pseudoholomorphic disks. This enables us to apply techniques from McDuff and Salamon [**22**], Section 4.5.

**3.3.1. Vortices on the punctured unit disk.** Let $B \subset \mathbb{C}$ be the closed unit disk and write the complex coordinate on $\mathbb{C}$ as $z = s + \mathrm{i}\,t$. Fix a continuous family $I\colon B \to \mathcal{J}(M, \omega)$ of $\omega$-compatible almost complex structures on $M$, a smooth function $\lambda\colon B \to (0, \infty)$ and a Hamiltonian perturbation $H\colon TB \to C^\infty(M)^G$. Consider the vortex equations

$$\overline{\partial}_{I,A,H}(u) = \frac{1}{2}\left(\mathrm{d}_{A,H}u + I(u) \circ \mathrm{d}_{A,H}u \circ \mathrm{i}\right) = 0, \qquad F_A + \lambda^2\,\mu(u)\,\mathrm{d}s \wedge \mathrm{d}t = 0 \tag{3.60}$$

on the punctured unit disk for pairs $(A, u)$ consisting of a connection $A \in \Omega^1(B \setminus \{0\}, \mathfrak{g})$ of class $C^1$ and a map $u\colon B \setminus \{0\} \to M$ of class $C^1$. As in Section 2.2, we denote by

$$\mathrm{d}_{A,H}u := \mathrm{d}u + X_{A,H}(u), \qquad X_{A,H}(u) := X_A(u) + X_H(u)$$

the derivative of $u$, twisted by the connection $A$ and the perturbation $H$. To simplify notation, we shall use the abbreviation

$$X_{A,H(v)} := X_{A(v)} + X_{H(v)}$$

for the evaluation at a tangent vector $v \in TB$. Solutions of equations (3.60) are called vortices. The Yang-Mills-Higgs energy density of such a vortex $(A, u)$ is the function

$$e(A, u) = \left(\frac{1}{2}\,|\mathrm{d}_{A,H}u|_I^2 + |\mu(u)|_{\mathfrak{g}}^2\right)\lambda^2,$$

where the norms are understood with respect to the Euclidean metric on $B$, the family of Riemannian metrics $\langle\cdot,\cdot\rangle_I := \omega(\cdot, I\cdot)$ on $M$ and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. The Yang-Mills-Higgs energy of $(A,u)$ on an open subset $U \subset B$ is then given by

$$E(A,u;U) = \int_U e(A,u)\ \mathrm{d}s \wedge \mathrm{d}t.$$

Note that this energy may be infinite.

REMARK 3.3.1. Note that equations (3.60) are not invariant under the action of the group of gauge transformations $C^2(B,G)$. This is because the almost complex structure $I$ is not required to be $G$-invariant. However, equations (3.60) will appear naturally in the proof of Gromov compactness for non-local vortices in Section 3.4.

**3.3.2. Main result.** We are now ready to state the main result of this section.

THEOREM 3.3.2 (Removal of singularities). *Fix a continuous family $I\colon B \to \mathcal{J}(M,\omega)$ of $\omega$-compatible almost complex structures on $M$ and a smooth function $\lambda\colon B \to (0,\infty)$. Let $(A,u)$ be a vortex of class $C^1$ solving equations (3.60) on the punctured unit disk $B \setminus \{0\}$, and assume that the following holds.*

(i) *$A$ extends continuously to all of $B$.*
(ii) *$(A,u)$ has finite Yang-Mills-Higgs energy $E(A,u;B) < \infty$.*
(iii) *$(A,u)$ satisfies an a priori estimate of the following form. There exist constants $\delta > 0$ and $C > 0$ such that the following holds. For all $z_0 \in B$ and all $r > 0$ such that $B_r(z_0) \subset B$ the Yang-Mills-Higgs energy density $e := e(A,u)$ satisfies*

$$E\big(A,u;B_r(z_0)\big) < \delta \quad \Longrightarrow \quad e(A,u)(z_0) \leq \frac{C}{r^2} \cdot E\big(A,u;B_r(z_0)\big).$$

*Then $u$ is of class $W^{1,p}$ on $B$ for every real number $p > 2$.*

The actual proof of Theorem 3.3.2 will be given in Section 3.3.6 below. It requires some prelimiaries which we now discuss. Our approach is inspired by the proof of removal of singularities for punctured pseudoholomorphic curves in McDuff and Salamon [**22**], Section 4.5.

Fix a continuous family $I\colon B \to \mathcal{J}(M,\omega)$ of $\omega$-compatible almost complex structures on $M$ and a smooth function $\lambda\colon B \to (0,\infty)$. Let $(A,u)$ be a vortex of class $C^1$ solving equations (3.60) on the punctured unit disk $B \setminus \{0\}$ and satisfying assumptions (i)–(iii) of Theorem 3.3.2.

**3.3.3. The graph construction.** Let $\widetilde{M} := B \times M$ denote the total space of the trivial symplectic fiber bundle over the disk $B$ with fiber $M$. The map $u\colon B \setminus \{0\} \to M$ then gives rise to a section of this bundle given by

$$\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}, \qquad \tilde{u}(z) := \big(z, u(z)\big).$$

The family $I\colon B \to \mathcal{J}(M,\omega)$ of almost complex structures on $M$ gives rise to an almost complex structure $\widetilde{I}$ on the manifold $\widetilde{M}$ defined by

$$\widetilde{I}(v,w) := \big(\mathrm{i}\, v, Iw + I\, X_{A,H(v)}(x) - X_{A,H(\mathrm{i}\, v)}(x)\big) \tag{3.61}$$

for $(z,x) \in B \times M$ and $v \in T_zB$, $w \in T_xM$. In fact, we have

$$\begin{aligned}
\widetilde{I}^2(v,w) &= \big(\mathrm{i}^2\, v, I(Iw + I\, X_{A,H(v)}(x) - X_{A,H(\mathrm{i}\, v)}(x)) + I\, X_{A,H(\mathrm{i}\, v)}(x) - X_{A,H(\mathrm{i}^2\, v)}(x)\big)\\
&= \big(-v, -w - X_{A,H(v)}(x) - IX_{A,H(\mathrm{i}\, v)}(x) + I\, X_{A,H(\mathrm{i}\, v)}(x) + X_{A,H(v)}(x)\big)\\
&= -(v,w).
\end{aligned}$$

Note at this point that the almost complex structure $\widetilde{I}$ is only continuous on $\widetilde{M}$. This follows directly from (3.61) since $I$ and $A$ are only assumed to be continuous on $B$.

The first step towards the proof of Theorem 3.3.2 is the following observation.

Lemma 3.3.3. *The map $\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}$ is $(\mathrm{i}, \widetilde{I})$-holomorphic.*

Proof. We begin by noting that the differential of the map $\tilde{u}$ is given by

$$\mathrm{d}\tilde{u}(v) = \big(v, \mathrm{d}u(v)\big)$$

for $v \in TB$. By the first vortex equation (3.60), we have

$$I\ \mathrm{d}_{A,H}u(v) = \mathrm{d}_{A,H}u(\mathrm{i}\, v).$$

Whence

$$\begin{aligned}
\widetilde{I}\ \mathrm{d}\tilde{u}(v) &= \widetilde{I}\big(v, \mathrm{d}u(v)\big)\\
&= \big(\mathrm{i}\, v, I\, \mathrm{d}u(v) + IX_{A,H(v)}(u) - X_{A,H(\mathrm{i}\, v)}(u)\big)\\
&= \big(\mathrm{i}\, v, I\, \mathrm{d}_{A,H}u(v) - X_{A,H(\mathrm{i}\, v)}(u)\big)\\
&= \big(\mathrm{i}\, v, \mathrm{d}_{A,H}u(\mathrm{i}\, v) - X_{A,H(\mathrm{i}\, v)}(u)\big)\\
&= \big(\mathrm{i}\, v, \mathrm{d}u(\mathrm{i}\, v)\big)\\
&= \mathrm{d}\tilde{u}(\mathrm{i}\, v).
\end{aligned}$$

This implies that

$$\overline{\partial}_{\widetilde{I}}(\tilde{u}) = \frac{1}{2}\big(\mathrm{d}\tilde{u} + \widetilde{I}(\tilde{u}) \circ \mathrm{d}\tilde{u} \circ \mathrm{i}\big) = 0,$$

which proves the lemma. □

Next we define a symplectic form $\widetilde{\omega}$ on the manifold $\widetilde{M}$ that tames the almost complex structure $\widetilde{I}$. It will be defined in terms of the standard symplectic form $\omega_0 := \mathrm{d}s \wedge \mathrm{d}t$ on the disk $B$ and the symplectic form $\omega$ on $M$ in the following way. We fix a constant $c_{A,H} > 1$ such that

$$\big|X_{A,H(v)}(x)\big|_J \le \frac{1}{5}\, c_{A,H} \cdot |v| \tag{3.62}$$

uniformly for all tangent vectors $v \in TB$ and all points $x \in M$, where $|\cdot|_I$ and $|\cdot|$ denote the norms associated to the metric $\langle\cdot,\cdot\rangle_I$ on $M$ and the Euclidean metric on $B$, respectively. Such a constant $c_{A,H}$ exists by compactness of $B$ and $M$. We then define

$$\widetilde{\omega} := c_{A,H}^2 \cdot \omega_0 \,\oplus\, \omega.$$

LEMMA 3.3.4. *The symplectic form $\widetilde{\omega}$ tames the almost complex structure $\widetilde{I}$.*

PROOF. Let $(v,w) \in T\widetilde{M}$ be such that $(v,w) \neq 0$. Applying inequality (3.62), the Cauchy-Schwarz inequality and Young's inequality we obtain

$$\begin{aligned}
\widetilde{\omega}\big((v,w),\widetilde{I}(v,w)\big) &= c_{A,H}^2 \cdot \omega_0(v,\mathrm{i}\,v) + \omega\big(w + X_{A,H(v)}, I(w + X_{A,H(v)})\big) \\
&\quad - \omega\big(X_{A,H(v)}, I(w+X_{A,H(v)})\big) - \omega\big(I(w+X_{A,H(v)}), IX_{A,H(\mathrm{i}\,v)}\big) \\
&\quad + \omega\big(IX_{A,H(v)}, IX_{A,H(\mathrm{i}\,v)}\big) \\
&= c_{A,H}^2 \cdot |v|^2 + \big|w + X_{A,H(v)}\big|_I^2 - \big\langle X_{A,H(v)}, w + X_{A,H(v)}\big\rangle_I \\
&\quad - \big\langle I(w + X_{A,H(v)}), X_{A,H(\mathrm{i}\,v)}\big\rangle_I + \big\langle IX_{A,H(v)}, X_{A,H(\mathrm{i}\,v)}\big\rangle_I \\
&\geq c_{A,H}^2 \cdot |v|^2 + \big|w + X_{A,H(v)}\big|_I^2 - \big|X_{A,H(v)}\big|_I \big|w + X_{A,H(v)}\big|_I \\
&\quad - \big|I(w + X_{A,H(v)})\big|_I \big|X_{A,H(\mathrm{i}\,v)}\big|_I - \big|IX_{A,H(v)}\big|_I \big|X_{A,H(\mathrm{i}\,v)}\big|_I \\
&\geq c_{A,H}^2 \cdot |v|^2 + \big|w + X_{A,H(v)}\big|_I^2 - 4\big|X_{A,H(v)}\big|_I^2 - \frac{1}{4}\big|w + X_{A,H(v)}\big|_I^2 \\
&\quad - \frac{1}{4}\big|w + X_{A,H(v)}\big|_I^2 - 4\big|X_{A,H(\mathrm{i}\,v)}\big|_I^2 - \big|X_{A,H(v)}\big|_I \big|X_{A,H(\mathrm{i}\,v)}\big|_I \\
&\geq c_{A,H}^2 \cdot |v|^2 + \frac{1}{2}\big|w + X_{A,H(v)}\big|_I^2 - 5\big|X_{A,H(v)}\big|_I^2 - 5\big|X_{A,H(\mathrm{i}\,v)}\big|_I^2 \\
&\geq c_{A,H}^2 \cdot |v|^2 + \frac{1}{2}\big|w + X_{A,H(v)}\big|_I^2 - \frac{1}{5}\,c_{A,H}^2 \cdot |v|^2 - \frac{1}{5}\,c_{A,H}^2 \cdot |\mathrm{i}\,v|^2 \\
&= c_{A,H}^2 \cdot |v|^2 + \frac{1}{2}\big|w + X_{A,H(v)}\big|_I^2 - \frac{1}{5}\,c_{A,H}^2 \cdot |v|^2 - \frac{1}{5}\,c_{A,H}^2 \cdot |v|^2 \\
&\geq \frac{1}{2}\Big(c_{A,H}^2 \cdot |v|^2 + \big|w + X_{A,H(v)}\big|_I^2\Big) > 0.
\end{aligned}$$

$\square$

Let us denote by $\langle\cdot,\cdot\rangle_{\widetilde{I}}$ the Riemannian metric on $\widetilde{M}$ associated to the symplectic form $\widetilde{\omega}$ and the almost complex structure $\widetilde{I}$, that is,

$$\big\langle (v_1,w_1),(v_2,w_2)\big\rangle_{\widetilde{I}} := \frac{1}{2}\Big(\widetilde{\omega}\big((v_1,w_1),\widetilde{I}(v_2,w_2)\big) - \widetilde{\omega}\big(\widetilde{I}(v_1,w_1),(v_2,w_2)\big)\Big)$$

for $(v_1,w_1),(v_2,w_2) \in T\widetilde{M}$. The norm associated to this metric is then given by

$$\big|(v,w)\big|_{\widetilde{I}}^2 = \widetilde{\omega}\big((v,w),\widetilde{I}(v,w)\big)$$

for $(v,w) \in T\widetilde{M}$. The next result relates this norm with the metrics on $B$ and $M$.

LEMMA 3.3.5. *The metric on $\widetilde{M}$ determined by $\widetilde{\omega}$ and $\widetilde{I}$ satisfies the relation*

$$\frac{1}{2}\left(|v|^2 + \left|w + X_{A,H(v)}\right|_I^2\right) \le \left|(v,w)\right|_{\widetilde{I}}^2 \le 3\, c_{A,H}^2 \cdot \left(|v|^2 + \left|w + X_{A,H(v)}\right|_I^2\right)$$

*for all tangent vectors $(v,w) \in T\widetilde{M}$, where $c_{A,H}$ is the constant from (3.62).*

PROOF. We have already seen in the proof of Lemma 3.3.4 that

$$\left|(v,w)\right|_{\widetilde{I}}^2 = \widetilde{\omega}\big((v,w), \widetilde{I}(v,w)\big) \ge \frac{1}{2}\left(|v|^2 + \left|w + X_{A,H(v)}\right|_I^2\right).$$

A similar computation yields the estimate

$$\begin{aligned}
\widetilde{\omega}\big((v,w), \widetilde{I}(v,w)\big) &= c_{A,H}^2 \cdot |v|^2 + \left|w + X_{A,H(v)}\right|_I^2 - \left\langle X_{A,H(v)}, w + X_{A,H(v)}\right\rangle_I \\
&\quad - \left\langle I(w + X_{A,H(v)}), X_{A,H(\mathrm{i}\,v)}\right\rangle_I + \left\langle I X_{A,H(v)}, X_{A,H(\mathrm{i}\,v)}\right\rangle_I \\
&\le c_{A,H}^2 \cdot |v|^2 + \left|w + X_{A,H(v)}\right|_I^2 + \left|X_{A,H(v)}\right|_I \left|w + X_{A,H(v)}\right|_I \\
&\quad + \left|I(w + X_{A,H(v)})\right|_I \left|X_{A,H(\mathrm{i}\,v)}\right|_I + \left|I X_{A,H(v)}\right|_I \left|X_{A,H(\mathrm{i}\,v)}\right|_I \\
&\le c_{A,H}^2 \cdot |v|^2 + \left|w + X_{A,H(v)}\right|_I^2 + \left|X_{A,H(v)}\right|_I^2 + \left|w + X_{A,H(v)}\right|_I^2 \\
&\quad + \left|w + X_{A,H(v)}\right|_I^2 + \left|X_{A,H(\mathrm{i}\,v)}\right|_I^2 + \left|X_{A,H(v)}\right|_I \left|X_{A,H(\mathrm{i}\,v)}\right|_I \\
&\le c_{A,H}^2 \cdot |v|^2 + 3\left|w + X_{A,H(v)}\right|_I^2 + 2\left|X_{A,H(v)}\right|_I^2 + 2\left|X_{A,H(\mathrm{i}\,v)}\right|_I^2 \\
&\le c_{A,H}^2 \cdot |v|^2 + 3\left|w + X_{A,H(v)}\right|_I^2 + c_{A,H}^2 \cdot |v|^2 + c_{A,H}^2 \cdot |\mathrm{i}\,v|^2 \\
&= c_{A,H}^2 \cdot |v|^2 + 3\left|w + X_{A,H(v)}\right|_I^2 + c_{A,H}^2 \cdot |v|^2 + c_{A,H}^2 \cdot |v|^2 \\
&\le 3\left(c_{A,H}^2 \cdot |v|^2 + \left|w + X_{A,H(v)}\right|_I^2\right) \\
&\le 3\, c_{A,H}^2\left(|v|^2 + \left|w + X_{A,H(v)}\right|_I^2\right).
\end{aligned}$$

Here we used that $c_{A,H} > 1$ by definition. □

**3.3.4. Mean value inequality.** We derive a mean value inequality for the $\widetilde{I}$-holomorphic curve $\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}$ from the a priori estimate for the vortex $(A,u)$ provided by assumption (iii) of Theorem 3.3.2. We emphasize that the mean value inequality of Lemma 4.3.1 in McDuff and Salamon [**22**] does not apply to the curve $\tilde{u}$ since the almost complex structure $\widetilde{I}$ is only continuous.

Recall first that the energy of the curve $\tilde{u}$ on an open subset $U \subset B$ is given by

$$E(\tilde{u}; U) = \frac{1}{2}\int_U |\mathrm{d}\tilde{u}|_{\widetilde{I}}^2 \cdot \lambda^2 \,\mathrm{d}s \wedge \mathrm{d}t,$$

where the operator norm $|\mathrm{d}\tilde{u}|_{\widetilde{I}}$ is understood with respect to the standard metric on $B$ and the metric on $\widetilde{M}$ determined by $\widetilde{\omega}$ and $\widetilde{I}$ (see [**22**], Section 2.2).

LEMMA 3.3.6. *The energy of the curve* $\tilde{u}\colon B\setminus\{0\}\to\widetilde{M}$ *is finite, that is,*

$$E(\tilde{u},B)<\infty.$$

*Moreover, there exist constants* $\delta>0$, $C_{A,H}>0$ *and* $r_0>0$ *such that for all* $z_0\in B\setminus\{0\}$ *and all* $0<r<r_0$ *such that* $B_r(z_0)\subset B\setminus\{0\}$, *the curve* $\tilde{u}$ *satisfies the mean value inequality*

$$E\big(\tilde{u};B_r(z_0)\big)<\delta \qquad\Longrightarrow\qquad \big|\mathrm{d}\tilde{u}(z_0)\big|_{\widetilde{I}}^2\le\frac{C_{A,H}}{r^2}\cdot E\big(\tilde{u};B_r(z_0)\big)+C_{A,H}.$$

PROOF. We start by comparing the energy of the map $\tilde{u}\colon B\setminus\{0\}\to\widetilde{M}$ and the Yang-Mills-Higgs energy of the vortex $(A,u)$. Recall that the Yang-Mills-Higgs energy of $(A,u)$ on $U\subset B$ is given by

$$E(A,u;U)=\int_U\left(\frac{1}{2}\left|\mathrm{d}_{A,H}u\right|_I^2+|\mu(u)|^2\right)\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t.$$

By Lemma 3.3.5 we have estimates

$$\begin{aligned}|\mathrm{d}\tilde{u}(v)|_{\widetilde{I}}^2&=\big|\big(v,\mathrm{d}u(v)\big)\big|_{\widetilde{I}}^2\\&\ge\frac{1}{2}\Big(|v|^2+\big|\mathrm{d}u(v)+X_{A,H(v)}(u)\big|_I^2\Big)=\frac{1}{2}\Big(|v|^2+|\mathrm{d}_{A,H}u(v)|_I^2\Big)\end{aligned}$$

and

$$\begin{aligned}|\mathrm{d}\tilde{u}(v)|_{\widetilde{I}}^2&=\big|\big(v,\mathrm{d}u(v)\big)\big|_{\widetilde{I}}^2\\&\le 3\,c_{A,H}^2\cdot\Big(|v|^2+\big|\mathrm{d}u(v)+X_{A,H(v)}(u)\big|_I^2\Big)=3\,c_{A,H}^2\cdot\Big(|v|^2+|\mathrm{d}_{A,H}u(v)|_I^2\Big),\end{aligned}$$

for all $v\in TB$. Rewriting these estimates in terms of operator norms we get

$$\frac{1}{2}\Big(1+|\mathrm{d}_{A,H}u|_I^2\Big)\le|\mathrm{d}\tilde{u}|_{\widetilde{I}}^2\le 3\,c_{A,H}^2\cdot\Big(1+|\mathrm{d}_{A,H}u|_I^2\Big). \tag{3.63}$$

Hence we may estimate the energy of $\tilde{u}$ in terms of the Yang-Mills-Higgs energy of $(A,u)$ by

$$\begin{aligned}E(\tilde{u};B)&=\frac{1}{2}\int_B|\mathrm{d}\tilde{u}|_{\widetilde{I}}^2\,\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t\\&\le\frac{3}{2}\,c_{A,H}^2\cdot\int_B\Big(1+|\mathrm{d}_{A,H}u|_I^2\Big)\,\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t\\&=3\,c_{A,H}^2\int_B\left(\frac{1}{2}\,|\mathrm{d}_{A,H}u|_I^2+|\mu(u)|^2\right)\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t\\&\qquad\qquad-3\,c_{A,H}^2\cdot\int_B\left(|\mu(u)|^2-\frac{1}{2}\right)\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t\\&\le 3\,c_{A,H}^2\cdot E(A,u)+3\,c_{A,H}^2\cdot\int_B\left(|\mu(u)|^2+\frac{1}{2}\right)\lambda^2\ \mathrm{d}s\wedge\mathrm{d}t.\end{aligned}$$

The first term on the right-hand side of this inequality is finite as $E(\Phi,\Psi,u)<\infty$ by assumption (ii). The second term is finite because $|\mu(u)|$ is bounded since $M$ is compact. Hence $E(\tilde{u};B)<\infty$, which proves the first assertion of the lemma.

To prove the mean value inequality, we fix $z_0\in B\setminus\{0\}$ and $r>0$ such that $B_r(z_0)\subset B\setminus\{0\}$. By assumption (iii) of the theorem there exist constants $\delta'>0$ and $C>0$ such that the vortex $(\Phi,\Psi,u)$ satisfies the a priori estimate

$$E\big(A,u;B_r(z_0)\big)<\delta' \quad\Longrightarrow\quad e(A,u)(z_0)\le\frac{C}{r^2}\cdot E\big(A,u;B_r(z_0)\big). \tag{3.64}$$

Define a constant

$$K:=\frac{\pi}{2}\cdot\big\|\lambda\big\|^2_{C^0(B)}\cdot\big\|\mu\big\|^2_{C^0(M)}\ge 0.$$

Note that $K$ is finite by compactness of $B$ and $M$. We next define the constants $\delta$ and $C_{A,H}$ by

$$\delta:=\frac{\delta'}{4}>0 \qquad\text{and}\qquad C_{A,H}:=12\,c_{A,H}^2\cdot C\cdot(K+1)\cdot\big\|\lambda\big\|^{-2}_{C^0(B)}>0,$$

where $c_{A,H}$ is the constant from (3.62). We also set

$$r_0:=\min\left\{\sqrt{\frac{\delta'}{2K}},1\right\}>0.$$

Assume now that $r<r_0$ and

$$E\big(\tilde{u};B_r(z_0)\big)<\delta. \tag{3.65}$$

We then obtain from (3.63) the inequality

$$\begin{aligned}
E\big(\tilde{u};B_r(z_0)\big) &= \frac{1}{2}\int_{B_r(z_0)}|\mathrm{d}\tilde{u}|^2\,\lambda^2\,\mathrm{d}s\wedge\mathrm{d}t\\
&\ge \frac{1}{4}\int_{B_r(z_0)}\Big(|\mathrm{d}_{A,H}u|_I^2+1\Big)\,\lambda^2\,\mathrm{d}s\wedge\mathrm{d}t\\
&= \frac{1}{2}\int_{B_r(z_0)}\Big(\frac{1}{2}\,|\mathrm{d}_{A,H}u|_I^2+|\mu(u)|^2\Big)\,\lambda^2\,\mathrm{d}s\wedge\mathrm{d}t\\
&\qquad\qquad -\frac{1}{2}\int_{B_r(z_0)}\Big(|\mu(u)|^2-\frac{1}{2}\Big)\,\lambda^2\,\mathrm{d}s\wedge\mathrm{d}t\\
&\ge \frac{1}{2}\,E\big(A,u;B_r(z_0)\big)-\frac{1}{2}\,K\,r^2.
\end{aligned}$$

We have thus proved that

$$E\big(A,u;B_r(z_0)\big)\le 2\,E\big(\tilde{u};B_r(z_0)\big)+K\,r^2. \tag{3.66}$$

Hence it follows from assumption (3.65) that

$$E\big(A,u;B_r(z_0)\big)<\frac{\delta'}{2}+K\,r^2.$$

Since $r < r_0$ it follows from the definition of $r_0$ and the previous inequality that

$$E\big(A, u; B_r(z_0)\big) < \delta',$$

and so the a priori estimate (3.64) implies that

$$e(A, u)(z_0) \leq \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big).$$

Combining this with the second inequality in (3.63), we obtain

$$\begin{aligned}
|\mathrm{d}\tilde{u}(z_0)|^2 &\leq 3\, c_{A,H}^2 \cdot \Big(1 + |\mathrm{d}_{A,H} u(z_0)|_I^2\Big) \\
&\leq 6\, c_{A,H}^2 \cdot \Big(\frac{1}{2}\, |\mathrm{d}_{A,H} u(z_0)|_I^2 + \big|\mu\big(u(z_0)\big)\big|^2\Big) + 3\, c_{A,H}^2 \\
&= 6\, c_{A,H}^2 \cdot e(A, u)(z_0) \cdot \lambda^{-2}(z_0) + 3\, c_{A,H}^2 \\
&\leq \frac{6\, c_{A,H}^2\, C \cdot \lambda^{-2}(z_0)}{r^2} \cdot E\big(A, u; B_r(z_0)\big) + 3\, c_{A,H}^2.
\end{aligned}$$

Here we used the relation

$$\frac{1}{2}\, |\mathrm{d}_{A,H} u(z_0)|^2 + |\mu(u(z_0))|^2 = e(A, u)(z_0) \cdot \lambda^{-2}(z_0)$$

which holds by formula (2.19). Applying inequality (3.66) again, we obtain from this an estimate

$$\begin{aligned}
|\mathrm{d}\tilde{u}(z_0)|^2 &\leq \frac{12\, c_{A,H}^2\, C \cdot \lambda^{-2}(z_0)}{r^2} \cdot E\big(\tilde{u}; B_r(z_0)\big) + 3\big(4\, CK \cdot \lambda^{-2}(z_0) + 1\big) \cdot c_{A,H}^2 \\
&\leq \frac{C_{A,H}}{r^2} \cdot E\big(\tilde{u}; B_r(z_0)\big) + C_A.
\end{aligned} \tag{3.67}$$

This finishes the proof of Lemma 3.3.6. □

**3.3.5. Isoperimetric inequality.** We shall also need the following isoperimetric inequality for the almost Kähler manifold $\widetilde{M}$ which is taken from Theorem 4.4.1 in [**22**]. Before stating this inequality, we briefly recall some notation from [**22**], Section 4.4. For any smooth loop $\gamma\colon\ \partial B \to \widetilde{M}$ we denote by

$$\ell(\gamma) := \int_0^{2\pi} |\dot{\gamma}(\theta)|\, \mathrm{d}\theta$$

its length with respect to the metric $\langle \cdot, \cdot \rangle_{\widetilde{I}}$. If the length $\ell(\gamma)$ is smaller than the injectivity radius of $\widetilde{M}$, then $\gamma$ admits a smooth local extension $u_\gamma\colon B \to \widetilde{M}$ such that

$$u_\gamma\big(e^{i\theta}\big) = \gamma(\theta)$$

for all $\theta \in [0, 2\pi]$. The local symplectic action of $\gamma$ is then defined by

$$a(\gamma) := -\int_B u_\gamma^*\, \widetilde{\omega}.$$

Note that it does not depend on the choice of local extension $u_\gamma$ as long as the length of $\gamma$ is smaller than the injectivity radius of $\widetilde{M}$. Then the isoperimetric inequality from Theorem 4.4.1 in [**22**] applies to the manifold $\widetilde{M}$ even though the almost complex structure $\widetilde{I}$ is only continuous. In fact, a careful analysis of the proof of Theorem 4.4.1 shows that it does not require the almost complex structure to have more regularity, whence we have proved the following lemma.

LEMMA 3.3.7. *For every constant $c > 1/4\pi$ there exists a constant $\ell_0 > 0$ such that*

$$\ell(\gamma) < \ell_0 \qquad \Longrightarrow \qquad |a(\gamma)| \leq c \cdot \ell(\gamma)^2$$

*for every smooth loop $\gamma\colon \partial B \to \widetilde{M}$.*

The next result shows how this isoperimetric inequality can be used to estimate the energy of the curve $\tilde{u}$. It is adapted from Lemma 4.5.1 in McDuff and Salamon [**22**]. First of all, we introduce some more notation. Let $r_0$ be the constant of Lemma 3.3.6. Define a smooth function

$$\varepsilon\colon (0, r_0] \to [0, \infty), \qquad \varepsilon(r) := E\big(\tilde{u}; B_r(0)\big) = \frac{1}{2}\int_{B_r(0)} |\mathrm{d}\tilde{u}|^2\,,$$

which assigns to every $r$ the energy of the punctured curve $\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}$ on the disk $B_r(0)$, and let $\gamma_r\colon \partial B \to \widetilde{M}$ denote the loop defined by

$$\gamma_r(\theta) := \tilde{u}\big(re^{i\theta}\big)$$

for $\theta \in [0, 2\pi]$. Note that the function $\varepsilon$ is nondecreasing and that $\lim_{r\to 0} \varepsilon(r) = 0$.

LEMMA 3.3.8. *For every constant $c > 1/4\pi$ there exists a constant $r_1 > 0$ such that*

$$0 < r < r_1 \qquad \Longrightarrow \qquad \varepsilon(r) \leq c \cdot \ell(\gamma_r)^2.$$

PROOF. The proof is adapted from that of Lemma 4.5.1 in [**22**]. So let $c > 1/4\pi$, let $\ell_0$ be the constant from Lemma 3.3.7, and let $\delta$, $C_{A,H}$ and $r_0$ be the constants from Lemma 3.3.6. We fix a constant $r_1 > 0$ such that

$$r_1 \leq \min\left\{r_0, \frac{1}{2}\right\} \qquad \text{and} \qquad \varepsilon(2r_1) < \min\left\{\delta, \frac{\ell_0^2 - 2\pi^2\, C_{A,H}\, r_1^2}{2\pi^2\, C_{A,H}}\right\}. \tag{3.68}$$

This is possible since by Lemma 3.3.6 the energy $\varepsilon(1) = E(\tilde{u})$ is finite and because the function $\varepsilon$ is smooth and nondecreasing with $\lim_{r\to 0}\varepsilon(r) = 0$.

Let now $0 < r \leq r_1$. Then $E(\tilde{u}; B_{2r}(0)) = \varepsilon(2r) < \delta$ by inequality (3.68), so the mean value inequality of Lemma 3.3.6 yields

$$\big|\mathrm{d}\tilde{u}\big(re^{i\theta}\big)\big|^2 \leq \frac{C_{A,H}}{r^2} \cdot \varepsilon(2r) + C_{A,H}.$$

Hence the norm of the derivative of $\gamma_r$ in the direction of $\theta$ satisfies an estimate

$$|\dot{\gamma}_r(\theta)| = \frac{r}{\sqrt{2}} \cdot \big|\mathrm{d}\tilde{u}\big(re^{i\theta}\big)\big| \leq \sqrt{\frac{C_{A,H}}{2} \cdot \varepsilon(2r) + \frac{C_{A,H}}{2} \cdot r^2}.$$

Using inequality (3.68) we thus infer that the length of the loop $\gamma_r$ satisfies

$$\ell(\gamma_r) = \int_0^{2\pi} |\dot{\gamma}_r(\theta)|\, \mathrm{d}\theta \leq \sqrt{2\pi^2\, C_{A,H} \cdot \varepsilon(2r) + 2\pi^2\, C_{A,H} \cdot r^2} < \ell_0. \tag{3.69}$$

Let $0 < \rho \leq r \leq r_1$, and denote by $u_\rho \colon B \to \widetilde{M}$ the local extension of the loop $\gamma_\rho$ defined by the formula

$$u_\rho\big(\rho' e^{i\theta}\big) := \exp_{\gamma_\rho(0)}\big(\rho'\, \xi(\theta)\big)$$

for $0 < \rho' < \rho$ and $\theta \in [0, 2\pi]$, where the map $\xi \colon [0, 2\pi] \to T_{\gamma_\rho(0)}\widetilde{M}$ is determined by the condition

$$\exp_{\gamma_\rho(0)}\big(\xi(\theta)\big) = \gamma_\rho(\theta).$$

We consider the sphere $v_{\rho r} \colon S^2 \to \widetilde{M}$ that is obtained from the restriction of the curve $\tilde{u} \colon B \setminus \{0\} \to \widetilde{M}$ to the annulus $B_r(0) \setminus B_\rho(0)$ by filling in the boundary circles $\gamma_\rho$ and $\gamma_r$ with the disks $u_\rho$ and $u_r$. Since the symplectic form $\widetilde{\omega}$ tames the almost complex structure $\widetilde{I}$ by Lemma 3.3.4, it follows from Lemma 2.2.1 in [**22**] that the $\widetilde{I}$-holomorphic curve $\tilde{u} \colon B_r(0) \setminus B_\rho(0) \to \widetilde{M}$ satisfies the energy identity

$$E\big(\tilde{u}; B_r(0) \setminus B_\rho(0)\big) = \int_{B_r(0)\setminus B_\rho(0)} \tilde{u}^*\, \widetilde{\omega}.$$

The sphere $v_{\rho r} \colon S^2 \to \widetilde{M}$ is contractible because it is the boundary of the 3-ball consisting of the union of the disks $u_s \colon B \to \widetilde{M}$ for $\rho \leq s \leq r$. Whence

$$0 = \int_{S^2} v_{\rho r}^*\, \widetilde{\omega} = \int_{B_r(0)\setminus B_\rho(0)} \tilde{u}^*\, \widetilde{\omega} + \int_B u_\rho^*\, \widetilde{\omega} - \int_B u_r^*\, \widetilde{\omega}.$$

To understand the minus sign on the right-hand side of this identity note that the disks $u_\rho$ and $u_r$ have different orientation considered as submanifolds of the sphere $S^2$. Hence we obtain the identity

$$E\big(\tilde{u}; B_r(0) \setminus B_\rho(0)\big) + \int_B u_\rho^*\, \widetilde{\omega} = \int_B u_r^*\, \widetilde{\omega}.$$

Taking the limit $\rho \to 0$, we obtain from this an equality

$$\varepsilon(r) = E\big(\tilde{u}; B_r(0)\big) = \int_B u_r^*\, \widetilde{\omega} = a(\gamma_r),$$

which expresses the energy of the punctured curve $\tilde{u} \colon B \setminus \{0\} \to \widetilde{M}$ in terms of the local symplectic action of the loop $\gamma_r$. By inequality (3.69) we may apply the isoperimetric inequality of Lemma 3.3.7, whence

$$\varepsilon(r) \leq c \cdot \ell(\gamma_r)^2.$$

This proves Lemma 3.3.8. $\square$

We are now ready for the actual proof of Theorem 3.3.2.

**3.3.6. Proof of Theorem 3.3.2.** This proof is modeled on the proof of Theorem 4.1.2 in McDuff and Salamon [**22**], Section 4.5.

Let $p > 2$ be a real number. Choose $r > 0$ such that

$$r < r_1 \leq \frac{1}{2} \qquad \text{and} \qquad \varepsilon(2r) < \delta,$$

where $r_1$ and $\delta$ are the constants from Lemma 3.3.8 and Lemma 3.3.6, respectively. Note that the existence of $r$ is guaranteed because $\varepsilon(1) = E(\tilde{u}; B) < \infty$ by Lemma 3.3.6 and the function $\varepsilon$ is smooth and nondecreasing with $\lim_{r\to 0} \varepsilon(r) = 0$.

Let $0 < \rho \leq r$. First of all, note that we may write the function $\varepsilon$ in the form

$$\varepsilon(\rho) = \frac{1}{2} \cdot \int_{B_\rho(0)} |\mathrm{d}\tilde{u}|^2 = \frac{1}{2} \cdot \int_0^\rho \rho' \int_0^{2\pi} \big|\mathrm{d}\tilde{u}\big(\rho' e^{i\theta}\big)\big|^2 \,\mathrm{d}\theta \,\mathrm{d}\rho'.$$

Choose the constant $c > 1/4\pi$ from Lemma 3.3.8 sufficiently small such that

$$c < \frac{1}{4\pi} + \frac{1}{2\pi\, p}. \tag{3.70}$$

The isoperimetric inequality of Lemma 3.3.8 then yields

$$\begin{aligned}
\varepsilon(\rho) &\leq c \cdot \ell(\gamma_\rho)^2 \\
&= \frac{c\,\rho^2}{2} \cdot \left( \int_0^{2\pi} \big|\mathrm{d}\tilde{u}\big(\rho e^{i\theta}\big)\big| \,\mathrm{d}\theta \right)^2 \\
&\leq \pi\, c\, \rho^2 \cdot \int_0^{2\pi} \big|\mathrm{d}\tilde{u}\big(\rho e^{i\theta}\big)\big|^2 \,\mathrm{d}\theta \\
&= 2\pi\, c\, \rho \cdot \dot{\varepsilon}(\rho).
\end{aligned}$$

Here we used Hölder's inequality and the explicit formula for $\varepsilon(\rho)$ given above. Setting

$$\alpha := \frac{1}{4\pi\, c} < 1$$

we may rewrite the previous inequality in the form

$$\frac{2\alpha}{\rho} \leq \frac{\dot{\varepsilon}(\rho)}{\varepsilon(\rho)}.$$

Let $\rho < \rho_1 < r_1$. Integrating this differential inequality from $\rho$ to $\rho_1$ then yields

$$\left(\frac{\rho_1}{\rho}\right)^{2\alpha} \leq \frac{\varepsilon(\rho_1)}{\varepsilon(\rho)},$$

whence

$$\varepsilon(\rho) \leq c_1 \cdot \rho^{2\alpha}$$

for some constant $c_1 := \rho_1^{-2\alpha} \cdot \varepsilon(\rho_1)$. Applying the mean value inequality of Lemma 3.3.6 we obtain the estimate

$$\big|\mathrm{d}\tilde{u}\big(\rho e^{i\theta}\big)\big|^2 \leq \frac{C_{A,H}}{\rho^2} \cdot \varepsilon(2\rho) + C_{A,H} \leq c_2 \cdot \rho^{-2(1-\alpha)} + C_{A,H},$$

where $C_{A,H}$ is the constant of Lemma 3.3.6 and $c_2 > 0$ is some constant (not depending on $\rho$ and $\theta$). Furthermore

$$\big|\mathrm{d}\tilde{u}\big(\rho e^{i\theta}\big)\big| \leq \sqrt{c_2} \cdot \rho^{-(1-\alpha)} + \sqrt{C_{A,H}}. \tag{3.71}$$

This inequality has two important consequences. First, applying Minkowski's inequality we obtain

$$\bigg(\int_{B_r(0)} \big|\mathrm{d}\tilde{u}\big|^p\bigg)^{\frac{1}{p}} \leq \sqrt{c_2} \cdot \bigg(\int_{B_r(0)} \rho^{-p(1-\alpha)}\bigg)^{\frac{1}{p}} + \sqrt{C_{A,H}} \cdot \big(\pi\, r^2\big)^{\frac{1}{p}},$$

where the integral

$$\int_{B_r(0)} \rho^{-p(1-\alpha)} = \int_0^r \int_0^{2\pi} \rho^{1-p(1-\alpha)}\, \mathrm{d}\theta\, \mathrm{d}\rho = 2\pi \cdot \int_0^r \rho^{1-p(1-\alpha)}\, \mathrm{d}\rho$$

is finite because

$$p < \frac{2}{1-\alpha}$$

by (3.70) and hence $1 - p\,(1-\alpha) > -1$. Thus we conclude that

$$\bigg(\int_{B_r(0)} \big|\mathrm{d}\tilde{u}\big|^p\bigg)^{\frac{1}{p}} < \infty.$$

This shows that the derivative of the punctured curve $\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}$ is of class $L^p$ on the disk $B_r(0)$. Second, it follows from (3.71) that the punctured curve $\tilde{u}\colon B \setminus \{0\} \to \widetilde{M}$ extends continuously over zero. In fact, consider any two points $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ in the punctured disk $B_r(0) \setminus \{0\}$ such that $0 < r_1 < r_2 \leq r$ and $\theta_1, \theta_2 \in [0, 2\pi]$. Then

$$\begin{aligned}
\big|\tilde{u}(z_2) - \tilde{u}(z_1)\big| &\leq \big|\tilde{u}\big(r_2 e^{i\theta_2}\big) - \tilde{u}\big(r_1 e^{i\theta_2}\big)\big| + \big|\tilde{u}\big(r_1 e^{i\theta_2}\big) - \tilde{u}\big(r_1 e^{i\theta_1}\big)\big| \\
&\leq \bigg|\int_{r_1}^{r_2} \mathrm{d}\tilde{u}\big(\rho\, e^{i\theta_2}\big)\, \mathrm{d}\rho\bigg| + \bigg|\int_{\theta_1}^{\theta_2} \mathrm{d}\tilde{u}\big(r_1 e^{i\theta}\big)\, \mathrm{d}\theta\bigg| \\
&\leq \int_{r_1}^{r_2} \big|\mathrm{d}\tilde{u}\big(\rho\, e^{i\theta_2}\big)\big|\, \mathrm{d}\rho + \int_{\theta_1}^{\theta_2} \big|\mathrm{d}\tilde{u}\big(r_1 e^{i\theta}\big)\big|\, \mathrm{d}\theta \\
&\leq c_3 \cdot \int_{r_1}^{r_2} \Big(\rho^{-(1-\alpha)} + 1\Big)\, \mathrm{d}\rho + c_4 \cdot \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \\
&= c_3 \cdot \Big(r_2^\alpha + r_2 - r_1^\alpha - r_1\Big) + c_4 \cdot \big(\theta_2 - \theta_1\big) \\
&\leq c_3 \cdot \Big((r_2 - r_1)^\alpha + r_2 - r_1\Big) + c_5 \cdot \big|e^{i\theta_2} - e^{i\theta_1}\big|
\end{aligned}$$

for constants $c_3, c_4, c_5 > 0$ not depending on $z_1$ and $z_2$. As the manifold $\widetilde{M}$ is compact and hence complete, we conclude that $\tilde{u}$ extends continuously over zero. Since the curve $\tilde{u}$ is smooth on the punctured disk $B\setminus\{0\}$, Exercise 4.5.4 in [**22**] now shows that the weak first derivatives of $\tilde{u}$ exist on $B_r(0)$ and agree with the strong derivatives on $B_r(0)\setminus\{0\}$. Thus $\tilde{u}$ is of class $W^{1,p}$ on the disk $B_r(0)$ and hence on all of $B$.

This completes the proof of Theorem 3.3.2.

## 3.4. Convergence modulo bubbling

The aim of this section is to prove a compactness result for non-local vortices, ignoring any bubbling phenomena (Theorem 3.4.1). This is the first step towards Gromov compactness for non-local vortices.

**3.4.1. Main result.** The main result of this section is the following theorem. Recall that we denote by $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(P,M)^G)$ the space of Hamiltonian perturbations of class $C^\ell$, for any positive integer $\ell$.

THEOREM 3.4.1 (Convergence modulo bubbling). *Fix a real constant $E>0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p>2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$.*

*Let $(A_\nu, u_\nu)$ be a sequence of non-local vortices solving equations (2.16) such that the Yang-Mills-Higgs energy satisfies a uniform bound*

$$\sup_\nu E\big(A_\nu, u_\nu\big) < \infty.$$

*Then there exist a non-local vortex $(A,u)$ of class $C^{\ell-1}$ that solves equations (2.16), a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$, and a finite set $Z=\{z_1,\dots,z_N\}$ of distinct points on $\Sigma$ such that, after passing to a subsequence if necessary, the following holds.*

(i) *The sequence $g_\nu^*A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on $\Sigma$;*
(ii) *the sequence $(g_\nu^*A_\nu, g_\nu^{-1}u_\nu)$ converges to $(A,u)$ weakly in the $W^{2,p}$-topology and strongly in the $C^1$-topology on compact subsets of $\Sigma\setminus Z$;*
(iii) *for every $j\in\{1,\dots,N\}$ and every $\varepsilon>0$ such that $B_\varepsilon(z_j)\cap Z=\{z_j\}$, the limit*

$$m_\varepsilon(z_j) := \lim_{\nu\to\infty} E\big(g_\nu^*A_\nu, g_\nu^{-1}u_\nu; B_\varepsilon(z_j)\big)$$

*exists and is a continuous function of $\varepsilon$, and*

$$m(z_j) := \lim_{\varepsilon\to 0} m_\varepsilon(z_j) \geq \hbar,$$

*where $\hbar$ is the constant of Theorem 3.2.1;*

(iv) *for every compact subset $K \subset \Sigma$ such that $Z$ is contained in the interior of $K$,*

$$E\big(A, u; K\big) + \sum_{j=1}^{N} m(z_j) = \lim_{\nu\to\infty} E\big(g_\nu^* A_\nu, g_\nu^{-1} u_\nu; K\big).$$

Under the additional hypothesis that no bubbling occurs, we get the following stronger compactness result, which we state here for later reference.

