On Fano Threefolds with $b_2 \geq 2$

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Contents

1 Introduction

The goal of my diploma thesis is the complete classification of primitive Fano threefolds whose second Betti numbers are not less than two.

By a *Fano threefold* we mean a nonsingular projective variety over $\mathbb{C}$ of dimension 3 and with ample anticanonical divisor. It is called *primitive* if it is not isomorphic to the blowing-up of a Fano threefold along a nonsingular irreducible curve.

I will prove the following theorem. For any Fano threefold X, $\mathrm{b}_2(X)$ denotes its second Betti number and K_X its canonical divisor.

Theorem. Primitive Fano threefolds X with $\mathrm{b}_2(X) \geq 2$ are classified as follows.

$\mathrm{b}_2(X)$	$(-K_X)^3$	type of X
2	6	a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ whose branch locus is a divisor of bidegree $(2,4)$
2	12	a double covering of W_6, a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$, whose branch locus is a member of $\lvert -K_{W_6}\rvert$, or a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2,2)$
2	14	a double covering of $V_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ whose branch locus is a member of $\lvert -K_{V_7}\rvert$
2	24	a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ whose branch locus is a divisor of bidegree $(2,2)$
2	30	a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$
2	48	W_6, a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$
2	54	$\mathbb{P}^2 \times \mathbb{P}^1$
2	56	$V_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$
2	62	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$
3	12	a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(2,2,2)$
3	14	a nonsingular member of the complete linear system $\lvert\mathcal{O}_P(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3)\rvert$ on the $\mathbb{P}^2$-bundle $\pi : P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)^{\oplus 2}) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$
3	48	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
3	52	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$

This work is motivated as follows. In the early 1980's S. Mori and S. Mukai presented in [MM81] a list which was claimed to be the complete classification of Fano threefolds with $\mathrm{b}_2 \geq 2$.

However, a thorough proof of these results has never been published. Instead, Mori and Mukai explained the general principle of how to classify such Fano threefolds in [MM83], giving only outlines of proofs for parts of their list. Some

20 years later it turned out that at least one type was missing in this list (cf. [MM03]).

Hence it seems worthwhile to work out this classification in full detail and to check whether any types are still missing. Of course it would be far beyond the scope of a diploma thesis to go through the whole classification