COROLLARY 3.4.2. *Fix a real constant $E > 0$ and an $E$-admissible area form* $\mathrm{dvol}_\Sigma$ *on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some integer $\ell \geq 4$. Let $(A_\nu, u_\nu)$ be a sequence of non-local vortices solving equations (2.16) such that*

$$\sup_\nu E\big(A_\nu, u_\nu\big) < \infty \quad \textit{and} \quad \sup_\nu \big\|\mathrm{d}_{A_\nu,H} u_\nu\big\|_{L^\infty(\Sigma)} < \infty.$$

*Then there exist a non-local vortex $(A, u)$ of class $C^{\ell-1}$ that solves equations (2.16), and a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that, after passing to a subsequence if necessary, the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ converges to $(A, u)$ weakly in the $W^{2,p}$-topology and strongly in the $C^1$-topology on $\Sigma$.*

The proof of Theorem 3.4.1 and its Corollary 3.4.2 will occupy the remainder of this section. It is inspired by the proofs of Theorem 4.6.1 in McDuff and Salamon [**22**] and Theorem 3.2 in Cieliebak et. al. [**3**].

Throughout this section, let us make the following assumptions. We fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for $\ell \geq 4$. Consider a sequence

$$(A_\nu, u_\nu) \in \mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$$

of solutions of the non-local vortex equations (2.16) such that the Yang-Mills-Higgs energy satisfies a uniform bound

$$\sup_\nu E\big(A_\nu, u_\nu\big) < \infty.$$

**3.4.2. Singular points.** The first step in the proof Theorem 3.4.1 is to investigate those points on $\Sigma$ where bubbling occurs. Following the terminology in McDuff and Salamon [**22**], Section 4.6, we call a point $z \in \Sigma$ **singular** for the sequence $(A_\nu, u_\nu)$ if there exists a sequence $z^\nu$ of points in $\Sigma$ converging to $z$ such that

$$\big|\mathrm{d}_{A_\nu,H} u_\nu(z^\nu)\big|_{J_\Theta} \to \infty.$$

The main result of this subsection is the observation that the sequence $(A_\nu, u_\nu)$ can have only finitely many singular points. This is the content of Lemma 3.4.4 below. It relies on the following technical lemma.

LEMMA 3.4.3. *Let $z$ be a singular point of the sequence $(A_\nu, u_\nu)$. Then*

$$\liminf_{\nu\to\infty} E\big(A_\nu, u_\nu; B_\varepsilon(z)\big) \geq \hbar$$

*for every $0 < \varepsilon \leq R$, where $\hbar$ and $R$ are the constants from Theorem 3.2.1.*

PROOF. Our proof follows the lines of the proof of Theorem 2.1 in Wehrheim [**35**]. Let $z$ be a singular point of the sequence $(A_\nu, u_\nu)$, and assume for contradiction that

$$\liminf_{\nu\to\infty} E\big(A_\nu, u_\nu; B_\varepsilon(z)\big) < \hbar$$

for some $0 < \varepsilon \leq R$. Since $z$ is singular there exists a sequence $z^\nu \to z$ such that

$$\big|\mathrm{d}_{A_\nu,H} u_\nu(z^\nu)\big|_{J_\Theta} \to \infty.$$

Hence there exists $\nu_0$ such that

$$z^{\nu_0} \in B_{\varepsilon/2}(z), \quad \big|\mathrm{d}_{A_\nu,H} u_\nu(z^{\nu_0})\big|_{J_\Theta} > \frac{8\,C\hbar}{\varepsilon^2} \quad \text{and} \quad E\big(A_\nu, u_\nu; B_{\varepsilon/2}(z^{\nu_0})\big) < \hbar. \tag{3.72}$$

We may therefore apply the a priori estimate from Theorem 3.2.1 to the vortex $(A_\nu, u_\nu)$ on the disk $B_{\varepsilon/2}(z^{\nu_0})$, obtaining an estimate

$$\frac{1}{2}\big|\mathrm{d}_{A_\nu,H} u_\nu(z^{\nu_0})\big|^2_{J_\Theta} \leq \frac{1}{2}\big|\mathrm{d}_{A_\nu,H} u_\nu(z^{\nu_0})\big|^2_{J_\Theta} + \big|\mu(u_\nu(z^{\nu_0}))\big|^2 \leq \frac{C}{(\varepsilon/2)^2} \cdot E\big(A_\nu, u_\nu; B_{\varepsilon/2}(z^{\nu_0})\big).$$

Using the third inequality in (3.72), it now follows from this that

$$\big|\mathrm{d}_{A_\nu,H} u_\nu(z^{\nu_0})\big|^2_{J_\Theta} \leq \frac{8\,C\hbar}{\varepsilon^2},$$

contradicting the second inequality in (3.72). □

Since $\sup_\nu E(A_\nu, u_\nu) < \infty$ by assumption, it follows from Lemma 3.4.3 that the sequence $(A_\nu, u_\nu)$ has finitely many singular points. More specifically, we have the following result.

LEMMA 3.4.4. *After passing to a subsequence if necessary, the sequence $(A_\nu, u_\nu)$ has a finite set $Z = \{z_1, \ldots, z_N\} \subset \Sigma$ of singular points and satisfies*

$$\sup_\nu \big\|\mathrm{d}_{A_\nu,H} u_\nu\big\|_{L^\infty(K)} < \infty$$

*for every compact set $K \subset \Sigma \setminus Z$.*

PROOF. The proof of this lemma is exactly the same as that of the Claim in the proof of Theorem 4.6.1 in [**22**]. However, for the sake of completeness we record it here anyway. By Lemma 3.4.3 the sequence $(A_\nu, u_\nu)$ has finitely many singular points. We may hence assume by induction that, after passing to a subsequence if necessary, the set of singular points of the sequence $(A_\nu, u_\nu)$ contains the set

$$Z_k = \{z_1, \ldots, z_k\}.$$

(We start with $k = 0$ and $Z_k = \emptyset$.) If the sequence $\|\mathrm{d}_{A_\nu} u_\nu\|_{L^\infty(K)}$ is bounded for every compact set $K \subset \Sigma \setminus Z_k$ then there is nothing to prove. Otherwise, we can choose a compact subset $K \subset \Sigma \setminus Z_k$ such that the sequence $\|\mathrm{d}_{A_\nu} u_\nu\|_{L^\infty(K)}$ is unbounded. Then we may take a sequence of points $z_{k+1}^\nu$ in $K$ such that

$$\left\|\mathrm{d}_{A_\nu,H} u_\nu\right\|_{L^\infty(K)} = \left|\mathrm{d}_{A_\nu,H} u_\nu(z_{k+1}^\nu)\right|_{J_\Theta}.$$

After passing to a further subsequence if necessary, we may assume that the sequence $z_{k+1}^\nu$ converges to a point $z_{k+1} \in \Sigma \setminus Z_k$ and

$$\left|\mathrm{d}_{A_\nu,H} u_\nu(z_{k+1}^\nu)\right|_{J_\Theta} \to \infty.$$

Whence the singular set of this latter subsequence contains the set

$$Z_{k+1} := Z_k \cup \{z_{k+1}\}.$$

Since the number of possible singular points is finite, this completes the proof. □

By Lemma 3.4.4 above, it is no loss of generality to assume that the sequence $(A_\nu, u_\nu)$ has finitely many singular points

$$Z := \{z_1, \dots, z_N\} \subset \Sigma$$

and satisfies

$$\sup_\nu \left\|\mathrm{d}_{A_\nu,H} u_\nu\right\|_{L^\infty(K)} < \infty \tag{3.73}$$

for every compact set $K \subset \Sigma \setminus Z$ contained in the complement of the singular points.

We are now ready to investigate the convergence properties of the sequence $(A_\nu, u_\nu)$ more closely.

**3.4.3. Uhlenbeck compactness and Coulomb gauge.** We apply Uhlenbeck's weak compactness theorem, the local slice theorem for the action of the group of gauge transformations, and the Banach-Alaoglu theorem (see [**34**] and [**38**] for details) to the sequence $(A_\nu, u_\nu)$.

LEMMA 3.4.5. *There exist a pair $(A, u)$ consisting of a connection $A \in \mathcal{A}^{1,p}(P)$ and a section $u \in W^{1,p}_{\mathrm{loc}}(\Sigma \setminus Z, P(M))$ defined on $\Sigma \setminus Z$, a smooth reference connection $A_0 \in \mathcal{A}(P)$, and a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that the following holds.*

(i) *The connection $A$ is in Coulomb gauge relative to $A_0$, that is,*

$$\mathrm{d}_{A_0}^*(A - A_0) = 0.$$

(ii) *After passing to a subsequence if necessary, the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ converges to $(A, u)$ in the following sense.*

   (a) *The sequence $g_\nu^* A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on $\Sigma$.*

(b) *The sequence* $g_\nu^{-1}u_\nu$ *converges to* $u$ *weakly in the* $W^{1,p}$*-topology and strongly in the* $C^0$*-topology on compact subsets of* $\Sigma \setminus Z$.

(c) *Every* $g_\nu^* A_\nu$ *is in Coulomb gauge relative to* $A$*, that is,*

$$\mathrm{d}_A^*\big(g_\nu^* A_\nu - A\big) = 0.$$

(iii) *The Yang-Mills-Higgs energy of the sequence* $(g_\nu^* A_\nu, g_\nu^{-1}u_\nu)$ *satisfies a uniform bound*

$$\sup_\nu E(g_\nu^* A_\nu, g_\nu^{-1}u_\nu) < \infty.$$

Proof. Our argument is taken from the proof of Theorem 3.2 in Cieliebak et. al. [**3**]. By assumption, $(A_\nu, u_\nu)$ is a solution of the second vortex equation

$$F_{A_\nu} = -\mu(u_\nu)\, \mathrm{dvol}_\Sigma\,.$$

Since $M$ is compact, the right-hand side of this equation is uniformly bounded in the $L^p$-norm on $\Sigma$, whence the sequence $F_{A_\nu}$ is uniformly bounded in the $L^p$-norm on $\Sigma$. Thus, by weak Uhlenbeck compactness ([**34**], Thm. A) there exists a sequence of gauge transformations $h_\nu \in \mathcal{G}^{2,p}(P)$ such that the sequence $h_\nu^* A_\nu$ is uniformly bounded in $\mathcal{A}^{1,p}(P)$ with respect to the $W^{1,p}$-topology. It follows from the Banach-Alaoglu theorem ([**38**], Thm. V.2.1) that there exists a connection $\tilde{A} \in \mathcal{A}^{1,p}(P)$ such that, after passing to a subsequence if necessary, the sequence $h_\nu^* A_\nu$ converges to $\tilde{A}$ weakly in the $W^{1,p}$-topology.

Now we apply the local slice theorem ([**34**], Thm. F, see also [**3**], Thm. B.1). We take $\tilde{A}$ as reference connection and choose a smooth connection $A_0 \in \mathcal{A}(P)$ such that $\|\tilde{A} - A_0\|_{W^{1,p}(\Sigma)}$ (and hence also $\|\tilde{A} - A_0\|_{L^p(\Sigma)}$) is sufficiently small. Then the local slice theorem (taking $q = p$) asserts the existence of a gauge transformation $h \in \mathcal{G}^{2,p}(P)$ such that

$$\mathrm{d}_{\tilde{A}}^*\big(h_* A_0 - \tilde{A}\big) = 0.$$

This implies (see [**34**], Lemma 8.4 (ii)) that

$$\mathrm{d}_{h_* A_0}^*\big(\tilde{A} - h_* A_0\big) = 0,$$

whence

$$\mathrm{d}_{A_0}^*\big(h^*\tilde{A} - A_0\big) = h^*\, \mathrm{d}_{h_* A_0}^*\big(\tilde{A} - h_* A_0\big) = 0.$$

Define the connection

$$A := h^*\tilde{A} \in \mathcal{A}^{1,p}(P).$$

Then $A$ is in Coulomb gauge relative to $A_0$. This proves (i).

We have seen above that $h_\nu^* A_\nu$ is uniformly bounded in $\mathcal{A}^{1,p}(P)$, so $h^* h_\nu^* A_\nu$ is uniformly bounded in $\mathcal{A}^{1,p}(P)$ as well (by continuity of the action of the group of gauge transformations, see [**34**], Lemma A.6). Hence it follows from the Banach-Alaoglu theorem and Rellich's theorem ([**34**], Thm. B.2 (iii)) that, after passing to a subsequence if necessary, the sequence $h^* h_\nu^* A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology. Now we apply the local slice theorem a second time, taking $A$ as reference connection. By what we just proved, we have

$$\lim_{\nu\to\infty} \|h^*h_\nu^*A_\nu - A\|_{L^p(\Sigma)} = 0, \qquad \sup_\nu \|h^*h_\nu^*A_\nu - A\|_{W^{1,p}(\Sigma)} < \infty.$$

Then the local slice theorem (again taking $q = p$) asserts the existence of a sequence of gauge transformations $\hat{h}_\nu \in \mathcal{G}^{2,p}(P)$ such that

$$\mathrm{d}_A^*\big(\hat{h}_\nu^*h^*h_\nu^*A_\nu - A\big) = 0 \tag{3.74}$$

and

$$\lim_{\nu\to\infty} \Big\|\hat{h}_\nu^*h^*h_\nu^*A_\nu - A\Big\|_{L^p(\Sigma)} = 0, \qquad \sup_\nu \Big\|\hat{h}_\nu^*h^*h_\nu^*A_\nu - A\Big\|_{W^{1,p}(\Sigma)} < \infty. \tag{3.75}$$

We finally define the desired sequence of gauge transformations by

$$g_\nu := h_\nu h^* \hat{h}_\nu \in \mathcal{G}^{2,p}(P)$$

(see [**34**], Lemma A.5). Then (3.74) proves (c) in (ii).

Furthermore, by the Banach-Alaoglu theorem and Rellich's theorem it follows from the second estimate in (3.75) that the sequence $g_\nu^*A_\nu$ is uniformly bounded in the $W^{1,p}$-norm, and that, after passing to a subsequence if necessary, the sequence $g_\nu^*A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology. This proves part (a) in (ii).

It remains to show convergence for the sequence of sections $g_\nu^{-1}u_\nu$. By Hölder's inequality it follows from (3.73) that

$$\sup_\nu \big\|\mathrm{d}_{g_\nu^*A_\nu,H}\big(g^{-1}u_\nu\big)\big\|_{L^p(K)} = \sup_\nu \|\mathrm{d}_{A_\nu,H}u_\nu\|_{L^p(K)} \le \sup_\nu \|\mathrm{d}_{A_\nu,H}u_\nu\|_{L^\infty(K)} < \infty$$

for every compact subset $K \subset \Sigma \setminus Z$. Hence, by compactness of $M$ it follows that the sequence $g_\nu^{-1}u_\nu$ is uniformly bounded in the $W^{1,p}$-norm on compact subsets of $\Sigma \setminus Z$. Thus we conclude from the Banach-Alaoglu theorem and Rellich's theorem that there exists a section $u \in W^{1,p}_{\mathrm{loc}}(\Sigma \setminus Z, P(M))$ that is of class $W^{1,p}$ on compact subsets of $\Sigma \setminus Z$, such that, after passing to a subsequence if necessary, the sequence $g_\nu^{-1}u_\nu$ converges to $u$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on compact subsets of $\Sigma \setminus Z$. This proves (a) and (b) in (ii).

Lastly, Assertion (iii) follows from the assumption $\sup_\nu E(A_\nu, u_\nu) < \infty$ by gauge invariance of the Yang-Mills-Higgs energy. The lemma is proved. □

By Lemma 3.4.5 and gauge invariance of the vortex equations (2.16) we may henceforth assume without loss of generality that there exists a pair $(A, u)$ consisting of a connection $A \in \mathcal{A}^{1,p}(P)$ on $P$ and a section $u \in W^{1,p}_{\mathrm{loc}}(\Sigma \setminus Z, P(M))$ of $P(M)$ defined on $\Sigma \setminus Z$, and a smooth reference connection $A_0 \in \mathcal{A}(P)$, such that the following holds.

(i') The connection $A$ is in Coulomb gauge relative to $A_0$, that is,

$$\mathrm{d}^*_{A_0}(A - A_0) = 0.$$

(ii') The sequence $(A_\nu, u_\nu)$ converges to $(A, u)$ in the following sense.
  (a) The sequence $A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on $\Sigma$.
  (b) The sequence $u_\nu$ converges to $u$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on compact subsets of $\Sigma \setminus Z$.
  (c) Every $A_\nu$ is in Coulomb gauge relative to $A$, that is,

$$\mathrm{d}^*_A\big(A_\nu - A\big) = 0.$$

(iii') The Yang-Mills-Higgs energy of the sequence $(A_\nu, u_\nu)$ satisfies a uniform bound

$$\sup_\nu E(A_\nu, u_\nu) < \infty.$$

Let us note at this point that part (a) of (ii') above proves assertion (i) of Theorem 3.4.1.

**3.4.4. The limit equations.** This subsection is devoted to the study of the nonlocal vortex equations

$$\overline{\partial}_{J,A_\nu,\Theta,H}(u_\nu) = 0, \qquad F_{A_\nu} + \mu(u_\nu)\,\mathrm{dvol}_\Sigma = 0$$

in the limit $\nu \to \infty$. This yields equations for the limit pair $(A, u)$. Since the sequence of sections $u_\nu$ only converges on compact subsets of $\Sigma \setminus Z$, these limit equations will only be defined on $\Sigma \setminus Z$. However, using the Removable Singularity Theorem 3.3.2 we will later be able to write down equations for $(A, u)$ that hold on all of $\Sigma$.

Lemma 3.4.6. *There exists a $G$-equivariant map*

$$\Theta_\infty \colon P \to EG^N$$

*of class $W^{1,p}$ such that, after passing to a subsequence if necessary, the sequence of maps*

$$\Theta_{(A_\nu,u_\nu)} \colon P \to EG^N$$

*converges to $\Theta_\infty$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology.*

Proof. By part (a) of (ii') on p. 106 the sequence $A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology on $\Sigma$ and is therefore uniformly bounded in the $W^{1,p}$-topology on $\Sigma$ by the Banach-Steinhaus theorem. Hence the sequence $F_{A_\nu}$ is uniformly bounded in $L^p$ on $\Sigma$, whence part (i) of the (Estimates) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) implies that the sequence of maps $\Theta_{(A_\nu,u_\nu)}$ is uniformly bounded in the $W^{1,p}$-topology on $\Sigma$ as well. Hence we conclude from the theorems of Banach-Alaoglu and Rellich that there exists a $G$-equivariant map $\Theta_\infty\colon P \to EG^N$ of class $W^{1,p}$ such that, after passing to a subsequence if necessary, the sequence $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_\infty$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology. □

The limit classifying map $\Theta_\infty\colon P \to EG^N$ gives rise to a $G$-equivariant family of $\omega$-compatible almost complex structures on $M$, denoted by

$$J_{\Theta_\infty} := J \circ \Theta_\infty\colon P \to \mathcal{J}(M,\omega).$$

Note that, by Lemma 3.4.6 above, the family $J_{\Theta_\infty}$ is of class $W^{1,p}$ and hence continuous by Rellich's theorem.

Lemma 3.4.7. *The pair $(A,u)$ is a solution of class $W^{1,p}_{\mathrm{loc}}$ of the vortex equations*

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) := \frac{1}{2}\Big(\mathrm{d}_{A,H}u + J_{\Theta_\infty}(u)\circ \mathrm{d}_{A,H}u \circ j_\Sigma\Big) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0 \quad (3.76)$$

*on the complement $\Sigma \setminus Z$ of the singular points.*

Proof. By Lemma 3.4.6 we may assume without loss of generality that the sequence of maps $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_\infty$ strongly in the $C^0$-topology. Hence the sequence of families of almost complex structures $J_{\Theta(A_\nu,u_\nu)}\colon P \to \mathcal{J}(M,\omega)$ converges to the family $J_{\Theta_\infty}$ strongly in the $C^0$-topology as well. Moreover, by parts (a) and (b) of (ii') on p. 106 we know that the sequence $(A_\nu,u_\nu)$ converges to $(A,u)$ weakly in the $W^{1,p}$-topology on compact subsets of $\Sigma$. We thus conclude that the sequence of 1-forms

$$\overline{\partial}_{J,A_\nu,\Theta,H}(u_\nu) = \frac{1}{2}\big(\mathrm{d}_{A_\nu,H}u_\nu + J_{\Theta(A_\nu,u_\nu)}(u_\nu)\circ \mathrm{d}_{A_\nu,H}u_\nu \circ j_\Sigma\big)$$

converges weakly in the $L^p$-norm on compact subsets of $\Sigma \setminus Z$ to the 1-form

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) := \frac{1}{2}\big(\mathrm{d}_{A,H}u + J_{\Theta_\infty}(u)\circ \mathrm{d}_{A,H}u \circ j_\Sigma\big).$$

Likewise, it follows that the sequence of 2-forms

$$F_{A_\nu} + \mu(u_\nu)\,\mathrm{dvol}_\Sigma$$

converges weakly in the $L^p$-norm on compact subsets of $\Sigma \setminus Z$ to the 2-form

$$F_A + \mu(u)\,\mathrm{dvol}_\Sigma\,.$$

On the other hand, every vortex $(A_\nu,u_\nu)$ solves the non-local vortex equations

$$\overline{\partial}_{J,A_\nu,\Theta,H}(u_\nu) = 0, \qquad F_{A_\nu} + \mu(u_\nu)\,\mathrm{dvol}_\Sigma = 0.$$

Passing to weak $L^p$-limits, it thus follows that the pair $(A,u)$ is a solution of class $W^{1,p}_{\mathrm{loc}}$ of the vortex equations

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0$$

on the open subset $\Sigma \setminus Z$. □

In order to apply the Removable Singularity Theorem 3.3.2 we need to improve the convergence properties of the sequence $(A_\nu,u_\nu)$.

LEMMA 3.4.8. *The limit pair $(A, u)$ has the following properties.*

(i) *$(A, u)$ is of class $W^{2,p}_{loc}$ and of class $C^1$ on $\Sigma \setminus Z$.*
(ii) *After passing to a subsequence if necessary, the sequence $(A_\nu, u_\nu)$ converges to $(A, u)$ weakly in the $W^{2,p}$-topology and strongly in the $C^1$-topology on compact subsets of $\Sigma \setminus Z$.*

PROOF. The proof is by standard elliptic bootstrapping and is adapted from the proof of Theorem 3.1 in Cieliebak et. al. [**3**].

Proof of (i): By Lemma 3.4.7 the pair $(A, u)$ solves the limit vortex equations (3.76).

Consider the connection $A$. Set $\alpha := A - A_0 \in W^{k,p}(\Sigma, T^*\Sigma_0 \otimes P(\mathfrak{g}))$. By (i') on p. 106, there exists a smooth reference connection $A_0 \in \mathcal{A}(P)$ such that $A$ is in Coulomb gauge relative to $A_0$, that is,

$$\mathrm{d}^*_{A_0}\alpha = 0. \tag{3.77}$$

Combining this equation with the second vortex equation in (3.76) we obtain

$$\begin{aligned}
\mathrm{d}_{A_0}\alpha &= \mathrm{d}\alpha + [A_0 \wedge \alpha] \\
&= \mathrm{d}A - \mathrm{d}A_0 + [A_0 \wedge A] - [A_0 \wedge A_0] \\
&= -\left(\mathrm{d}A_0 + \frac{1}{2}[A_0 \wedge A_0]\right) - \frac{1}{2}[A_0 \wedge A_0] + [A_0 \wedge A] + \mathrm{d}A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] + \frac{1}{2}[A \wedge A] + \mathrm{d}A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] + F_A \\
&= -F_{A_0} - \frac{1}{2}[\alpha \wedge \alpha] - \mu(u)\, \mathrm{dvol}_\Sigma . 
\end{aligned} \tag{3.78}$$

Since $A$ and $u$ are of class $W^{1,p}_{\mathrm{loc}}$ on $\Sigma \setminus Z$, it follows from formula (3.78) that $\mathrm{d}_{A_0}\alpha$ is of class $W^{1,p}_{\mathrm{loc}}$ on $\Sigma \setminus Z$. Furthermore, it is obvious from formula (3.77) that $\mathrm{d}^*_{A_0}\alpha$ is of class $W^{1,p}_{\mathrm{loc}}$ on $\Sigma \setminus Z$. Hence it follows from elliptic regularity for the Hodge-Laplace operator $\mathrm{d}_{A_0}\mathrm{d}^*_{A_0} + \mathrm{d}^*_{A_0}\mathrm{d}_{A_0}$ that $\alpha$ is of class $W^{2,p}_{\mathrm{loc}}$ on $\Sigma \setminus Z$. Thus $A = A_0 + \alpha$ is of class $W^{2,p}_{\mathrm{loc}}$ on $\Sigma \setminus Z$. By Rellich's theorem ([**34**], Thm. B.2 (iii)), it follows that $A$ is of class $C^1$ on the subset $\Sigma \setminus Z$.

Next we prove that $u$ is of class $C^1$ on $\Sigma \setminus Z$. We shall work in local coordinates. Choose a holomorphic coordinate chart $\mathbb{C} \supset D \to \Sigma \setminus Z$ and a Darboux chart on $M$. We may thus assume without loss of generality that $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$, where $\omega_0$ is the standard symplectic form on $\mathbb{R}^{2n}$. By (2.18), with respect to these local coordinates, the first of the limit vortex equations (3.76) takes the form

$$\partial_s u + X_\Phi(u) + X_F(u) + J_{\Theta_\infty}(u)\big(\partial_t u + X_\Psi(u) + X_G(u)\big) = 0 \tag{3.79}$$

where we express the connection $A$ as

$$A = \Phi \,\mathrm{d}s + \Psi \,\mathrm{d}t, \qquad \Phi, \Psi \colon D \to \mathfrak{g},$$

the Hamiltonian perturbation $H$ as

$$H = F \,\mathrm{d}s + G \,\mathrm{d}t, \qquad F, G \colon D \times M \to \mathbb{R},$$

and consider $J_{\Theta_\infty}(u)$ as a matrix-valued map

$$J_{\Theta_\infty}(u) \colon D \to M_{2n,2n}(\mathbb{R}). \tag{3.80}$$

We may then rewrite equation (3.79) as

$$\partial_s u + J_{\Theta_\infty}(u)\,\partial_t u = -X_\Phi(u) - X_F(u) - J_{\Theta_\infty}(u)\big(X_\Psi(u) + X_G(u)\big). \tag{3.81}$$

Let us investigate the regularity of the right-hand side of this equation. Firstly, we have already proved that $A$ is of class $C^1$ on $\Sigma \setminus Z$, whence $\Phi$ and $\Psi$ are of class $C^1$ on $D$. It follows that the families of smooth vector fields $X_\Phi$ and $X_\Psi$ on $M$ are of class $C^1$ on $D$. Likewise, $H$ and hence $F$ and $G$ are of class $C^\ell$ by assumption. Thus we see from (2.15) that the families of smooth vector fields $X_F$ and $X_G$ on $M$ are of class $C^1$ on $D$. Since $u$ is of class $W^{1,p}_{\mathrm{loc}}$ on $D$ by assumption, it follows that each of the sections $X_\Phi(u)$, $X_\Psi(u)$, $X_F(u)$ and $X_G(u)$ is of class $C^1$ on $D$ ([**34**], Lemma B.8). Secondly, we have seen above that Lemma 3.4.6 implies that the family $J_{\Theta_\infty}$ is of class $W^{1,p}$. Hence, since $u$ is of class $W^{1,p}_{\mathrm{loc}}$ on $D$ by assumption, it follows that the map (3.80) is of class $W^{1,p}_{\mathrm{loc}}$ on $D$ ([**34**], Lemma B.8). We therefore conclude that the right-hand side of equation (3.81) is of class $W^{1,p}_{\mathrm{loc}}$ on $D$ (see [**34**], Lemma B.3). Applying Proposition B.4.9 (i) in [**22**] (see also Lemma 3.3 in [**3**]) it follows that $u$ is of class $W^{2,p}_{\mathrm{loc}}$ on $D$. By Rellich's theorem we infer that $u$ is of class $C^1$ on $D$.

Proof of (ii): We first show that the sequence $(A_\nu, u_\nu)$ is uniformly bounded in $W^{2,p}$ on compact subsets of $\Sigma \setminus Z$.

Let $K \subset \Sigma \setminus Z$ be an arbitrary compact subset. By parts (a) and (b) of (ii') on p. 106 the sequence $(A_\nu, u_\nu)$ converges weakly in the $W^{1,p}$-topology on $K$. Hence it follows from the Banach-Steinhaus theorem that $(A_\nu, u_\nu)$ satisfies a uniform $W^{1,p}$-bound on $K$.

Consider the sequence $A_\nu$. Set $\alpha_\nu := A_\nu - A \in W^{1,p}(\Sigma \setminus Z, T^*(\Sigma \setminus Z) \otimes P(\mathfrak{g}))$. By part (c) of (ii') on p. 106, every $A_\nu$ is in Coulomb gauge relative to $A$, that is,

$$\mathrm{d}_A^* \alpha_\nu = 0. \tag{3.82}$$

Combining this equation with the second vortex equation $F_{A_\nu} + \mu(u_\nu)\,\mathrm{dvol}_\Sigma = 0$ we obtain

$$\begin{aligned}
\mathrm{d}_A\alpha_\nu &= \mathrm{d}\alpha_\nu + [A\wedge\alpha_\nu] \\
&= \mathrm{d}A_\nu - \mathrm{d}A + \big[A\wedge A_\nu\big] - [A\wedge A] \\
&= -\left(\mathrm{d}A + \frac{1}{2}[A\wedge A]\right) - \frac{1}{2}[A\wedge A] + \big[A\wedge A_\nu\big] + \mathrm{d}A_\nu \\
&= -F_A - \frac{1}{2}[\alpha_\nu\wedge\alpha_\nu] + \frac{1}{2}\big[A_\nu\wedge A_\nu\big] + \mathrm{d}A_\nu \\
&= -F_A - \frac{1}{2}[\alpha_\nu\wedge\alpha_\nu] + F_{A_\nu} \\
&= -F_A - \frac{1}{2}[\alpha_\nu\wedge\alpha_\nu] - \mu(u_\nu)\,\mathrm{dvol}_\Sigma\,. \qquad (3.83)
\end{aligned}$$

Since $A_\nu$ and $u_\nu$ are uniformly bounded in $W^{1,p}$ on $K$, it follows from formula (3.83) that $\mathrm{d}_A\alpha_\nu$ is uniformly bounded in $W^{1,p}$ on $K$. Furthermore, it is obvious from formula (3.82) that $\mathrm{d}_A^*\alpha_\nu$ is uniformly bounded in $W^{1,p}$ on $K$. Since $A$ is of class $W^{2,p}_{\mathrm{loc}}$ on $\Sigma\setminus Z$ by assertion (i), it follows from elliptic regularity for the Hodge-Laplace operator $\mathrm{d}_A\mathrm{d}_A^* + \mathrm{d}_A^*\,\mathrm{d}_A$ that $\alpha_\nu$ is uniformly bounded in $W^{2,p}$ on $K$. Thus the sequence $A_\nu = A+\alpha_\nu$ is uniformly bounded in $W^{2,p}$ on $K$.

Next we consider the sequence of sections $u_\nu$. As in the proof of assertion (i) above we shall work in a holomorphic chart $\mathbb{C}\supset D\to\Sigma\setminus Z$ and a Darboux chart on $M$, that is, we assume without loss of generality that $(M,\omega) = (\mathbb{R}^{2n},\omega_0)$. By (2.18), with respect to these local coordinates, the first of the vortex equations (2.16) for $(A_\nu,u_\nu)$ takes the form

$$\partial_s u_\nu + X_{\Phi_\nu}(u_\nu) + X_F(u_\nu) + J_{\Theta(A_\nu,u_\nu)}(u_\nu)\big(\partial_t u_\nu + X_{\Psi_\nu}(u_\nu) + X_G(u_\nu)\big) = 0 \qquad (3.84)$$

where we express the connection $A_\nu$ as

$$A_\nu = \Phi_\nu\,\mathrm{d}s + \Psi_\nu\,\mathrm{d}t, \qquad \Phi_\nu,\Psi_\nu\colon D\to\mathfrak{g},$$

the Hamiltonian perturbation $H$ as

$$H = F\,\mathrm{d}s + G\,\mathrm{d}t, \qquad F,G\colon D\times M\to\mathbb{R},$$

and consider $J_{\Theta(A_\nu,u_\nu)}(u_\nu)$ as a matrix-valued map

$$J_{\Theta(A_\nu,u_\nu)}(u_\nu)\colon D\to M_{2n,2n}(\mathbb{R}). \qquad (3.85)$$

We may then rewrite equation (3.84) as

$$\begin{aligned}
&\partial_s u_\nu + J_{\Theta(A_\nu,u_\nu)}(u_\nu)\,\partial_t u_\nu \\
&\qquad = -X_{\Phi_\nu}(u_\nu) - X_F(u_\nu) - J_{\Theta(A_\nu,u_\nu)}(u_\nu)\big(X_{\Psi_\nu}(u_\nu) + X_G(u_\nu)\big). \qquad (3.86)
\end{aligned}$$

Let us consider the right-hand side of this equation.

Since the sequence $A_\nu$ is uniformly bounded in $W^{2,p}$ on compact subsets of $\Sigma \setminus Z$, Rellich's theorem implies that $A_\nu$, whence $\Phi_\nu$ and $\Psi_\nu$, are uniformly bounded in $C^1$ on $K$. Hence the vector fields $X_{\Phi_\nu}$ and $X_{\Psi_\nu}$ are uniformly bounded in $C^1$ on $K$.

Moreover, by Lemma 3.4.6 and the Banach-Steinhaus theorem we may assume without loss of generality that the sequence of maps $\Theta_{(A_\nu,u_\nu)}$ is uniformly bounded in the $W^{1,p}$-topology on $K$. Hence part (i) of the (Estimates) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) implies that the sequence $\Theta_{(A_\nu,u_\nu)}$ is uniformly bounded in the $W^{1,p}$-topology on $K$ as well. Since $u_\nu$ is uniformly bounded in $W^{1,p}$ on compact subsets of $\Sigma \setminus Z$, it thus follows from Lemmata B.8 and B.1 in [**34**] that the maps (3.85) are uniformly bounded in $W^{1,p}$ on $K$. Hence we conclude that the right-hand side of equation (3.86) is uniformly bounded in $W^{1,p}$ on $K$. Applying Proposition B.4.9 (i) and (ii) in [**22**] (see also Lemma 3.3 in [**3**]), it follows that $u_\nu$ satisfies a uniform $W^{2,p}$-bound on the subset $K$. This shows that the sequence $u_\nu$ is uniformly bounded in $W^{2,p}$ on compact subsets of $\Sigma \setminus Z$.

Since $(A_\nu, u_\nu)$ is uniformly bounded in $W^{2,p}$ on compact subsets of $\Sigma \setminus Z$, it follows from the theorems of Alaoglu and Rellich that there exists a connection $A_\infty$ on $P$ and a $G$-equivaraint map $u_\infty \colon P \to M$, both of class $W^{2,p}_{\mathrm{loc}}$ and of class $C^1$ on $\Sigma \setminus Z$, such that, after passing to a subsequence if necessary, the sequence $(A_\nu, u_\nu)$ converges to $(A_\infty, u_\infty)$ weakly in the $W^{2,p}$-topology and strongly in the $C^1$-topology on compact subsets of $\Sigma \setminus Z$. Since $(A_\nu, u_\nu)$ converges to $(A, u)$ strongly in the $C^0$-topology on compact subsets of $\Sigma \setminus Z$ by (ii'), parts (a) and (b), we must necessarily have $A = A_\infty$ and $u = u_\infty$.

This completes the proof of Lemma 3.4.8. □

**3.4.5. Removal of singularities.** We verify that the limit pair $(A, u)$ satisfies the assumptions of the Removable Singularity Theorem 3.3.2.

LEMMA 3.4.9. *The limit pair $(A, u)$ has the following properties.*

(i) *It has finite Yang-Mills-Higgs energy $E(A, u) < \infty$.*
(ii) *It satisfies an a priori estimate of the following form.*
*There exist constants $\hbar > 0$, $C > 0$ and $R > 0$ such that for all $z_0 \in \Sigma \setminus Z$ and all $0 < r \leq R$ such that $B_r(z_0) \subset \Sigma \setminus Z$, the following holds. If*

$$E_{J_{\Theta_\infty}}\big(A, u; B_r(z_0)\big) < \hbar,$$

*then*

$$\frac{1}{2}\big|\mathrm{d}_{A,H} u(z_0)\big|^2_{J_{\Theta_\infty}} + \big|\mu(u(z_0))\big|^2 \leq \frac{C}{r^2} \cdot E_{J_{\Theta_\infty}}\big(A, u; B_r(z_0)\big).$$

PROOF. Let $K_\mu \subset \Sigma \setminus Z$ be an exhausting sequence of compact subsets such that

$$K_\mu \subset K_{\mu+1} \quad \text{and} \quad \bigcup_\mu K_\mu = \Sigma \setminus Z.$$

By Lemma 3.4.8 (ii) we may without loss of generality assume that the sequence $(A_\nu, u_\nu)$ converges to $(A, u)$ strongly in the $C^1$-topology on compact subsets of $\Sigma \setminus Z$. By Lemma 3.4.6 we may without loss of generality assume that the sequence of maps $\Theta_{(A_\nu,u_\nu)}$ converges to $\Theta_\infty$ strongly in the $C^0$-topology. Hence the sequence of families of almost complex structures $J_{\Theta(A_\nu,u_\nu)}\colon P \to \mathcal{J}(M,\omega)$ converges to the family $J_{\Theta_\infty}$ strongly in the $C^0$-topology as well. It follows that the sequence of functions

$$e(A_\nu, u_\nu) := \frac{1}{2}\left|\mathrm{d}_{A_\nu,H} u_\nu\right|^2_{J_{\Theta(A_\nu,u_\nu)}} + \left|\mu(u_\nu)\right|^2$$

converges to

$$e(A, u) := \frac{1}{2}\left|\mathrm{d}_{A,H} u\right|^2_{J_{\Theta_\infty}} + \left|\mu(u)\right|^2$$

strongly in the $C^0$-topology on every compact set $K_\mu$.

Proof of (i): We conclude further that

$$\int_{K_\mu} e(A,u)\,\mathrm{dvol}_\Sigma = \lim_{\nu\to\infty}\int_{K_\mu} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma$$

for every $\mu$. Moreover, since $(A_\nu, u_\nu)$ is smooth on $\Sigma$,

$$\lim_{\mu\to\infty}\int_{K_\mu} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma = \int_{\Sigma\setminus Z} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma = \int_\Sigma e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma = E(A_\nu,u_\nu)$$

for every $\nu$. By Fatou's lemma we therefore have

$$\begin{aligned}
E(A,u) &= \int_\Sigma e(A,u)\,\mathrm{dvol}_\Sigma \\
&= \int_{\Sigma\setminus Z} e(A,u)\,\mathrm{dvol}_\Sigma \\
&\le \liminf_{\mu\to\infty}\int_{K_\mu} e(A,u)\,\mathrm{dvol}_\Sigma \\
&= \liminf_{\mu\to\infty}\left(\lim_{\nu\to\infty}\int_{K_\mu} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma\right) \\
&\le \sup_\nu\left(\lim_{\mu\to\infty}\int_{K_\mu} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma\right) = \sup_\nu E(A_\nu,u_\nu).
\end{aligned}$$

In the last inequality we used that, for $\nu$ fixed, the sequence $\int_{K_\mu} e(A_\nu,u_\nu)\,\mathrm{dvol}_\Sigma$ is non-decreasing. Since $\sup_\nu E(A_\nu,u_\nu) < \infty$ by (iii') on p. 106, the claim follows.

Proof of (ii): Let $\hbar$, $C$ and $R$ be the constants of Theorem 3.2.1. Let $z_0 \in \Sigma\setminus Z$ and let $0 < r \le R$ such that $B_r(z_0) \subset \Sigma\setminus Z$. Assume that

$$E_{J_{\Theta_\infty}}\big(A,u;B_r(z_0)\big) < \hbar.$$

There exists $\mu_0$ such that $B_r(z_0) \subset K_{\mu_0}$. Fix $\epsilon_0 = \epsilon_0(r) > 0$ so small that

$$\epsilon_0 \leq \frac{1}{2}\big(\hbar - E_{J_{\Theta_\infty}}\big(A, u; B_r(z_0)\big)\big). \tag{3.87}$$

We have seen above that $e(A_\nu, u_\nu)$ converges to $e(A, u)$ strongly in $C^0$ on $K_{\mu_0}$. It follows that

$$\begin{aligned}\lim_{\nu\to\infty} E_{J_{\Theta(A_\nu,u_\nu)}}\big(A_\nu, u_\nu; B_r(z_0)\big) &= \lim_{\nu\to\infty} \int_{B_r(z_0)} e(A_\nu, u_\nu)\,\mathrm{dvol}_\Sigma \\ &= \int_{B_r(z_0)} e(A, u)\,\mathrm{dvol}_\Sigma = E_{J_{\Theta_\infty}}\big(A, u; B_r(z_0)\big)\end{aligned}$$

and

$$\lim_{\nu\to\infty} e(A_\nu, u_\nu)(z_0) = e(A, u)(z_0).$$

To simplify notation we will drop the subscripts indicating the dependency on the almost complex structure in the remainder of the proof. It follows that there exists $\nu_0 = \nu_0(\epsilon_0)$ such that

$$E\big(A_\nu, u_\nu; B_r(z_0)\big) \leq E\big(A, u; B_r(z_0)\big) + \epsilon$$

and

$$e(A, u)(z_0) \leq e(A_\nu, u_\nu)(z_0) + \epsilon$$

for all $0 < \epsilon < \epsilon_0$ and all $\nu \geq \nu_0$. Then by the choice of $\epsilon_0$ in inequality (3.87) we have

$$E\big(A_\nu, u_\nu; B_r(z)\big) < \hbar,$$

so the a priori estimate of Theorem 3.2.1 implies that

$$e(A_\nu, u_\nu)(z_0) \leq \frac{C}{r^2} \cdot E\big(A_\nu, u_\nu; B_r(z_0)\big)$$

for all $\nu \geq \nu_0$. We conclude from this that

$$\begin{aligned}e(A, u)(z_0) &\leq e(A_\nu, u_\nu)(z_0) + \epsilon \\ &\leq \frac{C}{r^2} \cdot E\big(A_\nu, u_\nu; B_r(z_0)\big) + \epsilon \\ &\leq \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big) + \epsilon \cdot \left(\frac{C}{r^2} + 1\right)\end{aligned}$$

for all $0 < \epsilon < \epsilon_0$. We must therefore have

$$e(A, u)(z_0) \leq \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big).$$

Lemma 3.4.9 is proved. $\square$

We may now apply the Removable Singularity Theorem 3.3.2 to the limit pair $(A, u)$.

LEMMA 3.4.10. *The pair* $(A, u)$ *is a solution of the vortex equations*

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0$$

*on all of* $\Sigma$.

PROOF. We apply the Removable Singularity Theorem 3.3.2 to each of the finitely many singular points in $Z$.

Let $z_j \in Z$ be a singular point of the sequence $(A_\nu, u_\nu)$, and choose a holomorphic disk $B \to \Sigma$ such that the singular point $z_j$ gets identified with the origin $0 \in B$, where $B \subset \mathbb{C}$ denotes the closed unit disk. As we have seen in Section 2.2.5, locally in this chart the vortex $(A, u)$ gets identified with a pair $(A, u)$ consisting of a connection $A \in \Omega^1(B \setminus \{0\}, \mathfrak{g})$ and a map $u\colon B \setminus \{0\} \to M$ satisfying the vortex equations

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) = 0, \qquad F_A + \lambda^2\,\mu(u)\;\mathrm{d}s \wedge \mathrm{d}t = 0,$$

where $\lambda\colon B \to (0, \infty)$ is a smooth function determined by

$$\mathrm{dvol}_\Sigma = \lambda^2\;\mathrm{d}s \wedge \mathrm{d}t.$$

Let us now check that the pair $(A, u)$ satisfies the assumptions of the Removable Singularity Theorem.

First of all, Lemma 3.4.6 implies that the family $J_{\Theta_\infty}$ is continuous on $\Sigma$. Furthermore, by Lemma 3.4.8 the pair $(A, u)$ is of class $C^1$ on the punctured disk $B \setminus \{0\}$.

Next, since $A$ is of class $W^{1,p}$ on $\Sigma$ it follows by Rellich's theorem that $A$ is continuous on all of $B$.

Moreover, by Lemma 3.4.9 (i) the Yang-Mills-Higgs energy $E(A, u)$ of the non-local vortex $(A, u)$ is finite. Hence it follows from identity (2.20) that the Yang-Mills-Higgs energy $E(A, u; B)$ of $(A, u)$ on $B$ is also finite.

Lastly, it follows from Lemma 3.4.9 (ii) and identity (2.20) that the pair $(A, u)$ satisfies an a priori estimate of the following form: There exist constants $\delta > 0$ and $C > 0$ such that the following holds. For all $z_0 \in B$ and all $r > 0$ such that $B_r(z_0) \subset B$ the Yang-Mills-Higgs energy density $e := e(A, u)$ satisfies

$$E\big(A, u; B_r(z_0)\big) < \delta \quad \Longrightarrow \quad e(A, u)(z_0) \le \frac{C}{r^2} \cdot E\big(A, u; B_r(z_0)\big).$$

The Removable Singularity Theorem now implies that the map $u$ is of class $W^{1,p}$ on all of $B$. By finiteness of the set $Z$ of singular points we have thus proved that the section $u\colon \Sigma \setminus Z \to P(M)$ extends to a section $u\colon \Sigma \to P(M)$ of class $W^{1,p}$ on $\Sigma$. By Lemma 3.76, the pair $(A, u)$ solves the vortex equations

$$\overline{\partial}_{J_{\Theta_\infty},A,H}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0$$

on all of $\Sigma$. □

**3.4.6. The limit non-local vortex.** In this subsection we study the properties of the limit vortex $(A, u)$. First of all, we check that it is $\mathrm{dvol}_\Sigma$-tame.

LEMMA 3.4.11. *The limit pair $(A, u)$ satisfies the taming condition (2.1).*

PROOF. First of all, every vortex $(A_\nu, u_\nu)$ satisfies the taming condition (2.1)

$$\int_\Sigma |\mu(u_\nu)|^2 \,\mathrm{dvol}_\Sigma < E$$

by assumption. Now by part (b) of (ii') on p. 106, the sequence $u_\nu(z)$ converges to $u(z)$ for all points $z \in \Sigma \setminus Z$. Since $Z$ is finite, it follows that the sequence $|\mu(u_{nu})|$ converges to $|\mu(u)|$ pointwise almost everywhere on $\Sigma$. Moreover, since $M$ is compact the sequence $|\mu(u_{nu})|$ is uniformly bounded in the $C^0$-topology. Hence Lebesgue's Dominated Convergence Theorem implies that

$$\int_\Sigma |\mu(u)|^2 \,\mathrm{dvol}_\Sigma = \lim_{\nu\to\infty} \int_\Sigma |\mu(u_\nu)|^2 \,\mathrm{dvol}_\Sigma \leq E.$$

□

By the previous lemma, the map

$$\Theta_{(A,u)} \colon P \to EG^N$$

is well-defined. It is of class $W^{1,p}$ and hence also of class $C^0$. Since $A_\nu$ converges to $A$ strongly in the $C^0$-topology on $\Sigma$ by part (a) of (ii') on p. 106, it is an immediate consequence of the (Continuity) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) that both maps $\Theta_{(A,u)}$ and $\Theta_\infty \colon P \to EG^N$ appear as the $C^0$-limit of the sequence $\Theta_{(A_\nu,u_\nu)}$, whence

$$\Theta_{(A,u)} = \Theta_\infty.$$

In particular, we have

$$J_{\Theta_\infty} = J_{\Theta(A,u)} = J \circ \Theta_{(A,u)} \colon P \to \mathcal{J}(M, \omega).$$

This shows that the limit pair $(A, u)$ is a non-local vortex of class $W^{1,p}$ solving the non-local vortex equations

$$\overline{\partial}_{J,A,\Theta,H}(u) = 0, \qquad F_A + \mu(u)\,\mathrm{dvol}_\Sigma = 0. \tag{3.88}$$

We may hence apply elliptic bootstrapping as in Section 2.2.6 in order to improve the regularity of the pair $(A, u)$.

LEMMA 3.4.12. *The limit vortex $(A, u)$ is of class $C^{\ell-1}$.*

PROOF. By (3.88) above, the pair $(A, u)$ solves the non-local vortex equations (2.16). By Lemma 3.4.5 (i), the connection $A$ is in Coulomb gauge with respect to a smooth reference connection $A_0$. Thus Lemma 2.2.9 applies to the pair $(A, u)$ and we conclude that $(A, u)$ is of class $C^{\ell-1}$. □

This completes the proof of assertions (i) and (ii) of Theorem 3.4.1.

**3.4.7. Convergence of the energy.** It remains to prove parts (iii) and (iv) of Theorem 3.4.1. We will proceed along the lines of the proofs of assertions (ii) and (iii) of Theorem 4.6.1 in McDuff and Salamon [**22**].

Fix numbers $\varepsilon_j > 0$ for $j = 1, \dots, N$ such that the geodesic disks $B_{\varepsilon_j}(z_j)$ are pairwise disjoint. Then, after passing to a subsequence if necessary, the limits

$$m_{\varepsilon_j}(z_j) := \lim_{\nu\to\infty} E\big(A_\nu, u_\nu; B_{\varepsilon_j}(z_j)\big)$$

exist for $j = 1, \dots, N$. The function $\varepsilon \mapsto m_\varepsilon(z_j)$ is continuous for $0 < \varepsilon \leq \varepsilon_j$ since $(A_\nu, u_\nu)$ converges in $C^1$ on the annulus $B_{\varepsilon_j}(z_j) \setminus B_\varepsilon(z_j)$ by Lemma 3.4.8. Recall from Lemma 3.4.3 that

$$\liminf_{\nu\to\infty} E\big(A_\nu, u_\nu; B_{\varepsilon_j}(z_j)\big) \geq \hbar.$$

Hence it follows that

$$m(z_j) := \lim_{\varepsilon\to 0} m_\varepsilon(z_j) \geq \hbar.$$

This proves (iii) in Theorem 3.4.1. To prove (iv), fix a number $\varepsilon \leq \min_j \varepsilon_j$ and note that

$$\begin{aligned} E\left(A, u; K \setminus \bigcup_{j=1}^N B_\varepsilon(z_j)\right) &= \lim_{\nu\to\infty} E\big(A_\nu, u_\nu; K\big) - \sum_{j=1}^N \lim_{\nu\to\infty} E\big(A_\nu, u_\nu; B_\varepsilon(z_j)\big) \\ &= \lim_{\nu\to\infty} E\big(A_\nu, u_\nu; K\big) - \sum_{j=1}^N m_\varepsilon(z_j). \end{aligned}$$

Taking the limit $\varepsilon \to 0$,

$$E(A, u; K) = \lim_{\nu\to\infty} E\big(A_\nu, u_\nu; K\big) - \sum_{j=1}^N m(z_j),$$

which proves assertion (iv).

This finishes the proof of Theorem 3.4.1.

**3.4.8. Proof of Corollary 3.4.2.** Let us now assume in addition that the sequence $(A_\nu, u_\nu)$ satisfies

$$\sup_\nu \big\| \mathrm{d}_{A_\nu, H} u_\nu \big\|_{L^\infty(\Sigma)} < \infty.$$

This implies that there are no singular points, that is, $Z = \emptyset$. Hence the corollary is an immediate consequence of assertion (ii) of Theorem 3.4.1.

## 3.5. Vortices vs. pseudoholomorphic maps

Inspired by work of Mundet i Riera [**26**] and Gaio and Salamon [**11**] we apply Gromov's graph construction in order to transform vortices into pseudoholomorphic curves. We deduce a mean value inequality for these curves from the a priori estimate for vortices proved in Section 3.2. The results of this section (Proposition 3.5.3) are in preparation for the proof of Theorem 3.1.7 in Section 3.6.

**3.5.1. The graph construction.** Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

Recall that we denote by $p\colon P(M) = P \times_G M \to \Sigma$ the fiber bundle associated to the $G$-bundle $\pi\colon P \to \Sigma$ and the $G$-manifold $M$. We may equivalently think of a $G$-equivariant map $u\colon P \to M$ as a section $\tilde{u}\colon \Sigma \to P(M)$ of this fiber bundle, defined by

$$\tilde{u}(z) := [p, u(p)], \quad \pi(p) = z. \tag{3.89}$$

The goal of this section is to endow the total space $P(M)$ with the structure of an almost complex symplectic manifold in such a way that every section $\tilde{u}\colon \Sigma \to P(M)$ coming from a non-local vortex $(A, u)$ becomes a pseudoholomorphic curve.

We begin with a general review of symplectic forms and almost complex structures on the fiber bundle $P(M)$. Our exposition follows the work of Mundet i Riera [**26**], Mundet i Riera and Tian [**27**], Cieliebak, Gaio, and Salamon [**4**], Gaio and Salamon [**11**], González and Woodward [**13**] and González, Woodward, and Ziltener [**12**].

**3.5.2. Splitting of the tangent bundle.** Let $A \in \mathcal{A}^{1,p}(P)$ be an arbitrary connection of class $W^{1,p}$. We denote the points of $P(M) = P \times_G M$ as equivalence classes $[p, x]$, where $p \in P$ and $x \in M$. The tangent space $T_{[p,x]}P(M)$ of $P(M)$ at any such point is then given by

$$T_{[p,x]}P(M) = \big(T_pP \times T_xM\big)\big/\big\{(p\xi, -X_\xi(x)) \,|\, \xi \in \mathfrak{g}\big\},$$

where $p.\xi \in T_pP$ denotes the infinitesimal action of $\xi$ on $P$ at the point $p$. The elements of this tangent space will be denoted as equivalence classes $[v, w]$, where $v \in T_pP$ and $w \in T_xM$.

The connection $A$ gives rise to a splitting

$$TP \cong TP^{\mathrm{hor}} \oplus TP^{\mathrm{vert}}$$

of the tangent bundle of $P$ into horizontal and vertical subbundles. The horizontal component of a tangent vector $v \in T_pP$ is then given by the relation

$$v = v^{\mathrm{hor}} + p.A_p(v).$$

The splitting of $TP$ further induces a splitting

$$TP(M) \cong TP(M)^{\mathrm{hor}} \oplus TP(M)^{\mathrm{vert}} \tag{3.90}$$

of the tangent bundle of $P(M)$ in the following way. The horizontal subbundle is the image of $TP^{\mathrm{hor}}$ in $TP(M)$ and is given by

$$TP(M)^{\mathrm{hor}} = TP^{\mathrm{hor}} \times_G M \cong p^*T\Sigma,$$

where $p\colon P(M) \to \Sigma$ denotes the bundle projection. The vertical subbundle is the kernel of the differential $\mathrm{d}p\colon TP(M) \to T\Sigma$ and is given by

$$TP(M)^{\mathrm{vert}} = P \times_G TM.$$

The decomposition of any tangent vector $[v,w] \in T_{[p,x]}P(M)$ corresponding to (3.90) is then given by

$$[v,w]^{\mathrm{hor}} := \bigl[v^{\mathrm{hor}}, 0\bigr]$$

and

$$[v,w]^{\mathrm{vert}} := \bigl[p.A_p(v), w\bigr] = \bigl[0, w + X_{A_p(v)}(x)\bigr].$$

It will later be useful to write this decomposition in the form

$$[v,w] = \bigl[v^{\mathrm{hor}}, w + X_{A_p(v)}(x)\bigr]. \tag{3.91}$$

**3.5.3. Symplectic form.** Assume that the connection $A$ is smooth. We explain how $A$ and $H$ induce a symplectic form on the total space $P(M)$. Our exposition follows Guillemin and Sternberg [**16**], Section 9.5, and Gaio and Salamon [**11**], Appendix A.

Denote by $p_1\colon P \times M \to P$ and $p_2\colon P \times M \to M$ the canonical projections. Define a 1-form $\alpha_H$ on the product $P \times M$ by

$$\alpha_H(v,w) := H_{\mathrm{d}_p\pi(v)}(x)$$

for all $(p,x) \in P \times M$ and $(v,w) \in T_pP \times T_xM$. Note that this form is $G$-invariant. Consider the 2-form

$$\widetilde{\sigma}_{A,H} := p_2^*\omega - \mathrm{d}\bigl\langle p_1^*A, \mu \circ p_2\bigr\rangle_{\mathfrak{g}} - \mathrm{d}\alpha_H \in \Omega^2(P \times M).$$

To simplify notation, we will henceforth drop the projections $p_1$ and $p_2$ and simply write

$$\widetilde{\sigma}_{A,H} = \omega - \mathrm{d}\langle A, \mu\rangle - \mathrm{d}\alpha_H.$$

Claim. *The 2-form $\widetilde{\sigma}_{A,H}$ is closed, $G$-invariant, and vanishes along the orbits of the $G$-action on $P \times M$.*

Proof of Claim. The 2-form $\widetilde{\sigma}_{A,H}$ is closed because $\omega$ is closed. It is $G$-invariant since $\omega$ and $\alpha_H$ are $G$-invariant and $A$ and $\mu$ are $G$-equivariant. To prove that it vanishes along the orbits of the $G$-action on $P \times M$, we first note that

$$\widetilde{\sigma}_{A,H} = \omega - \langle \mathrm{d}A, \mu\rangle + \langle A \wedge \mathrm{d}\mu\rangle - \mathrm{d}\alpha_H.$$

More explicitly, this formula can be written in terms of the curvature forms of $A$ and $H$ as follows. Let $(p,x)\in P\times M$ and $(v_1,w_1),(v_2,w_2)\in T_pP\times T_xM$. Then

$$\begin{aligned}\widetilde{\sigma}_{A,H}\big((v_1,w_1),(v_2,w_2)\big) &= \omega\big(w_1+X_{H(\mathrm{d}\pi(v_1))}(x),w_2+X_{H(\mathrm{d}\pi(v_2))}(x)\big)\\ &\quad -\big\langle F_A(v_1,v_2)-[A(v_1),A(v_2)],\mu(x)\big\rangle+\big\langle A(v_1),\mathrm{d}\mu(w_2)\big\rangle-\big\langle A(v_2),\mathrm{d}\mu(w_1)\big\rangle\\ &\quad -\Omega_H(\pi(p),x)\cdot\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1),\mathrm{d}\pi(v_2)\big).\end{aligned}\tag{3.92}$$

Assume now that $(v_1,w_1)=(p.\xi,-X_\xi(x))$ is the infinitesimal action of $\xi\in\mathfrak{g}$ on $(p,x)$. Then $\mathrm{d}\pi(v_1)=0$. Hence, using the relations

$$\omega\big(X_\xi(x),w_2\big)=\big\langle\mathrm{d}\mu(w_2),\xi\big\rangle,\quad \mathrm{d}\mu\big(X_\xi(x)\big)=[\xi,\mu(x)],\quad \big\langle[\xi_1,\xi_2],\xi_3\big\rangle=\big\langle\xi_1,[\xi_2,\xi_3]\big\rangle$$

where $\xi_1,\xi_2,\xi_3\in\mathfrak{g}$, and the fact that $F_A$ is horizontal, we obtain from (3.92)

$$\begin{aligned}&\widetilde{\sigma}_A\big((p.\xi,-X_\xi(x)),(v_2,w_2)\big)\\ =\;&-\omega\big(X_\xi(x),w_2\big)-\big\langle F_A(p.\xi,v_2)-[A(p.\xi),A(v_2)],\mu(x)\big\rangle+\big\langle A(p.\xi),\mathrm{d}\mu(w_2)\big\rangle\\ &\quad+\big\langle A(v_2),\mathrm{d}\mu(X_\xi(x))\big\rangle\\ =\;&-\big\langle\mathrm{d}\mu(w_2),\xi\big\rangle+\big\langle[A(p.\xi),A(v_2)],\mu(x)\big\rangle+\big\langle\xi,\mathrm{d}\mu(w_2)\big\rangle+\big\langle A(v_2),\mathrm{d}\mu(X_\xi(x))\big\rangle\\ =\;&\big\langle[\xi,A(v_2)],\mu(x)\big\rangle+\big\langle A(v_2),[\xi,\mu(x)]\big\rangle\\ =\;&-\big\langle A(v_2),[\xi,\mu(x)]\big\rangle+\big\langle A(v_2),[\xi,\mu(x)]\big\rangle=0.\end{aligned}$$

□

By the Claim above, $\widetilde{\sigma}_{A,H}$ descends to a closed 2-form

$$\sigma_{A,H}\in\Omega^2(P(M)).$$

Note that for $H=0$ this form agrees with the coupling form in [**16**]. We obtain from (3.92) an explicit formula for the 2-form $\sigma_{A,H}$. For tangent vectors $[v_1,w_1],[v_2,w_2]\in T_{[p,x]}P(M)$, $[p,x]\in P(M)$, we have

$$\begin{aligned}&\sigma_{A,H}\big([v_1,w_1],[v_2,w_2]\big)\\ =\;&\sigma_{A,H}\Big(\big[v_1^{\mathrm{hor}},w_1+X_{A(v_1)}(x)\big],\big[v_2^{\mathrm{hor}},w_2+X_{A(v_2)}(x)\big]\Big)\\ =\;&\omega\big(w_1+X_{A(v_1)}(x)+X_{H(\mathrm{d}\pi(v_1^{\mathrm{hor}}))}(x),w_2+X_{A(v_2)}(x)+X_{H(\mathrm{d}\pi(v_2^{\mathrm{hor}}))}(x)\big)\\ &\quad-\big\langle F_A\big(v_1^{\mathrm{hor}},v_2^{\mathrm{hor}}\big),\mu(x)\big\rangle+\big\langle\big[A\big(v_1^{\mathrm{hor}}\big),A\big(v_2^{\mathrm{hor}}\big)\big],\mu(x)\big\rangle\\ &\quad+\big\langle A\big(v_1^{\mathrm{hor}}\big),\mathrm{d}\mu\big(w_2+X_{A(v_2)}(x)\big)\big\rangle-\big\langle A\big(v_2^{\mathrm{hor}}\big),\mathrm{d}\mu\big(w_1+X_{A(v_2)}(x)\big)\big\rangle\\ &\quad-\Omega_H(\pi(p),x)\cdot\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi\big(v_1^{\mathrm{hor}}\big),\mathrm{d}\pi\big(v_2^{\mathrm{hor}}\big)\big)\\ =\;&\omega\big(w_1+X_{A(v_1)}(x)+X_{H(\mathrm{d}\pi(v_1))}(x),w_2+X_{A(v_2)}(x)+X_{H(\mathrm{d}\pi(v_2))}(x)\big)\\ &\quad-\big\langle F_A(v_1,v_2),\mu(x)\big\rangle-\Omega_H(\pi(p),x)\cdot\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1),\mathrm{d}\pi(v_2)\big)(x)\end{aligned}\tag{3.93}$$

since $A$ vanishes on horizontal tangent vectors and $F_A$ is horizontal. In particular, we see from this formula that the 2-form $\sigma_{A,H}$ may be degenerate in the horizontal direction. However, we can make it into a symplectic form on $P(M)$ by adding on a sufficiently large multiple of the pull-back of the area form $\mathrm{dvol}_\Sigma$ along the bundle projection $p\colon P(M) \to \Sigma$. This leads us to define the symplectic form on $P(M)$ by

$$\omega_{A,H} := (1 + c_{A,H}) \cdot p^* \mathrm{dvol}_\Sigma + \sigma_{A,H} \in \Omega^2(P(M)), \tag{3.94}$$

where $c_{A,H} > 0$ is some sufficiently large constant such that

$$\begin{aligned} &\big|\big\langle F_A(v_1, v_2), \mu(x)\big\rangle + \Omega_H(\pi(p), x) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1), \mathrm{d}\pi(v_2)\big)\big| \\ &\qquad\qquad\qquad\qquad \leq c_{A,H} \cdot |\mathrm{d}\pi(v_1)| \cdot |\mathrm{d}\pi(v_2)| \end{aligned} \tag{3.95}$$

for all $v_1, v_2 \in T_pP$ and all $(p, x) \in P \times M$. Note that such a constant exists by compactness of $P \times M$ and because the curvature $F_A$ is horizontal.

Lemma 3.5.1. *The form $\omega_{A,H}$ is a symplectic form on $P(M)$.*

Proof. It follows from the defining relation (3.94) that $\omega_A$ is closed since both $\mathrm{dvol}_\Sigma$ and the 2-form $\sigma_{A,H}$ are closed. To check that $\omega_{A,H}$ is non-degenerate, let $[v_1, w_1] \in T_{[p,x]}P(M)$ and suppose that

$$\omega_{A,H}\big([v_1, w_1], [v_2, w_2]\big) = 0 \tag{3.96}$$

for all $[v_2, w_2] \in T_{[p,x]}P(M)$.

Firstly, let $v_2 = 0$. Plugging the explicit Formula (3.93) into (3.94) we obtain

$$\omega\big(w_1 + X_{A(v_1)}(x) + X_{H(\mathrm{d}\pi(v_1))}(x), w_2\big) = \omega_{A,H}\big([v_1, w_1], [v_2, w_2]\big) = 0$$

for all $w_2 \in T_xM$. Since $\omega$ is non-degenerate this implies that

$$w_1 + X_{A(v_1)}(x) + X_{H(\mathrm{d}\pi(v_1))}(x) = 0. \tag{3.97}$$

Secondly, let $v_2 = (\pi^* j_\Sigma)v_1$ be the horizontal lift of $j_\Sigma\, \mathrm{d}_p\pi(v_1)$ to $T_pP$ and let $w_2 = 0$. Plugging (3.93) into (3.94) and using (3.97) and (3.95) we then obtain

$$\begin{aligned} 0 &= -\big\langle F_A(v_1, v_2), \mu(x)\big\rangle - \Omega_H(\pi(p), x) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1), \mathrm{d}\pi(v_2)\big) \\ &\quad + (1 + c_{A,H}) \cdot p^*\mathrm{dvol}_\Sigma(v_1, v_2) \\ &\geq -c_{A,H} \cdot |\mathrm{d}_p\pi(v_1)| \cdot |\mathrm{d}_p\pi(v_2)| + (1 + c_{A,H}) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1), \mathrm{d}_p\pi(v_2)\big) \\ &= -c_{A,H} \cdot |\mathrm{d}_p\pi(v_1)| \cdot |j_\Sigma\, \mathrm{d}\pi(v_1)| + (1 + c_{A,H}) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v_1), j_\Sigma\, \mathrm{d}\pi(v_1)\big) \\ &= |\mathrm{d}\pi(v_1)|^2 . \end{aligned}$$

It follows that $\mathrm{d}\pi(v_1) = 0$, whence $v_1^{\mathrm{hor}} = 0$. Combining this with (3.97) we obtain

$$[v_1, w_1] = \big[v_1^{\mathrm{hor}}, w_1 + X_{A(v_1)}(x)\big] = 0.$$

Hence $\sigma_{A,H}$ is non-degenerate. □

**3.5.4. Almost complex structure.** Let $(A, u)$ be a non-local vortex of class $W^{1,p}$ solving equations (2.16). We use the splitting (3.90) of the tangent bundle $TP(M)$ induced by $A$ and $H$ to exhibit an almost complex structure $I_{\Theta,A,\tilde{u},H}$ on $P(M)$ in terms of the complex structure $j_\Sigma$ on $\Sigma$ and the almost complex structure $I$ on $M$. We emphasize that the almost complex structure $I_{\Theta,A,\tilde{u},H}$ depends on the vortex $(A, u)$.

First, we note that the complex structure $j_\Sigma$ lifts to a $G$-equivariant complex structure

$$\pi^* j_\Sigma \colon TP^{\mathrm{hor}} \to TP^{\mathrm{hor}}$$

on the horizontal subbundle. We then define an endomorphism $I_{\Theta,A,\tilde{u},H}$ of the tangent bundle $TP(M)$ by

$$I_{\Theta,A,\tilde{u},H}[v, w] := \Big[(\pi^* j_\Sigma)v^{\mathrm{hor}}, J_{\Theta(A,u)}\big(w + X_{A(v)} + X_{H(\mathrm{d}\pi(v))}\big) - X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\Big]. \tag{3.98}$$

More explicitly, for $[p, x] \in P(M)$ and $[v, w] \in T_{[p,x]}P(M)$ this formula reads

$$\begin{aligned}&I_{\Theta,A,\tilde{u},H}[v, w]\\ &:= \Big[(\pi^* j_\Sigma)_p v^{\mathrm{hor}}, J_{\Theta(A,u)(p)}\big(w + X_{A_p(v)}(x) + X_{H(\mathrm{d}_p\pi(v))}(x)\big) - X_{H(j_\Sigma\,\mathrm{d}_p\pi(v))}(x)\Big].\end{aligned}$$

By the next lemma, this endomorphism is in fact an almost complex structure.

Lemma 3.5.2. *$I_{\Theta,A,\tilde{u},H}$ is an almost complex structure on $P(M)$.*

Proof. Let $[v, w] \in TP(M)$. Then formula (3.98) yields

$$\begin{aligned}
I^2_{\Theta,A,\tilde{u},H}[v, w] &= I_{\Theta,A,\tilde{u},H}\Big[(\pi^* j_\Sigma)v^{\mathrm{hor}}, J_{\Theta(A,u)}\big(w + X_{A(v)} + X_{H(\mathrm{d}\pi(v))}\big) - X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\Big]\\
&= \Big[(\pi^* j_\Sigma)^2 v^{\mathrm{hor}}, J_{\Theta(A,u)}\big(J_{\Theta(A,u)}\big(w + X_{A(v)} + X_{H(\mathrm{d}\pi(v))}\big) - X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\\
&\qquad + X_{A((\pi^* j_\Sigma)v^{\mathrm{hor}})} + X_{H(\mathrm{d}\pi((\pi^* j_\Sigma)v^{\mathrm{hor}}))}\big) - X_{H(j_\Sigma\,\mathrm{d}\pi((\pi^* j_\Sigma)v^{\mathrm{hor}}))}\Big]\\
&= \Big[-v^{\mathrm{hor}}, -w - X_{A(v)} - X_{H(\mathrm{d}\pi(v))} - J_{\Theta(A,u)}X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\\
&\qquad + J_{\Theta(A,u)}X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\big) - X_{H(j_\Sigma^2\,\mathrm{d}\pi(v))}\Big]\\
&= \Big[-v^{\mathrm{hor}}, -w - X_{A(v)} - X_{H(\mathrm{d}\pi(v))} - J_{\Theta(A,u)}X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\\
&\qquad + J_{\Theta(A,u)}X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\big) + X_{H(\mathrm{d}\pi(v))}\Big]\\
&= -\Big[v^{\mathrm{hor}}, w + X_{A(v)}\Big]\\
&= -[v, w].
\end{aligned}$$

In the third equality we used that $A$ vanishes on horizontal vectors, and in the last equality we used relation (3.91). ∎

**3.5.5. Main result.** The main result of this section is the following proposition.

PROPOSITION 3.5.3. *Fix a real constant $E > 0$ and an $E$-admissible area form* $\mathrm{dvol}_\Sigma$ *on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.*

*Let $(A_0, u_0)$ be a smooth non-local vortex solving equations (2.16). Then the following holds.*

(i) *The 2-form $\omega_{A_0,H}$ defined by (3.94) is a well-defined symplectic form on $P(M)$.*
(ii) *There exists a constant $c' > 0$ such that the following holds.*
*For every non-local vortex $(A, u)$ of class $W^{1,p}$ solving equations (2.16), the almost complex structures $I_{\Theta,A,\tilde{u},H}$ and $I_{\Theta,A_0,\tilde{u}_0,H}$ defined by (3.98) satisfy*

$$\big\|I_{\Theta,A,\tilde{u},H} - I_{\Theta,A_0,\tilde{u}_0,H}\big\|_{C^0(\Sigma)} \le c' \cdot \big\|A - A_0\big\|_{C^0(\Sigma)} \cdot \big\|J_{\Theta(A,u)} - J_{\Theta(A_0,u_0)}\big\|_{C^0(\Sigma)}.$$

*There exist constants $c, c'' > 0$, $r_0 > 0$, and $\delta, C > 0$ such that for all non-local vortices $(A, u)$ of class $W^{1,p}$ solving equations (2.16) and satisfying*

$$\big\|A - A_0\big\|_{C^0(\Sigma)} \le c \qquad \text{and} \qquad \big\|J_{\Theta(A,u)} - J_{\Theta(A_0,u_0)}\big\|_{C^0(\Sigma)} \le c$$

*the following holds.*

(iii) *The almost complex structure $I_{\Theta,A,\tilde{u},H}$ defined by (3.98) is tamed by the symplectic form $\omega_{A_0,H}$.*
(iv) *The section $\tilde{u}\colon \Sigma \to P(M)$ defined by (3.89) is a $(j_\Sigma, I_{\Theta,A,\tilde{u},H})$-holomorphic curve.*

*Denote by $\langle\cdot,\cdot\rangle_{I_{\Theta,A,\tilde{u},H}} := \omega_{A_0,H}(\cdot\,, I_{\Theta,A,\tilde{u},H}\cdot)$ the Riemannian metric on $P(M)$ determined by the symplectic form $\omega_{A_0,H}$ and the almost complex structure $I_{\Theta,A,\tilde{u},H}$, and recall that the energy of the curve $\tilde{u}\colon \Sigma \to P(M)$ is given by*

$$E_{I_{\Theta,A,H}}(\tilde{u}) := \frac{1}{2}\int_\Sigma |\mathrm{d}\tilde{u}|^2_{I_{\Theta,A,H}}\, \mathrm{dvol}_\Sigma,$$

*where the norm $|\mathrm{d}\tilde{u}|_{I_{\Theta,A,H}}$ is understood with respect to the metric $\langle\cdot,\cdot\rangle_{I_{\Theta,A,\tilde{u},H}}$ on $P(M)$ and the Kähler metric on $\Sigma$ determined by* $\mathrm{dvol}_\Sigma$ *and $j_\Sigma$.*

(v) *The energy of the curve $\tilde{u}\colon \Sigma \to P(M)$ and the Yang-Mills-Higgs energy of the vortex $(A, u)$ are related by*

$$E_{I_{\Theta,A,H}}(\tilde{u}) \le c'' \cdot \Big(E_{J_\Theta}(A, u) + \mathrm{Vol}(\Sigma)\Big).$$

(vi) *For all $z_0 \in \Sigma$ and all $0 < r < r_0$, the curve $\tilde{u}\colon \Sigma \to P(M)$ satisfies a mean value inequality of the form*

$$\begin{aligned} &E_{I_{\Theta,A,H}}\big(\tilde{u}; B_r(z_0)\big) < \delta \\ \Longrightarrow \qquad &\big|\mathrm{d}\tilde{u}(z_0)\big|^2_{I_{\Theta,A,H}} \le \frac{C}{r^2} \cdot E_{I_{\Theta,A,H}}\big(\tilde{u}; B_r(z_0)\big) + C. \end{aligned}$$

**3.5.6. Proof of Proposition 3.5.3.** Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. Let $(A_0, u_0)$ be a non-local vortex solving equations (2.16).

**Proof of (i):** By assumption, the connection $A_0$ is smooth. Hence the 2-form $\omega_{A_0,H}$ given by formula (3.94) is well-defined. By Lemma 3.5.1 above, it is a symplectic form on $P(M)$.

**Proof of (ii):** First of all, by Lemma 3.5.2 above $I_{\Theta,A_0,\tilde{u}_0,H}$ and $I_{\Theta,A,u,H}$ are almost complex structures on $P(M)$. By assumption, the pair $(A_0, u_0)$ is smooth and the pair $(A, u)$ is of class $W^{1,p}$. In particular, $A_0$ and $A$ are of class $C^0$ by Rellich's theorem ([**34**], Theorem B.2). Hence it follows from part (i) of the (Regularity) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) and formula (3.98) that $I_{\Theta,A_0,\tilde{u}_0,H}$ and $I_{\Theta,A,u,H}$ are both of class $C^0$. The estimate is straightforward from formula (3.98).

**Proof of (iii):** This is a consequence of the following lemma.

LEMMA 3.5.4. *There exists a constant $c > 0$ such that the following holds. For every non-local vortex $(A, u)$ of class $W^{1,p}$ satisfying*

$$\big\|A - A_0\big\|_{C^0(\Sigma)} \leq c \qquad \text{and} \qquad \big\|J_{\Theta(A,u)} - J_{\Theta(A_0,u_0)}\big\|_{C^0(\Sigma)} \leq c$$

*the almost complex structure $I_{\Theta,A,\tilde{u},H}$ given by (3.98) is tamed by the symplectic form $\omega_{A_0,H}$.*

PROOF. Suppose that $[v, w] \in TP(M)$ such that $[v, w] \neq 0$. We have to prove that

$$\omega_{A_0,H}\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big) > 0.$$

By formula (3.91) we may without loss of generality assume that $v$ is $A$-horizontal. Then (3.98) yields

$$I_{\Theta,A,\tilde{u},H}[v,w] = \big[(\pi^* j_\Sigma)v, J_{\Theta(A,u)}w + J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))} - X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\big].$$

Combining this with (3.93) and (3.94), we then obtain

$$\begin{aligned}
&\omega_{A_0,H}\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big)\\
= \;& \sigma_{A_0,H}\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big) + (1 + c_{A_0,H}) \cdot p^*\mathrm{dvol}_\Sigma\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big)\\
= \;& \omega\big(w + X_{A_0(v)} + X_{H(\mathrm{d}\pi(v))}, J_{\Theta(A,u)}w + J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))} - X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\\
&\qquad\qquad + X_{A_0(\pi^*(j_\Sigma)v)} + X_{H(\mathrm{d}\pi(\pi^* j_\Sigma)v)}\big)\\
& - \big\langle F_{A_0}(v, (\pi^* j_\Sigma)v), \mu\big\rangle - \Omega_H\,\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v), \mathrm{d}\pi((\pi^* j_\Sigma)v)\big)\\
& + (1 + c_{A_0,H}) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v), j_\Sigma\,\mathrm{d}\pi(v)\big)\\
= \;& \omega\big(w + X_{A_0(v)} + X_{H(\mathrm{d}\pi(v))}, J_{\Theta(A,u)}w + J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))} + X_{A_0(\pi^*(j_\Sigma)v)}\big)\\
& - \big\langle F_{A_0}(v, (\pi^* j_\Sigma)v), \mu\big\rangle - \Omega_H\,\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v), j_\Sigma\,\mathrm{d}\pi(v)\big)\\
& + (1 + c_{A_0,H}) \cdot \mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v), j_\Sigma\,\mathrm{d}\pi(v)\big).
\end{aligned} \tag{3.99}$$

The last three terms on the right-hand side of this equation may be estimated using inequality (3.95) by

$$\begin{aligned} & -\big\langle F_{A_0}(v,(\pi^*j_\Sigma)v),\mu\big\rangle - \Omega_H\,\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v),j_\Sigma\,\mathrm{d}\pi(v)\big) \\ & + (1+c_{A_0,H})\cdot\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v),j_\Sigma\,\mathrm{d}\pi(v)\big) \\ \geq\ & -c_{A_0,H}\cdot|\mathrm{d}\pi(v)|\cdot|j_\Sigma\,\mathrm{d}\pi(v)| + (1+c_{A_0,H})\cdot|\mathrm{d}\pi(v)|^2 \\ \geq\ & |\mathrm{d}\pi(v)|^2. \end{aligned} \tag{3.100}$$

In order to estimate the first term in (3.99), we express it in the form

$$\begin{aligned} & \omega\big(w + X_{A_0(v)} + X_{H(\mathrm{d}\pi(v))}, J_{\Theta(A,u)}w + J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))} + X_{A_0((\pi^*j_\Sigma)v)}\big) \\ = \ & \omega(w,J_{\Theta(A,u)}w) + \omega\big(X_{(A-A_0(\mathrm{d}\pi(v))},J_{\Theta(A,u)}w\big) + \omega\big(X_{H(\mathrm{d}\pi(v))},J_{\Theta(A,u)}w\big) \\ & + \omega\big(w,J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big) + \omega\big(X_{(A-A_0(\mathrm{d}\pi(v))},J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big) \\ & + \omega\big(X_{H(\mathrm{d}\pi(v))},J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big) + \omega\big(w,X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big) \\ & + \omega\big(X_{(A-A_0)(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big) + \omega\big(X_{H(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big). \end{aligned}$$

Here we used that $A-A_0$ is horizontal and hence descends to the base $\Sigma$, and that $A(v)=0$ since $v$ is $A$-horizontal by assumption. Now we have

$$\omega(w,J_{\Theta(A,u)}w) = |w|^2_{J_{\Theta(A,u)}}, \qquad \omega\big(X_{H(\mathrm{d}\pi(v))},J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big) = \big|X_{H(\mathrm{d}\pi(v))}\big|^2_{J_{\Theta(A,u)}}$$

and

$$\omega\big(X_{H(\mathrm{d}\pi(v))},J_{\Theta(A,u)}w\big) = -\omega\big(J_{\Theta(A,u)}w,X_{H(\mathrm{d}\pi(v))}\big) = -\omega\big(w,J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big).$$

Moreover, there exists a constant $\tilde c > 1$ (not depending on $(A,u)$) such that

$$\begin{aligned} \big|\omega\big(X_{(A-A_0)(\mathrm{d}\pi(v))},J_{\Theta(A,u)}w\big)\big| &\leq \tilde c\,|\mathrm{d}\pi(v)|\cdot|w|_{J_{\Theta(A,u)}}\cdot\big\|A-A_0\big\|_{C^0(\Sigma)}, \\ \big|\omega\big(X_{(A-A_0(\mathrm{d}\pi(v))},J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}\big)\big| &\leq \tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|_{C^0(\Sigma)}, \\ \big|\omega\big(w,X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\big| &\leq \tilde c\,|\mathrm{d}\pi(v)|\cdot|w|_{J_{\Theta(A,u)}}\cdot\big\|A-A_0\big\|_{C^0(\Sigma)}, \\ \big|\omega\big(X_{(A-A_0)(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\big| &\leq \tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|^2_{C^0(\Sigma)}, \\ \big|\omega\big(X_{H(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\big| &\leq \tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|_{C^0(\Sigma)}. \end{aligned}$$

Using Young's inequality we therefore obtain an estimate

$$\begin{aligned} & \big|\omega\big(w + X_{A_0(v)} + X_{H(\mathrm{d}\pi(v))}, J_{\Theta(A,u)}w + J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))} + X_{A_0((\pi^*j_\Sigma)v)}\big)\big| \\ \geq\ & |w|^2_{J_{\Theta(A,u)}} - 2\tilde c\,|\mathrm{d}\pi(v)|\cdot|w|_{J_{\Theta(A,u)}}\cdot\big\|A-A_0\big\|_{C^0(\Sigma)} \\ & \qquad - 2\tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|_{C^0(\Sigma)} - \tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|^2_{C^0(\Sigma)} \\ \geq\ & |w|^2_{J_{\Theta(A,u)}} - \frac{1}{2}\,|w|^2_{J_{\Theta(A,u)}} - 2\tilde c^2\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|^2_{C^0(\Sigma)} \\ & \qquad - 2\tilde c\,|\mathrm{d}\pi(v)|^2\cdot\big\|A-A_0\big\|_{C^0(\Sigma)} \end{aligned}$$

$$\geq \frac{1}{2}|w|^2_{J_{\Theta(A,u)}} - 2\tilde{c}^2\,|\mathrm{d}\pi(v)|^2 \cdot \left\|A - A_0\right\|_{C^0(\Sigma)} \cdot \Big(1 + \left\|A - A_0\right\|_{C^0(\Sigma)}\Big).$$

Combining this with (3.100) it follows from (3.99) that

$$\begin{aligned}\omega_{A_0,H}\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big) \\ \geq \frac{1}{2}|w|^2_{J_{\Theta(A,u)}} + |\mathrm{d}\pi(v)|^2 \cdot \Big(1 - 2\tilde{c}^2\left\|A - A_0\right\|_{C^0(\Sigma)} \cdot \Big(1 + \left\|A - A_0\right\|_{C^0(\Sigma)}\Big)\Big).\end{aligned}$$

Hence, setting $c := (\sqrt{1 + 1/\tilde{c}^2} - 1)/2 > 0$, we get

$$\omega_{A_0,H}\big([v,w], I_{\Theta,A,\tilde{u},H}[v,w]\big) \geq \frac{1}{2}\Big(|w|^2_{J_{\Theta(A,u)}} + |\mathrm{d}\pi(v)|^2\Big) > 0$$

whenever

$$\left\|A - A_0\right\|_{C^0(\Sigma)} \leq c.$$

Note that the constant $c$ does not depend on $(A,u)$. □

**Proof of (iv):** Since $(A,u)$ is a non-local vortex, assertion (iv) follows from the following lemma.

LEMMA 3.5.5. *Let $(A,u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P,M)^G$ be a pair of class $W^{1,p}$. Let $\tilde{u}\colon \Sigma \to P(M)$ be the section associated to $u$ defined by (3.89). If the pair $(A,u)$ satisfies the equation*

$$\overline{\partial}_{J,A,\Theta,H}(u) := \frac{1}{2}\left(\mathrm{d}_{A,H}u + J_{\Theta(A,u)}(u) \circ \mathrm{d}_{A,H}u \circ j_\Sigma\right) = 0,$$

*then*

$$\overline{\partial}_{I_{\Theta,A,H}}(\tilde{u}) := \frac{1}{2}\left(\mathrm{d}\tilde{u} + I_{\Theta,A,\tilde{u},H}(\tilde{u}) \circ \mathrm{d}\tilde{u} \circ j_\Sigma\right) = 0,$$

*that is, $\tilde{u}$ is a $(j_\Sigma, I_{\Theta,A,\tilde{u},H})$-holomorphic curve.*

PROOF. First, we express the derivative of the curve $\tilde{u}$ in terms of the derivative of the map $u$. For $z \in \Sigma$ and $v \in T_z\Sigma$ we have

$$\mathrm{d}\tilde{u}(z)(v) = \big[\tilde{v}, \mathrm{d}u(p)(\tilde{v})\big] = \big[\tilde{v}, \mathrm{d}_{A,H}u(p)(\tilde{v}) - X_{H(v)}(u(p))\big]$$

for $p \in P$ such that $\pi(p) = z$, where $\tilde{v} \in T_pP$ denotes the $A$-horizontal lift of $v$. Here we used the definition

$$\mathrm{d}_{A,H}u = \mathrm{d}u + X_A(u) + X_H(u),$$

and the fact that $\mathrm{d}_Au(p)(\tilde{v}) = \mathrm{d}u(p)(\tilde{v})$ since $\tilde{v}$ is $A$-horizontal. By assumption, $(A,u)$ satisfies the equation

$$\mathrm{d}_{A,H}u + J_{\Theta(A,u)}(u) \circ \mathrm{d}_{A,H}u \circ j_\Sigma = 0.$$

It follows that

$$J_{\Theta(A,u)}(u)\,\mathrm{d}_{A,H}u(\tilde{v}) = \mathrm{d}_{A,H}u\big((\pi^*j_\Sigma)\tilde{v}\big).$$

Using formula (3.98) we therefore obtain

$$
\begin{aligned}
I_{\Theta,A,\tilde{u},H}(\tilde{u})\,\mathrm{d}\tilde{u}(v) &= I_{\Theta,A,\tilde{u},H}(\tilde{u})\left[\tilde{v}, \mathrm{d}_{A,H}u(p)(\tilde{v}) - X_{H(v)}(u(p))\right] \\
&= \Big[(\pi^* j_\Sigma)\tilde{v}, J_{\Theta(A,u)}(u)\big(\mathrm{d}_{A,H}u(p)(\tilde{v}) - X_{H(v)}(u(p)) + X_{A(\tilde{v})}(u(p)) \\
&\qquad + X_{H(v)}(u(p))\big) - X_{H(j_\Sigma v)}(u(p))\Big] \\
&= \Big[(\pi^* j_\Sigma)\tilde{v}, J_{\Theta(A,u)}(u)\,\mathrm{d}_{A,H}u(p)(\tilde{v}) - X_{H(j_\Sigma v)}(u(p))\Big] \\
&= \Big[(\pi^* j_\Sigma)\tilde{v}, \mathrm{d}_{A,H}u(p)\big((\pi^* j_\Sigma)\tilde{v}\big) - X_{H(j_\Sigma v)}(u(p))\Big] \\
&= \big[(\pi^* j_\Sigma)\tilde{v}, \mathrm{d}u(p)\big((\pi^* j_\Sigma)\tilde{v}\big)\big] \\
&= \mathrm{d}\tilde{u}\big(j_\Sigma\, v\big).
\end{aligned}
$$

This implies

$$
\mathrm{d}\tilde{u} + I_{\Theta,A,\tilde{u},H}(\tilde{u}) \circ \mathrm{d}\tilde{u} \circ j_\Sigma = 0,
$$

which proves the lemma. □

**Proof of (v) and (vi):** Let $(A, u)$ be a non-local vortex of class $W^{1,p}$ such that

$$
\big\|A - A_0\big\|_{C^0(\Sigma)} \leq c \qquad \text{and} \qquad \big\|J_{\Theta(A,u)} - J_{\Theta(A_0,u_0)}\big\|_{C^0(\Sigma)} \leq c,
$$

where $c$ is the constant of Lemma 3.5.4 above. By Lemma 3.5.4 the symplectic form $\omega_{A_0,H}$ tames the almost complex structure $I_{\Theta,A,\tilde{u},H}$. We shall denote by $\langle\cdot,\cdot\rangle_{I_{\Theta,A,\tilde{u},H}}$ the Riemannian metric on $P(M)$ determined by $\omega_{A_0,H}$ and $I_{\Theta,A,\tilde{u},H}$. It is given by

$$
\begin{aligned}
&\big\langle [v_1, w_1], [v_2, w_2]\big\rangle_{I_{\Theta,A,\tilde{u},H}} \\
&:= \frac{1}{2}\Big(\omega_{A_0,H}\big([v_1, w_1], I_{\Theta,A,\tilde{u},H}[v_2, w_2]\big) + \omega_{A_0,H}\big([v_2, w_2], I_{\Theta,A,\tilde{u},H}[v_1, w_1]\big)\Big)
\end{aligned}
$$

for $[v_1, w_1], [v_2, w_2] \in TP(M)$. The norm associated to this metric is then given by

$$
\big|[v, w]\big|^2_{I_{\Theta,A,\tilde{u},H}} = \omega_{A_0,H}\big([v, w], I_{\Theta,A,\tilde{u},H}[v, w]\big)
$$

for $[v, w] \in TP(M)$. The next lemma relates this norm with the Kähler metric on $\Sigma$ determined by $\mathrm{dvol}_\Sigma$ and $j_\Sigma$ and the metric on $M$ determined by $\omega$ and $J_{\Theta(A,u)}$.

Lemma 3.5.6. *Fix a smooth reference connection $A_0 \in \mathcal{A}(P)$. There exist constants $c > 0$ and $C_{A_0,H} > 0$ such that for all non-local vortices $(A, u)$ of class $W^{1,p}$ satisfying*

$$
\big\|A - A_0\big\|_{C^0(\Sigma)} \leq c \qquad \textit{and} \qquad \big\|J_{\Theta(A,u)} - J_{\Theta(A_0,u_0)}\big\|_{C^0(\Sigma)} \leq c
$$

*the following holds.*

*The Riemannian metric* $\langle\cdot,\cdot\rangle_{I_{\Theta,A,\tilde{u},H}}$ *on* $P(M)$ *determined by* $\omega_{A_0,H}$ *and* $I_{\Theta,A,\tilde{u},H}$ *satisfies the relation*

$$\frac{1}{2}\left(|\mathrm{d}\pi(v)|^2+\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}\right) \le \left|[v,w]\right|^2_{I_{\Theta,A,\tilde{u},H}} \le C_{A_0,H}\cdot\left(|\mathrm{d}\pi(v)|^2+\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}\right)$$

*for all* $[v,w]\in TP(M)$ *such that* $v$ *is* $A$*-horizontal, where* $\pi\colon P\to\Sigma$ *denotes the bundle projection.*

PROOF. Let $[v,w]\in TP(M)$. By formula (3.91) we may without loss of generality assume that $v$ is $A$-horizontal. By (3.98) we have

$$I_{\Theta,A,\tilde{u},H}[v,w]=\left[(\pi^*j_\Sigma)v,\,J_{\Theta(A,u)}w+J_{\Theta(A,u)}X_{H(\mathrm{d}\pi(v))}-X_{H(j_\Sigma\,\mathrm{d}\pi(v))}\right].$$

A calculation similar to the one in the proof of Lemma 3.5.4 above yields

$$\begin{aligned}\left|[v,w]\right|^2_{I_{\Theta,A,\tilde{u},H}} \;=\;& \omega\big(w+X_{H(\mathrm{d}\pi(v))},J_{\Theta(A,u)}\big(w+X_{H(\mathrm{d}\pi(v))}\big)\big)\\ &+\omega\big(X_{(A-A_0(\mathrm{d}\pi(v))},J_{\Theta(A,u)}\big(w+X_{H(\mathrm{d}\pi(v))}\big)\big)\\ &+\omega\big(w+X_{H(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\\ &+\omega\big(X_{(A-A_0)(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\\ &-\big\langle F_{A_0}(v,(\pi^*j_\Sigma)v),\mu\big\rangle-\Omega_H\,\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v),j_\Sigma\,\mathrm{d}\pi(v)\big)\\ &+(1+c_{A_0,H})\cdot\mathrm{dvol}_\Sigma\big(\mathrm{d}\pi(v),j_\Sigma\,\mathrm{d}\pi(v)\big)\end{aligned}$$

Recall further from the proof of Lemma 3.5.4 that there exists a constant $c_1>1$ (not depending on $(A,u)$) such that

$$\begin{aligned}&\left|\omega\big(X_{(A-A_0(\mathrm{d}\pi(v))},J_{\Theta(A,u)}\big(w+X_{H(\mathrm{d}\pi(v))}\big)\big)\right|,\ \left|\omega\big(w+X_{H(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\right|\\ &\qquad\le c_1\,|\mathrm{d}\pi(v)|\cdot\left|w+X_{H(\mathrm{d}\pi(v))}\right|_{J_{\Theta(A,u)}}\cdot\left\|A-A_0\right\|_{C^0(\Sigma)}\end{aligned}$$

and

$$\left|\omega\big(X_{(A-A_0)(\mathrm{d}\pi(v))},X_{(A-A_0)(j_\Sigma\,\mathrm{d}\pi(v))}\big)\right|\le c_1\,|\mathrm{d}\pi(v)|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}.$$

Using Young's inequality we therefore obtain estimates

$$\begin{aligned}\left|[v,w]\right|^2_{I_{\Theta(A,u),A,\tilde{u},H}} \;\ge\;& \left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}\\ &-2c_1\,|\mathrm{d}\pi(v)|\cdot\left|w+X_{H(\mathrm{d}\pi(v))}\right|_{J_{\Theta(A,u)}}\cdot\left\|A-A_0\right\|_{C^0(\Sigma)}\\ &-c_1\,|\mathrm{d}\pi(v)|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}+|\mathrm{d}\pi(v)|^2 \qquad (3.101)\\ \ge\;& \left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}-\frac{1}{2}\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}\\ &-2c_1^2\,|\mathrm{d}\pi(v)|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}\\ &-2c_1^2\,|\mathrm{d}\pi(v)|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}+|\mathrm{d}\pi(v)|^2\end{aligned}$$

$$\begin{aligned} \geq\ & \frac{1}{2}\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}+\left|\mathrm{d}\pi(v)\right|^2 \\ & -4c_1^2\left|\mathrm{d}\pi(v)\right|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)} \end{aligned}$$

and

$$\begin{aligned} \left|[v,w]\right|^2_{I_{\Theta,A,\tilde{u},H}} \leq\ & \left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}} \\ & -2c_1\left|\mathrm{d}\pi(v)\right|\cdot\left|w+X_{H(\mathrm{d}\pi(v))}\right|_{J_{\Theta(A,u)}}\cdot\left\|A-A_0\right\|_{C^0(\Sigma)} \\ & -c_1\left|\mathrm{d}\pi(v)\right|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}+(1+2c_{A_0,H})\left|\mathrm{d}\pi(v)\right|^2 \qquad (3.102) \\ \leq\ & \left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}+\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}} \\ & +2c_1^2\left|\mathrm{d}\pi(v)\right|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)} \\ & +2c_1^2\left|\mathrm{d}\pi(v)\right|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}+(1+2c_{A_0,H})\left|\mathrm{d}\pi(v)\right|^2 \\ \leq\ & 2\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}+(1+2c_{A_0,H})\left|\mathrm{d}\pi(v)\right|^2 \\ & +4c_1^2\left|\mathrm{d}\pi(v)\right|^2\cdot\left\|A-A_0\right\|^2_{C^0(\Sigma)}. \end{aligned}$$

Whence there exists a constant $C_{A_0,H}>0$ (not depending on $(A,u)$) such that

$$\left|[v,w]\right|^2_{I_{\Theta,A,\tilde{u},H}}\geq\frac{1}{2}\cdot\Big(\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}+\left|\mathrm{d}\pi(v)\right|^2\Big)$$

and

$$\left|[v,w]\right|^2_{I_{\Theta,A,\tilde{u},H}}\leq C_{A_0,H}\cdot\Big(\left|w+X_{H(\mathrm{d}\pi(v))}\right|^2_{J_{\Theta(A,u)}}+\left|\mathrm{d}\pi(v)\right|^2\Big)$$

whenever the norm $\|A-A_0\|_{C^0(\Sigma)}$ is sufficiently small. Thus, taking $c$ to be the constant of Lemma 3.5.4 and shrinking it if necessary, the lemma follows. □

We now complete the proof of assertions (v) and (vi) of Proposition 3.5.3 by proving the following lemma.

Lemma 3.5.7. *Fix a smooth reference connection $A_0\in\mathcal{A}(P)$, and let $C_{A_0,H}$ be the constant of Lemma 3.5.6. There exist constants $r_0>0$ and $\delta,C>0$ such that for all non-local vortices $(A,u)$ of class $W^{1,p}$ satisfying*

$$\left\|A-A_0\right\|_{C^0(\Sigma)}\leq c \qquad \text{and} \qquad \left\|J_{\Theta(A,u)}-J_{\Theta(A_0,u_0)}\right\|_{C^0(\Sigma)}\leq c,$$

*where $c$ is the constant of Lemma 3.5.6, the following holds.*

(i) *The energy of the curve $\tilde{u}\colon\Sigma\to P(M)$ and the Yang-Mills-Higgs energy of the vortex $(A,u)$ are related by*

$$E_{I_{\Theta,A,H}}(\tilde{u})\leq C_{A_0,H}\cdot\big(E_{J_\Theta}(A,u)+\mathrm{Vol}(\Sigma)\big).$$

(ii) *For all $z_0 \in \Sigma$ and all $0 < r < r_0$, the curve $\tilde{u}\colon \Sigma \to P(M)$ satisfies a mean value inequality of the form*

$$\begin{aligned} &E_{I_{\Theta,A,H}}\big(\tilde{u}; B_r(z_0)\big) < \delta \\ \Longrightarrow \qquad &\big|\mathrm{d}\tilde{u}(z_0)\big|^2_{I_{\Theta,A,H}} \le \frac{C}{r^2} \cdot E_{I_{\Theta,A,H}}\big(\tilde{u}; B_r(z_0)\big) + C. \end{aligned}$$

Proof. We first compare the Yang-Mills-Higgs energy of the vortex $(A, u)$ with the energy of the curve $\tilde{u}\colon \Sigma \to P(M)$. For $z \in \Sigma$ and $v \in T_z\Sigma$ we have

$$\mathrm{d}\tilde{u}(z)(v) = \big[\tilde{v}, \mathrm{d}u(p)(\tilde{v})\big] = \big[\tilde{v}, \mathrm{d}_A u(z)(\tilde{v})\big] \tag{3.103}$$

for all $p \in P$ such that $\pi(p) = z$, where $\tilde{v} \in T_pP$ denotes the $A$-horizontal lift of $v$. Recall that

$$\mathrm{d}_{A,H}u = \mathrm{d}_A u + X_H(u).$$

Hence, by Lemma 3.5.6 and formula (3.103) we have an inequality

$$\frac{1}{2}\Big(|v|^2 + |\mathrm{d}_{A,H}u(v)|^2_{J_\Theta}\Big) \le \big|\mathrm{d}\tilde{u}(v)\big|^2_{I_{\Theta,A,H}} \le C_{A_0,H} \cdot \Big(|v|^2 + |\mathrm{d}_{A,H}u(v)|^2_{J_\Theta}\Big)$$

for all $v \in T\Sigma$, where $C_{A_0,H}$ is the constant of Lemma 3.5.6. Rewriting this in terms of operator norms we therefore get an estimate

$$\frac{1}{2}\Big(1 + |\mathrm{d}_{A,H}u|^2_{J_\Theta}\Big) \le \big|\mathrm{d}\tilde{u}\big|^2_{I_{\Theta,A,H}} \le C_{A_0,H} \cdot \Big(1 + |\mathrm{d}_{A,H}u|^2_{J_\Theta}\Big). \tag{3.104}$$

Let $U \subset \Sigma$ be an open subset. Recall from [**22**], Section 2.2, that the energy of the curve $\tilde{u}$ on $U$ is given by

$$E_{I_{\Theta,A,H}}(\tilde{u}; U) = \frac{1}{2}\int_U |\mathrm{d}\tilde{u}|^2_{I_{\Theta,A,H}}\,\mathrm{dvol}_\Sigma,$$

and recall from Remark 2.2.5 that the Yang-Mills-Higgs energy of the vortex $(A, u)$ on $U$ may be expressed as

$$E_{J_\Theta}(A, u; U) = \frac{1}{2}\int_U \left(\frac{1}{2}\,|\mathrm{d}_{A,H}u|^2_{J_\Theta} + |\mu(u)|^2\right)\mathrm{dvol}_\Sigma\,.$$

Proof of (i): We use inequality (3.104) to estimate the energy of the curve $\tilde{u}$ in terms of the Yang-Mills-Higgs energy of the vortex $(A, u)$ by

$$
\begin{aligned}
E_{I_{\Theta,A,H}}(\tilde{u}) &= \frac{1}{2}\int_\Sigma |\mathrm{d}\tilde{u}|^2_{I_{\Theta,A,H}}\,\mathrm{dvol}_\Sigma \\
&\leq \frac{C_{A_0,H}}{2}\cdot\int_\Sigma\Big(1+|\mathrm{d}_{A,H}u|^2_{J_\Theta}\Big)\,\mathrm{dvol}_\Sigma \\
&= C_{A_0,H}\cdot\int_\Sigma\left(\frac{1}{2}\,|\mathrm{d}_{A,H}u|^2_{J_\Theta}+|\mu(u)|^2\right)\mathrm{dvol}_\Sigma \\
&\qquad\qquad -\frac{C_{A_0,H}}{2}\cdot\int_\Sigma\Big(2\,|\mu(u)|^2-1\Big)\,\mathrm{dvol}_\Sigma \\
&\leq C_{A_0,H}\cdot E_{J_\Theta}(A,u)+C_{A_0,H}\cdot\int_\Sigma\mathrm{dvol}_\Sigma \\
&= C_{A_0,H}\cdot\Big(E_{J_\Theta}(A,u)+\mathrm{Vol}(\Sigma)\Big).
\end{aligned}
$$

This proves (i).

Proof of (ii): Let $z_0 \in \Sigma$. By Theorem 3.2.1 there exist constants $\hbar > 0$, $C' > 0$ and $R > 0$ such that for all $0 < r < R$ the vortex $(A, u)$ satisfies the a priori estimate

$$
\begin{aligned}
&E_{J_\Theta}\big(A,u;B_r(z_0)\big)<\hbar \\
\Longrightarrow\qquad &\frac{1}{2}\big|\mathrm{d}_{A,H}u(z_0)\big|^2_{J_\Theta}+\big|\mu(u(z_0))\big|^2\leq\frac{C'}{r^2}\cdot E_{J_\Theta}\big(A,u;B_r(z_0)\big). \qquad (3.105)
\end{aligned}
$$

We set

$$K := \big\|\mu\big\|^2_{C^0(M)} \geq 0.$$

Note that $K$ is finite by compactness of $M$. We then define constants $\delta$ and $C$ by

$$\delta := \frac{\hbar}{4} > 0 \quad\text{and}\quad C := 4\,C_{A_0}\cdot(1+C')\cdot\left(1+K\cdot\sup_{0<r<R}\frac{\mathrm{Vol}(B_r(z_0))}{r^2}\right),$$

where $C_{A_0,H}$ is the constant of Lemma 3.5.6. Here $B_r(z_0)$ denotes the closed geodesic disk in $\Sigma$ around $z_0$ of radius $r$, understood with respect to the Kähler metric determined by the area for $\mathrm{dvol}_\Sigma$ and the complex structure $j_\Sigma$. Further, fix a constant $0 < r_0 < R$ such that

$$\mathrm{Vol}\big(B_r(z_0)\big) \leq \frac{\hbar}{2K}$$

for all $0 < r < r_0$.

Assume now that $r < r_0$ and

$$E_{I_{\Theta,A,H}}\big(\tilde{u}; B_r(z_0)\big) < \delta. \qquad (3.106)$$

We then obtain from (3.104) the inequality

$$\begin{aligned}
E_{I_{\Theta,A,H}}\big(\tilde u; B_r(z_0)\big) &= \frac{1}{2}\int_{B_r(z_0)} |\mathrm{d}\tilde u|^2_{I_{\Theta,A,H}}\,\mathrm{dvol}_\Sigma \\
&\geq \frac{1}{4}\int_{B_r(z_0)} \Big(1+|\mathrm{d}_{A,H}u|^2_{J_\Theta}\Big)\,\mathrm{dvol}_\Sigma \\
&= \frac{1}{2}\int_{B_r(z_0)} \bigg(\frac{1}{2}\,|\mathrm{d}_{A,H}u|^2_{J_\Theta} + |\mu(u)|^2\bigg)\,\mathrm{dvol}_\Sigma \\
&\qquad\qquad - \frac{1}{4}\int_{B_r(z_0)} \Big(2\,|\mu(u)|^2 - 1\Big)\,\mathrm{dvol}_\Sigma \\
&\geq \frac{1}{2}\,E_{J_\Theta}\big(A,u;B_r(z_0)\big) - \frac{1}{2}\,K\cdot \mathrm{Vol}\big(B_r(z_0)\big).
\end{aligned}$$

We have thus proved that

$$E_{J_\Theta}\big(A,u;B_r(z_0)\big) \leq 2\,E_{I_{\Theta,A,H}}\big(\tilde u;B_r(z_0)\big) + K\cdot\mathrm{Vol}\big(B_r(z_0)\big). \tag{3.107}$$

Hence by (3.106),

$$E_{J_\Theta}\big(A,u;B_r(z_0)\big) < \frac{\hbar}{2} + K\cdot\mathrm{Vol}\big(B_r(z_0)\big).$$

Since $r < r_0$ it follows from the definition of $r_0$ and the last inequality that

$$E_{J_\Theta}\big(A,u;B_r(z_0)\big) < \hbar,$$

so the a priori estimate (3.105) implies

$$\frac{1}{2}\big|\mathrm{d}_{A,H}u(z_0)\big|^2_{J_\Theta} + \big|\mu(u(z_0))\big|^2 \leq \frac{C'}{r^2}\cdot E_{J_\Theta}\big(A,u;B_r(z_0)\big). \tag{3.108}$$

Combining this with the second inequality in (3.104) we get

$$\begin{aligned}
\big|\mathrm{d}\tilde u(z_0)\big|^2_{I_{\Theta,A,H}} &\leq C_{A_0,H}\cdot\Big(1+\big|\mathrm{d}_{A,H}u(z_0)\big|^2_{J_\Theta}\Big) \\
&\leq 2\,C_{A_0,H}\cdot\bigg(\frac{1}{2}\big|\mathrm{d}_{A,H}u(z_0)\big|^2_{J_\Theta} + \big|\mu(u(z_0))\big|^2\bigg) + C_{A_0,H} \\
&\leq \frac{2\,C'\,C_{A_0,H}}{r^2}\cdot E_{J_\Theta}\big(A,u;B_r(z_0)\big) + C_{A_0,H}.
\end{aligned}$$

Once again applying inequality (3.107) we obtain

$$\begin{aligned}
\big|\mathrm{d}\tilde u(z_0)\big|^2_{I_{\Theta,A,H}} &\leq \frac{4\,C'\,C_{A_0,H}}{r^2}\cdot E_{I_{\Theta,A,H}}\big(\tilde u;B_r(z_0)\big) + 2\,C_{A_0,H}\cdot\bigg(C'\cdot K\cdot\frac{\mathrm{Vol}(B_r(z_0))}{r^2}+1\bigg) \\
&\leq \frac{C}{r^2}\cdot E_{I_{\Theta,A,H}}\big(\tilde u;B_r(z_0)\big) + C.
\end{aligned}$$

This proves (ii) and finishes the proof of the lemma. □

The proof of Proposition 3.5.3 is now complete.

## 3.6. Proof of Gromov compactness

We are now ready to prove Theorem 3.1.7 on Gromov compactness for vortices. Our strategy is to adapt the proof of Theorem 5.3.1 on Gromov convergence for pseudoholomorphic curves in McDuff and Salamon [**22**], replacing the statements of Theorem 4.6.1 and Propositions 4.7.1 and 4.7.2 in [**22**] with the corresponding statements of Theorem 3.4.1 and Proposition A.2.1.

Fix a real constant $E > 0$ and an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. Let $n$ be a nonnegative integer.

Let $(A_\nu, u_\nu, \mathbf{z}_\nu)$ be a sequence of $n$-marked non-local vortices solving equations (2.16) such that the Yang-Mills-Higgs energy satisfies a uniform bound

$$\sup_\nu E(A_\nu, u_\nu) < \infty.$$

Our goal is to construct a rooted $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ and a polystable non-local vortex

$$\big(A, \mathbf{u}, \mathbf{z}\big) = \big((A, u_0), \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big) \tag{3.109}$$

of combinatorial type $T$ such that the sequence $(A_\nu, u_\nu, \mathbf{z}_\nu)$ Gromov converges to $(A, \mathbf{u}, \mathbf{z})$ in the sense of Definition 3.1.5. We shall proceed in nine steps.

**Step 1** We fix a root vertex 0 and associate to it the principal component $\Sigma_0 := \Sigma$.

**Step 2** We apply Theorem 3.4.1 to the sequence of non-local vortices $(A_\nu, u_\nu)$. The conclusion is that there exists a smooth vortex $(A, u_0)$, a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$, and a finite set $Z_0 = \{\zeta_1, \ldots, \zeta_N\}$ of bubbling points on $\Sigma_0$ such that, after passing to a subsequence if necessary,

(i) the sequence $g_\nu^* A_\nu$ converges to $A$ weakly in the $W^{1,p}$-topology and strongly in the $C^0$-topology on $\Sigma_0$;
(ii) the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ converges to $(A, u_0)$ strongly in the $C^1$-topology on compact subsets of $\Sigma_0 \setminus Z_0$;
(iii) for every $j \in \{1, \ldots, N\}$ and every $\varepsilon > 0$ such that $B_\varepsilon(\zeta_j) \cap Z_0 = \{\zeta_j\}$ the limit

$$m_\varepsilon(\zeta_j) := \lim_{\nu \to \infty} E\big(g_\nu^* A_\nu, g_\nu^{-1} u_\nu; B_\varepsilon(\zeta_j)\big)$$

exists and is a continuous function of $\varepsilon$, and

$$m(\zeta_j) := \lim_{\varepsilon \to 0} m_\varepsilon(\zeta_j) \ge \hbar,$$

where $\hbar$ is the constant of Theorem 3.2.1;

(iv) for every compact subset $K \subset \Sigma_0$ such that $Z_0$ is contained in the interior of $K$,

$$E\big(A, u_0; K\big) + \sum_{j=1}^{N} m(\zeta_j) = \lim_{\nu\to\infty} E\big(g_\nu^* A_\nu, g_\nu^{-1} u_\nu; K\big).$$

**Step 3** We apply Proposition 3.5.3 to the non-local vortex $(A, u_0)$. It follows that there exists a symplectic form $\omega_{A,H}$ and an almost complex structure

$$I := I_{\Theta,A,\tilde{u}_0,H}$$

on the closed manifold $P(M) := P \times_G M$ such that $I$ is tamed by $\omega_{A,H}$ and the section $u_0 \colon \Sigma_0 \to P(M)$ (see Remark 3.1.1) becomes a $(j_{\Sigma_0}, I)$-holomorphic curve.

**Step 4** After passing to a subsequence if necessary, we apply Proposition 3.5.3 to the sequence of non-local vortices $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$.

In fact, by (i) and (ii) in Step 2, after passing to a subsequence if necessary each element of the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ does satisfy the assumptions of Proposition 3.5.3. To see this note that it follows from (ii) in Step 2 and the (Continuity) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3) that $J_{\Theta(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)}$ converges to $J_{\Theta(A,u)}$ in the $C^0$-topology on $\Sigma$.

We thus obtain a sequence of $\omega_{A,H}$-tame almost complex structures

$$I_\nu := I_{\Theta, g_\nu^* A_\nu, g_\nu^{-1}\tilde{u}_\nu, H}$$

on $P(M)$ that converges to $I$ strongly in the $C^0$-topology and such that

(v) the curve $g_\nu^{-1} u_\nu \colon \Sigma_0 \to P(M)$ is $(j_{\Sigma_0}, I_\nu)$-holomorphic for every $\nu$;

(vi) the energy of the curve $g_\nu^{-1} u_\nu$ satisfies a uniform bound

$$\sup_\nu E_{I_\nu}\big(g_\nu^{-1} u_\nu\big) < \infty;$$

(vii) there exist constants $r_0 > 0$ and $\delta, C > 0$ (not depending on $\nu$) such that for every $\nu$ the curve $u_\nu$ satisfies a mean value inequality of the following form.

For all $z_0 \in \Sigma_0$ and all $0 < r < r_0$,

$$\begin{aligned} &E_{I_\nu}\big(g_\nu^{-1} u_\nu; B_r(z_0)\big) < \delta \\ \Longrightarrow \qquad &\big|\mathrm{d}\big(g_\nu^{-1} u_\nu\big)(z_0)\big|_{I_\nu}^2 \le \frac{C}{r^2} \cdot E_{I_\nu}\big(g_\nu^{-1} u_\nu; B_r(z_0)\big) + C. \end{aligned}$$

Note that in order to obtain (vi) we used gauge invariance of the Yang-Mills-Higgs energy and the fact that $\sup_\nu E(A_\nu, u_\nu) < \infty$ by assumption.

**Step 5** We rephrase assertions (iii) and (iv) in Step 2 above in terms of the energy of the sequence $g_\nu^{-1}u_\nu$. More precisely, we claim that

(iii') for every $j \in \{1, \dots, N\}$ and every $\varepsilon > 0$ such that $B_\varepsilon(\zeta_j) \cap Z_0 = \{\zeta_j\}$ the limit
$$m'_\varepsilon(\zeta_j) := \lim_{\nu\to\infty} E_{I_\nu}\big(g_\nu^{-1}u_\nu; B_\varepsilon(\zeta_j)\big)$$
exists and is a continuous function of $\varepsilon$, and
$$m'(\zeta_j) := \lim_{\varepsilon\to 0} m'_\varepsilon(\zeta_j) \geq \hbar,$$
where $\hbar$ is the constant of Theorem 3.2.1;

(iv') for every compact subset $K \subset \Sigma_0$ such that $Z_0$ is contained in the interior of $K$,
$$E\big(A, u_0; K\big) + \sum_{j=1}^N m'(\zeta_j) = \lim_{\nu\to\infty} E\big(g_\nu^* A_\nu, g_\nu^{-1}u_\nu; K\big).$$

To see this, we first note that, after passing to a subsequence if necessary, it follows as in the proof of Theorem 4.6.1 in [**22**] that the limit $m'_\varepsilon(z_j)$ exists and is a continuous function of $\varepsilon$, for $j = 1, \dots, N$. Assertions (iii') and (iv') then follow from assertions (iii) and (iv) in Step 2 once we have shown that
$$m'(\zeta_j) = m(\zeta_j) \text{ for } j = 1, \dots, N,$$
which we will do now.

To simplify notation, we write $\hat{A}_\nu := g_\nu^* A_\nu$ and $\hat{u}_\nu := g_\nu^{-1}u_\nu$. Let $j \in \{1, \dots, N\}$ and let $\varepsilon > 0$. The Yang-Mills-Higgs energy of the vortex $(\hat{A}_\nu, \hat{u}_\nu)$ on $B_\varepsilon(\zeta_j)$ is given by
$$E\big(\hat{A}_\nu, \hat{u}_\nu; B_\varepsilon(\zeta_j)\big) = \frac{1}{2}\int_{B_\varepsilon(\zeta_j)} \big|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\big|^2_{J_\Theta}\,\mathrm{dvol}_{\Sigma_0} + \int_{B_\varepsilon(\zeta_j)} \big|\mu(\hat{u}_\nu)\big|^2\,\mathrm{dvol}_{\Sigma_0}, \tag{3.110}$$
and the energy of the curve $\hat{u}_\nu\colon \Sigma_0 \to P(M)$ on $B_\varepsilon(\zeta_j)$ is given by
$$E_{I_\nu}\big(\hat{u}_\nu; B_\varepsilon(\zeta_j)\big) = \frac{1}{2}\int_{B_\varepsilon(\zeta_j)} \big|\mathrm{d}\hat{u}_\nu\big|^2_{I_\nu}\,\mathrm{dvol}_{\Sigma_0}. \tag{3.111}$$

It follows from (3.103) in the proof of Lemma 3.5.7 and estimates (3.101) and (3.102) in the proof of Lemma 3.5.6 that there exists a constant $c > 0$ (not depending on $(\hat{A}_\nu, \hat{u}_\nu)$) such that
$$\begin{aligned}\big|\mathrm{d}\hat{u}_\nu(v)\big|^2_{I_\nu} \leq \big|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\big|^2_{J_\Theta} &+ 2c\,|v|\cdot\big|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\big|_{J_\Theta}\cdot\big\|\hat{A}_\nu - A\big\|_{C^0(\Sigma_0)}\\ &+ 2c\,|v|^2\cdot\big\|\hat{A}_\nu - A\big\|^2_{C^0(\Sigma_0)} + (1 + 2c_{A,H})\,|v|^2\end{aligned}$$
and
$$\begin{aligned}\big|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\big|^2_{J_\Theta} \geq \big|\mathrm{d}\hat{u}_\nu(v)\big|^2_{I_\nu} &+ 2c\,|v|\cdot\big|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\big|_{J_\Theta}\cdot\big\|\hat{A}_\nu - A\big\|_{C^0(\Sigma_0)}\\ &+ 2c\,|v|^2\cdot\big\|\hat{A}_\nu - A\big\|^2_{C^0(\Sigma_0)} + |v|^2\end{aligned}$$

for $v \in T\Sigma_0$, where $\tilde{v} \in TP$ denotes the $A_\nu$-horizontal lift of $v$. By Young's inequality, we have

$$\begin{aligned} 2c\,|v| \cdot \left|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\right|_{J_\Theta} \cdot \left\|\hat{A}_\nu - A\right\|_{C^0(\Sigma_0)} \le c\,|v|^2 \\ + c\left|\mathrm{d}_{\hat{A}_\nu}\hat{u}_\nu(\tilde{v}) + X_{H(v)}\right|^2_{J_\Theta} \cdot \left\|\hat{A}_\nu - A\right\|^2_{C^0(\Sigma_0)}. \end{aligned}$$

Hence, passing to operator norms we obtain inequalities

$$\left|\mathrm{d}\hat{u}_\nu\right|^2_{I_\nu} \le \left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} + R_\nu \qquad \text{and} \qquad \left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} \le \left|\mathrm{d}\hat{u}_\nu\right|^2_{I_\nu} + R_\nu, \tag{3.112}$$

where we abbreviate

$$R_\nu := c + 2c_{A,H} + c\left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} \cdot \left\|\hat{A}_\nu - A\right\|^2_{C^0(\Sigma_0)} + 2c\left\|\hat{A}_\nu - A\right\|^2_{C^0(\Sigma_0)}.$$

Now we have

$$\begin{aligned} \int_{B_\varepsilon(\zeta_j)} R_\nu \,\mathrm{dvol}_{\Sigma_0} = 2c\left\|\hat{A}_\nu - A\right\|^2_{C^0(\Sigma_0)} \cdot \int_{B_\varepsilon(\zeta_j)} \left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} \mathrm{dvol}_{\Sigma_0} \\ + \left(c + 2c_{A,H} + 2c\left\|\hat{A}_\nu - A\right\|^2_{C^0(\Sigma_0)}\right) \cdot \mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big). \end{aligned}$$

Taking the limit $\nu \to \infty$, we get

$$\lim_{\nu\to\infty} \int_{B_\varepsilon(\zeta_j)} R_\nu \,\mathrm{dvol}_{\Sigma_0} = \big(c + 2\,c_{A,H}\big) \cdot \mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big).$$

Here we used that $\hat{A}_\nu$ converges to $A$ strongly in the $C^0$-topology on $\Sigma_0$ by (i) in Step 2 above, and that

$$\sup_\nu \int_{B_\varepsilon(\zeta_j)} \left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} \mathrm{dvol}_{\Sigma_0} < \infty$$

since $\sup_\nu E(\hat{A}_\nu, \hat{u}_\nu) < \infty$ by assumption. Combining the above limit with (3.112), (3.110) and (3.111) we conclude that

$$\begin{aligned} m_\varepsilon(\zeta_j) &= \lim_{\nu\to\infty} \frac{1}{2} \int_{B_\varepsilon(\zeta_j)} \left|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\right|^2_{J_\Theta} \mathrm{dvol}_{\Sigma_0} + \lim_{\nu\to\infty} \int_{B_\varepsilon(\zeta_j)} \left|\mu(\hat{u}_\nu)\right|^2 \mathrm{dvol}_{\Sigma_0} \\ &\le \lim_{\nu\to\infty} \frac{1}{2} \int_{B_\varepsilon(\zeta_j)} \left|\mathrm{d}\hat{u}_\nu\right|^2_{I_\nu} \mathrm{dvol}_{\Sigma_0} + \left(c + 2\,c_{A,H} + \left\|\mu\right\|^2_{C^0(M)}\right) \cdot \mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big) \\ &= m'_\varepsilon(\zeta_j) + \left(c + 2\,c_{A,H} + \left\|\mu\right\|^2_{C^0(M)}\right) \cdot \mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big) \end{aligned}$$

and

$$\begin{aligned} m'_\varepsilon(\zeta_j) &= \lim_{\nu\to\infty} \frac{1}{2}\int_{B_\varepsilon(\zeta_j)} \big|\mathrm{d}\hat{u}_\nu\big|^2_{I_\nu}\,\mathrm{dvol}_{\Sigma_0} \\ &\leq \lim_{\nu\to\infty}\frac{1}{2}\int_{B_\varepsilon(\zeta_j)} \big|\mathrm{d}_{\hat{A}_\nu,H}\hat{u}_\nu\big|^2_{J_\Theta}\,\mathrm{dvol}_{\Sigma_0} + \lim_{\nu\to\infty}\int_{B_\varepsilon(\zeta_j)}\big|\mu(\hat{u}_\nu)\big|^2\,\mathrm{dvol}_{\Sigma_0} \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \big(c + 2\,c_{A,H}\big)\cdot \mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big) \\ &= m_\varepsilon(\zeta_j) + \big(c + 2\,c_{A,H}\big)\cdot\mathrm{Vol}\big(B_\varepsilon(\zeta_j)\big). \end{aligned}$$

Finally letting $\varepsilon \to 0$, we thus obtain

$$m(\zeta_j) \leq m'(\zeta_j) \qquad \text{and} \qquad m'(\zeta_j) \leq m(\zeta_j).$$

Whence $m'(\zeta_j) = m(\zeta_j)$. This finishes the proof of assertions (iii') and (iv').

**Step 6** We prove that the sequence of $n$-marked $I_\nu$-holomorphic curves $(g_\nu^{-1}u_\nu, \mathbf{z}_\nu)$ Gromov converges to a stable map $(\mathbf{u}, \mathbf{z})$ of combinatorial type $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ in the sense of Definition 5.2.1 in [**22**], with the only exception that

(viii) we assume that $\Sigma_0$ is of arbitrary genus but does not admit any automorphisms other than the identity.

We claim that the proof is the same as the proof of Theorem 5.3.1 in [**22**], except for the modification (viii). In fact, the proof of Theorem 5.3.1 in [**22**] is based on Theorem 4.6.1 and Propositions 4.7.1 and 4.7.2 in [**22**]. Therefore, all we need to do is to replace the statements of Theorem 4.6.1 with the corresponding statements (i), (ii), (iii') and (iv') above, and the statements of Propositions 4.7.1 and 4.7.2 with the corresponding statements of Proposition A.2.1, applied to the sequence of $I_\nu$-holomorphic curves $g_\nu^{-1}u_\nu$. We shall appeal to Proposition A.2.1 in the following situation. We take (see Step 4)

- the closed symplectic manifold $(M, \omega)$ to be the manifold $(P(M), \omega_{A,H})$;
- the almost complex structures $I_\nu$ and $I$ on $M$ to be the almost complex structures $I_\nu$ and $I$ on $P(M)$, respectively;
- the sequence of $I_\nu$-holomorphic curves $u_\nu$ to be the sequence of $I_\nu$-holomorphic curves
$$g_\nu^{-1}u_\nu\colon \Sigma_0 \to P(M).$$

Note that it follows from (ii), (iii') and Step 4 that the assumptions of Proposition A.2.1 are indeed satisfied by the sequence of $I_\nu$-holomorphic curves $g_\nu^{-1}u_\nu$.

So the proof of Theorem 5.3.1 in [**22**] carries over except for the modification (viii). In our case the principal component $\Sigma_0$ will be a distinguished component of the stable map $\mathbf{u}$. As a consequence, we have to modify the base step in the induction argument in the proof of Theorem 5.3.1 in [**22**] (see p. 119 in [**22**]) in the following way.

We define the set $Z_1$ in [**22**] to be the set

$$Z_1 := Z_0 = \{\zeta_1, \dots, \zeta_N\}$$

of bubbling points on the principal component $\Sigma_0$ obtained in Step 2. Then

- there exists a positive real number $r$ such that $B_r(\zeta_j) \cap Z_1 = \{\zeta_j\}$ for each $j = 1, \dots, N$;
- there exist holomorphic disks $\varphi_j \colon \mathbb{C} \supset B \to B_r(\zeta_j)$ such that $\varphi_j(0) = \zeta_j$ for each $j = 1, \dots, N$, where $B$ denotes the closed unit disk.

Moreover, we define the sequence of automorphisms $\phi_1^\nu$ of $\mathbb{C}$ in [**22**] to be trivial, that is,

$$\phi_1^\nu := \mathrm{id}_{\mathbb{C}}$$

for all $\nu$. Then it follows from Assertion (ii) above that, for every $j = 1, \dots, N$, the sequence

$$\big(g_\nu^{-1} u_\nu\big) \circ \varphi_j = \big(g_\nu^{-1} u_\nu\big) \circ \varphi_j \circ \phi_1^\nu \colon B \to P(M) \tag{3.113}$$

converges to

$$u_0 \circ \varphi_j \colon B \to P(M)$$

strongly in the $C^\infty$-topology on every compact subset of the punctured disk $B \setminus \{0\}$. Hence we may apply Proposition A.2.1 to the sequence of $I_\nu$-holomorphic curves (3.113), for $j \in \{1, \dots, N\}$. This shows that the inductive step of the proof of Theorem 5.3.1 in [**22**] is valid for the sequence of curves (3.113).

It follows that the sequence $(g_\nu^{-1} u_\nu, \mathbf{z}_\nu)$ Gromov converges to a stable map

$$(\mathbf{u}, \mathbf{z}) = \big(u_0, \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big)$$

of combinatorial type $T = (V = \{0\} \sqcup V_S, E, \Lambda)$. This stable map has a distinguished principal component $\Sigma_0$ corresponding to the root vertex 0 of the tree $T$.

**Step 7** We define the polystable vortex (3.109) to be the polystable vortex that consists of the vortex $(A, u_0)$ on the principal component $\Sigma_0$ obtained in Step 2 and the stable map ($\mathbf{u}$,$\mathbf{z}$) obtained in Step 6.

**Step 8** It follows from the convergence of $(g_\nu^* A_\nu, u_\nu^{-1} u_\nu)$ against $(A, u)$ proved in (i) and (ii) in Step 2 and from Gromov convergence of the sequence $(g_\nu^{-1} u_\nu, \mathbf{z}_\nu)$ against the stable curve $(\mathbf{u}, \mathbf{z})$ proved in Step 6, that the sequence of marked vortices $(A_\nu, u_\nu, \mathbf{z}_\nu)$ Gromov converges against the polystable vortex (3.109) in the sense of Definition 3.1.5, with the only exception that the (Energy) axiom in this definition remains to be verified.

**Step 9** We check that the (Energy) axiom in Definition 3.1.5 is also satisfied. In fact, since $(g_\nu^{-1}u_\nu, \mathbf{z}_\nu)$ Gromov converges against the stable map $(\mathbf{u}, \mathbf{z})$ it follows from Definition 5.2.1 in [**22**] that

$$\sum_{\gamma \in T_{0\alpha}} E_I(u_\gamma) = m_{0\alpha}(\mathbf{u}) = \lim_{\varepsilon\to 0}\lim_{\nu\to\infty} E_{I_\nu}\big(g_\nu^{-1}u_\nu; B_\varepsilon(z_{0\alpha})\big)$$

for every $\alpha \in V_S$ such that $0E\alpha$. Here we used that $u_0^\nu = (g_\nu^{-1}u_\nu)\circ\phi_0^\nu = g_\nu^{-1}u_\nu$. On the other hand, we know from (iv') in Step 5 that

$$E(A, u_0) + \sum_{\alpha\in V_S, 0E\alpha} \lim_{\varepsilon\to 0}\lim_{\nu\to\infty} E_{I_\nu}\big(g_\nu^{-1}u_\nu; B_\varepsilon(z_{0\alpha})\big) = \lim_{\nu\to\infty} E\big(g_\nu^* A_\nu, g_\nu^{-1}u_\nu\big).$$

Hence by gauge invariance of the Yang-Mills-Higgs energy we get

$$\lim_{\nu\to\infty} E(A_\nu, u_\nu) = E(A, u_0) + \sum_{\alpha\in V_S} E_I(u_\alpha).$$

Now we see from the definition of the almost complex structure $I := I_{\Theta,A,\tilde{u}_0,H}$ on $P(M)$ in (3.98) in Section 3.5 that $I$ and $J_{\Theta(A,u_0)}$ agree on the fibers of $P(M)$. Since the curve $u_\alpha$ maps into the fiber $P(M)_{z_{0\alpha}}$ over the bubbling point $z_{0\alpha} \in \Sigma_0$, it thus follows that $E_I(u_\alpha) = E_{J_{\Theta(A,u_0)}(z_{0\alpha})}(u_\alpha)$. We conclude that

$$\lim_{\nu\to\infty} E(A_\nu, u_\nu) = E(A, u_0) + \sum_{\alpha \in V_S} E_{J_{\Theta(A,u_0)}(z_{0\alpha})}(u_\alpha) = E(A, \mathbf{u}).$$

This proves the (Energy) axiom.

Thus we conclude that, after passing to a subsequence if necessary, the sequence of $n$-marked vortices $(A_\nu, u_\nu, \mathbf{z}_\nu)$ Gromov converges to the polystable vortex (3.109) in the sense of Definition 3.1.5.

The proof of Theorem 3.1.7 is now complete.

CHAPTER 4

# Moduli spaces and transversality

In this chapter we study the moduli space of polystable non-local vortices. We first prove a Fredholm theorem for the linearized non-local vortex equations, in Section 4.1. Next, in Section 4.2 we consider the moduli space of irreducible non-local vortices and prove that this moduli space is an oriented finite-dimensional manifold for generic $J$ and $H$. Our proof is adapted from Cieliebak et. el. [**3**]. In Section 4.3 we finally define the moduli space of simple polystable non-local vortices and prove that its strata are oriented finite dimensional manifolds of the expected dimension for generic $J$ and $H$. Note that we have to restrict to simple polystable vortices in order to obtain transversality. Here we follow the approach to the definition of the moduli space of pseudoholomorphic curves due to McDuff and Salamon [**22**].

## 4.1. Fredholm theory for non-local vortices

The aim of this section is to provide a Fredholm theorem for non-local vortices. Our argument follows the lines of Section 4.2 in Cieliebak et. el. [**3**].

**4.1.1. The linearized non-local vortex equations.** Fix a real constant $E > 0$, an $E$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, a real number $p > 2$ and a positive integer $k$. We denote by $\mathcal{A}^{k,p}(P)$ the space of connections on $P$ of class $W^{k,p}$, and by $W^{k,p}(P, M)^G$ the space of $G$-equivariant maps $u\colon P \to M$ of class $W^{k,p}$ (see [**34**], Appendices A and B for details on these spaces). Following the approach of Cieliebak et. el. [**3**], Section 4.2, we consider the Banach manifold

$$\mathcal{B}^{k,p} = \mathcal{B}^{k,p}(P, M; E, \mathrm{dvol}_\Sigma) = \big\{(A, u) \in \mathcal{A}^{k,p}(P) \times W^{k,p}(P, M)^G \,\big|\, u \text{ satisfies (2.1)}\big\}.$$

The group $\mathcal{G}^{k+1,p}(P)$ of gauge transformations of $P$ of class $W^{k+1,p}$ acts on this space by formula (2.2) from the right (see [**34**], Lemmata A.6 and B.3 for details). The tangent space of $\mathcal{B}^{k,p}$ at $(A, u) \in \mathcal{B}^{k,p}$ is given by

$$T_{(A,u)}\mathcal{B}^{k,p} = W^{k,p}\big(\Sigma, T^*\Sigma \otimes P(\mathfrak{g})\big) \oplus W^{k,p}\big(\Sigma, E_u\big), \qquad E_u := u^*TM/G.$$

Fix a perturbation datum $(\Theta, J) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J}$. As in Remark 2.2.1, the $G$-equivariant family of almost complex structures $J_{\Theta(A,u)}\colon P \to \mathcal{J}(M, \omega)$ determines a complex structure on the vector bundle $E_u \to \Sigma$.

We define a vector bundle $\mathcal{E}^{k-1,p} := \mathcal{E}^{k-1,p}_{J_\Theta} \to \mathcal{B}^{k,p}$ with fibers given by

$$\mathcal{E}^{k-1,p}_{(A,u)} := W^{k-1,p}\big(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u\big) \oplus W^{k-1,p}\big(\Sigma, \Lambda^2 T^*\Sigma \otimes P(\mathfrak{g})\big),$$

where $P(\mathfrak{g}) := P \times_G \mathfrak{g}$ denotes the adjoint bundle. The action of the group of gauge transformations $\mathcal{G}^{k+1,p}(P)$ on $\mathcal{B}^{k,p}$ lifts to an action on this bundle.

Fix a positive integer $\ell$ and denote by $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(M)^G)$ the space of Hamiltonian perturbations of class $C^\ell$ (see Section 2.2.2). For $H \in \mathcal{H}^\ell$ consider the $\mathcal{G}^{k+1,p}(P)$-equivariant section

$$\mathcal{F}_{J,\Theta,H} \colon \mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p}$$

given by

$$\mathcal{F}_{J,\Theta,H}(A,u) := \Big((A,u), \overline{\partial}_{J,A,\Theta,H}(u), F_A + \mu(u)\,\mathrm{dvol}_\Sigma\Big).$$

Note that the operators defining this section are precisely the operators appearing on the left-hand side of the perturbed non-local vortex equations (2.16). Thus a pair $(A,u)$ solves equations (2.16) if and only if $\mathcal{F}_{J,\Theta,H}(A,u) = 0$.

Consider the augmented vertical derivative

$$\mathcal{D}_{A,u} \colon T_{(A,u)}\mathcal{B}^{k,p} \to \mathcal{E}^{k-1,p}_{(A,u)} \oplus W^{k-1,p}\big(\Sigma, P(\mathfrak{g})\big) \tag{4.1}$$

of the section $\mathcal{F}_{J,\Theta,H}$ at a zero $(A,u)$. It is given in explicit terms by

$$\mathcal{D}_{A,u}\begin{pmatrix}\alpha\\ \xi\end{pmatrix} = \begin{pmatrix} D_A\alpha + D_u\xi \\ \mathrm{d}_A\alpha + \mathrm{d}\mu(u)\xi\,\mathrm{dvol}_\Sigma \\ -\,\mathrm{d}_A^*\alpha + L_u^*\xi \end{pmatrix}. \tag{4.2}$$

Let us explain this operator in more detail. The first component on the right-hand side of (4.2) is defined as follows. The operator

$$D_A \colon W^{k,p}\big(\Sigma, T^*\Sigma \otimes P(\mathfrak{g})\big) \to W^{k-1,p}\big(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u\big) \tag{4.3}$$

is obtained by linearizing the Cauchy-Riemann operator

$$\overline{\partial}_{J,A,\Theta,H}(u) = \frac{1}{2}\big(d_{A,H}u + J_{\Theta(A,u)}(u) \circ d_{A,H}u \circ j_\Sigma\big)$$

with respect to $A$. It is given in explicit terms by the formula

$$D_A\,\alpha = (L_u\alpha)^{0,1} - \frac{1}{2} J_{\Theta(A,u)}(u)\Big(\mathrm{d}_{\Theta(A,u)}J\left(\mathrm{d}_{(A,u)}\Theta(\alpha,0)\right)\Big)\,\partial_{J,A,\Theta,H}(u),$$

where

$$(L_u\alpha)^{0,1} := \frac{1}{2}\big(L_u\alpha + J_{\Theta(A,u)}(u) \circ L_u\alpha \circ j_\Sigma\big)$$

is the complex antilinear part of the 1-form $L_u\alpha \in W^{k-1,p}(\Sigma, T^*\Sigma \otimes E_u)$. Here we denote by $L_x \colon \mathfrak{g} \to T_xM$ the infinitesimal action of $G$ on $M$ at the point $x \in M$. Moreover, the operator

$$\mathrm{d}_{(A,u)}\Theta \colon T_{(A,u)}\mathcal{B}^{k-1} \to W^{k-1,p}\big(P, T_{\Theta(A,u)}EG^N\big)^G, \qquad (\alpha,\xi) \mapsto \mathrm{d}_{(A,u)}\Theta(\alpha,\xi)$$

is the derivative at $(A,u)$ of the regular classifying map $\Theta\colon \mathcal{B}^{k,p} \to W^{k,p}(P, EG^N)^G$, and the operator

$$\mathrm{d}_{\Theta(A,u)} J\colon T_{\Theta(A,u)} EG^N \to T\mathcal{J}(M,\omega)$$

is the derivative at $\Theta_{(A,u)}$ of the $G$-equivariant family $J\colon EG^N \to \mathcal{J}(M,\omega)$ of almost complex structures on $M$. Recall at this point that the tangent space $T_{J_e}\mathcal{J}(M,\omega)$ at an almost complex structure $J_e \in \mathcal{J}(M,\omega)$, $e \in EG^N$, consists of all those sections $Y \in \Omega^0(M, \mathrm{End}(TM))$ that satisfy

$$YJ_e + J_e Y = 0 \quad \text{and} \quad \omega(Yv, w) + \omega(v, Yw) = 0.$$

Lastly, the operator

$$\partial_{J,A,\Theta,H}(u) := \frac{1}{2}\big(\mathrm{d}_{A,H} u - J_{\Theta(A,u)}(u) \circ \mathrm{d}_{A,H} u \circ j_\Sigma\big)$$

is the complex linear part of the derivative $\mathrm{d}_{A,H} u$. Likewise, the operator

$$D_u\colon W^{k,p}\big(\Sigma, E_u\big) \to W^{k-1,p}\big(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u\big) \tag{4.4}$$

is obtained by linearizing the Cauchy-Riemann operator $\overline{\partial}_{J,A,\Theta,H}(u)$ with respect to $u$, and is given in explicit terms by the formula

$$D_u\,\xi = (\nabla_{A,H}\xi)^{0,1} - \frac{1}{2} J_{\Theta(A,u)}(u)\Big(\nabla_\xi J_{\Theta(A,u)} + \mathrm{d}_{\Theta(A,u)} J\,\big(\mathrm{d}_{(A,u)}\Theta(0,\xi)\big)\Big)\,\partial_{J,A,\Theta,H}(u).$$

We see from this formula that the operator (4.4) is a real linear Cauchy-Riemann operator in the sense of McDuff and Salamon [**22**], Definition C.1.5. Here the operator

$$(\nabla_{A,H}\xi)^{0,1} := \frac{1}{2}\big(\nabla_{A,H}\xi + J_{\Theta(A,u)}(u) \circ \nabla_{A,H}\xi \circ j_\Sigma\big)$$

denotes the complex antilinear part of the 1-form

$$\nabla_{A,H}\xi := \nabla\xi + \nabla_\xi X_{A,H}(u),$$

where the 1-form $X_{A,H}\colon TP \to \mathrm{Vect}(M,\omega)$ is given by

$$X_{A,H}(v) := X_{A(v)} + X_{H_{\mathrm{d}\pi(v)}}$$

for $v \in T_pP$.

The operator

$$\begin{aligned} \mathrm{d}_A + \mathrm{d}\mu(u)\,\mathrm{dvol}_\Sigma\colon T_{(A,u)}\mathcal{B}^{k,p} &\to W^{k-1,p}\big(\Sigma \otimes \Lambda^2 T^*\Sigma, P(\mathfrak{g})\big) \\ (\alpha,\xi) &\mapsto \mathrm{d}_A\alpha + \mathrm{d}\mu(u)\xi\,\mathrm{dvol}_\Sigma \end{aligned}$$

in the second component on the right-hand side of (4.2) is obtained by linearizing the operator

$$F_A + \mu(u)\,\mathrm{dvol}_\Sigma\colon \mathcal{B}^{k,p} \to W^{k-1,p}\big(\Sigma \otimes \Lambda^2 T^*\Sigma, P(\mathfrak{g})\big)$$

(see [**3**], Section 4.2, formula (23) for details).

The augmentation operator

$$-\,\mathrm{d}_A^* + L_u^*\colon T_{(A,u)}\mathcal{B}^{k,p} \to W^{k-1,p}(\Sigma, P(\mathfrak{g})), \qquad (\alpha,\xi) \mapsto -\,\mathrm{d}_A^*\alpha + L_u^*\xi$$

in the third component on the right-hand side of (4.2) is introduced for gauge fixing (see [**3**], Section 4.2, formula (23) for details). It imposes an additional local slice condition on the action of the group of gauge transformations $\mathcal{G}^{k+1,p}(P)$ on $\mathcal{B}^{k,p}$. Here we denote by $L_x^*\colon T_xM \to \mathfrak{g}$ the adjoint of the infinitesimal action of $G$ on $M$ at the point $x \in M$.

**4.1.2. Main result.** The main result of this section is the following proposition.

PROPOSITION 4.1.1. *Fix a real constant $E > 0$ and an $E$-admissible area form* $\mathrm{dvol}_\Sigma$ *on $\Sigma$. Let $p > 2$ be a real number, and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$ for some positive integer $\ell$. Then the operator (4.1) is Fredholm for every pair $(A,u) \in \mathcal{B}^{k,p}$. Its (real) index is given by*

$$\operatorname{ind}\mathcal{D}_{A,u} = \frac{1}{2}\big(\dim M - 2\,\dim G\big)\cdot\chi(\Sigma) + 2\,\big\langle c_1^G(TM), [u]^G\big\rangle,$$

*where $\chi(\Sigma)$ denotes the Euler characteristic of the surface $\Sigma$, $c_1^G(TM)$ denotes the equivariant first Chern class of the tangent bundle of $M$, and $[u]^G$ denotes the equivariant degree of the map $u$.*

PROOF. The proof is adapted from the proof of Proposition 4.6 in Cieliebak et. al. [**3**]. Let us consider the operator

$$\begin{aligned} W^{k,p}\big(\Sigma, T^*\Sigma\otimes P(\mathfrak{g})\big) &\to W^{k-1,p}\big(\Sigma, P(\mathfrak{g})\big) \oplus W^{k-1,p}\big(\Sigma\otimes\Lambda^2T^*\Sigma, P(\mathfrak{g})\big) \\ \alpha &\mapsto \big(d_A^*\alpha, d_A\alpha\big) \end{aligned} \tag{4.5}$$

first. It is Fredholm of index

$$-\dim G\cdot\chi(\Sigma).$$

By the Riemann-Roch theorem ([**22**], Them. C.1.10) the real linear Cauchy-Riemann operator (4.4) is Fredholm of index

$$\operatorname{ind} D_{A,u} = \frac{1}{2}\dim X\cdot\chi(\Sigma) + 2\,\big\langle c_1(E_u), [\Sigma]\big\rangle,$$

where

$$\begin{aligned} \big\langle c_1(E_u), [\Sigma]\big\rangle &= \big\langle c_1^G(u^*TM), \pi_G^*[\Sigma]\big\rangle \\ &= \big\langle u_G^*\, c_1^G(TM), \pi_G^*[\Sigma]\big\rangle \\ &= \big\langle c_1^G(TM), u_*^G\,\pi_G^*[\Sigma]\big\rangle = \big\langle c_1^G(TM), [u]^G\big\rangle. \end{aligned}$$

We see from (4.2) that the operator (4.1) is a compact perturbation of the direct sum of the Fredholm operators (4.5) and (4.4). In fact, the operator (4.3) and the operators $\mathrm{d}\mu(u)\,\mathrm{dvol}_\Sigma$ and $L_u^*$ are compact since they factor through the inclusion $W^{k,p} \hookrightarrow C^{k-1}$ which is compact by Rellich's theorem ([**34**], Thm. B.2). Here we used that the derivative

$$\mathrm{d}_{(A,u)}\Theta\colon T_{(A,u)}\mathcal{B}^{k-1,p} \to W^{k-1,p}\big(P, T_{\Theta(A,u)}EG^N\big)^G$$

is bounded by part (iii) of the (Regularity) axiom for the regular classifying map $\Theta$ (see Definition 2.1.3).

Summing up we conclude that the operator (4.1) is Fredholm and its index is given by the sum of the indices of the operators (4.5) and (4.4) ([**22**], Thm. A.1.5). $\square$

**4.1.3. The adjoint operator.** The aim of this subsection is to provide the technical Lemma 4.1.2 below. We begin by considering the formal $L^2$-adjoint

$$\mathcal{D}^*_{A,u}\colon \mathcal{E}^{k+1,p}_{(A,u)} \oplus W^{k+1,p}\big(\Sigma, P(\mathfrak{g})\big) \to T_{(A,u)}\mathcal{B}^{k,p} \tag{4.6}$$

of the operator (4.1). It is given by the formula

$$\mathcal{D}^*_{A,u}\begin{pmatrix}\eta\\ \phi\\ \psi\end{pmatrix} = \begin{pmatrix} D^*_A\eta + \mathrm{d}^*_A\phi - \mathrm{d}_A\psi \\ D^*_u\eta + JL_u{*}\phi + L_u\psi\end{pmatrix}, \tag{4.7}$$

where $*$ denotes the Hodge star operator determined by the area form $\mathrm{dvol}_\Sigma$ on $\Sigma$.

Following the terminology in [**3**], Section 4.1, a pair $(A,u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P,M)^G$ is called **regular** if the infinitesimal action

$$W^{2,p}\big(\Sigma, P(\mathfrak{g})\big) \to W^{1,p}\big(\Sigma, T^*\Sigma \otimes P(\mathfrak{g})\big) \oplus W^{1,p}\big(\Sigma, u^*TM/G\big), \qquad \psi \mapsto (-\,\mathrm{d}_A\psi, L_u\psi)$$

of the group of gauge transformations $\mathcal{G}^{2,p}(P)$ at $(A,u)$ is injective. Here $L_x\colon \mathfrak{g} \to T_xM$ denotes the infinitesimal action of $G$ on $M$ at the point $x \in M$.

We may now state the technical lemma, which is a straightforward generalization of Proposition 4.8 (i) in [**3**].

LEMMA 4.1.2. *If $(A,u)$ is a regular solution of the vortex equations (2.16), then $\mathcal{D}^*_{A,u}(\eta,\phi,\psi) = 0$ implies $\psi = 0$.*

PROOF. The proof is adapted from Cieliebak et. al. [**3**], Section 4.2. Let $(A,u)$ be a regular solution of the vortex equations (2.16). We begin with the following

CLAIM. *$\mathcal{D}_{A,u}\mathcal{D}^*_{A,u}(\eta,\phi,\psi) = (\tilde\eta,\tilde\phi,\tilde\psi)$, where $\tilde\psi = \mathrm{d}^*_A\,\mathrm{d}_A\psi + L^*_uL_u\psi$.*

PROOF OF CLAIM. First of all, it follows from a straightforward computation that

$$\mathcal{D}_{A,u}\mathcal{D}^*_{A,u}(\eta,\phi,\psi) = (\tilde\eta,\tilde\phi,\tilde\psi),$$

where

$$\begin{aligned}\tilde\psi &= -\,\mathrm{d}^*_A\big(D^*_A\eta + \mathrm{d}^*_A\phi - \mathrm{d}_A\psi\big) + L^*_u\big(D^*_u\eta + JL_u{*}\phi + L_u\psi\big)\\ &= \mathrm{d}^*_A\,\mathrm{d}_A\psi + L^*_uL_u\psi - \mathrm{d}^*_A\,D^*_A\eta + L^*_uD^*_u\eta + [{*}F_A + \mu(u), {*}\phi].\end{aligned}$$

By the first vortex equation (2.16), we have

$$-\,\mathrm{d}_A^*\,D_A^*\eta + L_u^* D_u^*\eta = 0.$$

In fact, as we have seen above, the operator $(A,u) \mapsto \overline{\partial}_{J,A,\Theta,H}(u)$ is a $\mathcal{G}^{2,p}(P)$-equivariant section of the vector bundle over $\mathcal{B}^{k,p}$ with fibers $W^{k-1,p}(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u)$. Thus its vertical derivative at $(A,u)$, given by the operator $(\alpha,\xi) \mapsto D_A\alpha + D_u\xi$, vanishes on the infinitesimal action of any gauge transformation $\psi \in \Omega^0(\Sigma, P(\mathfrak{g}))$ at $(A,u)$, given by $(-\,\mathrm{d}_A\psi, L_u\psi)$, that is,

$$-D_A\,\mathrm{d}_A\psi + D_u L_u\psi = 0.$$

It follows that

$$\left\langle -\,\mathrm{d}_A^*\,D_A^*\eta + L_u^* D_u^*\eta, \psi \right\rangle = \left\langle \eta, -D_A\,\mathrm{d}_A\psi + D_u L_u\psi \right\rangle = 0$$

for every $\eta \in \Omega^{0,1}_{J_\Theta}(\Sigma, E_u)$, whence $-\,\mathrm{d}_A^*\,D_A^*\eta + L_u^* D_u^*\eta = 0$. By the second vortex equation (2.16), we have

$$[*F_A + \mu(u), *\phi] = 0.$$

This proves the Claim. □

Suppose now that $\mathcal{D}^*_{A,u}(\eta,\phi,\psi) = 0$. By the above Claim, if follows that

$$\mathrm{d}_A^*\,\mathrm{d}_A\psi + L_u^* L_u\psi = 0,$$

whence

$$\left|\mathrm{d}_A\psi\right|^2 + \left|L_u\psi\right|^2 = \left\langle \mathrm{d}_A\psi, \mathrm{d}_A\psi \right\rangle + \left\langle L_u\psi, L_u\psi \right\rangle = \left\langle \mathrm{d}_A^*\,\mathrm{d}_A\psi + L_u^* L_u\psi, \psi \right\rangle = 0.$$

This implies $\mathrm{d}_A\psi = L_u\psi = 0$, so regularity of $(A,u)$ yields $\psi = 0$. The lemma is proved. □

## 4.2. Irreducible non-local vortices

The goal of this section is introduce the moduli space of irreducible non-local vortices. We prove that, for generic Hamiltonian perturbations, this moduli space carries the structure of an oriented finite-dimensional manifold. Our arguments are adpated from the definition of the moduli space of irreducible vortices in Cieliebak et. al. [**3**], Section 4.1.

**4.2.1. Irreducible vortices.** We begin by recalling from Cieliebak et. al. [**3**], Section 4.1, the notion of irreducible vortices. Fix a real number $p > 2$. A pair $(A,u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P,M)^G$ is called **irreducible** if there exists a point $p_0 \in P$ such that the following holds.

(i) $G_{u(p_0)} = \{1\}$, where $1 \in G$ denotes the unit element;
(ii) $\mathrm{im}\, L_{u(p_0)} \cap \mathrm{im}\, I\, \mathrm{L}_{u(p_0)} = \{0\}$ for every almost complex structure $I \in \mathcal{J}(M,\omega)$.

**4.2.2. Regular area forms.** We introduce a regularity condition for area forms on $\Sigma$, which sharpens the admissibility condition of Definition 2.1.1. By definition, this condition implies that all non-local vortices are irreducible in the sense of Section 4.2.1. It will later play a central role in the proof of transversality for polystable non-local vortices in Section 4.3.

DEFINITION 4.2.1 (Regular area forms). Fix a real constant $E > 0$. An area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ is called **$E$-regular** if it is $E$-admissible and all elements of the configuration space $\mathcal{B}^{1,p}(P, M; E, \mathrm{dvol}_\Sigma)$ are irreducible.

The next lemma ensures that $E$-regular area forms always exist.

LEMMA 4.2.2. *For every real constant $E > 0$ there exists an $E$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$.*

PROOF. The proof is similar to the proof of Lemma 4.2 in Cieliebak et. al. [**3**]. □

**4.2.3. Moduli space.** In this subsection, we introduce the moduli space of non-local vortices. Fix a class $B \in H_2^G(M; \mathbb{Z})$ in the integral equivariant homology of $M$. Firstly, the class $B$ descends to a class $b \in H_2(BG; \mathbb{Z})$, which in turn determines an isomorphism class of principal $G$-bundles over $\Sigma$ such that the characteristic class of these bundles agrees with $b$ (see [**4**], Prop. 2.1 for details). Let us fix a principal $G$-bundle $\pi\colon P \to \Sigma$ with characteristic class $b$. Secondly, we denote the equivariant symplectic area of the class $B$ by

$$E_B := \big\langle [\omega - \mu]^G, B \big\rangle.$$

Here $\omega - \mu$ denotes the equivariant symplectic form (see [**16**] for details). Let us fix an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$.

Let $\ell$ be a positive integer. Recall the notation $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(M)^G)$. Given a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}^\ell$, we denote by

$$\widetilde{\mathcal{M}}(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$$

the space of all vortices $(A, u) \in \mathcal{B}^{1,p}(P, M; E_B, \mathrm{dvol}_\Sigma)$ of degree $B$ (see Section 2.2.4) that solve the perturbed non-local vortex equations (2.16), and by

$$\widetilde{\mathcal{M}}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$$

the subspace of irreducible vortices. The group of gauge transformations $\mathcal{G}^{2,p}(P)$ acts on both spaces by (2.2). The quotient

$$\mathcal{M}(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H) := \widetilde{\mathcal{M}}(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)/\mathcal{G}^{2,p}(P)$$

is called the **moduli space of vortices of degree** $B$. Its subspace of irreducible vortices is denoted by

$$\mathcal{M}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H) := \widetilde{\mathcal{M}}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)/\mathcal{G}^{2,p}(P).$$

Note that these two moduli spaces agree whenever the area form $\mathrm{dvol}_\Sigma$ is $E_B$-regular (see Definition 4.2.1).

**4.2.4. Regular Hamiltonian perturbations.** Fix an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ and a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. Following the terminology in [**3**], Definition 4.9, the Hamiltonian perturbation $H$ is called **regular for the quadruple** $(\mathrm{dvol}_\Sigma, \Theta, J, B)$ if the augmented vertical derivative $\mathcal{D}_{A,u}$ (see Section 4.1.1) is surjective for every irreducible vortex $(A, u) \in \widetilde{\mathcal{M}}^*(P, X; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$. We shall denote by

$$\mathcal{H}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta, J; B) \subset \mathcal{H}$$

the set of Hamiltonian perturbations such that $H$ is regular for $(\mathrm{dvol}_\Sigma, \Theta, J, B)$.

**4.2.5. Main result.** We are now in a position to state the main result of this section in Theorem 4.2.3 below. This theorem generalizes Theorems 4.10 (i) and 5.1 (ii) in Cieliebak et. al. [**3**] to the non-local case.

Fix an equivariant homology class $B \in H_2^G(M; \mathbb{Z})$ and let $E_B = \langle [\omega - \mu]^G, B\rangle$ be the equivariant symplectic area of $B$. Fix an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ and a real number $p > 2$. Recall from Section 2.1.1 the definition of the configuration space

$$\mathcal{B}^{1,p} := \mathcal{B}^{1,p}(P, M; E_B, \mathrm{dvol}_\Sigma) := \big\{(A, u) \in \mathcal{A}^{1,p}(P) \times W^{1,p}(P, M)^G \,\big|\, u \text{ satisfies (2.1)}\big\}$$

of pairs $(A, u)$ of class $W^{1,p}$ satisfying the taming condition (2.1). Note that this space is a smooth Banach manifold. By Lemma 2.1.2, the group $\mathcal{G}^{2,p} := \mathcal{G}^{2,p}(P)$ of gauge transformations acts freely on $\mathcal{B}^{1,p}$, and the quotient $\mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ is a smooth Banach manifold.

THEOREM 4.2.3. *Fix a class $B \in H_2^G(M; \mathbb{Z})$ and let $E_B := \langle [\omega - \mu]^G, B\rangle$. Fix an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, a real number $p > 2$, and a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.*

(i) *If $H \in \mathcal{H}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta, J; B)$, then the moduli space*

$$\mathcal{M}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$$

*is an oriented smooth submanifold of $\mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ of (real) dimension*

$$\frac{1}{2}\big(\dim M - 2\dim G\big) \cdot \chi(\Sigma) + 2\,\big\langle c_1^G(TM), B\big\rangle,$$

*where $\chi(\Sigma)$ denotes the Euler characteristic of the surface $\Sigma$ and $c_1^G(TM)$ is the equivariant first Chern class of the tangent bundle of $M$.*

(ii) *The set $\mathcal{H}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta, J; B)$ is a countable intersection of open and dense subsets of $\mathcal{H}$.*

The proof of Theorem 4.2.3 will occupy the remainder of this section. It is adapted from the proofs of Theorems 4.10 (i) and 5.1 (ii) in Cieliebak et. al. [**3**] and from the proof of Theorem 3.1.5 in McDuff and Salamon [**22**].

Let us fix a class $B \in H_2^G(M; \mathbb{Z})$ and let $E_B := \langle [\omega - \mu]^G, B\rangle$. Fix moreover an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$, a real number $p > 2$, and a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

**4.2.6. Proof of Theorem 4.2.3 (i).** Let $H \in \mathcal{H}$ be a regular Hamiltonian perturbation. Recall from Section 4.1.1 that we have a smooth Banach vector bundle $\mathcal{E}^p \to \mathcal{B}^{1,p}$. An irreducible pair $(A, u)$ is contained in the space $\widetilde{\mathcal{M}}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$ if and only if it is a zero of the $\mathcal{G}^{2,p}$-equivariant section

$$\mathcal{F}_{J,\Theta,H} \colon \mathcal{B}^{1,p} \to \mathcal{E}^p, \quad \mathcal{F}_{J,\Theta,H}(A,u) := \Big((A,u), \overline{\partial}_{J,A,\Theta,H}(u), F_A + \mu(u)\,\mathrm{dvol}_\Sigma\Big).$$

This section is a smooth map between smooth Banach manifolds. The subset $\mathcal{B}^*$ of irreducible pairs in $\mathcal{B}^{1,p}$ is an open smooth Banach submanifold. We therefore have

$$\widetilde{\mathcal{M}}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H) = \mathcal{F}^{-1}_{\Theta,J,H}(0) \cap \mathcal{B}^*.$$

Recall from Section 4.1.1 that the augmented vertical derivative of the section $\mathcal{F}_{\Theta,J,H}$ at a zero $(A, u)$ is given by the operator

$$\mathcal{D}_{A,u} \colon T_{(A,u)}\mathcal{B}^{1,p} \to \mathcal{E}^p_{(A,u)} \oplus W^p\big(\Sigma, P(\mathfrak{g})\big).$$

Since $H$ is regular, this operator is surjective (see Section 4.2.4). By Proposition 4.1.1 it is Fredholm of (real) index

$$\frac{1}{2}\big(\dim M - 2\,\dim G\big) \cdot \chi(\Sigma) + 2\,\big\langle c_1^G(TM), B\big\rangle.$$

Hence it has a right inverse by [**22**], Lemma A.3.6. It thus follows from the infinite dimensional implicit function theorem ([**22**], Thm. A.3.3) that $\mathcal{M}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$ is a smooth submanifold of the Banach manifold $\mathcal{B}^{1,p}/\mathcal{G}^{2,p}$ of the claimed dimension (see [**22**], proof of Theorem 3.1.5 (i), for further details).

The manifold $\mathcal{M}^*(P, M; B, \mathrm{dvol}_\Sigma; \Theta, J, H)$ carries a natural orientation. The proof of this fact is the same as the proof of Proposition 4.13 in [**3**].

This proves part (i) of Theorem 4.2.3.

**4.2.7. Universal moduli space.** The proof of part (ii) of Theorem 4.2.3 is based on the following proposition. Recall that $\mathcal{H}^\ell := C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(M)^G)$ denotes the Banach space of Hamiltonian perturbations of class $C^\ell$, for any positive integer $\ell$. Consider the space

$$\widetilde{\mathcal{M}}^*\big(B, J; \mathcal{H}^\ell\big) := \big\{(A, u, H) \,\big|\, (A, u) \in \widetilde{\mathcal{M}}^*(P, X; B, \mathrm{dvol}_\Sigma; \Theta, J, H) \text{ and } H \in \mathcal{H}^\ell\big\}$$

consisting of all triples $(A, u, H)$, where $(A, u)$ is an irreducible perturbed non-local vortex of degree $B$ and the Hamiltonian perturbation $H$ varies over the space $\mathcal{H}^\ell$. This space is invariant under the action of the group of gauge transformations $\mathcal{G}^{2,p}(P)$. Passing to the quotient we obtain the universal moduli space

$$\mathcal{M}^*\big(B, J; \mathcal{H}^\ell\big) := \widetilde{\mathcal{M}}^*\big(B, J; \mathcal{H}^\ell\big)/\mathcal{G}^{2,p}(P) = \mathcal{M}^*(P, X; B, \mathrm{dvol}_\Sigma; \Theta, J, H) \times \mathcal{H}^\ell.$$

PROPOSITION 4.2.4. *For every real number $p > 2$ and every integer $\ell \geq 4$, the universal moduli space $\mathcal{M}^*(B, J; \mathcal{H}^\ell)$ is a separable $C^{\ell-1}$-Banach submanifold of $\mathcal{B}^{1,p}/\mathcal{G}^{2,p} \times \mathcal{H}^\ell$.*

PROOF. The proof is adapted from the proofs of Theorem 4.10 (i) in Cieliebak et. al. [**3**] and Proposition 3.2.1 in McDuff and Salamon [**22**]. Fix an almost complex structure $J \in \mathcal{J}$. Consider the $C^{\ell-1}$-Banach space bundle

$$\mathcal{E}^p := \mathcal{E}^p_{J_\Theta} \to \mathcal{B}^{1,p} \oplus \mathcal{H}^\ell$$

with fibers given by

$$\mathcal{E}^p_{(A,u,H)} := L^p\big(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u\big) \oplus L^p\big(\Sigma, \Lambda^2 T^*\Sigma \otimes P(\mathfrak{g})\big), \quad E_u := u^*TM/G$$

(see [**22**], Section 3.2 for details on the Banach manifold structure on this bundle). The action of the group of gauge transformations $\mathcal{G}^{2,p}$ on $\mathcal{B}^{1,p}$ lifts to an action on this bundle. Consider the $\mathcal{G}^{2,p}$-equivariant section

$$\mathcal{F}_{J,\Theta} \colon \mathcal{B}^{1,p} \oplus \mathcal{H}^\ell \to \mathcal{E}^p$$

given by

$$\mathcal{F}_{J,\Theta}(A, u, H) := \Big((A, u, H), \overline{\partial}_{J,A,\Theta,H}(u), F_A + \mu(u)\,\mathrm{dvol}_\Sigma\Big).$$

This section is a map of class $C^{\ell-1}$ between $C^{\ell-1}$-Banach manifolds. Its augmented vertical derivative

$$\mathcal{D}_{A,u,H} \colon T_{(A,u)}\mathcal{B}^{1,p} \oplus T_H\mathcal{H}^\ell \to \mathcal{E}^p_{(A,u)} \oplus L^p\big(\Sigma, P(\mathfrak{g})\big)$$

at a zero $(A, u, H)$ is given by

$$\mathcal{D}_{A,u,H}\begin{pmatrix}\alpha\\ \xi\\ \hat{H}\end{pmatrix} = \mathcal{D}_{A,u}\begin{pmatrix}\alpha\\ \xi\end{pmatrix} + \begin{pmatrix}(X_{\hat{H}}(u))^{0,1}\\ 0\\ 0\end{pmatrix}. \tag{4.8}$$

It is a map of class $C^{\ell-1}$ between Banach spaces. Here $\mathcal{D}_{A,u}$ is the augmented vertical derivative (4.1) of the section $\mathcal{F}_{J,\Theta,H}$ at a zero $(A, u)$ introduced in Section 4.1, and

$$(X_{\hat{H}}(u))^{0,1} := \frac{1}{2}\big(X_{\hat{H}}(u) + J_{\Theta(A,u)}(u) \circ X_{\hat{H}}(u) \circ j_\Sigma\big)$$

denotes the complex antilinear part of the 1-form $X_{\hat{H}}(u) \in C^\ell(\Sigma, T^*\Sigma \otimes E_u)$. Note that the tangent space of $\mathcal{H}^\ell$ at $H$ is given by $T_H\mathcal{H}^\ell \cong C^\ell(\Sigma, T^*\Sigma \otimes C^\infty(P, M)^G)$.

A triple $(A, u, H)$ is contained in the space $\widetilde{\mathcal{M}}^*(B, J; \mathcal{H}^\ell)$ if and only if $(A, u)$ is irreducible and $\mathcal{F}_{\Theta,J}(A, u, H) = 0$. Recall from Section 4.2.1 that irreducibility of pairs is an open condition. Thus the assertion of the proposition will follow from the infinite dimensional implicit function theorem ([**22**], Thm. A.3.3) once we have shown that the augmented vertical derivative $\mathcal{D}_{A,u,H}$ is surjective and has a right inverse for every triple $(A, u, H) \in \widetilde{\mathcal{M}}^*(B, J; \mathcal{H}^\ell)$.

We prove that the operator $\mathcal{D}_{A,u,H}$ is surjective: Formula (4.8) shows that the operator $\mathcal{D}_{A,u,H}$ is a compact perturbation of the operator $\mathcal{D}_{A,u}$. By Proposition 4.1.1, the operator $\mathcal{D}_{A,u}$ is Fredholm. Hence it follows that the operator $\mathcal{D}_{A,u,H}$ is Fredholm as well ([**22**], Thm. A.1.5). In particular, it has a closed image. In order to prove surjectivity it will therefore suffice to show that the image of $\mathcal{D}_{A,u,H}$ is dense.

Assume for contradiction that the image of $\mathcal{D}_{A,u,H}$ were not dense. Let $q$ be a real number such that $1/p + 1/q = 1$. Then, by the Hahn-Banach theorem, there exists a nonzero triple

$$(\eta, \phi, \psi) \in L^q\big(\Sigma, \Lambda^{0,1}_{J_{\Theta(A,u)}} T^*\Sigma \otimes E_u\big) \oplus L^q\big(\Sigma, \Lambda^2 T^*\Sigma \otimes P(\mathfrak{g})\big) \oplus L^q\big(\Sigma, P(\mathfrak{g})\big)$$

that is $L^2$-orthogonal to the image of the operator $\mathcal{D}_{A,u,H}$. We see from formula (4.8) that the image of the operator $\mathcal{D}_{A,u}$ is contained in the image of the operator $\mathcal{D}_{A,u,H}$. Hence the triple $(\eta, \phi, \psi)$ is $L^2$-orthogonal to the image of the operator $\mathcal{D}_{A,u}$. This has two consequences. Firstly, by Proposition 2.2.8 we may assume without loss of generality that the vortex $(A, u)$ is of class $W^{\ell,p}$. Hence it follows from standard elliptic operator theory ([**22**], Thm C.2.3) that $(\eta, \phi, \psi)$ is of class $W^{\ell,p}$ and

$$\mathcal{D}^*_{A,u}(\eta, \phi, \psi) = 0. \tag{4.9}$$

Secondly, we have

$$\int_\Sigma \big\langle \eta, X_{\hat{H}}(u) \big\rangle \, \mathrm{dvol}_\Sigma = 0 \tag{4.10}$$

for every $\hat{H} \in \mathcal{H}^\ell$.

By irreducibility of $(A, u)$ there exists a point $p_0 \in P$ and an open $G$-invariant neighborhood $U = U(p_0) \subset P$ such that

$$G_{u(p)} = \{1\} \qquad \text{and} \qquad \mathrm{im} L_{u(p)} \cap \mathrm{im} J L_{u(p)} = \{0\}$$

for all $p \in U$, where $1 \in G$ denotes the unit element. We claim that this implies that

$$(\eta, \phi, \psi) = 0,$$

contradicting our assumption. In fact, it follows from formula (4.10) that $\eta(p) = 0$ for all $p \in U$. A proof of this may be found in the proof of Theorem 4.10 (i) in Cieliebak et. al. [**3**], and this proof still applies in our case after replacing the family of almost complex structures $J\colon \Sigma \to \mathcal{J}_G(M, \omega)$ by the family $J_{\Theta(A,u)}\colon P \to \mathcal{J}(M, \omega)$. Furthermore, by Lemma 4.1.2 it follows from (4.9) that $\psi = 0$. Combining formulas (4.7) and (4.9) we therefore conclude that $L_{u(p)}\phi = 0$ for all $p \in U$. Since $G_{u(p)} = \{1\}$ this implies $\phi(p) = 0$ for all $p \in U$. We have thus proved that $\eta$ and $\phi$ vanish simultaneously on some open subset of $P$. Since

$$\mathcal{D}^*_{A,u}(\eta, \phi, 0) = 0$$

by (4.9) it follows by unique continuation for first order elliptic operators (see McDuff and Salamon [**22**], Section 2.3 and Remark 3.2.3 for details) that $\eta = 0$ and $\phi = 0$.

We prove that the operator $\mathcal{D}_{A,u,H}$ has a right inverse: This is an immediate consequence of Lemma A.3.6 in McDuff and Salamon [**22**] since $\mathcal{D}_{A,u,H}$ is Fredholm and surjective by the above.

Thus the infinite dimensional implicit function theorem implies that $\mathcal{M}^*(B,J;\mathcal{H}^\ell)$ is a $C^{\ell-1}$-Banach submanifold of $\mathcal{B}^{1,p}/\mathcal{G}^{2,p}\times\mathcal{H}^\ell$. It is separable since $\mathcal{B}^{1,p}/\mathcal{G}^{2,p}\times\mathcal{H}^\ell$ has this property. This proves Proposition 4.2.4. □

**4.2.8. Proof of Theorem 4.2.3 (ii).** We are now ready for the proof of part (ii) of Theorem 4.2.3. Recall from Section 4.1.1 the definition of the $\mathcal{G}^{2,p}$-equivariant section

$$\mathcal{F}_{J,\Theta,H}\colon \mathcal{B}^{1,p}\to\mathcal{E}^p,\quad \mathcal{F}_{J,\Theta,H}(A,u):=\Big((A,u),\overline{\partial}_{J,A,\Theta,H}(u),F_A+\mu(u)\,\mathrm{dvol}_\Sigma\Big)$$

and of the augmented vertical derivative

$$\mathcal{D}_{A,u}\colon T_{(A,u)}\mathcal{B}^{1,p}\to\mathcal{E}^p_{(A,u)}\oplus W^p\big(\Sigma,P(\mathfrak{g})\big)$$

of $\mathcal{F}_{J,\Theta,H}$ at the zero $(A,u)$. Denote by

$$\mathcal{H}^\ell_{\mathrm{reg}}:=\big\{H\in\mathcal{H}^\ell\,\big|\,\mathcal{D}_{A,u}\text{ is surjective for all }(A,u)\in\widetilde{\mathcal{M}}^*(P,M;B,\mathrm{dvol}_\Sigma;\Theta,J,H)\big\}$$

the set of all Hamiltonian perturbations of class $C^\ell$ such that $(J,H)$ is regular.

The proof is in two steps.

**Step 1** We prove that the set $\mathcal{H}^\ell_{\mathrm{reg}}$ is a countable intersection of open and dense subsets of $\mathcal{H}^\ell$ for $\ell$ sufficiently large.

Let $\ell\geq 4$. Fix an almost complex structure $J\in\mathcal{J}$, and consider the canonical projection

$$\mathcal{P}\colon\mathcal{M}^*\big(B,J;\mathcal{H}^\ell\big)\to\mathcal{H}^\ell.$$

By Proposition 4.2.4, this map is a $C^{\ell-1}$-map between separable $C^{\ell-1}$-Banach manifolds. Fix an element $([A,u],H)\in\mathcal{M}^*(B,J;\mathcal{H}^\ell)$ and consider the derivative

$$\mathrm{d}_{([A,u],H)}\mathcal{P}\colon T_{([A,u],H)}\mathcal{M}^*\big(B,J;\mathcal{H}^\ell\big)\to T_H\mathcal{H}^\ell.$$

Claim. *The derivative $\mathrm{d}_{([A,u],H)}\mathcal{P}$ is a Fredholm operator. Its index agrees with the index of the augmented vertical derivative $\mathcal{D}_{A,u}$. It is onto if and only if $\mathcal{D}_{A,u}$ is onto.*

Proof of Claim. By formula (4.8) in the proof of Proposition 4.2.4 above, the tangent space

$$T_{([A,u],H)}\mathcal{M}^*\big(B,J;\mathcal{H}^\ell\big)\subset T_{[A,u]}\big(\mathcal{B}^{1,p}/\mathcal{G}^{2,p}\big)\oplus C^\ell\big(\Sigma,T^*\Sigma\otimes C^\infty(P,M)^G\big)$$

consists of all triples $(\alpha,\xi,\hat{H})$ such that

$$\mathcal{D}_{A,u,H}\begin{pmatrix}\alpha\\ \xi\\ \hat{H}\end{pmatrix}=\mathcal{D}_{A,u}\begin{pmatrix}\alpha\\ \xi\end{pmatrix}+\begin{pmatrix}(X_{\hat{H}}(u))^{0,1}\\ 0\\ 0\end{pmatrix}=0. \tag{4.11}$$

The derivative $\mathrm{d}_{([A,u],H)}\mathcal{P}$ is given by the projection $([\alpha,\xi],\hat{H}) \mapsto \hat{H}$. Hence the claim follows from Lemma A.3.6 in [**22**]. □

It follows from the Claim that a regular value $H$ of the projection $\mathcal{P}$ is a Hamiltonian perturbation such that $\mathcal{D}_{A,u}$ is surjective for every irreducible vortex $[A,u]$ in the fiber of $\mathcal{P}$ over $H$. Equivalently, the set $\mathcal{H}^{\ell}_{\mathrm{reg}}$ of regular Hamiltonian perturbations of class $C^{\ell}$ is exactly the set of regular values of the projection $\mathcal{P}$. By the Sard-Smale theorem ([**22**], Theorem A.5.1) this set is a countable intersection of open and dense subsets of $\mathcal{H}^{\ell}$ whenever

$$\ell - 2 \geq \operatorname{ind}\mathcal{P} = \operatorname{ind}\mathcal{D}_{A,u}.$$

Here we used that the projection $\mathcal{P}$ is a map of class $C^{\ell-1}$ between separable Banach spaces.

**Step 2** We prove that the set $\mathcal{H}_{\mathrm{reg}}$ is a countable intersection of open and dense subsets of $\mathcal{H}$.

For $H \in \mathcal{H}^{\ell}$ and any real number $c > 0$ we consider the space $\widetilde{\mathcal{M}}_c(H)$ of vortices $(A,u) \in \widetilde{\mathcal{M}}(P,M;B,\mathrm{dvol}_{\Sigma};\Theta,J,H)$ satisfying the conditions

$$\left\|\mathrm{d}_{A,H}u\right\|_{L^{\infty}(\Sigma)} \leq c \tag{4.12}$$

and

$$\inf_{G_x \neq \{1\}} \mathrm{d}_M\big(u(p_0),x\big) \geq \frac{1}{c}, \qquad |\eta_1| + |\eta_2| \leq c\left|L_{u(p_0)}\eta_1 + JL_{u(p_0)}\eta_2\right| \tag{4.13}$$

for some point $p_0 \in P$ and all $\eta_1,\eta_2 \in \mathfrak{g}$, where $1 \in G$ denotes the unit element and $\mathrm{d}_M\colon M \times M \to \mathbb{R}$ denotes the Riemannian distance function on $M$ determined by the metric $\langle\cdot,\cdot\rangle_{J_{\Theta(A,u)}}$. Condition (4.13) ensures that the space $\widetilde{\mathcal{M}}_c(H)$ consists entirely of irreducible vortices. Consider the set

$$\mathcal{H}_{\mathrm{reg},c} := \big\{H \in \mathcal{H} \,\big|\, \mathcal{D}_{A,u} \text{ is surjective for all vortices } (A,u) \in \widetilde{\mathcal{M}}_c(H)\big\}.$$

Note that, for varying $c > 0$, the spaces $\widetilde{\mathcal{M}}_c(H)$ exhaust the space of irreducible vortices $\widetilde{\mathcal{M}}^*(P,X;B,\mathrm{dvol}_{\Sigma};\Theta,J,H)$. Hence

$$\mathcal{H}_{\mathrm{reg}} = \bigcap_{c>0} \mathcal{H}_{\mathrm{reg},c}.$$

It therefore suffices to prove that each set $\mathcal{H}_{\mathrm{reg},c}$ is open and dense in $\mathcal{H}$ with respect to the $C^{\infty}$-topology.

We prove that the complement of $\mathcal{H}_{\mathrm{reg},c}$ in $\mathcal{H}$ is closed: Let $H_{\nu}$ be a sequence in $\mathcal{H} \setminus \mathcal{H}_{\mathrm{reg},c}$ that converges to $H \in \mathcal{H}$ in the $C^{\infty}$-topology. Then there exists a sequence of vortices $(A_{\nu},u_{\nu})$ such that $(A_{\nu},u_{\nu}) \in \widetilde{\mathcal{M}}_c(H_{\nu})$ and the operator $\mathcal{D}_{A_{\nu},u_{\nu}}$ is not surjective for all $\nu$.

Combining Proposition 2.2.6, Corollary 3.4.2 and Proposition 3.1.6 we conclude that there exists a vortex $(A,u) \in \widetilde{\mathcal{M}}(P,M;B,\mathrm{dvol}_\Sigma;\Theta,J,H)$ and a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}$ such that, after passing to a subsequence if necessary, the sequence $(g_\nu^* A_\nu, g_\nu^{-1} u_\nu)$ converges to $(A,u)$ strongly in the $C^1$-topology on $\Sigma$.

This has two consequences. Firstly, the limit vortex $(A,u)$ is contained in the space $\widetilde{\mathcal{M}}_c(H)$. Secondly, since the operators $\mathcal{D}_{A_\nu,u_\nu}$ are not surjective for all $\nu$ it follows that the operator $\mathcal{D}_{A,u}$ is not surjective either. Hence $H \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{reg},c}$.

We prove that $\mathcal{H}_{\mathrm{reg},c}$ is dense in $\mathcal{H}$: Let $H \in \mathcal{H}$. Consider the set

$$\mathcal{H}^\ell_{\mathrm{reg},c} := \big\{ H \in \mathcal{H}^\ell \,\big|\, \mathcal{D}_{A,u} \text{ is surjective for all vortices } (A,u) \in \widetilde{\mathcal{M}}_c(H) \text{ of class } C^\ell \big\}.$$

Note that

$$\mathcal{H}_{\mathrm{reg},c} = \mathcal{H}^\ell_{\mathrm{reg},c} \cap \mathcal{H}.$$

By Step 1 the set $\mathcal{H}^\ell_{\mathrm{reg}}$ is dense in $\mathcal{H}^\ell$ for $\ell$ sufficiently large. It follows that there exists a sequence $H_\ell$ in $\mathcal{H}^\ell_{\mathrm{reg}}$ such that

$$\big\| H - H_\ell \big\|_{C^\ell} \le 2^{-\ell} \tag{4.14}$$

for large $\ell$. Since $H_\ell \in \mathcal{H}^\ell_{\mathrm{reg}}$ we have, in particuar, that $H_\ell \in \mathcal{H}^\ell_{\mathrm{reg},c}$. As above it follows that $\mathcal{H}^\ell_{\mathrm{reg},c}$ is open in $\mathcal{H}^\ell$ with respect to the $C^\ell$-topology. Whence there exists $\varepsilon_\ell > 0$ such that

$$\big\| H' - H_\ell \big\|_{C^\ell} < \varepsilon_\ell \qquad \Longrightarrow \qquad H' \in \mathcal{H}^\ell_{\mathrm{reg},c}$$

for every $H' \in \mathcal{H}^\ell$. Now choose $H'_\ell \in \mathcal{H}$ such that

$$\big\| H_\ell - H'_\ell \big\|_{C^\ell} \le \min\big\{\varepsilon_\ell, 2^{-\ell}\big\}. \tag{4.15}$$

Then

$$H'_\ell \in \mathcal{H}^\ell_{\mathrm{reg},c} \cap \mathcal{H} = \mathcal{H}_{\mathrm{reg},c},$$

and it follows from (4.14) and (4.15) that

$$\big\| H - H'_\ell \big\|_{C^k} \le \big\| H - H_\ell \big\|_{C^k} + \big\| H_\ell - H'_\ell \big\|_{C^k} \le 2^{-\ell}$$

for $\ell \ge k$ and all integers $k \ge 1$. Hence $H'_\ell \in \mathcal{H}_{\mathrm{reg},c}$ converges to $H$ in the $C^\infty$-topology.

This finishes the proof of Theorem 4.2.3 (ii).

## 4.3. Polystable non-local vortices

In this section, we introduce the moduli space of irreducible polystable non-local vortices. We prove that, for generic almost complex structure and generic Hamiltonian perturbation, certain strata of this moduli space consisting of simple polystable non-local vortices of fixed combinatorial type and prescribed degrees are oriented finite-dimensional manifolds. These strata will play an essential role in the definition of the gauged Gromov-Witten invariants in Chapter 5. We will adapt the approach taken in McDuff and Salamon [**22**], Section 8.5.

**4.3.1. Compatible homology classes.** The canonical projection

$$EG \times_G M \to BG$$

from the Borel construction onto the classifying space induces a characteristic projection

$$c\colon H_2^G(M;\mathbb{Z}) \to H_2(BG;\mathbb{Z})$$

in homology.

DEFINITION 4.3.1. Two equivariant homology classes $B_1, B_2 \in H_2^G(M;\mathbb{Z})$ are called **compatible** if $c(B_1) = c(B_2)$.

Let $P \to \Sigma$ be a principal $G$-bundle. A class $B \in H_2^G(M;\mathbb{Z})$ is called **representable by** $P$ if there exists a $G$-equivariant map $u\colon P \to M$ such that $[u]^G = B$. We have the following criterion for representability:

Two classes $B_1, B_2 \in H_2^G(M;\mathbb{Z})$ are representable by one and the same principal $G$-bundle if and only if they are compatible. This principal $G$-bundle is then determined up to isomorphism by the characteristic class $c(B_1) = c(B_2) \in H_2(BG;\mathbb{Z})$.

**4.3.2. Automorphisms of polystable vortices.** Fix an equivariant homology class $B \in H_2^G(M;\mathbb{Z})$ and let $E_B = \langle [\omega - \mu]^G, B\rangle$ be the equivariant symplectic area of $B$. Let $p > 2$ be a real number. Fix an $E_B$-admissible area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ and a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

Fix an $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$. The **group of reparametrizations of combinatorial type T** is the group $\mathfrak{R}_T$ of all tuples $\big(\tau, (\phi_\alpha)_{\alpha \in V}\big)$, where

- $\tau\colon T \to T$ is an isomorphism of $n$-labeled trees satisfying $\tau(0) = 0$ (see [**22**], Section D.2 for details);
- $\phi_\alpha\colon \Sigma_\alpha \to \Sigma_{\tau(\alpha)}$ is an automorphism of $\mathbb{P}^1$, for every $\alpha \in V_S$;
- $\phi_0 = \mathrm{id}_{\Sigma_0}$,

and the group multiplication is given by

$$(\tau', \{\phi'_\alpha\}) \circ (\tau, \{\phi_\alpha\}) := (\tau' \circ \tau, \{\phi'_{\tau(\alpha)} \circ \phi_\alpha\}).$$

Recall at this point that the automorphism group of $\mathbb{P}^1$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$. The group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ acts from the right on the set of all polystable non-local vortices of degree $B$ and of combinatorial type $T$ by

$$(A, \mathbf{u}, \mathbf{z}) \cdot (\tau, \{\phi_\alpha\}, g) := \big(\tilde{A}, \tilde{\mathbf{u}}, \tilde{\mathbf{z}}\big) \tag{4.16}$$

where

- $\tilde{z}_{\tau(\alpha)\tau(\beta)} = \phi_\alpha^{-1}(z_{\alpha\beta})$ for $\alpha, \beta \in V$ satisfying $\alpha E \beta$;
- $\tilde{z}_i = \phi_{\alpha_i}^{-1}(z_i)$ for $i = 1, \ldots, n$;
- $\tilde{A} = g^* A = g^{-1} A g + g^{-1}\,\mathrm{d}g$;
- $\tilde{u} = g^{-1} u$;
- $\tilde{u}_{\tau(\alpha)} = \big(g_{z_{0\alpha}}^{-1} u_\alpha\big) \circ \phi_\alpha$ for every $\alpha \in V_S$

(see Section 2.1.1 for the action of the group of gauge transformations and Section 3.1.2 for polystable vortices). Note that it follows from this definition that $\Lambda_\alpha = \tilde{\Lambda}_{\tau(\alpha)}$. In particular, the marked point $\tilde{z}_i$ lies on the component $\Sigma_{\tilde{\alpha}_i}$, where $\tilde{\alpha}_i = \tau(\alpha_i)$. Furthermore, it follows that $\tilde{z}_{0\tau(\alpha)} = z_{0\alpha}$ since the group of reparametrizations acts trivially on the principal component. Thus we see that the special points on the principal component $\Sigma_0$ are invariant under reparametrization.

**4.3.3. The moduli space of polystable non-local vortices.** For every $n$-labeled tree $T$, we denote the set of all polystable non-local vortices of degree $B$ and of combinatorial type $T$ by

$$\widetilde{\mathcal{M}}_T(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H).$$

Taking the quotient with respect to the action (4.16) of the automorphism group we obtain the moduli space

$$\mathcal{M}_T(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) := \widetilde{\mathcal{M}}_T(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \big/ \big(\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)\big)$$

of polystable non-local vortices of degree $B$ and of combinatorial type $T$. The **moduli space of $n$-marked polystable non-local vortices of degree** $B$ is then obtained by taking the union

$$\overline{\mathcal{M}}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) := \bigcup_T \mathcal{M}_T(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$$

over all isomorphism classes of $n$-labelled trees. Note that only finitely many strata in this union are nonempty. The top stratum of this moduli space is denoted by

$$\mathcal{M}_n(P, M; \mathrm{dvol}_\Sigma; B, n; \Theta, J, H) := \mathcal{M}_{\{0\}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$$

and will be called the **moduli space of $n$-marked non-local vortices of degree** $B$.

**4.3.4. Simple polystable non-local vortices.** Let $n$ be a nonnegative integer, and let $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ be an $n$-labeled tree. Following the terminolgy in McDuff and Salamon [**22**], Sections 6.1 and 8.5, a polystable non-local vortex

$$(A, \mathbf{u}, \mathbf{z}) = \big((A, u_0), \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big)$$

is called **simple** if every non-constant fiber bubble $u_\alpha$, $\alpha \in V_S$, is a simple pseudo-holomorphic curve in the sense of [**22**], Section 2.5, and $u_\alpha(\mathbb{P}^1) \neq u_\beta(\mathbb{P}^1)$ for any two vertices $\alpha \neq \beta$ such that $u_\alpha$ and $u_\beta$ are non-constant. Let $\{B_\alpha\}_{\alpha \in V}$ be a collection of equivariant homology classes in $H_2^G(M; \mathbb{Z})$ such that

$$\sum_{\alpha \in V} B_\alpha = B.$$

We denote by

$$\widetilde{\mathcal{M}}^*_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big)$$

the set of all simple polystable non-local vortices

$$(A, \mathbf{u}, \mathbf{z}) = \big((A, u_0), \{u_\alpha\}_{\alpha \in V_S}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n}\big)$$

of combinatorial type $T$ such that

$$[u_\alpha]^G = B_\alpha \text{ for all } \alpha \in V.$$

The automorphism group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ acts on this set. For technical reasons, we take the quotient in two stages (which is possible since the actions of the reparametrization group and of the group of gauge transformations commute). The quotient by the action of the group of gauge transformations will be denoted by

$$\widehat{\mathcal{M}}^*_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big) := \widetilde{\mathcal{M}}_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big)\big/\mathcal{G}^{2,p}(P),$$

and the quotient by the action of the full automorphism group will be denoted by

$$\mathcal{M}^*_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big) := \widehat{\mathcal{M}}_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big)\big/\mathfrak{R}_T.$$

The latter space is called the **moduli space of simple polystable vortices of combinatorial type $T$ and prescribed degrees** $\{B_\alpha\}$.

**4.3.5. Universal fiber bubbles.** The construction of the moduli space of simple polystable non-local vortices requires us to study the moduli space of pseudoholomorphic spheres in the fibers of the Borel construction $EG \times_G M \to BG$. For technical reasons, we shall be working with finite dimensional approximations of $EG$ (see Section 2.1.3). We will therefore consider the symplectic fiber bundle

$$\widetilde{M} := M^N_G := EG^N \times_G M \to BG^N$$

associated to the finite dimensional approximation $EG^N \to BG^N$ of the universal bundle $EG \to BG$. The $G$-invariant symplectic form $\omega$ on $M$ induces a fiberwise symplectic form $\widetilde{\omega}$ on $\widetilde{M}$. We may think of the $G$-equivariant family of $\omega$-compatible almost complex structures $J \colon EG^N \to \mathcal{J}(M, \omega)$ as an $\widetilde{\omega}$-compatible vertical almost complex structure $J \in \mathcal{J}^{\mathrm{Vert}}(\widetilde{M}, \widetilde{\omega})$ on the bundle $\widetilde{M}$. An equivariant homology class $B \in H^G_2(M; \mathbb{Z})$ is called a **fiberwise spherical class** if the class $B$ is a fiberwise spherical class when considered as a homology class in $H_2(\widetilde{M}; \mathbb{Z})$ (see Section A.3.1).

Let now $B \in H^G_2(M; \mathbb{Z})$ be a fiberwise spherical equivariant homology class. We define the **moduli space of of simple fiber bubbles of degree $B$ in $\widetilde{M}$** to be

$$\mathcal{M}^*\big(M^N_G; A; J\big) := \big\{(b, v) \,\big|\, b \in BG^N \text{ and } v \in \mathcal{M}^*\big(\widetilde{M}_b; A_b; J_b\big)\big\}$$

(see Section A.3.3 for details), thinking of its elements as pseudoholomorphic spheres in the fibers of the Borel construction. We shall denote by $\mathcal{J}_{\mathrm{reg}}(B)$ the set of almost complex structures in $\mathcal{J}$ that are regular for $B$ in the sense of Section A.3.2.

Applying Theorem A.3.3 we obtain the following transversality result.

PROPOSITION 4.3.2. *Let $B \in H_2^G(M;\mathbb{Z})$ be a fiberwise spherical equivariant homology class.*

(i) *If $J \in \mathcal{J}_{\mathrm{reg}}(B)$, then the moduli space $\mathcal{M}^*(M_G^N;B;J)$ is a smooth oriented manifold of (real) dimension*

$$\dim BG^N + \dim M + 2\left\langle c_1^G(TM), B\right\rangle,$$

*where $c_1^G(TM)$ denotes the equivariant first Chern class of the tangent bundle of $M$.*

(ii) *The set $\mathcal{J}_{\mathrm{reg}}(B)$ is a countable intersection of open and dense subsets of $\mathcal{J}$.*

REMARK 4.3.3. We emphasize that the dimension of the base $BG^N$, and hence the degree $B$, enters into the formula for the dimension of the moduli space in Proposition 4.3.2 (i). This is the reason why it is not possible to define gauged Gromov-Witten invariants for strongly semipositive Hamiltonian manifolds following the approach of McDuff and Salamon [**22**], Chapter 8. Rather, we have to restrict ourselves to the class of monotone manifolds.

**4.3.6. The edge evaluation map.** Let us fix an $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ and a collection $\{B_\alpha\}_{\alpha\in V}$ of equivariant homology classes in $H_2^G(M;\mathbb{Z})$ such that

$$\sum_{\alpha\in V} B_\alpha = B$$

and all classes $B_\alpha$, $\alpha \in V_S$, are fiberwise spherical. We begin by constructing an evaluation map on the moduli space of irreducible vortices of degree $B_0$ with one marked point. Consider the set of **framed vortices**

$$\widetilde{\mathcal{M}}(P, M; B_0, \mathrm{dvol}_\Sigma; \Theta, J, H) \times P.$$

Elements of this set are triples $(A, u, p)$ consisting of a non-local vortex $(A, u)$ of degree $B_0$ and a framing by some point $p$ in $P$. This set comes with two natural group actions. First, the automorphism group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ acts on this set by

$$(A, u, p) \cdot (\tau, \{\phi_\alpha\}, g) := \Big(\big(A, u, \pi(p)\big) \cdot \big(\tau, \{\phi_\alpha\}, g\big), p.g(p)^{-1}\Big), \tag{4.17}$$

where the triple $(A, u, \pi(p))$ is considered as a vortex with one marked point, and the action of $(\tau, \{\phi_\alpha\}, g)$ on this triple is given by formula (4.16). Second, the group $G$ acts via an action on the framing by

$$(A, u, p) \cdot h := (A, u, p.h) \tag{4.18}$$

for $h \in G$. Note that these two group actions commute. In fact, for $p \in P$, $g \in \mathcal{G}^{2,p}(P)$ and $h \in G$ we have

$$(p.h).(g(p.h))^{-1} = p.\big(hh^{-1}g(p)^{-1}h\big) = \big(p.g(p)^{-1}\big).h.$$

The set of framed vortices admits an evaluation map

$$\widetilde{\mathrm{ev}}_0\colon \widetilde{\mathcal{M}}(P, M; B_0, \mathrm{dvol}_\Sigma; \Theta, J, H) \times P \to EG^N \times M$$

defined by

$$\widetilde{\mathrm{ev}}_0(A, u, p) := \big(\Theta_{(A,u)}(p), u(p)\big).$$

This evaluation map is invariant under the action (4.17) of the automorphism group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ and equivariant with respect to the $G$-action (4.18) and the diagonal action of $G$ on the product $EG^N \times M$. In fact, by $\mathcal{G}^{2,p}(P)$-equivariance of the classifying map $\Theta$ we have

$$\begin{aligned}
\widetilde{\mathrm{ev}}_0\big(g^*A, g^{-1}u, p.g(p)^{-1}h\big) &= \Big(\Theta_{(g^*A, g^{-1}u)}\big(p.g(p)^{-1}h\big), g^{-1}u\big(p.g(p)^{-1}h\big)\Big) \\
&= \Big(\Theta_{(g^*A, g^{-1}u)}(p) \cdot g(p)^{-1}h, h^{-1}g(p) \cdot \big(g^{-1}u(p)\big)\Big) \\
&= \Big(\Theta_{(A,u)}(p) \cdot g(p)g(p)^{-1}h, h^{-1}g(p)g(p)^{-1} \cdot u(p)\Big) \\
&= \Big(\Theta_{(A,u)}(p) \cdot h, h^{-1} \cdot u(p)\Big) \\
&= h^{-1} \cdot \widetilde{\mathrm{ev}}_0(A, u, p)
\end{aligned}$$

for $g \in \mathcal{G}^{2,p}(P)$ and $h \in G$. Hence the evaluation map descends to an evaluation map

$$\mathrm{ev}_0\colon \mathcal{M}(P, M; B_0, \mathrm{dvol}_\Sigma; \Theta, J, H) \times \Sigma \to EG^N \times_G M \tag{4.19}$$

on the moduli space of vortices with one marked point.

Consider now the moduli space

$$\mathcal{M}^*\big(\{B_\alpha\}; J, H\big) := \mathcal{M}\big(P, M; B_0, \mathrm{dvol}_\Sigma; \Theta, J, H\big) \times \prod_{\alpha \in V_S} \mathcal{M}^*\big(EG^N(M); B_\alpha; J\big)$$

consisting of tuples

$$(A, \mathbf{u}) := \big([A, u_0], (u_\alpha)_{\alpha \in V_S}\big),$$

where $[A, u_0]$ is a gauge equivalence class of vortices of degree $B_0$ and the $u_\alpha$ are simple pseudoholomorphic spheres in the fibers of $EG^N \times_G M$. Let us denote by $Z(T)$ the set of tuples

$$\mathbf{z} := \big(\{z_{\alpha\beta}\}_{\alpha E \beta}, \{z_i\}_{1 \le i \le n}\big)$$

such that the points $z_{\alpha\beta} \in \Sigma_\alpha$ for $\alpha E \beta$ and $z_i \in \Sigma_\alpha$ for $\alpha_i = \alpha$ (see Section 3.1.2) are pairwise distinct for every $\alpha \in V$.

We may then define an **edge evaluation map**

$$\mathrm{ev}^E\colon \mathcal{M}^*\big(\{B_\alpha\}; J, H\big) \times Z(T) \to (EG \times_G M)^E \tag{4.20}$$

by setting

$$\mathrm{ev}^E\big((A, \mathbf{u}), \mathbf{z}\big) := \big\{\mathrm{ev}_0([A, u_0], z_{0\beta}\big\}_{0E\beta} \bigcup \big\{u_\alpha(z_{\alpha\beta})\big\}_{\alpha E\beta, \alpha \ne 0}.$$

Denote by

$$\Delta^E := \big\{\{x_{\alpha\beta}\}_{\alpha E\beta} \in (EG^N \times_G M)^E \,\big|\, x_{\alpha\beta} = x_{\beta\alpha}\big\} \subset \big(EG^N \times_G M\big)^E$$

the diagonal determined by the edge relation. The space

$$\widehat{\mathcal{M}}^*_T\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big)$$

of gauge equivalence classes of simple polystable vortices of combinatorial type $T$ and prescribed degrees $B_\alpha$ then gets identified with the preimage of the diagonal $\Delta^E$ under the edge evaluation map $\mathrm{ev}^E$.

**4.3.7. Regular pairs.** We formulate a regularity condition for the pair $(J, H)$ that is sufficient for obtaining transversality for the moduli space of simple vortices of fixed combinatorial type and prescribed degrees. The following definition is adapted from McDuff and Salamon [**22**], Definition 8.5.2.

Definition 4.3.4 (Regular pairs). Fix an equivariant homology class $B \in H_2^G(M;\mathbb{Z})$ and let $E_B := \langle [\omega - \mu]^G, B\rangle$ be its equivariant symplectic area. Fix an $E_B$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ (see Definition 4.2.1). Let $p > 2$ be a real number and fix a regular classifying map $\Theta \in \mathcal{C}^p_{\mathrm{reg}}$ (see Definition 2.1.3). A pair $(J, H) \in \mathcal{J} \times \mathcal{H}$ is called **regular for the triple** $(\mathrm{dvol}_\Sigma, \Theta, B)$ if it satisfies the following conditions.

- **(J)** $J \in \mathcal{J}_{\mathrm{reg}}(A)$ for every fiberwise spherical class $A \in H_2^G(M;\mathbb{Z})$.
- **(H)** $H \in \mathcal{H}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta, J; A)$ for every equivariant homology class $A \in H_2^G(M;\mathbb{Z})$ such that $A$ is compatible with $B$ (see Definition 4.3.1) and $\langle [\omega-\mu]^G, A\rangle \le E_B$.
- **(E)** The edge evaluation map (4.20) is transverse to the diagonal $\Delta^E$ for every $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ and every collection $\{B_\alpha\}_{\alpha\in V}$ of equivariant homology classes in $H_2^G(M;\mathbb{Z})$ satisfying the following conditions:
  - (i) for every $\alpha \in V_S$, $B_\alpha$ is fiberwise spherical and the component $\Sigma_\alpha$ contains at least three special points (see Section 3.1.2) whenever $B_\alpha = 0$;
  - (ii) $B_0$ is compatible with $B$ and $\langle [\omega - \mu]^G, B_0\rangle \le E_B$.

We will denote the set of all pairs $(J, H) \in \mathcal{J} \times \mathcal{H}$ that are regular for $(\mathrm{dvol}_\Sigma, \Theta, B)$ by $\mathcal{J}\mathcal{H}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$.

**4.3.8. Main result.** We are now ready to state the main result of this section.

Theorem 4.3.5. *Fix a class $B \in H_2^G(M;\mathbb{Z})$, let $E_B := \langle [\omega - \mu]^G, B\rangle$, and fix an $E_B$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a regular classifying map $\Theta \in \mathcal{C}^p_{\mathrm{reg}}$. Let $n$ be a nonnegative integer. Fix an $n$-labeled tree $T = (V = \{0\} \sqcup V_S, E, \Lambda)$ and a collection $\{B_\alpha\}_{\alpha\in V}$ of equivariant homology classes in $H_2^G(M;\mathbb{Z})$ satisfying the following conditions.*

(a) $\sum_{\alpha \in V} B_\alpha = B$;
(b) $B_0$ *is compatible with* $B$ *(see Definition 4.3.1) and* $\langle [\omega - \mu]^G, B_0 \rangle \leq E_B$;
(c) *for every* $\alpha \in V_S$, $B_\alpha$ *is fiberwise spherical and the component* $\Sigma_\alpha$ *contains at least three special points (see Section 3.1.2) whenever* $B_\alpha = 0$.

*Then the following holds.*

(i) *If* $(J, H) \in \mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$, *then the moduli space*

$$\mathcal{M}_T^*\big(P, M; \mathrm{dvol}_\Sigma; \{B_\alpha\}; \Theta, J, H\big)$$

*is a smooth oriented manifold of (real) dimension*

$$\mu(B, T) := \frac{1}{2}\big(\dim M - 2\dim G\big) \cdot \chi(\Sigma) + 2\left\langle c_1^G(TM), B\right\rangle + 2n - 2e(T),$$

*where* $e(T)$ *denotes the number of unoriented edges of* $T$.

(ii) *The set* $\mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$ *is a countable intersection of open and dense subsets of* $\mathcal{J} \times \mathcal{H}$.

PROOF. Proof of (i): Since $(J, H)$ is assumed to be regular, the edge evaluation map

$$\mathrm{ev}^E \colon \mathcal{M}^*\big(\{B_\alpha\}; J, H\big) \times Z(T) \to \big(EG^N \times_G M\big)^E$$

is transverse to the diagonal $\Delta^E$. Let $e(T)$ be the number of unoriented edges of the tree $T$. The dimension of the target space of $\mathrm{ev}^E$ is given by

$$2e(T) \cdot \big(\dim BG^N + \dim M\big),$$

and the codimension of $\Delta^E$ in $(EG \times_G M)^E$ is given by

$$e(T) \cdot \big(\dim BG^N + \dim M\big).$$

Moreover, the dimension of the set $Z(T)$ is

$$4e(T) + 2n.$$

Lastly, we apply Theorem 4.2.3 (i) and Proposition 4.3.2 (i) on order to determine the dimension of the moduli space $\mathcal{M}^*(\{B_\alpha\}; J, H)$. We obtain

$$\begin{aligned}
&\dim \mathcal{M}(P, M; B_0, \mathrm{dvol}_\Sigma; \Theta, J, H) + \sum_{\alpha \in V_S} \dim \mathcal{M}^*\big(M_G^N; B_\alpha; J\big) \\
=\ & \frac{1}{2}\big(\dim M - 2\dim G\big) \cdot \chi(\Sigma) + 2\left\langle c_1^G(TM), B_0\right\rangle \\
&\qquad + \sum_{\alpha \in V_S} \Big(\dim BG^N + \dim M + 2\left\langle c_1^G(TM), B_\alpha\right\rangle\Big) \\
=\ & \frac{1}{2}\big(\dim M - 2\dim G\big) \cdot \chi(\Sigma) + 2\left\langle c_1^G(TM), B\right\rangle + e(T) \cdot \big(\dim BG^N + \dim M\big).
\end{aligned}$$

Here we used that

$$\mathcal{M}^*(P,M;B_0,\mathrm{dvol}_\Sigma;\Theta,J,H)=\mathcal{M}(P,M;B_0,\mathrm{dvol}_\Sigma;\Theta,J,H)$$

since the area form $\mathrm{dvol}_\Sigma$ is $E_B$-regular.

Recall from Section 4.3.6 above that the manifold

$$\widehat{\mathcal{M}}^*_T(P,M;\mathrm{dvol}_\Sigma;\{B_\alpha\};\Theta,J,H)$$

gets identified with the preimage of $\Delta^E$ under $\mathrm{ev}^E$. Hence its dimension is given by

$$\begin{aligned}&\mathcal{M}^*(\{B_\alpha\};J,H)+4e(T)+2n-e(T)\cdot\big(\dim BG^N+\dim M\big)-6e(T)\\ =\ &\frac{1}{2}\big(\dim M-2\,\dim G\big)\cdot\chi(\Sigma)+2\left\langle c_1^G(TM),B\right\rangle+2n+4e(T).\end{aligned}$$

Since $\mathcal{M}^*_T(P,M;\mathrm{dvol}_\Sigma;\{B_\alpha\};\Theta,J,H)$ is the quotient of this set by the proper action of the reparametrization group $\mathfrak{R}_T$, it follows that its dimension is given by

$$\frac{1}{2}\big(\dim M-2\,\dim G\big)\cdot\chi(\Sigma)+2\left\langle c_1^G(TM),B\right\rangle+2n-2e(T).$$

Here we used that the reparametrization group $\mathfrak{R}_T$ has dimension $6e(T)$.

Proof of (ii): The proof is similar to the proof of Theorem 8.5.3 in [**22**]. $\square$

CHAPTER 5

# Gauged Gromov-Witten invariants

This chapter is devoted to the definition of the gauged Gromov-Witten invariants. We begin by recalling from González and Woodward [**13**] the concept of framed polystable non-local vortices, in Section 5.1. This will allow us to define a formal evaluation map on the moduli space of polystable non-local vortices. In Section 5.2 we shall investigate its restriction to the top stratum of the moduli space. As it turns out, this restriction is a pseudocycle in the sense of McDuff and Salamon [**22**]. It will play a crucial role in the definition of the gauged Gromov-Witten invariants via intersection theory for pseudocycles. This will be carried out in the final Section 5.3.

## 5.1. Framed polystable non-local vortices

In this section we use framed polystable vortices to define an evaluation map on the moduli space of marked polystable non-local vortices. The concept of framed polystable vortices is taken from González and Woodward [**13**].

Throughout this section, let $B \in H_2^G(M;\mathbb{Z})$ be an equivariant homology class and denote by $E_B := \langle [\omega - \mu]^G, B\rangle$ its equivariant symplectic area. Fix moreover an $E_B$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number and fix a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$.

**5.1.1. Framings.** Let $T$ be an $n$-labeled tree. By a framed polystable non-local vortex of degree $B$ and of combinatorial type $T$ we mean a tuple $(A, \mathbf{u}, \mathbf{z}, \mathbf{p})$ consisting of a polystable non-local vortex $(A, \mathbf{u}, \mathbf{z})$ of degree $B$ and of combinatorial type $T$ and a **framing** $\mathbf{p} := (p_1, \dots, p_n)$ consisting of points $p_1, \dots, p_n \in P$ such that $\pi(p_i) = z_{0i}$ for $i = 1, \dots, n$ (see Section 3.1.2 for the notation).

**5.1.2. Moduli space.** We denote the set of all framed polystable non-local vortices of degree $B$ and of combinatorial type $T$ by

$$\widetilde{\mathcal{M}}^{\mathrm{fr}}_T(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H).$$

The automorphism group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ (see Section 4.3.2) acts on this set by

$$(A, \mathbf{u}, \mathbf{z}, \mathbf{p}) \cdot (\tau, \{\phi_\alpha\}, g) := \Big((A, \mathbf{u}, \mathbf{z}) \cdot (\tau, \{\phi_\alpha\}, g),\ \mathbf{p} \cdot g\Big),$$

where the automorphism group acts on the triple $(A, \mathbf{u}, \mathbf{z})$ by formula (4.16), and the action of the group of gauge transformations on the framing is defined by

$$\mathbf{p} \cdot g := \bigl(p_1 \cdot g(p_1)^{-1}, \dots, p_n \cdot g(p_n)^{-1}\bigr). \tag{5.1}$$

Taking the quotient with respect to this group action, we obtain a corresponding moduli space

$$\mathcal{M}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) := \widetilde{\mathcal{M}}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \big/ \bigl(\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)\bigr).$$

Taking the union

$$\overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) := \bigcup_T \mathcal{M}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$$

over all isomorphism classes of $n$-labelled trees we obtain the **moduli space of framed $n$-marked polystable non-local vortices of degree** $B$. Note that only finitely many sets in this union are nonempty.

**5.1.3. Residual $G^n$-action.** There is a residual right action of the group $G^n$ on the set $\widetilde{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$ defined as follows. Given $\mathbf{h} = (h_1, \dots, h_n) \in G^n$ and a framed $n$-marked polystable vortex $(A, \mathbf{u}, \mathbf{z}, \mathbf{p})$, this action is defined by

$$(A, \mathbf{u}, \mathbf{z}, \mathbf{p}) \cdot \mathbf{h} := (A, \mathbf{u}, \mathbf{z}, \mathbf{p} \cdot \mathbf{h}), \qquad \mathbf{p} \cdot \mathbf{h} := \bigl(p_1 \cdot h_1, \dots, p_n \cdot h_n\bigr).$$

A short calculation as in Section 4.3.6 shows that the residual $G^n$-action commutes with the action (5.1) of the group of gauge transformations $\mathcal{G}^{2,p}(P)$. Hence it descends to a residual right action of $G^n$ on the moduli space $\overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$.

**5.1.4. Gromov convergence.** We introduce the notion of Gromov convergence for framed vortices.

Definition 5.1.1. A sequence $(A_\nu, u_\nu, \mathbf{z}_\nu, \mathbf{p}_\nu)$ of framed $n$-marked non-local vortices is said to **Gromov converge** to a framed polystable non-local vortex $(A, \mathbf{u}, \mathbf{z}, \mathbf{p})$ if the underlying sequence of $n$-marked non-local vortices $(A_\nu, u_\nu, \mathbf{z}_\nu)$ Gromov converges to the underlying polystable non-local vortex $(A, \mathbf{u}, \mathbf{z})$ in the sense of Definition 3.1.5, and the sequence of framings $\mathbf{p}_\nu$ converges to $\mathbf{p}$ in the sense that

$$p_k = \lim_{\nu \to \infty} p_k^\nu \cdot g_\nu(p_k^\nu)^{-1} \tag{5.2}$$

for $k = 1, \dots, n$, where $g_\nu \in \mathcal{G}^{2,p}(P)$ is the sequence of gauge transformations introduced in Definition 3.1.5.

**5.1.5. Classifying map.** The regular classifying map $\Theta$ gives rise to a formal classifying map

$$\Phi\colon \overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to EG^n \tag{5.3}$$

for the principal $G^n$-bundle

$$\overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to \overline{\mathcal{M}}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H)$$

in the following way.

We first define this map on each stratum separately. Let $T$ be an $n$-labeled tree and let $1 \leq i \leq n$. Define a map

$$\widetilde{\Phi}_T^i\colon \widetilde{\mathcal{M}}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to EG, \qquad (A, \mathbf{u}, \mathbf{z}, \mathbf{p}) \mapsto \Theta_{(A,u_0)}(p_i).$$

This map is invariant under the action of the automorphism group $\mathfrak{R}_T \times \mathcal{G}^{2,p}(P)$ (see Section 4.3) and equivariant with respect to the residual right action of $G^n$. In fact, by $\mathcal{G}^{2,p}(P)$-equivariance of the classifying map $\Theta$ we have, for $g \in \mathcal{G}^{2,p}(P)$ and $\mathbf{h} = (h_1, \ldots, h_n) \in G^n$,

$$\begin{aligned}
\widetilde{\Phi}_T^i\big(g^*A, g^{-1}\mathbf{u}, \mathbf{z}, \mathbf{p} \cdot g \cdot \mathbf{h}\big) &= \Theta_{(g^*A, g^{-1}u_0)}\big(p_i\, g(p_i)^{-1} \cdot h_i\big) \\
&= \Theta_{(g^*A, g^{-1}u_0)}(p_i) \cdot g(p_i)^{-1} \cdot h_i \\
&= \Theta_{(A,u_0)}(p_i) \cdot g(p_i) \cdot g(p_i)^{-1} \cdot h_i \\
&= \Theta_{(A,u_0)}(p_i) \cdot h_i \\
&= \widetilde{\Phi}_T^i(A, \mathbf{u}, \mathbf{z}, \mathbf{p}) \cdot h_i.
\end{aligned}$$

Hence it descends and we obtain a $G^n$-equivariant map

$$\Phi_T^i\colon \mathcal{M}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to EG$$

on each $T$-stratum. Patching together these maps we obtain $G^n$-equivariant maps

$$\Phi^i\colon \overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to EG.$$

The desired map (5.3) is then defined as the product

$$\Phi := (\Phi^1, \ldots, \Phi^n)$$

of these maps.

Remark 5.1.2. Note that the classifying map (5.3) is defined only formally. It does in particular not rely on any regularity properties of the moduli space of framed marked polystable vortices since it is defined purely in terms of the classifying map $\Theta$.

**5.1.6. Evaluation map.** Recall from Section 4.3.6 that there is a well-defined evaluation map

$$\mathrm{ev}_0\colon \mathcal{M}(P,M;B_0,\mathrm{dvol}_\Sigma;\Theta,J,H)\times\Sigma\to EG\times_G M$$

on the moduli space of non-local vortices with one marked point. We shall now use framed polystable vortices in order to extend this evaluation map to a formal evaluation map

$$\overline{\mathrm{ev}}_{\mathrm{dvol}_\Sigma,\Theta,J,H}\colon \overline{\mathcal{M}}_n(P,M;\mathrm{dvol}_\Sigma;B;\Theta,J,H)\to(EG\times_G M)^n \tag{5.4}$$

on the moduli space of $n$-marked polystable non-local vortices.

Remark 5.1.3. Note that this evaluation map only exists formally. In particular, it does not rely on any regularity properties of the moduli space.

First of all, the moduli space of framed $n$-marked non-local polystable vortices admits a formal framed evaluation map

$$\mathrm{ev}_n^{\mathrm{fr}}\colon \overline{\mathcal{M}}^{\mathrm{fr}}(P,M;\mathrm{dvol}_\Sigma;B;\Theta,J,H)\to M^n \tag{5.5}$$

which is equivariant with respect to the residual right action of $G^n$ on the moduli space and the standard left action of $G^n$ on $M^n$. It is defined by evaluating a framed polystable non-local vortex at its marked points in the following way. For every point $p\in P$ there is a canonical inclusion of the fiber

$$\iota_p\colon M\to P(M)=P\times_G M,\qquad x\mapsto[p,x]. \tag{5.6}$$

It identifies $M$ with the fiber of $P(M)$ over the point $\pi(p)\in\Sigma$. For every framed polystable non-local vortex $(A,\mathbf{u},\mathbf{z},\mathbf{p})$, this inclusion gives rise to maps

$$\iota_{p_k}^{-1}\circ u_{\alpha_k}\colon \Sigma_{\alpha_k}\to M,\qquad k=1,\dots,n.$$

Here we think of the $G$-equivariant map $u_0\colon P\to M$ as a section of the fiber bundle $P(M)\to\Sigma$ (see Remark 3.1.1). Let $T$ be an $n$-labeled tree. We then obtain formal evaluation maps

$$\widetilde{\mathrm{ev}}_{k,T}^{\mathrm{fr}}\colon \widetilde{\mathcal{M}}_T^{\mathrm{fr}}(P,M;\mathrm{dvol}_\Sigma;B;\Theta,J,H)\to M,\qquad (A,\mathbf{u},\mathbf{z},\mathbf{p})\mapsto\iota_{p_k}^{-1}\circ u_{\alpha_k}(z_k). \tag{5.7}$$

These maps are invariant under the action of the automorphism group $\mathfrak{R}_T\times\mathcal{G}^{2,p}(P)$ and equivariant with respect to the residual right action of $G^n$ on the moduli space. In fact, using formula (4.16) for $(\tau,\{\phi_\alpha\},g)\in\mathfrak{R}_T\times\mathcal{G}^{2,p}(P)$ and $\mathbf{h}\in G^n$ we get

$$\begin{aligned}\widetilde{\mathrm{ev}}_{k,T}^{\mathrm{fr}}\big((A,\mathbf{u},\mathbf{z},\mathbf{p})\cdot(\tau,\{\phi_\alpha\},g)\big) &= \iota_{p_k\cdot g(p_k)^{-1}}^{-1}\circ\big(g(p_k)^{-1}\cdot u_{\alpha_k}\big)\circ\phi_{\alpha_k}^{-1}\circ\phi_{\alpha_k}(z_k)\\ &= \iota_{p_k}^{-1}\circ u_{\alpha_k}(z_k)\\ &= \widetilde{\mathrm{ev}}_{k,T}^{\mathrm{fr}}(A,\mathbf{u},\mathbf{z},\mathbf{p})\end{aligned}$$

and

$$\widetilde{\mathrm{ev}}_{k,T}^{\mathrm{fr}}\big((A,\mathbf{u},\mathbf{z},\mathbf{p})\cdot\mathbf{h}\big) = \iota_{p_k\cdot h_k}^{-1}\circ u_{\alpha_k}(z_k)$$

$$\begin{aligned} &= h_k^{-1} \cdot \left(\iota_{p_k}^{-1} \circ u_{\alpha_k}\right)(z_k) \\ &= h_k^{-1} \cdot \widetilde{\mathrm{ev}}_{k,T}^{\mathrm{fr}}(A, \mathbf{u}, \mathbf{z}, \mathbf{p}). \end{aligned}$$

Hence the maps (5.7) descend to $G^n$-equivariant framed evaluation maps

$$\mathrm{ev}_{k,T}^{\mathrm{fr}} \colon \overline{\mathcal{M}}_T^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to M$$

on each $T$-stratum. Patching together these maps we finally obtain evaluation maps

$$\mathrm{ev}_k^{\mathrm{fr}} \colon \overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to M$$

on the moduli space of framed $n$-marked non-local polystable vortices. We then define the framed evaluation map (5.5) as the product

$$\mathrm{ev}^{\mathrm{fr}} := \left(\mathrm{ev}_1^{\mathrm{fr}}, \ldots, \mathrm{ev}_n^{\mathrm{fr}}\right)$$

of these evaluation maps.

The actual evaluation map (5.4) is now defined as follows. The classifying map (5.3) and the framed evaluation map (5.5) give rise to a product map

$$\Theta \times \mathrm{ev}^{\mathrm{fr}} \colon \overline{\mathcal{M}}_n^{\mathrm{fr}}(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to EG^n \times M^n.$$

This map is equivariant with respect to the residual right action of $G^n$ on the moduli space and the standard diagonal action of $G^n$ on the product $EG^n \times M^n$, whence it descends to a formal evaluation map

$$\overline{\mathrm{ev}}_{\mathrm{dvol}_\Sigma, \Theta, J, H} \colon \overline{\mathcal{M}}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to (EG \times_G M)^n.$$

Here we used the natural identification

$$EG^n \times_{G^n} M^n \cong (EG \times_G M)^n.$$

## 5.2. The gauged Gromov-Witten pseudocycle

The formal evaluation map (5.4) on the moduli space of marked polystable non-local vortices constructed in Section 5.1.6 above restricts to an evaluation map

$$\mathrm{ev}_{\mathrm{dvol}_\Sigma, \Theta, J, H} \colon \mathcal{M}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to (EG \times_G M)^n$$

on the moduli space of marked non-local vortices. A priori, this evaluation map will only be defined formally. The goal of this section is to prove that it is in fact a pseudocycle in the sense of McDuff and Salamon [**22**] for generic $J$ and $H$.

**5.2.1. Pseudocycles.** We begin with a review of the theory of pseudocycles. Our exposition follows McDuff and Salamon [**22**], Section 6.5. Let $X$ be a finite dimensional smooth manifold. A subset $B \subset X$ is said to be of **dimension at most** $d$, denoted $\dim B \leq d$, if it is contained in the image of a map $W \to X$ of class $C^1$ which is defined on a manifold $W$ whose components have dimension less than or equal to $d$. A map

$$f \colon V \to X$$

defined on an oriented $d$-dimensional manifold $V$ is called a **pseudocycle of dimension** $d$ **in** $X$ if it is of class $C^1$, the image $f(V)$ has a compact closure, and

$$\dim \Omega_f \leq \dim V - 2,$$

where

$$\Omega_f := \bigcap_{\substack{K \subset V \\ K \text{compact}}} \overline{f(V \setminus K)}$$

denotes the omega limit set of $f$. Two $d$-dimensional pseudocycles $f_0 \colon V_0 \to X$ and $f_1 \colon V_1 \to X$ are said to be **bordant** if there exists a $(d+1)$-dimensional oriented manifold $W$ with boundary $\partial W = V_1 \cup (-V_0)$ and a map $F \colon W \to X$ of class $C^1$ such that

$$F|_{V_0} = f_0, \quad F|_{V_1} = f_1, \quad \dim \Omega_F \leq d - 1.$$

**5.2.2. Main result.** The main result of this section is the following proposition. Recall from the introduction that $(M, \omega)$ is monotone if there exists a number $\tau > 0$ such that

$$\langle [\omega], A \rangle = \tau \cdot \langle c_1(TM), A \rangle$$

for every spherical homology class $A \in H_2(M; \mathbb{Z})$. Recall moreover from Section 4.3.7 the definition of the set $\mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$ of regular pairs $(J, H)$.

PROPOSITION 5.2.1. *Assume that $(M, \omega)$ is monotone. Fix a class $B \in H_2^G(M; \mathbb{Z})$ and let $E_B := \langle [\omega - \mu]^G, B \rangle$. Fix an $E_B$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$ and a perturbation datum $(\Theta, J, H) \in \mathcal{C}^p_{\mathrm{reg}} \times \mathcal{J} \times \mathcal{H}$. Let $n$ be a nonnegative integer.*

(i) *If $(J, H) \in \mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$, then the evaluation map*

$$\mathrm{ev}_{\mathrm{dvol}_\Sigma, \Theta, J, H} \colon \mathcal{M}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to (EG \times_G M)^n \tag{5.8}$$

*obtained by restricting the evaluation map (5.4) to the top stratum of the moduli space is a pseudocycle of (real) dimension*

$$\frac{1}{2}\Big(\dim M - 2 \dim G\Big) \cdot \chi(\Sigma) + 2 \left\langle c_1^G(TM), B \right\rangle + 2n.$$

(ii) *Let $(J_i, H_i) \in \mathcal{JH}_{\mathrm{reg}}(\Theta_i)$ for $i = 0, 1$. Then the corresponding pseudocycles $\mathrm{ev}_{\mathrm{dvol}_\Sigma, \Theta, J_0, H_0}$ and $\mathrm{ev}_{\mathrm{dvol}_\Sigma, \Theta, J_1, H_1}$ are bordant.*

The map (5.8) will be called the **gauged Gromov-Witten pseudocycle** on the moduli space of marked non-local vortices.

**5.2.3. Proof of Proposition 5.2.1.** The remainder of this section is devoted to the proof of Proposition 5.2.1. We start with the following lemma.

LEMMA 5.2.2. *The formal evaluation map*

$$\overline{\mathrm{ev}}_{\mathrm{dvol}_\Sigma,\Theta,J,H} \colon \overline{\mathcal{M}}_n(P,M;\mathrm{dvol}_\Sigma;B;\Theta,J,H) \to (EG\times_G M)^n$$

*is continuous with respect to Gromov convergence of a sequence of $n$-marked non-local vortices against a polystable non-local vortex. More specifically, let $[A_\nu,u_\nu,\boldsymbol{z}_\nu]$ be a sequence of marked non-local vortices that Gromov converges to a polystable non-local vortex $[A,\boldsymbol{u},\boldsymbol{z}]$ in the sense of Definition (3.1.5). Then*

$$\lim_{\nu\to\infty} \mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}[A_\nu,u_\nu,\boldsymbol{z}_\nu] = \overline{\mathrm{ev}}_{\mathrm{dvol}_\Sigma,\Theta,J,H}[A,\boldsymbol{u},\boldsymbol{z}].$$

PROOF. It follows from the definition of the framed evaluation map $\mathrm{ev}^{\mathrm{fr}}$ as a quotient of the map

$$\Theta\times\mathrm{ev}^{\mathrm{fr}} \colon \overline{\mathcal{M}}_n^{\mathrm{fr}}(P,M;\mathrm{dvol}_\Sigma;B;\Theta,J,H) \to EG^n\times M^n$$

that it will suffice to prove that

$$\lim_{\nu\to\infty} \Phi\big([A_\nu,u_\nu,\mathbf{z}_\nu,\mathbf{p}_\nu]\big) = \Phi\big([A,\mathbf{u},\mathbf{z},\mathbf{p}]\big) \tag{5.9}$$

and

$$\lim_{\nu\to\infty} \mathrm{ev}^{\mathrm{fr}}\big([A_\nu,u_\nu,\mathbf{z}_\nu,\mathbf{p}_\nu]\big) = \mathrm{ev}^{\mathrm{fr}}\big([A,\mathbf{u},\mathbf{z},\mathbf{p}]\big) \tag{5.10}$$

for every sequence $(A_\nu,u_\nu,\mathbf{z}_\nu,\mathbf{p}_\nu)$ of framed vortices Gromov converging to a framed polystable vortex $(A,\mathbf{u},\mathbf{z},\mathbf{p})$ in the sense of Definition 5.1.1.

Let us therefore assume that the sequence of $n$-marked non-local vortices $(A_\nu,u_\nu,\mathbf{z}_\nu)$ Gromov converges to the polystable non-local vortex $(A,\mathbf{u},\mathbf{z})$ in the sense of Definition 3.1.5, and that the sequence of framings $\mathbf{p}_\nu$ converges to $\mathbf{p}$ in the sense that

$$p_k = \lim_{\nu\to\infty} p_k^\nu\cdot g_\nu(p_k^\nu)^{-1}$$

for $k=1,\dots,n$, where $g_\nu\in\mathcal{G}^{2,p}(P)$ is the sequence of gauge transformations introduced in Definition 3.1.5.

Proof of (5.9): We see from the definition of the classifying map $\Phi$ in (5.3) that it suffices to prove that

$$\lim_{\nu\to\infty}\Theta_{(g_\nu^*A_\nu,g_\nu^{-1}u_\nu)}\big(p_i^\nu\, g(p_i^\nu)^{-1}\big) = \Theta_{(A,u_0)}(p_i).$$

But this follows from the (Continuity) axiom of the regular classifying map $\Theta$ (see Definition 2.1.3) and the fact that each map $\Theta_{(g_\nu^*A_\nu,g_\nu^{-1}u_\nu)}\colon P\to EG$ is uniformly continuous by compactness of $P$.

Proof of (5.10): It follows from (5.7) above that it will be enough to prove that

$$\lim_{\nu\to\infty} \iota_{p_k^\nu}^{-1} \circ u_\nu(z_k^\nu) = \iota_{p_k}^{-1} \circ u_{\alpha_k}(z_k) \tag{5.11}$$

for $k = 1, \dots, n$. Note that convergence of the framing

$$p_k = \lim_{\nu\to\infty} p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}$$

implies that the sequence of inclusions $\iota_{p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}} \colon M \to P(M)_{z_k^\nu}$ converges to the map $\iota_{p_k} \colon M \to P(M)_{z_{0k}}$ in the $C^0$-topology. By definition (5.6) we further have

$$\begin{aligned}\iota_{p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}}^{-1} \circ \big(g_\nu^{-1} u_\nu\big)(z_k^\nu) &= \iota_{p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}}^{-1}\big(\big[p_k^\nu, g_\nu(p_k^\nu)^{-1} \cdot u_\nu(p_k^\nu)\big]\big) \\ &= \iota_{p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}}^{-1}\big(\big[p_k^\nu \cdot g_\nu(p_k^\nu)^{-1}, u_\nu(p_k^\nu)\big]\big) = \iota_{p_k^\nu}^{-1} \circ u_\nu(z_k^\nu).\end{aligned} \tag{5.12}$$

We have to distinguish whether or not there is bubbling at the point $z_{0k}$.

Case 1: *There is no bubbling at* $z_{0k}$.

In this case, $\alpha_k = 0$ and $z_{0k} = z_k$. By (Map) in Definition 3.1.5 there exists a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that $g_\nu^{-1} u_\nu$ converges to $u$ in the $C^0$-topology on compact subsets of $\Sigma_0 \setminus Z_0$. Since $Z_0$ is finite and $z_k \notin Z_0$, it follows that $g_\nu^{-1} u_\nu$ converges to $u = u_0$ in the $C^0$-topology on a small compact neighborhood of $z_k$. Moreover, by (Marked point) in Definition 3.1.5 we have

$$\lim_{\nu\to\infty} z_k^\nu = z_k.$$

It follows from this that

$$\lim_{\nu\to\infty} \big(g_\nu^{-1} u_\nu\big)(z_k^\nu) = u_0(z_k).$$

Using (5.12) we therefore conclude that

$$\lim_{\nu\to\infty} \iota_{p_k^\nu}^{-1} \circ u_\nu(z_k^\nu) = \iota_{p_k}^{-1} \circ u_0(z_k).$$

Case 2: *There is bubbling at* $z_{0k}$.

In this case, $z_{0k} = z_{0\alpha}$ for some $\alpha \in V_{0\alpha_k}$ such that $0E\alpha$. By (Map) in Definition 3.1.5 there exists a sequence of gauge transformations $g_\nu \in \mathcal{G}^{2,p}(P)$ such that

$$u_{\alpha_k}^\nu = \big(g_\nu^{-1} u_\nu\big) \circ \varphi_{z_{0\alpha_k}} \circ \phi_{\alpha_k}^\nu \colon \Omega_{\alpha_k}^\nu \to P(M)$$

converges to

$$u_{\alpha_k} \colon \Sigma_\alpha \to P(M)_{z_{0\alpha_k}}$$

in the $C^0$-topology on compact subsets of $\Sigma_{\alpha_k} \setminus Z_{\alpha_k}$. Moreover, by (Marked point) in Definition 3.1.5 we have

$$\lim_{\nu\to\infty} \big(\varphi_{z_{0\alpha_k}} \circ \phi_{\alpha_k}^\nu\big)^{-1} z_k^\nu = z_k.$$

It follows that

$$\begin{aligned}\lim_{\nu\to\infty} \big(g_\nu^{-1} u_\nu\big)(z_k^\nu) &= \lim_{\nu\to\infty} \big((g_\nu^{-1} u_\nu) \circ \varphi_{z_{0\alpha_k}} \circ \phi_{\alpha_k}^\nu \circ (\varphi_{z_{0\alpha_k}} \circ \phi_{\alpha_i}^\nu)^{-1}\big)(z_k^\nu) \\ &= \lim_{\nu\to\infty} u_{\alpha_k}^\nu\big(\big(\varphi_{z_{0\alpha_k}} \circ \phi_{\alpha_i}^\nu\big)^{-1}(z_k^\nu)\big)\end{aligned}$$

$$= \quad u_{\alpha_k}(z_k).$$

Using (5.12) we therefore conclude that

$$\lim_{\nu\to\infty} \iota_{p_k^\nu}^{-1} \circ u_\nu(z_k^\nu) = \iota_{p_k}^{-1} \circ u_{\alpha_k}(z_k).$$

This proves (5.11) and completes the proof of the lemma. □

The proof of Proposition 5.2.1 now follows the lines of the proof of Theorem 8.5.1 in McDuff and Salamon [**22**].

## 5.3. Definition of the invariants

We are now in a position to actually define the gauged Gromov-Witten invariants. Our definition will be by means of intersection theory for pseudocycles as developed in McDuff and Salamon [**22**].

**5.3.1. Intersection theory for pseudocycles.** We recall some basic facts from intersection theory for pseudocycles as developed in McDuff and Salamon [**22**], Section 6.5.

Let $X$ be a smooth manifold of dimension $m$. For any two pseudocycles $e\colon U \to X$ and $f\colon V \to X$ there exists a well-defined **intersection number** $e \cdot f$ which only depends on the bordism classes of $e$ and $f$. The pseudocycles $e$ and $f$ are called **strongly transverse** if

$$\Omega_e \cap \overline{f(V)} = \emptyset, \qquad \overline{e(U)} \cap \Omega_f = \emptyset,$$

and

$$e(u) = f(v) = x \qquad \Longrightarrow \qquad T_xX = \operatorname{im} \mathrm{d}_e e + \operatorname{im} \mathrm{d}_v f.$$

In this case the intersection number can be expressed as a finite sum

$$e \cdot f = \sum_{\substack{u\in U, v\in V\\ e(u)=f(v)}} \nu(u,v),$$

where $\nu(u,v)$ denotes the intersection number of $e(U)$ and $f(V)$ at the point $e(u) = f(v)$. Let us write

$$H^*(X) := H^*(X;\mathbb{Z})/\text{torsion}$$

for the torsion free part of the integral cohomology of $X$. We will later need the following general result about intersection numbers of pseudocycles.

Lemma 5.3.1. *Let $\iota : X_1 \hookrightarrow X_2$ be an embedding of compact manifolds. Let $e\colon U \to X_1$ be a pseudocycle of dimension $d$ in $X_1$, and let $a \in H^d(X_2)$. Let $f_2\colon V_2 \to X_2$ be a pseudocycle in $X_2$ Poincaré dual to $a$, and let $f_1\colon V_1 \to X_1$ be a pseudocycle in $X_1$ Poincaré dual to $\iota^*a$. Then*

$$e \cdot f_1 = \iota_* e \cdot f_2,$$

*where $\iota_* e$ denotes the pseudocycle $\iota \circ e\colon U \to X_2$ (see [**22**], Lemma 6.5.9).*

**5.3.2. Gauged Gromov-Witten invariants.** Assume that $(M, \omega)$ is monotone. Recall that this means that there exists a number $\tau > 0$ such that

$$\langle [\omega], A \rangle = \tau \cdot \langle c_1(TM), A \rangle$$

for every spherical homology class $A \in H_2(M; \mathbb{Z})$.

Fix a class $B \in H_2^G(M; \mathbb{Z})$ and let $E_B := \langle [\omega - \mu]^G, B \rangle$ be its equivariant symplectic area, and fix an $E_B$-regular area form $\mathrm{dvol}_\Sigma$ on $\Sigma$. Let $p > 2$ be a real number, and fix a regular classifying map $\Theta \in \mathcal{C}^p_{\mathrm{reg}}$ and a perturbation datum $(J, H) \in \mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$.

Let $n$ be a nonnegative integer. By Proposition 5.2.1 there exists a gauged Gromov-Witten pseudocycle

$$\mathrm{ev}_{\mathrm{dvol}_\Sigma, \Theta, J, H} \colon \mathcal{M}_n(P, M; \mathrm{dvol}_\Sigma; B; \Theta, J, H) \to \big(EG^N \times_G M\big)^n \tag{5.13}$$

of (real) dimension

$$\frac{1}{2}\big(\dim M - 2\,\dim G\big) \cdot \chi(\Sigma) + 2\left\langle c_1^G(TM), B \right\rangle + 2n.$$

This pseudocycle gives rise to a homomorphism

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,\mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}} \colon H_G^*(M)^{\otimes n} \to \mathbb{Z} \tag{5.14}$$

in the following way. Let us abbreviate $M_G := EG \times_G M$. There is a natural isomorphism

$$H_G^*(M)^{\otimes n} \xrightarrow{\cong} H^*(M_G^n), \qquad a_1 \otimes \cdots \otimes a_n \mapsto \pi_1^* a_1 \cup \cdots \cup \pi_n^* a_n,$$

where $\pi_i \colon M_G^n \to M_G$ denotes the projection onto the $i$-th factor. Given equivariant cohomology classes $a_1, \ldots, a_n \in H_G^*(M)$, we may then identify the product $a_1 \otimes \ldots \otimes a_n$ with the cup product

$$a := \pi_1^* a_1 \cup \ldots \cup \pi_n^* a_n \in H^*\big(M_G^n\big)$$

and choose a pseudocycle

$$f_N \colon V \to \big(EG^N \times_G M\big)^n$$

Poincaré dual to $a$, where $N$ is sufficiently large. Then we define

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,\mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}}(a_1, \ldots, a_n) := f_N \cdot \mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}\,.$$

It follows from Lemma 5.3.1 and Lemma 6.5.6 in [**22**] that the homomorphism (5.14) is in fact independent of the pseudocycle $f_N$. It only depends on the bordism class of the pseudocycle $\mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}$.

**5.3.3. Main result.** We may now formulate the main result of this section.

THEOREM 5.3.2. *Assume that* $(M, \omega)$ *is monotone. Fix a class* $B \in H_2^G(M; \mathbb{Z})$ *and let* $E_B := \langle [\omega - \mu]^G, B \rangle$*, and fix an* $E_B$*-admissible area form* $\mathrm{dvol}_\Sigma$ *on* $\Sigma$*. Let* $p > 2$ *be a real number and fix a regular classifying map* $\Theta \in \mathcal{C}^p_{\mathrm{reg}}$ *and a regular perturbation datum* $(J, H) \in \mathcal{JH}_{\mathrm{reg}}(\mathrm{dvol}_\Sigma; \Theta; B)$*. Let* $n$ *be a positive integer. Then the homomorphism*

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,\mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}} \colon H^*_G(M)^{\otimes n} \to \mathbb{Z}$$

*defined in Section 5.3.2 above is independent of the perturbation datum* $(J, H)$*.*

PROOF. We have seen in the construction of the homomorphism (5.14) that this homomorphism only depends on the bordism class of the pseudocycle $\mathrm{ev}_{\mathrm{dvol}_\Sigma,\Theta,J,H}$. By Proposition 5.2.1 (ii), we know that this bordism class is independent of the perturbation datum $(J, H)$. □

The theorem shows that the homomorphism (5.14) is an invariant of the Hamiltonian $G$-manifold $(M, \omega, \mu)$ which only depends on the equivariant homology class $B$, the area form $\mathrm{dvol}_\Sigma$ and the classifying map $\Theta$. It will therefore be denoted by

$$\mathrm{GGW}^{M,\omega,\mu}_{B,n,\mathrm{dvol}_\Sigma,\Theta} \colon H^*_G(M)^{\otimes n} \to \mathbb{Z}$$

and called the $n$**-point gauged Gromov-Witten invariant in degree** $B$.

REMARK 5.3.3. We expect that the homomorphism (5.14) is also independent of the admissible area form $\mathrm{dvol}_\Sigma$ and the regular classifying map $\Theta$.

Theorem 5.3.2 completes the proof of the Main Theorem in the introduction.

APPENDIX A

# Auxiliary results

In this appendix we provide some basic results from symplectic geometry and gauge theory that will be used throughout this thesis.

Section A.1 provides useful formulas for covariant derivatives. In Section A.2 we extend the bubbling analysis of McDuff and Salamon [**22**], Section 4.7, to the case of almost complex structures that are only continuous. Finally, in Section A.3 we prove some basic facts about the moduli space of fiberwise pseudoholomorphic spheres in symplectic fiber bundles.

## A.1. Covariant derivatives

In this section we prove some basic formulas for covariant derivatives that will be needed for the calculations in the proof of the a apriori estimate in Section 3.2.4.

**A.1.1. The Levi-Civita connection.** We start by considering almost complex structures on symplectic manifolds. For any manifold $M$ we denote by $\mathrm{Vect}(M)$ the space of smooth vector fields on $M$.

LEMMA A.1.1. *Let $(M,\omega)$ be a symplectic manifold, and let $J$ be an $\omega$-compatible almost complex structure. Denote by $\langle\cdot,\cdot\rangle := \omega(\cdot, J\cdot)$ the Riemannian metric on $M$ determined by $\omega$ and $J$, and let $\nabla$ be the Levi-Civita connection on $M$ associated to this metric. Then*

(i) $(\nabla_X J)J + J(\nabla_X J) = 0$

(ii) $J(\nabla_{JX} J) = \nabla_X J$

(iii) $\nabla_X(JY) = J\nabla_X Y + (\nabla_X J)Y$

(iv) $\big\langle(\nabla_X J)Y, Z\big\rangle + \big\langle Y, (\nabla_X J)Z\big\rangle = 0$

(v) $\big\langle(\nabla_X J)\,Y, Z\big\rangle + \big\langle(\nabla_Y J)\,Z, X\big\rangle + \big\langle(\nabla_Z J)X, Y\big\rangle = 0$

*for $X, Y, Z \in$ Vect$(M)$. Let moreover $I \in \Omega^0(M, \mathrm{End}(TM))$ be an arbitrary endomorphism of the tangent bundle of $M$. Then*

(vi) $(\mathcal{L}_X I)Y = I\nabla_Y X + (\nabla_X I)Y - \nabla_{IY} X$

*for $X, Y \in$ Vect$(M)$.*

PROOF. For (i)–(v), see Lemma C.7.1 in [**22**]. For (vi), see Exercise 2.1.1 in [**22**]. $\square$

**A.1.2. Covariant derivatives.** Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and assume that $\mathfrak{g}$ is equipped with a $G$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. Let $(M,\omega,\mu)$ be a closed Hamiltonian $G$-manifold with symplectic form $\omega$ and moment map $\mu\colon M \to \mathfrak{g}^* \cong \mathfrak{g}$, where we identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by means of the invariant inner product on $\mathfrak{g}$. The action of $G$ on $M$ gives rise to an infinitesimal action of $\mathfrak{g}$ on $M$, and we denote by $X_\xi$ the Hamiltonian vector field on $M$ associated to the element $\xi \in \mathfrak{g}$. We will also us the notation

$$\mathrm{L}_x\colon \mathfrak{g} \to T_xM, \qquad \xi \mapsto \mathrm{L}_x\xi$$

for the infinitesimal action of $\mathfrak{g}$ on $M$.

Let $D \subset \mathbb{C}$ be an open subset of the complex plane. Fix a smooth connection $A \in \Omega^1(D,\mathfrak{g})$. The remainder of this section is concerned with differential operators acting on certain function spaces on $D$. We begin by defining a twisted covariant derivative

$$\nabla_A\colon \Omega^0(D,\mathfrak{g}) \to \Omega^1(D,\mathfrak{g}), \qquad \nabla_A\eta := \mathrm{d}\eta + [A,\eta]$$

acting on smooth $\mathfrak{g}$-valued functions on $D$. It is compatible with the inner product on $\mathfrak{g}$ in the following sense.

LEMMA A.1.2. *For all $\eta_1,\eta_2 \in \Omega^0(D,\mathfrak{g})$, we have*

$$\mathrm{d}\langle\eta_1,\eta_2\rangle_{\mathfrak{g}} = \langle\nabla_A\eta_1,\eta_2\rangle_{\mathfrak{g}} + \langle\eta_1,\nabla_A\eta_2\rangle_{\mathfrak{g}}.$$

PROOF. Let $v \in TD$ be a tangent vector. By the Leibniz rule,

$$\mathrm{d}\langle\eta_1,\eta_2\rangle(v) = \langle\mathrm{d}\eta_1(v),\eta_2\rangle + \langle\eta_1,\mathrm{d}\eta_2(v)\rangle.$$

Moreover, $G$-invariance of the inner product on $\mathfrak{g}$ implies that

$$\big\langle[A(v),\eta_1],\eta_2\big\rangle + \big\langle\eta_1,[A(v),\eta_2]\big\rangle = 0.$$

Combining both identities, the claimed formula follows. □

Denote by $\mathcal{J}(M,\omega)$ the space of smooth $\omega$-compatible almost complex structures on $M$, and fix a smooth family

$$J\colon D \to \mathcal{J}(M,\omega).$$

This family gives rise to a family

$$\langle\cdot,\cdot\rangle_J\colon D \ni z \mapsto \langle\cdot,\cdot\rangle_z := \langle\cdot,\cdot\rangle_{J_z} = \omega(\cdot,J_z\cdot) \tag{A.1}$$

of Riemannian metrics on $M$. We shall denote the adjoint of the operator $\mathrm{L}_x\colon \mathfrak{g} \to T_xM$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and the family of metrics $\langle\cdot,\cdot\rangle_J$ on $M$ by

$$\mathrm{L}_x^*\colon T_xM \to \mathfrak{g}, \qquad v \mapsto \mathrm{L}_x^*v.$$

We then have the following formula for the twisted covariant derivative of the moment map $\mu$.

LEMMA A.1.3. *Let $u \in C^\infty(D,M)$. Then*

$$\nabla_A\,\mu(u) = -\,\mathrm{L}_u^*J\,\mathrm{d}_Au.$$

PROOF. The proof is taken from Gaio and Salamon [**11**], Section 9. Let $v \in TD$ be a tangent vector. A short calculation shows that

$$\begin{aligned}\nabla_{A,v}\mu(u) &= \mathrm{d}(\mu(u))(v) + [A(v), \mu(u)] \\ &= \mathrm{d}\mu(u)\big(\mathrm{d}u(v)\big) + \mathrm{d}\mu(u)(X_{A(v)}(u)) = \mathrm{d}\mu(u)\big(\mathrm{d}_A u(v)\big).\end{aligned} \tag{A.2}$$

Moreover, the moment map property of $\mu$ yields

$$\begin{aligned}\big\langle \eta, \mathrm{L}_u^* \xi \big\rangle_{\mathfrak{g}} &= \big\langle \mathrm{L}_u \eta, \xi \big\rangle_J \\ &= \omega\big(X_\eta(u), J\xi\big) \\ &= \mathrm{d}\langle \mu, \eta\rangle_{\mathfrak{g}}(J\xi) = \big\langle \mathrm{d}\mu(u)(J\xi), \eta \big\rangle_{\mathfrak{g}}\end{aligned}$$

for all $\eta \in \Omega^0(D, \mathfrak{g})$ and $\xi \in \Omega^0(D, u^*TM)$. Hence $\mathrm{L}_u^* = \mathrm{d}\mu(u)J$, that is,

$$\mathrm{d}\mu(u) = -\,\mathrm{L}_u^* J.$$

On combining this with (A.2), the claim follows. □

Next we observe that the family (A.1) of Riemannian metrics on $M$ determines a family

$$\nabla\colon D \ni z \mapsto \nabla_z := \nabla_{J_z} \tag{A.3}$$

of associated Levi-Civita connections on $M$. Here $\nabla_z$ is the Levi-Civita connection

$$\nabla_z\colon \Omega^0(M, \mathrm{End}(TM)) \to \Omega^1(M, \mathrm{End}(TM))$$

determined by the Riemannian metric $\langle\cdot,\cdot\rangle_{J_z}$.

Let us fix a smooth connection 1-form $A \in \Omega^1(D, \mathfrak{g})$ on $D$. Now $A$ gives rise to a twisted covariant derivative

$$\nabla_A\colon \Omega^0(D, u^*TM) \to \Omega^1(D, u^*TM), \qquad \nabla_A\xi = \nabla\xi + \nabla_\xi X_A(u)$$

acting on sections of the vector bundle $u^*TM \to D$. In more explicit terms, this operator is given by

$$\nabla_A\xi\,(z) = \big(\nabla_z\xi\big)(z) + \big((\nabla_z)_{\xi(z)} X_A(u)\big)(z)$$

for $z \in B$. Furthermore, we define a twisted covariant derivative

$$\delta_A\colon \Omega^0\big(D, \Omega^0(M, \mathrm{End}(TM))\big) \to \Omega^1\big(D, \Omega^0(M, \mathrm{End}(TM))\big)$$

acting on families $D \to \Omega^0(M, \mathrm{End}(TM))$ by

$$\delta_A I := \mathrm{d}I - \mathcal{L}_{X_A} I,$$

where $\mathcal{L}_{X_A} I$ is the Lie derivative of $I$ along the vector field $X_A$. The 1-form $\delta_A I$ is given by

$$\delta_A I\,(v) = \mathrm{d}I(v) - \mathcal{L}_{X_{A(v)}} I_z$$

for $z \in B$ and $v \in T_zB$. In particular, $\mathcal{L}_{X_{A(v)}} I_z$ denotes the Lie derivative of the endomorphism $I_z$ in the direction of the vector field $X_{A(v)}$ on $M$.

**A.1.3. Compatibility.** The next result collects some basic properties of the operator $\delta_A$.

LEMMA A.1.4. *Let $J\colon D\to \mathcal{J}(M,\omega)$ be a smooth family of $\omega$-compatible almost complex structures on $M$ with associated family of Riemannian metrics $\langle\cdot,\cdot\rangle_J=\omega(\cdot,J\cdot)$. Let $A\in\Omega^1(D,\mathfrak{g})$ be a connection 1-form. Let $u\in C^\infty(D,M)$, and let $\xi_1,\xi_2\in\Omega^0(D,u^*TM)$ be sections. Then*

(i) $J(\delta_A J)+(\delta_A J)J=0$

(ii) $\big\langle\xi_1,J(\delta_A J)\xi_2\big\rangle-\big\langle J(\delta_A J)\xi_1,\xi_2\big\rangle=0$

(iii) $\big\langle\xi_1,(\delta_A J)\xi_2\big\rangle-\big\langle(\delta_A J)\xi_1,\xi_2\big\rangle=0.$

PROOF. First recall that the twisted covariant derivative $\delta_A$ is defined by

$$\delta_A J=\mathrm{d}J-\mathcal{L}_{X_A}J.$$

Proof of (i). This follows from $J^2=\mathrm{Id}$ by the Leibniz rule. More precisely, we have

$$0=\mathrm{d}\big(J^2\big)=(\mathrm{d}J)J+J(\mathrm{d}J)$$

and

$$0=\mathcal{L}_{X_A}\big(J^2\big)=(\mathcal{L}_{X_A}J)J+J(\mathcal{L}_{X_A}J).$$

Combining both identities, the claimed formula follows.

Proof of (ii). For $z\in B$ we compute

$$\begin{aligned}\big\langle\xi_1(z),J_z(\mathrm{d}J_z)\,\xi_2(z)\big\rangle_z&=-\frac{\partial}{\partial z'}\langle\xi_1(z),\xi_2(z)\rangle_{z'}\Big|_{z'=z}\\&=-\frac{\partial}{\partial z'}\langle\xi_2(z),\xi_1(z)\rangle_{z'}\Big|_{z'=z}\\&=\big\langle\xi_2(z),J_z(\mathrm{d}J_z)\,\xi_1(z)\big\rangle_z=\big\langle J_z(\mathrm{d}J_z)\,\xi_1(z),\xi_2(z)\big\rangle_z,\end{aligned}$$

that is,

$$\langle\xi_1,J(\mathrm{d}J)\xi_2\rangle=\langle J(\mathrm{d}J)\xi_1,\xi_2\rangle.$$

On the other hand note that, by $G$-invariance of $\omega$, we have $\mathcal{L}_{X_A}\omega=0$. Using this, we get

$$\begin{aligned}\langle J(\mathcal{L}_{X_A}J)\xi_1,\xi_2\rangle&=\omega\big((\mathcal{L}_{X_A}J)\xi_1,\xi_2\big)\\&=(\mathcal{L}_{X_A}\omega)(J\xi_1,\xi_2)-\mathcal{L}_{X_A}\big(\omega(J\xi_1,\xi_2)\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_1),\xi_2\big)\\&\quad-\omega\big(J\xi_1,\mathcal{L}_{X_A}\xi_2\big)\\&=-\mathcal{L}_{X_A}\big(\omega(J\xi_1,\xi_2)\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_1),\xi_2\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_2),\xi_1\big)\\&=-\mathcal{L}_{X_A}\big(\omega(J\xi_2,\xi_1)\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_2),\xi_1\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_1),\xi_2\big)\\&=(\mathcal{L}_{X_A}\omega)(J\xi_2,\xi_1)-\mathcal{L}_{X_A}\big(\omega(J\xi_2,\xi_1)\big)-\omega\big(J(\mathcal{L}_{X_A}\xi_2),\xi_1\big)\\&\quad+\omega\big(J(\mathcal{L}_{X_A}\xi_1),\xi_2\big)\end{aligned}$$

$$
\begin{aligned}
&= \omega\big((\mathcal{L}_{X_A}J)\xi_2, \xi_1\big) \\
&= \langle \xi_1, J(\mathcal{L}_{X_A}J)\xi_2\rangle.
\end{aligned}
$$

The formulas for the Lie derivatives that we used in this computation may be found in [**20**, p. 34 and Prop. 3.2(a)]. Combining both identities, the claimed formula follows.

Proof of (iii). This is a direct consequence of Formulas (i) and (ii) above. For, we have

$$
\begin{aligned}
\langle \xi_1, (\delta_A J)\xi_2\rangle &= \langle J\xi_1, J(\delta_A J)\xi_2\rangle \\
&= -\langle J\xi_1, (\delta_A J)J\xi_2\rangle \\
&= -\langle (\delta_A J)\xi_1, J^2\xi_2\rangle = \langle (\delta_A J)\xi_1, \xi_2\rangle.
\end{aligned}
$$

□

**A.1.4. Leibniz rule.** The next proposition provides a Leibniz rule for the covariant derivative $\nabla_A$.

Proposition A.1.5 (Leibniz rule). *Let $J\colon D \to \mathcal{J}(M,\omega)$ be a smooth family of $\omega$-compatible almost complex structures on $M$ with associated family of Riemannian metrics $\langle\cdot,\cdot\rangle_J = \omega(\cdot, J\cdot)$. Let $A \in \Omega^1(D,\mathfrak{g})$ be a connection 1-form. Let $u \in C^\infty(D,M)$ and let $\xi \in \Omega^0(D, u^*TM)$ be a section. Let $I \in \Omega^0(D, \Omega^0(M, \mathrm{End}(TM)))$. Then*

$$\nabla_A(I\xi) = I\nabla_A\xi + \big(\nabla_{\mathrm{d}_A u} I\big)\xi + \big(\delta_A I\big)\xi.$$

Proof. Let $v \in TD$ be a tangent vector. Using the Leibniz rule we compute

$$\nabla_v(I\xi) = I\nabla_v\xi + (\nabla_{\mathrm{d}u(v)}I)\xi + \mathrm{d}I(v)\,\xi.$$

By Lemma A.1.1 (vi) we further have

$$\nabla_{I\xi}X_{A(v)} = I\nabla_\xi X_{A(v)} + \big(\nabla_{X_{A(v)}}I\big)\xi - \big(\mathcal{L}_{X_{A(v)}}I\big)\xi.$$

Hence we get

$$
\begin{aligned}
\nabla_{A,v}(I\xi) &= \nabla_v(I\xi) + \nabla_{I\xi}X_{A(v)} \\
&= I\nabla_v\xi + I\nabla_\xi X_{A(v)} + (\nabla_{\mathrm{d}u(v)}I)\xi + (\nabla_{X_{A(v)}}I)\xi + \mathrm{d}I(v)\,\xi - (\mathcal{L}_{X_{A(v)}}I)\xi \\
&= I\big(\nabla_v\xi + \nabla_\xi X_{A(v)}\big) + \big(\nabla_{\mathrm{d}u(v)+X_{A(v)}}I\big)\xi + \big(\,\mathrm{d}I(v) - \mathcal{L}_{X_{A(v)}}I\big)\xi \\
&= I\nabla_{A,v}\xi + (\nabla_{\mathrm{d}_A u(v)}I)\xi + (\delta_{A,v}I)\xi.
\end{aligned}
$$

□

**A.1.5. Metric compatibility.** The next proposition makes precise the sense in which the operator $\nabla_A$ is compatible with the Riemannian metric $\langle\cdot,\cdot\rangle_J$ on $M$.

Proposition A.1.6 (Metric compatibility). *Let $J\colon D \to \mathcal{J}(M,\omega)$ be a smooth family of $\omega$-compatible almost complex structures on $M$ with associated Riemannian metric $\langle\cdot,\cdot\rangle_J = \omega(\cdot, J\cdot)$. Let $A \in \Omega^1(D,\mathfrak{g})$ be a connection 1-form. Let $u \in C^\infty(D,M)$, and let $\xi_1, \xi_2 \in \Omega^0(D, u^*TM)$ be sections. Then*

$$\mathrm{d}\langle \xi_1, \xi_2\rangle = \langle \nabla_A\xi_1, \xi_2\rangle + \langle \xi_1, \nabla_A\xi_2\rangle + \langle J\xi_1, (\delta_A J)\xi_2\rangle.$$

The proof of Proposition A.1.6 is based on the following two lemmas.

LEMMA A.1.7. *Under the assumptions of Proposition A.1.6 we have*

$$\mathrm{d}\langle \xi_1, \xi_2\rangle = \langle \nabla\xi_1, \xi_2\rangle + \langle \xi_1, \nabla\xi_2\rangle + \langle J\xi_1, (\mathrm{d}J)\xi_2\rangle.$$

PROOF. First recall that the function $\langle \xi_1, \xi_2\rangle \colon D \to \mathbb{R}$ is given explicitly by the formula $\langle \xi_1, \xi_2\rangle(z) = \langle \xi_1(z), \xi_2(z)\rangle_z$ for $z \in D$. Whence

$$\mathrm{d}_z\langle \xi_1, \xi_2\rangle = \frac{\partial}{\partial z'}\big\langle \xi_1(z'), \xi_2(z')\big\rangle_z\Big|_{z'=z} + \frac{\partial}{\partial z'}\big\langle \xi_1(z), \xi_2(z)\big\rangle_{z'}\Big|_{z'=z} \tag{A.4}$$

The first term on the right-hand side of this expression is computed using pointwise compatibility of the family of Levi-Civita connections $\nabla_z$ with the family of metrics $\langle\cdot,\cdot\rangle_{J_z}$ on $M$. So

$$\begin{aligned}\frac{\partial}{\partial z'}\big\langle \xi_1(z'), \xi_2(z')\big\rangle_z\Big|_{z'=z} &= \big\langle \nabla_z\xi_1(z'), \xi_2(z')\big\rangle_z + \big\langle \xi_1(z'), \nabla_z\xi_2(z')\big\rangle_z\Big|_{z'=z}\\ &= \big\langle \nabla_z\xi_1(z), \xi_2(z)\big\rangle_z + \big\langle \xi_1(z), \nabla_z\xi_2(z)\big\rangle_z.\end{aligned}$$

For the second term we have

$$\begin{aligned}\frac{\partial}{\partial z'}\big\langle \xi_1(z), \xi_2(z)\big\rangle_{z'}\Big|_{z'=z} &= \frac{\partial}{\partial z'}\,\omega_{u(z)}\big(\xi_1(z), I_{z'}\,\xi_2(z)\big)\Big|_{z'=z}\\ &= \omega_{u(z)}\big(\xi_1(z), (\mathrm{d}J)(z)\,\xi_2(z)\big)\\ &= -\omega_{u(z)}\big(\xi_1(z), J_z^2\,(\mathrm{d}J)(z)\,\xi_2(z)\big)\\ &= \big\langle J_z\xi_1(z), (\mathrm{d}J)(z)\,\xi_2(z)\big\rangle_z.\end{aligned}$$

Combining both results we obtain

$$\mathrm{d}_z\langle \xi_1, \xi_2\rangle = \big\langle \nabla_z\xi_1(z), \xi_2(z)\big\rangle_z + \big\langle \xi_1(z), \nabla_z\xi_2(z)\big\rangle_z + \big\langle J_z\,\xi_1(z), (\mathrm{d}J)(z)\,\xi_2(z)\big\rangle_z.$$

□

LEMMA A.1.8. *Let $h \in C^\infty_G(M, \mathbb{R})$ be a Hamiltonian function on $M$, with Hamiltonian vector field $X_h \in C^\infty_G(M, TM)$. Then, under the assumptions of Proposition A.1.6,*

$$\langle \xi_1, \nabla_{\xi_2}X_h\rangle + \langle \nabla_{\xi_1}X_h, \xi_2\rangle + \langle J\xi_1, (\mathcal{L}_{X_h}J)\,\xi_2\rangle = 0.$$

PROOF. The proof is adapted from the proof of Lemma B.2 in [**11**]. We first write

$$\langle \xi_1, \nabla_{\xi_2}X_h\rangle + \langle \nabla_{\xi_1}X_h, \xi_2\rangle = \langle J\xi_1, J\nabla_{\xi_2}X_h\rangle + \langle J\nabla_{\xi_1}X_h, J\xi_2\rangle.$$

Using the formulas

$$J\nabla_{\xi_2}X_h = \nabla_{J\xi_2}X_h - (\nabla_{X_h}J)\xi_2 + (\mathcal{L}_{X_h}J)\xi_2$$

and

$$J\nabla_{\xi_1}X_h = \nabla_{\xi_1}(JX_h) - (\nabla_{\xi_1}J)X_h,$$

which hold by Lemma A.1.1 (vi) and (iii), and rearranging terms we get

$$\begin{aligned}\langle \xi_1, \nabla_{\xi_2} X_h\rangle + \langle \nabla_{\xi_1} X_h, \xi_2\rangle = \langle J\xi_1, \nabla_{J\xi_2} X_h\rangle + \langle \nabla_{\xi_1}(JX_h), J\xi_2\rangle - \langle J\xi_1, (\nabla_{X_h} J)\xi_2\rangle \\ - \langle (\nabla_{\xi_1} J)X_h, J\xi_2\rangle + \langle J\xi_1, (\mathcal{L}_{X_h} J)\xi_2\rangle.\end{aligned}$$

We simplify this using

$$\langle J\xi_1, (\nabla_{X_h} J)\xi_2\rangle = -\langle \xi_1, J(\nabla_{X_h} J)\xi_2\rangle = \langle \xi_1, (\nabla_{X_h} J)J\xi_2\rangle$$

from Lemma A.1.1 (i), obtaining

$$\begin{aligned}\langle \xi_1, \nabla_{\xi_2} X_h\rangle + \langle \nabla_{\xi_1} X_h, \xi_2\rangle = \langle \nabla_{\xi_1}(JX_h), J\xi_2\rangle + \langle J\xi_1, \nabla_{J\xi_2} X_h\rangle - \big(\langle (\nabla_{\xi_1} J)X_h, J\xi_2\rangle \\ + \langle \xi_1, (\nabla_{X_h} J)J\xi_2\rangle\big) + \langle J\xi_1, (\mathcal{L}_{X_h} J)\xi_2\rangle.\end{aligned}$$

By Lemma A.1.1 (v),

$$\langle (\nabla_{\xi_1} J)X_h, J\xi_2\rangle + \langle (\nabla_{X_h} J)J\xi_2, \xi_1\rangle + \langle (\nabla_{J\xi_2} J)\xi_1, X_h\rangle = 0.$$

Hence

$$\begin{aligned}\langle \xi_1, \nabla_{\xi_2} X_h\rangle + \langle \nabla_{\xi_1} X_h, \xi_2\rangle = \langle \nabla_{\xi_1}(JX_h), J\xi_2\rangle + \langle J\xi_1, \nabla_{J\xi_2} X_h\rangle + \langle (\nabla_{J\xi_2} J)\xi_1, X_h\rangle \\ + \langle J\xi_1, (\mathcal{L}_{X_h} J)\xi_2\rangle.\end{aligned}$$

Since $X_h$ is the Hamiltonian vector field of $h$ we have

$$\mathrm{d}h = \omega(X_h, \cdot) = \langle JX_h, \cdot\rangle,$$

that is, $JX_h = \nabla h$ is the gradient vector field of $h$. But then

$$\begin{aligned}\langle \nabla_{\xi_1}(JX_h), J\xi_2\rangle &= \langle \nabla_{\xi_1}\nabla h, J\xi_2\rangle \\ &= \nabla\,\mathrm{d}h(\xi_1, J\xi_2) \\ &= \nabla\,\mathrm{d}h(J\xi_2, \xi_1) \\ &= \langle \nabla_{J\xi_2}\nabla h, \xi_1\rangle = \langle \nabla_{J\xi_2}(JX_h), \xi_1\rangle = \langle \xi_1, \nabla_{J\xi_2}(JX_h)\rangle\end{aligned}$$

since the Hessian $\nabla\,\mathrm{d}h$ is symmetric. Moreover, by Lemma A.1.1 (iv),

$$\langle (\nabla_{J\xi_2} J)\xi_1, X_h\rangle = -\langle \xi_1, (\nabla_{J\xi_2} J)X_h\rangle.$$

Thus we get

$$\begin{aligned}\langle \xi_1, \nabla_{\xi_2} X_h\rangle + \langle \nabla_{\xi_1} X_h, \xi_2\rangle = \langle \xi_1, \nabla_{J\xi_2}(JX_h) - J\nabla_{J\xi_2} X_h - (\nabla_{J\xi_2} J)X_h\rangle \\ + \langle J\xi_1, (\mathcal{L}_{X_h} J)\xi_2\rangle.\end{aligned}$$

Using

$$\nabla_{J\xi_2}(JX_h) - J\nabla_{J\xi_2} X_h - (\nabla_{J\xi_2} J)X_h = 0$$

from Lemma A.1.1 (iii) we finally obtain

$$\langle \xi_1, \nabla_{\xi_2} X_h\rangle + \langle \nabla_{\xi_1} X_h, \xi_2\rangle = \langle J\xi_1, (\mathcal{L}_{X_h} J)\xi_2\rangle.$$

The lemma is proved. □

We are now ready to prove Proposition A.1.6.

Proof of Proposition A.1.6. First note that

$$\langle \nabla_A \xi_1, \xi_2 \rangle = \langle \nabla \xi_1, \xi_2 \rangle + \langle \nabla_{\xi_1} X_A(u), \xi_2 \rangle$$

and

$$\langle \xi_1, \nabla_A \xi_2 \rangle = \langle \xi_1, \nabla \xi_2 \rangle + \langle \xi_1, \nabla_{\xi_2} X_A(u) \rangle.$$

Using Lemma A.1.7 we may write

$$\begin{aligned} \mathrm{d}\langle \xi_1, \xi_2 \rangle = {} & \langle \nabla_A \xi_1, \xi_2 \rangle + \langle \xi_1, \nabla_A \xi_2 \rangle - \big( \langle \nabla_{\xi_1} X_A(u), \xi_2 \rangle + \langle \xi_1, \nabla_{\xi_2} X_A(u) \rangle \big) \\ & + \langle J\xi_1, (\mathrm{d}J)\xi_2 \rangle. \end{aligned}$$

To compute the third term on the right-hand side of this expression, we note that for every tangent vector $v \in TD$, $X_{A(v)}(u)$ is a Hamiltonian vector field on $M$ along $u$ associated to the Hamiltonian function $\langle \mu, A(v) \rangle_{\mathfrak{g}} \in C^\infty_G(M, \mathbb{R})$. Hence Lemma A.1.8 yields

$$\langle \nabla_{\xi_1} X_{A(v)}(u), \xi_2 \rangle + \langle \xi_1, \nabla_{\xi_2} X_{A(v)}(u) \rangle = \langle J\xi_1, (\mathcal{L}_{X_{A(v)}} J)\, \xi_2 \rangle = -\langle \xi_1, J(\mathcal{L}_{X_{A(v)}} J)\, \xi_2 \rangle$$

for $v \in TD$. Whence

$$\langle \nabla_{\xi_1} X_A(u), \xi_2 \rangle + \langle \xi_1, \nabla_{\xi_2} X_A(u) \rangle = \langle J\xi_1, (\mathcal{L}_{X_A} J)\, \xi_2 \rangle.$$

Rearranging terms we thus obtain

$$\begin{aligned} \mathrm{d}\langle \xi_1, \xi_2 \rangle &= \langle \nabla_A \xi_1, \xi_2 \rangle + \langle \xi_1, \nabla_A \xi_2 \rangle + \big\langle J\xi_1, \big( \mathrm{d}J - \mathcal{L}_{X_A} J \big) \xi_2 \big\rangle \\ &= \langle \nabla_A \xi_1, \xi_2 \rangle + \langle \xi_1, \nabla_A \xi_2 \rangle + \langle J\xi_1, (\delta_A J)\xi_2 \rangle. \end{aligned}$$

This completes the proof of Proposition A.1.6. □

**A.1.6. Bilinear form.** We now define a $\mathfrak{g}$-valued bilinear form $\rho\colon u^*TM \otimes u^*TM \to \mathfrak{g}$ on the vector bundle $u^*TM \to D$ by the relation

$$\big\langle \eta, \rho(\xi_1, \xi_2) \big\rangle_{\mathfrak{g}} = \big\langle \nabla_{\xi_1} X_\eta(u), \xi_2 \big\rangle_J$$

for $\xi_1, \xi_2 \in \Omega^0(D, u^*TM)$ and $\eta \in \Omega^0(D, \mathfrak{g})$. We have the following result.

Lemma A.1.9. *Let $u \in C^\infty(D, M)$, let $\eta \in \Omega^0(D, \mathfrak{g})$ and let $\xi \in \Omega^0(D, u^*TM)$ be a section. Then*

(i) $\nabla_A \mathrm{L}_u \eta - \mathrm{L}_u \nabla_A \eta = \nabla_{\mathrm{d}_A u} X_\eta(u)$

(ii) $\nabla_A \mathrm{L}_u^* \xi - \mathrm{L}_u^* \nabla_A \xi = \rho(\mathrm{d}_A u, \xi) - \mathrm{L}_u^* J (\delta_A J) \xi.$

PROOF. Proof of (i). Let $v \in TD$ be a tangent vector. Because the Levi-Civita connection $\nabla$ is torsion free, we have

$$\begin{aligned}
&\nabla_{A,v}(\mathrm{L}_u\eta) - \mathrm{L}_u\nabla_{A,v}\eta \\
&= \nabla_v \mathrm{L}_u \eta + \nabla_{X_\eta(u)} X_\Phi(u) - \mathrm{L}_u\big(\mathrm{d}\eta(v) + [A(v),\eta]\big) \\
&= \nabla_{\mathrm{d}u(v)} X_\eta(u) + \mathrm{L}_u(\mathrm{d}\eta(v)) + \nabla_{X_{A(v)}(u)} X_\eta(u) + X_{[A(v),\eta]}(u) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \mathrm{L}_u(\mathrm{d}\eta(v)) - X_{[A(v),\eta]}(u) \\
&= \nabla_{\mathrm{d}u(v)} X_\eta(u) + \nabla_{X_{A(v)}(u)} X_\eta(u) \\
&= \nabla_{\mathrm{d}_A u(v)} X_\eta(u).
\end{aligned} \tag{A.5}$$

Proof of (ii). Our proof builds on the proof of Lemma C.2 in [**11**]. Let $v \in TD$ be a tangent vector. Using Lemmas A.1.2 and A.1.6 we compute

$$\begin{aligned}
&\big\langle \eta, \nabla_{A,v}(\mathrm{L}_u^*\xi) - \mathrm{L}_u^*\nabla_{A,v}\xi \big\rangle \\
&= \mathrm{d}\langle \eta, \mathrm{L}_u^*\xi\rangle(v) - \langle \nabla_{A,v}\eta, \mathrm{L}_u^*\xi\rangle - \langle \eta, \mathrm{L}_u^*\nabla_{A,s}\xi\rangle \\
&= \mathrm{d}\langle \mathrm{L}_u\eta, \xi\rangle(v) - \langle \mathrm{L}_u\nabla_{A,v}\eta, \xi\rangle - \langle \mathrm{L}_u\eta, \nabla_{A,v}\xi\rangle \\
&= \mathrm{d}\langle \mathrm{L}_u\eta, \xi\rangle(v) - \langle \mathrm{L}_u\nabla_{A,v}\eta, \xi\rangle - \mathrm{d}\langle \mathrm{L}_u\eta, \xi\rangle(v) + \langle \nabla_{A,v}(\mathrm{L}_u\eta), \xi\rangle - \langle \mathrm{L}_u\eta, J(\delta_{A,v}J)\xi\rangle \\
&= \langle \nabla_{A,v}(\mathrm{L}_u\eta) - \mathrm{L}_u\nabla_{A,v}\eta, \xi\rangle - \langle \eta, \mathrm{L}_u^* J(\delta_{A,v}J)\xi)\rangle.
\end{aligned}$$

We may simplify this further using Formula (A.5). By definition of the bilinear form $\rho$ we thus get

$$\begin{aligned}
\big\langle \eta, \nabla_{A,v}\, \mathrm{L}_u^*\xi - \mathrm{L}_u^*\nabla_{A,v}\xi \big\rangle &= \big\langle \nabla_{\mathrm{d}_A u(v)} X_\eta(u), \xi \big\rangle + \big\langle \mathrm{L}_u^* J(\delta_{A,v}J)\xi \big\rangle \\
&= \big\langle \eta, \rho(\mathrm{d}_A u(v), \xi) + \mathrm{L}_u^* J(\delta_{A,v}J)\xi \big\rangle.
\end{aligned}$$

This proves the lemma. □

**A.1.7. Weitzenböck type formula.** We close this section with Proposition A.1.10 below, which states a Weitzenböck type formula for the operator $\nabla_A$. As we shall see, in this context it will be convenient to think of the family (A.3) of Levi-Civita connections on $M$ in the following way: Fix a reference connection $\nabla_0$ on $M$ and write the affine linear space of all connections on $M$ as

$$\nabla_0 + \Omega^1(M, \mathrm{End}(TM)).$$

Define a map

$$\nabla^{\mathrm{LC}} \colon \mathcal{J}(M,\omega) \to \nabla_0 + \Omega^1(M, \mathrm{End}(TM)), \quad I \mapsto \nabla_I \tag{A.6}$$

that assigns to every $\omega$-compatible almost complex structure $I$ on $M$ the Levi-Civita connection of the Riemannian metric $\langle\cdot,\cdot\rangle_I := \omega(\cdot, I\cdot)$. The family (A.3) is then given by

$$\nabla \colon D \to \nabla_0 + \Omega^1(M, \mathrm{End}(TM)), \quad z \mapsto \nabla_{J_z},$$

where $J\colon D \to \mathcal{J}(M,\omega)$ is the fixed family of almost complex structures on $M$. We define the twisted derivative of the family $\nabla$ as follows. Denote by

$$\mathrm{d}\nabla^{\mathrm{LC}}\colon \Omega^0(M,\mathrm{End}(TM)) \to \Omega^1(M,\mathrm{End}(TM)), \quad I \mapsto \mathrm{d}\nabla^{\mathrm{LC}}(I)$$

the derivative of the map (A.6), and recall that the twisted derivative of the family $J\colon D \to \mathcal{J}(M,\omega)$ is given by

$$\delta_A J\colon TD \to \Omega^0(M,\mathrm{End}(TM)).$$

We then define the twisted derivative of the family (A.3) to be the map

$$\mathrm{D}_A\nabla := \mathrm{d}\nabla^{\mathrm{LC}} \circ \delta_A J\colon TD \to \Omega^1(M,\mathrm{End}(TM)). \tag{A.7}$$

The evaluation of this differential on a tangent vector $v \in TD$ will be denoted by

$$\mathrm{D}_{A,v}\nabla = \mathrm{d}\nabla^{\mathrm{LC}}\big(\delta_{A,v}J\big) \in \Omega^1(M,\mathrm{End}(TM)).$$

Furthermore, we denote by

$$R\colon D \ni z \mapsto R_z \in \Omega^2(M,\mathrm{End}(TM))$$

the family of Riemann curvature tensors on $M$ associated to the family of Levi-Civita connections (A.3). We shall use the abbreviations

$$\nabla_{A,s} := \nabla_{A,\partial s}, \quad \nabla_{A,t} := \nabla_{A,\partial t}, \quad v_s := \mathrm{d}_A u(\partial s), \quad v_t := \mathrm{d}_A u(\partial t),$$

where $\partial s$ and $\partial t$ are the coordinate vector fields on $D$.

With this notation we may now state the Weitzenböck type formula for $\nabla_A$.

Proposition A.1.10 (Weitzenböck type formula). *Let $J\colon D \to \mathcal{J}(M,\omega)$ be a smooth family of $\omega$-compatible almost complex structures on $M$. Let $A \in \Omega^1(D,\mathfrak{g})$ be a connection 1-form. Let $u \in C^\infty(D,M)$, and let $\xi \in \Omega^0(D,u^*TM)$ be a section. Then*

$$\begin{aligned}\big(\nabla_{A,s}\nabla_{A,t} - \nabla_{A,t}\nabla_{A,s}\big)\xi = R(v_s,v_t)\xi &+ \nabla_\xi X_{F_A(\partial_s,\partial_t)}(u)\\ &+ \big((\mathrm{D}_{A,s}\nabla)(v_t) - (\mathrm{D}_{A,t}\nabla)(v_s)\big)\xi.\end{aligned}$$

Proof. By the Leibniz rule,

$$\nabla_{A,v}(\nabla_{A,w}\xi) = \nabla_{A,v}\big((\nabla_z)_{A,w}\xi\big) + \mathrm{D}_{A,v}\nabla\big(\mathrm{d}_A u(w_z)\big)\xi$$

for $z \in D$, $v \in T_zD$ and $w \in \mathrm{Vect}(D)$. Hence

$$\begin{aligned}\big(\nabla_{A,s}\nabla_{A,t} - \nabla_{A,t}\nabla_{A,s}\big)\xi = \nabla_{A,s}\big((\nabla_z)_{A,t}\xi\big) &- \nabla_{A,t}\big((\nabla_z)_{A,s}\xi\big)\\ &+ \big((\mathrm{D}_{A,s}\nabla)(v_t) - (\mathrm{D}_{A,t}\nabla)(v_s)\big)\xi.\end{aligned}$$

By Lemma B.4 in [**11**] we now have

$$\nabla_{A,s}\big((\nabla_z)_{A,t}\xi\big) - \nabla_{A,t}\big((\nabla_z)_{A,s}\xi\big) = R(v_s,v_t)\xi + \nabla_\xi X_{F_A(\partial_s,\partial_t)}(u).$$

□

## A.2. Bubbles connect revisited

The aim of this section is to generalize the bubbling analysis in McDuff and Salamon [**22**], Section 4.7, to the case of almost complex structures that are only continuous. The result will be needed to carry out the bubbling analysis in the proof of Gromov compactness in Section 3.6. More precisely, we re-prove Propositions 4.7.1 and 4.7.2 in [**22**] under the weaker assumption the the almost complex structures only converge in the $C^0$-topology.

**A.2.1. Main result.** The main result of this section is Proposition A.2.1 below. Let $(M, \omega)$ be an arbitrary closed symplectic manifold. Moreover, for $z_0 \in \mathbb{C}$ and $r > 0$

$$B_r(z_0) := \big\{ z \in \mathbb{C} \,\big|\, |z - z_0| \leq r \big\}$$

denotes the closed disk of radius $r$ around $z_0$. We shall denote by $\mathcal{J}^0_\tau(M, \omega)$ the space of $\omega$-tame almost complex structures on $M$ of class $C^0$. Given an $\omega$-tame almost complex structure $I \in \mathcal{J}^0_\tau(M, \omega)$, we denote by $\langle \cdot, \cdot \rangle_I$ the corresponding Riemannian metric on $M$ determined by $I$ and $\omega$. Recall that the energy of an $I$-holomorphic curve $u\colon B_r(z_0) \to M$ is then given by

$$E_I\big(u, B_r(z_0)\big) = \frac{1}{2} \int_{B_r(z_0)} |\mathrm{d}u|_I^2 \,,$$

(see [**22**], Section 2.2 for details).

Proposition A.2.1 (Bubbles connect). *Let $(M, \omega)$ be a compact symplectic manifold. Let $I \in \mathcal{J}^0_\tau(M, \omega)$ be an $\omega$-tame almost complex structure on $M$ of class $C^0$. Suppose that $I_\nu \in \mathcal{J}^0_\tau(M, \omega)$ is a sequence of $\omega$-tame almost complex structures on $M$ of class $C^0$ that converges to $I$ in the $C^0$-topology. Fix a point $z_0 \in \mathbb{C}$ and a real number $r_0 > 0$. Suppose that $u_\nu\colon B_{r_0}(z_0) \to M$ is a sequence of $I_\nu$-holomorphic curves and $u\colon B_{r_0}(z_0) \to M$ is an $I$-holomorphic curve such that the following holds.*

(a) *The sequence $u_\nu$ converges to $u$ in the $C^\infty$-topology on every compact subset of $B_{r_0}(z_0) \setminus \{z_0\}$.*
(b) *The limit*

$$m_0 := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E_{I_\nu}\big(u_\nu; B_\varepsilon(z_0)\big)$$

*exists and is positive.*
(c) *There exist constants $\delta, C > 0$ such that for all $z \in B_{r_0}(z_0)$ and all $r > 0$ with $B_r(z) \subset B_{r_0}(z_0)$, the curve $u_\nu$ satisfies a mean value inequality of the form*

$$E_{I_\nu}\big(u; B_r(z)\big) < \delta \qquad \Longrightarrow \qquad \big|\mathrm{d}u_\nu(z)\big|^2_{I_\nu} \leq \frac{C}{r^2} \cdot E_{I_\nu}\big(u; B_r(z)\big) + C$$

*for every $\nu$.*

*Then there exist a sequence of automorphisms $\psi_\nu \in \mathrm{Aut}(\mathbb{P}^1)$, an $I$-holomorphic sphere $v\colon \mathbb{P}^1 \to M$ and finitely many distinct points $z_1, \dots, z_\ell, z_\infty \in \mathbb{P}^1$ such that, after passing to a subsequence if necessary, the following holds.*

(i) *The sequence* $\psi_\nu$ *converges to* $z_0$ *in the* $C^\infty$*-topology on every compact subset of* $\mathbb{P}^1 \setminus \{z_\infty\} \cong \mathbb{C}$.

(ii) *The sequence*

$$v_\nu := u_\nu \circ \psi_\nu \colon \psi_\nu^{-1}\big(B_{r_0}(z_0)\big) \to M$$

*converges to* $v$ *in the* $C^\infty$*-topology on every compact subset of* $\mathbb{P}^1 \setminus \{z_1, \dots, z_\ell, z_\infty\}$*, and the limits*

$$m_j := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E_{I_\nu}\big(v_\nu; B_\varepsilon(z_j)\big)$$

*exist and are positive for* $j = 1, \dots, \ell$.

(iii) *No energy gets lost in the limit, that is,*

$$E_I(v) + \sum_{j=1}^{\ell} m_j = m_0.$$

(iv) *If* $v$ *is constant, then* $\ell \geq 2$.

*Moreover, bubbles connect in the sense that*

$$u(z_0) = v(z_\infty),$$

*and, for every* $\epsilon > 0$*, there exist constants* $\delta_0 > 0$ *and* $\nu_0$ *such that*

$$\mathrm{d}(z, z_0) + \mathrm{d}\big((\psi_\nu)^{-1}(z), z_\infty\big) < \delta_0 \qquad \Longrightarrow \qquad \mathrm{d}\big(u_\nu(z), u(z_0)\big) < \epsilon$$

*for every* $\nu \geq \nu_0$ *and every* $z \in \mathbb{P}^1$.

**A.2.2. Proof of Proposition A.2.1.** Our proof is essentially the same as the proof of Propositions 4.7.1 and 4.7.2 in [**22**] except for two differences, which we now discuss.

First, the almost complex structures $I_\nu$ and $I$ are only of class $C^0$, and the sequence $I_\nu$ is assumed to converge to $I$ only in the $C^0$-topology ([**22**] requires $C^\infty$-convergence). This will cause problems in all arguments involving the derivatives of $I_\nu$ and $I$. Now the only part of the proof of Propositions 4.7.1 and 4.7.2 where the derivatives of $I_\nu$ and $I$ come into play is when the mean value inequality for pseudoholomorphic curves of Lemma 4.3.1 in [**22**] is used. In fact, a careful examination of the proof of Lemma 4.3.1 in [**22**] shows that the constant $\delta$ in the statement of this lemma depends on the first and second derivatives of the almost complex structure. Hence the only difference between our proof of Proposition A.2.1 and the proof of Propositions 4.7.1 and 4.7.2 resulting from the lack of regularity of $I_\nu$ and $I$ is that we cannot apply the mean value inequality of Lemma 4.3.1. Instead, as a substitute for this inequality, we have to incorporate the Mean Value Inequality (c) into the assumptions of Proposition A.2.1.

Second, the mean value inequality of assertion (c) in the assumptions of Proposition A.2.1 is slightly weaker than the mean value inequality of Lemma 4.3.1 in [**22**] since it contains an additive constant. Now the only step in the proof of Propositions 4.7.1 and 4.7.2 where Lemma 4.3.1 is used is in the proof of Lemma 4.7.3. We therefore need to amend the proof of Lemma 4.7.3, using the mean value inequality of (c) instead of the mean value inequality from Lemma 4.3.1. The result is formulated in the following lemma. For $r < R$ we shall denote by

$$A(r,R) := \big\{ z \in \mathbb{C} \,\big|\, r \le |z| \le R \big\}$$

the closed annulus in $\mathbb{C}$ of inner radius $r$ and outer radius $R$ centered at the origin.

Lemma A.2.2. *Let $(M,\omega)$ be a compact symplectic manifold and let $I \in \mathcal{J}^0_\tau(M,\omega)$ be an $\omega$-tame almost complex structure on $M$ of class $C^0$. Fix constants $\delta, C > 0$. Then, for every $\mu < 1$, there exist constants $R_0, \delta_0, c > 0$ such that the following holds.*

*Suppose that $0 < r < R < R_0$ with $R/r \ge 4$, and that $u\colon A(r,R) \to M$ is an $I$-holomorphic curve that satisfies a mean value inequality of the following form. For all $z \in A(r,R)$ and all $\rho > 0$ such that $B_\rho(z) \subset A(r,R)$,*

$$E_I\big(u; B_\rho(z)\big) < \delta \qquad \Longrightarrow \qquad \frac{1}{2}\big|\mathrm{d}u(z)\big|_I^2 \le \frac{C}{\rho^2} \cdot E_I\big(u; B_\rho(z)\big) + C. \tag{A.8}$$

*Then if the energy of $u$ is sufficiently small in the sense that*

$$E_I(u) = E_I\big(u; A(r,R)\big) < \delta_0$$

*we have estimates*

$$E_I\big(u; A(e^T r, e^{-T} R)\big) \le c \cdot e^{-2\mu T} \cdot E_I(u) \tag{A.9}$$

*and*

$$\sup_{z,z' \in A(e^T r, e^{-T} R)} \mathrm{d}\big(u(z), u(z')\big) \le c \cdot \Big( e^{-\mu T} \cdot \sqrt{E_I(u)} + R \Big) \tag{A.10}$$

*for* $\log 2 \le T \le \log \sqrt{R/r}$.

The proof of Lemma A.2.2 is deferred to the end of this subsection. We now continue with the proof of Proposition A.2.1.

Comparing the statement of Lemma A.2.2 with the statement of Lemma 4.7.3 in [**22**] we see that there are precisely two differences. For, in Lemma A.2.2 the annulus $A(r,R)$ is assumed to be sufficiently small in the sense that $R < R_0$, and inequality (A.10) contains an additional additive constant $R$.

A careful examination of the proofs of Propositions 4.7.1 and 4.7.2 in [**22**] now shows that Lemma 4.7.3 is used in these proofs exactly three times, once in the proof of Lemma 4.7.4, and twice in the proof of Proposition 4.7.1 (see p.103 in [**22**]).

Let us discuss the proof of Lemma 4.7.4 first. In this proof, Lemma 4.7.3 is applied to the $I_\nu$-holomorphic curve $u_\nu\colon A(\delta^\nu/\rho, \rho) \to M$ (we use our notation), where $\rho > 0$ is sufficiently small and $\nu$ sufficiently large. Now the Mean Value Inequality (c) in the hypothesis of Proposition A.2.1 ensures that the curve $u_\nu$ indeed satisfies the requirements of Lemma A.2.2. Then, replacing Lemma 4.7.3 by Lemma A.2.2, we proceed in the proof of Lemma 4.7.4 as follows. First, we have to assume that $\rho > 0$ is such that $\rho < R_0$, where $R_0 > 0$ is the constant from Lemma A.2.2. This is no loss of generality since we are only interested in the situation where $\rho \to 0$. If $2\rho \leq r$, then $A(\delta^\nu/2\rho, 2\rho) \subset A(\delta^\nu/r, r)$. Applying inequality (A.10) to the annulus $A(\delta^\nu/2\rho, 2\rho)$ with $T = \log 2$, we obtain constants $\delta', c' > 0$ such that

$$E^\nu(2\rho) < \delta' \quad \Longrightarrow \quad \sup_{z,z' \in A(\delta^\nu/\rho,\rho)} \mathrm{d}\big(u_\nu(z), u_\nu(z')\big) \leq c' \cdot \sqrt{E^\nu(2\rho)} + c' \cdot \rho.$$

Take the limit $\nu \to \infty$ to obtain

$$E(2\rho) < \delta' \quad \Longrightarrow \quad \mathrm{d}\big(u(\rho), v(1/\rho)\big) = \lim_{\nu\to\infty} \mathrm{d}\big(u_\nu(\rho), u_\nu(\delta^\nu/\rho)\big) \leq c' \cdot \sqrt{E^\nu(2\rho)} + c' \cdot \rho.$$

Here we use the notation

$$E^\nu(2\rho) := E_{I_\nu}\big(u_\nu; A(\delta^\nu/\rho,\rho)\big) \quad \text{and} \quad E(2\rho) := \lim_{\nu\to\infty} E^\nu(2\rho)$$

as in [**22**]. Finally we let $\rho \to 0$ to obtain $u(0) = v(\infty)$. The proof of Lemma 4.7.4 now proceeds exactly as in [**22**].

Next we discuss the proof of Proposition 4.7.1. In the proof on p. 103 in [**22**] the first estimate (4.7.1) of Lemma 4.7.3 is applied to the annuli $A(\delta^\nu, \varepsilon^\nu)$, where $\varepsilon^\nu, \delta^\nu \to 0$. Since estimate (4.7.1) is exactly the same as estimate (A.9) in Lemma A.2.2 above, it only remains to verify that Lemma A.2.2 does in fact apply to this situation. As already explained, the curve $u_\nu$ satisfies the assumptions of Lemma A.2.2 because of the Mean Value Inequality (c) in the hypothesis of Proposition A.2.1. Moreover, since $\varepsilon^\nu \to 0$, and as we are interested in the limit $\nu \to \infty$, we may, after passing to a subsequence if necessary, assume that $\varepsilon^\nu < R_0$, where $R_0 > 0$ is the constant of Lemma A.2.2. Then the proof of Proposition 4.7.1 proceeds as in [**22**].

Thus we have shown that, with the above modifications, the proofs of Propositions 4.7.1 and 4.7.2 in [**22**] carry over to our situation, and so the proof of Proposition A.2.1 is now complete modulo the proof of Lemma A.2.2.

**A.2.3. Proof of Lemma A.2.2.** The proof is adapted from the proof of Lemma 4.7.3 in McDuff and Salamon [**22**]. Let $0 < \mu < 1$ be fixed and define $c' = c'(\mu) := 1/4\pi\mu$.

We begin by recalling some notation from [**22**], Section 4.4. Let $B$ denote the closed unit disk in $\mathbb{C}$ centered at the origin. For any smooth loop $\gamma\colon \partial B \to M$ we denote by

$$\ell(\gamma) := \int_0^{2\pi} |\dot\gamma(\theta)|_I \, \mathrm{d}\theta$$

its length with respect to the Riemannian metric $\langle\cdot,\cdot\rangle_I$ on $M$ determined by $\omega$ and $I$. If the length $\ell(\gamma)$ is smaller than the injectivity radius of $M$, then $\gamma$ admits a smooth local extension $u_\gamma\colon B \to M$ such that

$$u_\gamma\big(e^{i\theta}\big) = \gamma(\theta)$$

for every $\theta \in [0, 2\pi]$, and the local symplectic action of $\gamma$ is hence defined by

$$a(\gamma) := -\int_B u_\gamma^* \, \omega.$$

Note that this does not depend on the choice of local extension $u_\gamma$ as long as the length of $\gamma$ is smaller than the injectivity radius of $M$. As we have already seen in Section 3.2, the isoperimetric inequality of Theorem 4.4.1 in [**22**] continues to hold in our situation where the almost complex structure is only of class $C^0$. Since $c' > 1/4\pi$ by assumption, we thus have an isoperimetric inequality of the following form: There exists a constant $0 < \delta_0 \leq \delta$ such that

$$\ell(\gamma) < 4\pi\sqrt{C} \cdot \sqrt{\delta_0} \qquad \Longrightarrow \qquad \left|a(\gamma)\right|_I \leq c' \cdot \ell(\gamma)^2 \tag{A.11}$$

for every smooth loop $\gamma\colon\ \partial B \to M$. Here $\delta$ and $C$ are the constants from the hypothesis of the lemma.

Now we are ready to start with the actual proof of inequality (A.9). For $r \leq \rho \leq R$ let $\gamma_\rho\colon\ \partial B \to M$ denote the loop defined by

$$\gamma_\rho(\theta) := u\big(\rho\, e^{i\theta}\big)$$

for $\theta \in [0, 2\pi]$. Furthermore, for $\log 2 \leq t \leq \log \sqrt{R/r}$ we define a smooth function $\varepsilon(t)$ that assigns to every $t$ the energy of the curve $u$ on the annulus $A(e^t r, e^{-t} R)$, that is,

$$\varepsilon(t) := E_I\Big(u; A\big(e^t r, e^{-t} R\big)\Big) = \frac{1}{2}\int_{A(e^t r, e^{-t} R)} |\mathrm{d}u|_I^2 \,.$$

Note that the condition $t \leq \log \sqrt{R/r}$ ensures that $e^t r \leq e^{-t} R$. Moreover, we will later need the condition $t \geq \log 2$ in order to be able to apply the Mean Value Inequality (A.8) from the assumptions.

Fix a number $\log 2 \leq T \leq \log \sqrt{R/r}$ and consider a point $z = \rho\, e^{i\theta} \in A(e^T r, e^{-T} R)$. Since $T \geq \log 2$ it follows that $2r \leq \rho \leq R/2$, whence the disk $B_\rho(z)$ is contained in the annulus $A(r, R)$. If $E_I(u) < \delta_0 \leq \delta$, we may then apply the Mean Value Inequality (A.8), obtaining

$$\frac{1}{2}\left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I^2 \leq \frac{C}{\rho^2} \cdot E_I\big(u; B_\rho(z)\big) + C \leq \frac{C}{r^2} \cdot E_I(u) + C. \tag{A.12}$$

Since $E_I(u) < \delta_0$ we get from this the following estimate for the norm of the derivative of $\gamma_\rho$ in the direction of $\theta$:

$$\left|\dot\gamma_\rho(\theta)\right|_I = \frac{r}{\sqrt{2}} \cdot \left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I \leq \sqrt{C \cdot E_I(u) + C \cdot r^2} \leq \sqrt{C} \cdot \Big(\sqrt{\delta_0} + r\Big)\,.$$

Next we define

$$R_0 := \sqrt{\delta_0},$$

and assume for the remainder of this proof that $R < R_0$. We then obtain for the length of the loop $\gamma_\rho$ the estimate

$$\ell(\gamma_\rho) = \int_0^{2\pi} \left|\dot\gamma_\rho(\theta)\right|_I \mathrm{d}\theta < 4\pi\sqrt{C}\cdot\sqrt{\delta_0}. \tag{A.13}$$

Hence it follows from the Isoperimetric Inequality (A.11) that

$$\left|a(\gamma_\rho)\right|_I \leq c'\cdot\ell(\gamma_\rho)^2 \tag{A.14}$$

for $2r \leq \rho \leq R/2$.

As in [**22**], Remark 4.4.2, we denote by $u_\rho\colon B \to M$ the local extension of an arbitrary loop $\gamma_\rho$, defined by the formula

$$u_\rho\big(\rho' e^{i\theta}\big) := \exp_{\gamma_\rho(0)}\big(\rho'\,\xi(\theta)\big)$$

for $0 < \rho' < \rho$ and $\theta \in [0, 2\pi]$, where the map $\xi\colon [0,2\pi] \to T_{\gamma_\rho(0)}M$ is determined by the condition

$$\exp_{\gamma_\rho(0)}\big(\xi(\theta)\big) = \gamma_\rho(\theta).$$

Let $\log 2 \leq t \leq \log\sqrt{R/r}$, and consider the sphere $v_t\colon S^2 \to M$ that is obtained from the restriction of the map $u$ to the annulus $A(e^t r, e^{-t}R)$ by filling in the boundary circles $\gamma_{e^t r}$ and $\gamma_{e^{-t}R}$ with the local extensions $u_{e^t r}$ and $u_{e^{-t}R}$. The sphere $v_t\colon S^2 \to M$ is contractible because it is the boundary of the 3-ball consisting of the union of the disks $u_\rho\colon B \to M$ for $e^t r \leq \rho \leq e^{-t}R$. Hence

$$0 = \int_{S^2} v_t^*\omega = \int_{A(e^t r, e^{-t}R)} u^*\omega - \int_B u_{e^t r}^*\,\omega + \int_B u_{e^{-t}R}^*\,\omega.$$

To understand the minus sign on the right-hand side of this identity note that the disks $u_{e^t r}$ and $u_{e^{-t}R}$ have different orientation considered as submanifolds of the sphere $S^2$. Since the symplectic form $\omega$ tames the almost complex structure $I$, it follows from the energy identity from Lemma 2.2.1 in [**22**] that

$$E_I\Big(u; A\big(e^t r, e^{-t}R\big)\Big) = \int_{A(e^t r, e^{-t}R)} u^*\omega.$$

We may therefore write the previous identity in terms of the local symplectic action as

$$\varepsilon(t) = E_I\Big(u; A\big(e^t r, e^{-t}R\big)\Big) = -a(\gamma_{e^t r}) + a(\gamma_{e^{-t}R}).$$

Thus, applying the Isoperimetric Inequality (A.13) and Hölder's inequality we obtain

$$\begin{aligned}
\varepsilon(t) &= -a(\gamma_{e^t r}) + a(\gamma_{e^{-t}R}) \\
&\leq c'\cdot\ell(\gamma_{e^t r})^2 + c'\cdot\ell(\gamma_{e^{-t}R})^2 \\
&\leq c'\cdot\left(\int_0^{2\pi}\left|\dot\gamma_{e^t r}(\theta)\right|_I \mathrm{d}\theta\right)^2 + c'\cdot\left(\int_0^{2\pi}\left|\dot\gamma_{e^{-t}R}(\theta)\right|_I \mathrm{d}\theta\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c'(e^t r)^2}{2} \cdot \left( \int_0^{2\pi} \left| \mathrm{d}u(e^t r e^{i\theta}) \right|_I \mathrm{d}\theta \right)^2 + \frac{c'(e^{-t}R)^2}{2} \cdot \left( \int_0^{2\pi} \left| \mathrm{d}u(e^{-t} R e^{i\theta}) \right|_I \mathrm{d}\theta \right)^2 \\
&\le 2\pi c' \cdot \left( \frac{1}{2}(e^t r)^2 \cdot \int_0^{2\pi} \left| \mathrm{d}u(e^t r e^{i\theta}) \right|_I^2 \mathrm{d}\theta + \frac{1}{2}(e^{-t}R)^2 \cdot \int_0^{2\pi} \left| \mathrm{d}u(e^{-t} R e^{i\theta}) \right|_I^2 \mathrm{d}\theta \right).
\end{aligned}
$$

To estimate this further, recall that

$$
\varepsilon(t) = \frac{1}{2} \int_{A(e^t r, e^{-t}R)} |\mathrm{d}u|_I^2 = \frac{1}{2} \int_{e^t r}^{e^{-t}R} \rho \int_0^{2\pi} \left| \mathrm{d}u\big(\rho e^{i\theta}\big) \right|_I^2 \mathrm{d}\theta \, \mathrm{d}\rho,
$$

whence

$$
\dot{\varepsilon}(t) = -\frac{1}{2}(e^t r)^2 \int_0^{2\pi} \left| \mathrm{d}u\big(e^t r e^{i\theta}\big) \right|_I^2 \mathrm{d}\theta - \frac{1}{2}(e^{-t}R)^2 \int_0^{2\pi} \left| \mathrm{d}u\big(e^{-t} R e^{i\theta}\big) \right|_I^2 \mathrm{d}\theta.
$$

We see from this computation that the function $\varepsilon(t)$ is of class $C^1$ in $t$ even though the almost complex structure $I$ and hence also the norm $|\cdot|_I$ on $M$ are only of class $C^0$. Hence we conclude that

$$
\varepsilon(t) \le -2\pi c' \cdot \dot{\varepsilon}(t).
$$

Because $\mu = 1/4\pi c'$ this implies

$$
\dot{\varepsilon}(t) \le -2\mu \cdot \varepsilon(t) < 0.
$$

Now let $\log 2 \le T \le \log \sqrt{R/r}$. Integrating this differential inequality from $\log 2$ to $T$, it follows that

$$
\varepsilon(T) \le e^{-2\mu(T - \log 2)} \cdot \varepsilon(\log 2) \le e^{-2\mu T} e^{2\mu} \cdot E_I(u). \tag{A.15}
$$

This proves inequality (A.9).

We next prove inequality (A.10), starting with the following observation.

Claim. *Let $r \le \rho_1 < \rho_2 \le R$ such that $\rho_2/\rho_1 \ge 4$, and set $\rho_0 := \sqrt{\rho_1\rho_2}$. Let also $\theta \in [0, 2\pi]$. Then the restriction of the map $u$ to the annulus $A(\rho_1, \rho_2)$ satisfies the following estimates.*

(i) *If $2\rho_1 \le \rho \le \rho_0$, then*

$$
\left| \mathrm{d}u\big(\rho e^{i\theta}\big) \right|_I^2 \le \frac{36\, C\, e^{2\mu}}{\rho^2} \cdot \left( \frac{\rho_1}{\rho} \right)^{2\mu} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C. \tag{A.16}
$$

(ii) *If $\rho_0 \le \rho \le \rho_2/2$, then*

$$
\left| \mathrm{d}u\big(\rho e^{i\theta}\big) \right|_I^2 \le \frac{36\, C\, e^{2\mu}}{\rho^2} \cdot \left( \frac{\rho}{\rho_2} \right)^{2\mu} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C. \tag{A.17}
$$

Proof of claim. Assume that $2\rho_1 \le \rho \le \rho_2/2$. We then have

$$\rho/\rho_1 \le \rho_2/\rho \quad \text{for} \quad \rho \le \rho_0$$

and

$$\rho/\rho_1 \ge \rho_2/\rho \quad \text{for} \quad \rho \ge \rho_0.$$

We distinguish four cases for $\rho$.

Case 1. $2\rho_1 \le \rho \le 2e\rho_1$.

The disk $B_{\rho/2}(\rho e^{i\theta})$ is contained in the annulus $A(\rho_1, \rho_2)$. If $E_I(u) < \delta_0 \le \delta$, we may therefore apply the Mean Value Inequality (A.8) obtaining

$$\begin{aligned}
\left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I^2 &\le \frac{4\,C}{\rho^2} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C \\
&\le \frac{4\,C}{\rho^2} \cdot \left(\frac{\rho}{\rho_1}\right)^{2\mu} \cdot \left(\frac{\rho_1}{\rho}\right)^{2\mu} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C \\
&\le \frac{16\,C\,e^{2\mu}}{\rho^2} \cdot \left(\frac{\rho_1}{\rho}\right)^{2\mu} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C.
\end{aligned}$$

In the last inequality we used that $\rho/\rho_1 \le 2e$, whence $(\rho/\rho_1)^{2\mu} \le 4e^{2\mu}$.

Case 2. $2e\rho_1 \le \rho \le \rho_0$.

The disk $B_{\rho/2}(\rho e^{i\theta})$ is contained in the annulus $A(\rho/e, e\rho)$. If $E_I(u) < \delta_0 \le \delta$, we may therefore apply the Mean Value Inequality (A.8) obtaining

$$\left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I^2 \le \frac{4\,C}{\rho^2} \cdot E_I\big(u; A(\rho/e, e\rho)\big) + C.$$

In order to estimate this further, we apply inequality (A.15) to the annulus

$$A\Big(e^{\log(\rho/\rho_1)-1} \cdot \rho_1, e^{-\log(\rho/\rho_1)+1} \cdot \rho_2\Big) \supset A(\rho/e, e\rho).$$

We get

$$E\bigg(u; A\Big(e^{\log(\rho/\rho_1)-1} \cdot \rho_1, e^{-\log(\rho/\rho_1)+1} \cdot \rho_2\Big)\bigg) \le e^{4\mu} \cdot e^{-2\mu\log(\rho/\rho_1)} \cdot E_I\big(u; A(\rho_1, \rho_2)\big).$$

Therefore

$$\left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I^2 \le \frac{36\,C\,e^{2\mu}}{\rho^2} \cdot \left(\frac{\rho_1}{\rho}\right)^{2\mu} \cdot E_I\big(u; A(\rho_1, \rho_2)\big) + C.$$

Case 3. $\rho_0 \le \rho \le \rho_2/2e$.

The disk $B_{\rho/2}(\rho e^{i\theta})$ is contained in the annulus $A(\rho/e, e\rho)$. If $E_I(u) < \delta_0 \le \delta$, we may therefore apply the Mean Value Inequality (A.8) obtaining

$$\left|\mathrm{d}u\big(\rho e^{i\theta}\big)\right|_I^2 \le \frac{4\,C}{\rho^2} \cdot E_I\big(u; A(\rho/e, e\rho)\big) + C.$$

In order to estimate this further, we again apply inequality (A.15) to the annulus

$$A\big(e^{\log(\rho_2/\rho)-1}\cdot\rho_1, e^{-\log(\rho_2/\rho)+1}\cdot\rho_2\big)\supset A(\rho/e, e\rho).$$

We get

$$E\Big(u; A\big(e^{\log(\rho_2/\rho)-1}\cdot\rho_1, e^{-\log(\rho_2/\rho)+1}\cdot\rho_2\big)\Big)\leq e^{4\mu}\cdot e^{-2\mu\log(\rho_2/\rho)}\cdot E_I\big(u; A(\rho_1,\rho_2)\big).$$

Therefore

$$\big|\mathrm{d}u\big(\rho e^{i\theta}\big)\big|_I^2\leq\frac{36\,C\,e^{2\mu}}{\rho^2}\cdot\left(\frac{\rho}{\rho_2}\right)^{2\mu}\cdot E_I\big(u; A(\rho_1,\rho_2)\big)+C.$$

Case 4. $\rho_2/2e\leq\rho\leq\rho_2/2$.

The disk $B_{\rho/2}(\rho e^{i\theta})$ is contained in the annulus $A(\rho_1,\rho_2)$. If $E_I(u)<\delta_0\leq\delta$, we may therefore apply the Mean Value Inequality (A.8) obtaining

$$\begin{aligned}\big|\mathrm{d}u\big(\rho e^{i\theta}\big)\big|_I^2&\leq\frac{4\,C}{\rho^2}\cdot E_I\big(u; A(\rho_1,\rho_2)\big)+C\\&\leq\frac{4\,C}{\rho^2}\cdot\left(\frac{\rho_2}{\rho}\right)^{2\mu}\cdot\left(\frac{\rho}{\rho_2}\right)^{2\mu}\cdot E_I\big(u; A(\rho_1,\rho_2)\big)+C\\&\leq\frac{16\,C\,e^{2\mu}}{\rho^2}\cdot\left(\frac{\rho}{\rho_2}\right)^{2\mu}\cdot E_I\big(u; A(\rho_1,\rho_2)\big)+C.\end{aligned}$$

Note that for the last inequality we used that $\rho_2/\rho\leq 2e$, so that $(\rho_2/\rho)^{2\mu}\leq 4e^{2\mu}$.

The Claim is proved. □

Let now $\log 2\leq t\leq\log(\rho_2/\rho_0)$. For $\rho_0\leq\rho\leq\rho_2/2$, inequality (A.17) implies that

$$\begin{aligned}\big|\mathrm{d}u\big(\rho e^{i\theta}\big)\big|_I&\leq\frac{18\sqrt{C}}{\rho}\cdot\left(\frac{\rho}{\rho_2}\right)^{\mu}\cdot\sqrt{E_I\big(u; A(\rho_1,\rho_2)\big)}+\sqrt{C}\\&=18\sqrt{C}\cdot\frac{\rho^{\mu-1}}{\rho_2^{\mu}}\cdot\sqrt{E_I\big(u; A(\rho_1,\rho_2)\big)}+\sqrt{C}.\end{aligned}$$

For any point $s_0e^{i\theta_0}$ such that $\rho_0\leq s_0\leq e^{-t}\rho_2$ and $0\leq\theta_0\leq 2\pi$, integrating the last inequality then yields

$$\begin{aligned}&\mathrm{d}_I\big(u(\rho_0), u(s_0e^{it_0})\big)\\\leq\;&\int_{\rho_0}^{s_0}\big|\partial_\rho u(\rho)\big|_I\,\mathrm{d}\rho+\int_0^{\theta_0}\big|\partial_\theta u\big(s_0e^{i\theta}\big)\big|_I\,\mathrm{d}\theta\\\leq\;&18\sqrt{C}\cdot\left(\int_{\rho_0}^{s_0}\frac{\rho^{\mu-1}}{\rho_2^{\mu}}\,\mathrm{d}\rho+\int_0^{\theta_0}\left(\frac{s_0}{\rho_2}\right)^{\mu}\mathrm{d}\theta\right)\cdot\sqrt{E_I\big(u; A(\rho_1,\rho_2)\big)}\\&\qquad\qquad+\big(s_0-\rho_0+s_0\theta_0\big)\cdot\sqrt{C}\end{aligned}$$

$$
\begin{aligned}
&\leq\ 18\sqrt{C}\cdot\left(\int_0^{e^{-t}\rho_2}\frac{\rho^{\mu-1}}{\rho_2^{\mu}}\,\mathrm{d}\rho+\int_0^{2\pi}\left(\frac{e^{-t}\rho_2}{\rho_2}\right)^{\mu}\mathrm{d}\theta\right)\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +e^{-t}\rho_2\cdot(1+2\pi)\cdot\sqrt{C}\\
&=\ 18\sqrt{C}\cdot\left(\frac{(e^{-t}\rho_2)^{\mu}}{\mu\cdot\rho_2^{\mu}}+2\pi\cdot e^{-\mu t}\right)\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\rho_2\cdot(1+2\pi)\cdot\sqrt{C}\\
&=\ 18\sqrt{C}\cdot\left(\frac{1}{\mu}+2\pi\right)\cdot e^{-\mu t}\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\rho_2\cdot(1+2\pi)\cdot\sqrt{C}.
\end{aligned}
$$

Similarly, for $2\rho_1\leq\rho\leq\rho_0$, inequality (A.16) implies that

$$
\begin{aligned}
\big|\mathrm{d}u\big(\rho e^{i\theta}\big)\big|_I &\leq \frac{18\sqrt{C}}{\rho}\cdot\left(\frac{\rho_1}{\rho}\right)^{\mu}\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\sqrt{C}\\
&=18\sqrt{C}\cdot\frac{\rho_1^{\mu}}{\rho^{\mu+1}}\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\sqrt{C},
\end{aligned}
$$

and for any point $s_1e^{i\theta_1}$ with $e^t\rho_1\leq s_1\leq\rho_0$ and $0\leq\theta_1\leq 2\pi$, integrating this inequality gives

$$
\begin{aligned}
&\mathrm{d}_I\big(u(\rho_0),u(s_1e^{it_1})\big)\\
&\leq\ \int_{s_1}^{\rho_0}\big|\partial_\rho u(\rho)\big|_I\,\mathrm{d}\rho+\int_0^{\theta_1}\big|\partial_\theta u\big(s_0e^{i\theta}\big)\big|_I\,\mathrm{d}\theta\\
&\leq\ 18\sqrt{C}\cdot\left(\int_{s_1}^{\rho_0}\frac{\rho_1^{\mu}}{\rho^{\mu+1}}\,\mathrm{d}\rho+\int_0^{\theta_1}\left(\frac{\rho_1}{s_1}\right)^{\mu}\mathrm{d}\theta\right)\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\big(\rho_0-s_1+s_1\theta_1\big)\cdot\sqrt{C}\\
&\leq\ 18\sqrt{C}\cdot\left(\int_{e^t\rho_1}^{0}\frac{\rho_1^{\mu}}{\rho^{\mu+1}}\,\mathrm{d}\rho+\int_0^{2\pi}\left(\frac{\rho_1}{e^t\rho_1}\right)^{\mu}\mathrm{d}\theta\right)\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +e^t\rho_1\cdot(2\pi-1)\cdot\sqrt{C}\\
&=\ 18\sqrt{C}\cdot\left(\frac{\rho_1^{\mu}}{\mu\cdot(e^t\rho_1)^{\mu}}+2\pi\cdot e^{-\mu t}\right)\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\rho_2\cdot(1+2\pi)\cdot\sqrt{C}\\
&=\ 18\sqrt{C}\cdot\left(\frac{1}{\mu}+2\pi\right)\cdot e^{-\mu t}\cdot\sqrt{E_I\big(u;A(\rho_1,\rho_2)\big)}+\rho_2\cdot(1+2\pi)\cdot\sqrt{C}.
\end{aligned}
$$

This proves inequality (A.10) and finishes the proof of Lemma A.2.2.

## A.3. Pseudoholomorphic spheres in the fiber

In this section, we study the moduli space of pseudoholomorphic spheres in the fibers of a symplectic fiber bundle over a closed base manifold of arbitrary dimension. Our approach is inspired by the construction of parametric Gromov-Witten invariants for trivial symplectic fiber bundles by Buşe [**2**] and the study of the moduli space of pseudoholomorphic spheres in the fibers of a Hamiltonian symplectic fiber bundle over a compact Riemann surface in McDuff and Salamon [**22**], Section 8.4.

**A.3.1. Symplectic fiber bundles.** Let $(M, \omega)$ be an arbitrary closed symplectic manifold. Let $B$ be a closed manifold, and consider a symplectic fiber bundle

$$\begin{tikzcd} (M,\omega) \arrow[r, hook, "i"] & (\widetilde{M},\widetilde{\omega}) \arrow[d, "\pi_{\widetilde{M}}"] \\ & B \end{tikzcd}$$

with typical fiber $(M, \omega)$. Any homology class $A \in H_2(\widetilde{M}; \mathbb{Z})$ contained in the image of the push forward map

$$i_*\colon H_2(M; \mathbb{Z}) \to H_2(\widetilde{M}; \mathbb{Z})$$

will be called **fiberwise**. For every fiberwise homology class $A \in H_2(\widetilde{M}; \mathbb{Z})$ there exists a unique collection of homology classes $A_b \in H_2(\widetilde{M}_b; \mathbb{Z})$ in the fibers of $\widetilde{M}$, indexed by $b \in B$, such that

$$i_{b*}A_b = A,$$

where $i_b\colon \widetilde{M}_b \to \widetilde{M}$ is the inclusion of the fiber.

**A.3.2. Vertical almost complex structures.** We denote by $\mathcal{J}^{\mathrm{Vert}} := \mathcal{J}^{\mathrm{Vert}}(\widetilde{M}, \widetilde{\omega})$ the space of vertical $\widetilde{\omega}$-compatible almost complex structures $J$ on the bundle $\widetilde{M} \to B$. More explicitly, this means that for every $b \in B$ the almost complex structure $J_b$ and the symplectic form $\widetilde{\omega}_b$ on the fiber $\widetilde{M}_b$ are compatible.

Remark A.3.1. In the applications in Section 4.3, $(M, \omega)$ will be a $G$-manifold equipped with a $G$-invariant symplectic form, and we will be interested in the particular case where

$$\widetilde{M} := Q \times_G M \to B$$

is the fiber bundle associated to some principal $G$-bundle $Q \to B$. We may then equivalently think of a vertical almost complex structure $J \in \mathcal{J}^{\mathrm{Vert}}$ as a smooth $G$-equivariant map

$$J\colon Q \to \mathcal{J}(M, \omega),$$

where $\mathcal{J}(M, \omega)$ denotes the space of $\omega$-compatible almost complex structures on $M$.

**A.3.3. Moduli space.** Let us now fix a fiberwise spherical homology class $A \in H_2(\widetilde{M};\mathbb{Z})$ and a vertical almost complex structure $J \in \mathcal{J}^{\mathrm{Vert}}$. We define the **moduli space of of simple fiberwise $J$-holomorphic spheres in $\widetilde{M}$ of degree** $A$ to be

$$\mathcal{M}^*\big(\widetilde{M};A;J\big) := \big\{(b,v)\,\big|\, b \in B \text{ and } v \in \mathcal{M}^*\big(\widetilde{M}_b;A_b;J_b\big)\big\}.$$

Here $\mathcal{M}^*(\widetilde{M}_b;A_b;J_b)$ denotes the moduli space of smooth simple $J_b$-holomorphic spheres of degree $A_b$ in the fiber $\widetilde{M}_b$ (see McDuff and Salamon [**22**], Section 3.1, for the definition of this moduli space).

**A.3.4. Fredholm theory.** Fix a real number $p > 2$, and consider the Banach manifold

$$\mathcal{B}^{1,p} := \big\{(b,v)\,\big|\, b \in B \text{ and } v \in W^{1,p}\big(\mathbb{P}^1,\widetilde{M}_b\big)\big\}$$

of spheres of class $W^{1,p}$ in the fibers of the bundle $\widetilde{M}$. Its tangent space at $(b,v) \in \mathcal{B}^{1,p}$ is given by

$$T_{(b,v)}\mathcal{B}^{1,p} = T_bB \oplus W^{1,p}\big(\mathbb{P}^1, v^*T\widetilde{M}_b\big).$$

For any positive integer $\ell$ we denote by $\mathcal{J}^\ell := \mathcal{J}^\ell(\widetilde{M},\widetilde{\omega})$ the space of $\widetilde{\omega}$-compatible vertical almost complex structures of class $C^\ell$ on $\widetilde{M} \to B$. Fix $J \in \mathcal{J}^\ell$, and define a vector bundle $\mathcal{E}^p := \mathcal{E}^p_J \to \mathcal{B}^{1,p}$ with fibers given by

$$\mathcal{E}^p_{(b,v)} := L^p\big(\mathbb{P}^1, \Lambda^{0,1}_{J_b} T^*\mathbb{P}^1 \otimes v^*T\widetilde{M}_b\big).$$

Consider the section $\mathcal{F}_J\colon \mathcal{B}^{1,p} \to \mathcal{E}^p$ given by

$$\mathcal{F}_J(b,v) := \big(b, \overline{\partial}_{J_b}(v)\big),$$

where

$$\overline{\partial}_{J_b}(v) := \frac{1}{2}\big(\mathrm{d}v + J_b \circ \mathrm{d}v \circ j_{\mathbb{P}^1}\big)$$

denotes the complex antilinear part of the derivative $\mathrm{d}v$. Thus a pair $(b,v)$ defines a fiberwise $J$-holomorphic sphere in $\widetilde{M}$ if and only if $\mathcal{F}_J(b,v) = 0$. The vertical derivative of the section $\mathcal{F}_J$ at a zero $(b,v)$ gives rise to an operator

$$\mathcal{D}_{b,v}\colon T_{(b,v)}\mathcal{B}^{1,p} \to \mathcal{E}^p_J, \tag{A.18}$$

which is given in explicit terms by

$$\mathcal{D}_{b,v}\begin{pmatrix}\beta\\ \xi\end{pmatrix} = D_b\beta + D_v\xi. \tag{A.19}$$

Let us explain this operator in more detail. The operator

$$D_b\colon T_bB \to L^p\big(\mathbb{P}^1, \Lambda^{0,1}_{J_b} T^*\mathbb{P}^1 \otimes v^*T\widetilde{M}_b\big) \tag{A.20}$$

is obtained by linearizing the Cauchy-Riemann operator

$$\overline{\partial}_{J_b}(v) := \frac{1}{2}\big(\mathrm{d}v + J_b \circ \mathrm{d}v \circ j_{\mathbb{P}^1}\big)$$

with respect to $b$. It is given in explicit terms by the formula

$$D_b\xi = -\frac{1}{2}J_b(v)\big(\mathrm{d}_b J_\bullet(\beta)\big)\,\partial_{J_b}(v).$$

Here the operator

$$\mathrm{d}_b J_\bullet\colon T_bB \to T_{J_b}\mathcal{J}\big(\widetilde{M}_b, \widetilde{\omega}_b\big)$$

is the derivative at $b$ of the vertical almost complex structure $J$ on $\widetilde{M}$, and

$$\partial_{J_b}(v) := \frac{1}{2}\big(\mathrm{d}v - J_b(v)\circ \mathrm{d}v \circ j_{\mathbb{P}^1}\big)$$

is the complex linear part of the derivative $\mathrm{d}v$. The operator

$$D_v\colon W^{1,p}\big(\mathbb{P}^1, v^*T\widetilde{M}_b\big) \to L^p\big(\mathbb{P}^1, \Lambda^{0,1}_{J_b}T^*\mathbb{P}^1 \otimes v^*T\widetilde{M}_b\big) \tag{A.21}$$

is obtained by linearizing the Cauchy-Riemann operator $\overline{\partial}_{J_b}(v)$ with respect to $v$. It is given in explicit terms by the formula

$$D_v\xi = (\nabla\xi)^{0,1} - \frac{1}{2}J_b\big(\nabla_\xi J_b\big)\,\partial_{J_b}(v).$$

We see from this formula that the operator (A.21) is a real linear Cauchy-Riemann operator in the sense of McDuff and Salamon [**22**], Definition C.1.5. Here the operator

$$(\nabla\xi)^{0,1} := \frac{1}{2}\big(\nabla\xi + J_b(v)\circ\nabla\xi\circ j_{\mathbb{P}^1}\big)$$

denotes the complex antilinear part of the 1-form $\nabla\xi$,

Proposition A.3.2. *Fix a vertical almost complex structure $J \in \mathcal{J}^\ell$ for some integer $\ell \geq 1$. Fix a real number $p > 2$. Then the operator (A.18) is Fredholm for every pair $(b, v) \in \mathcal{B}^{1,p}$. Its (real) index is given by*

$$\operatorname{ind}\mathcal{D}_{b,v} = \dim B + \dim M + 2\,\big\langle c_1(T\widetilde{M}_b), [v]\big\rangle,$$

*where $[v]$ denotes the degree of the sphere $v$.*

Proof. First, we note that the operator

$$T_bB \to 0, \qquad \beta \mapsto 0 \tag{A.22}$$

is Fredholm of index $\dim B$. Second, by the Riemann-Roch theorem (McDuff and Salamon [**22**], Thm. C.1.10) the real linear Cauchy-Riemann operator (A.21) is Fredholm of index

$$\operatorname{ind} D_{b,v} = \dim M + 2\,\big\langle c_1(T\widetilde{M}_b), [v]\big\rangle.$$

We see from (A.19) that the operator (A.18) is the direct sum of the Fredholm operators (A.22) and (A.21). Hence we conclude that the operator (A.18) is Fredholm and its index is given by the sum of the indices of each of the operators (A.22) and (A.21) ([**22**], Thm. A.1.5). □

**A.3.5. Regular vertical almost complex structures.** Fix a fiberwise spherical homology class $A \in H_2(\widetilde{M}; \mathbb{Z})$. A vertical almost complex structure $J \in \mathcal{J}^{\mathrm{Vert}}$ is called **regular for** $A$ if the vertical differential (A.18) is surjective for every $b \in B$ and every simple $J_b$-holomorphic sphere $v\colon \mathbb{P}^1 \to \widetilde{M}_b$ of degree $A_b$. We will henceforth denote by $\mathcal{J}_{\mathrm{reg}}^{\mathrm{Vert}}(A) := \mathcal{J}_{\mathrm{reg}}^{\mathrm{Vert}}(\widetilde{M}, \widetilde{\omega}; A)$ the set of all $\widetilde{\omega}$-compatible vertical almost complex structures on $\widetilde{M}$ that are regular for $A$.

**A.3.6. Main result.** The main result of this section is the following theorem.

THEOREM A.3.3. *Let $A \in H_2(M; \mathbb{Z})$ be a fiberwise spherical homology class.*

(i) *If $J \in \mathcal{J}_{\mathrm{reg}}^{\mathrm{Vert}}(A)$, then $\mathcal{M}^*(\widetilde{M}; A; J)$ is a smooth oriented manifold of (real) dimension*
$$\dim \mathcal{M}^*(\widetilde{M}; A; J) = \dim B + \dim M + 2\left\langle c_1(TM), A\right\rangle.$$
*Here we denote by*
$$\langle c_1(TM), A\rangle := \langle c_1(T\widetilde{M}_b), A_b\rangle, \quad b \in B$$
*the Poincaré pairing of the first Chern class of the fiber with the fiberwise homology class $A$, which is independent of the point $b$ in the base.*

(ii) *The set $\mathcal{J}_{\mathrm{reg}}^{\mathrm{Vert}}(A)$ is a countable intersection of open and dense subsets of $\mathcal{J}^{\mathrm{Vert}}$.*

PROOF. The proof of this theorem is similar to the proof of Theorem 8.4.1 in [**22**]. $\square$

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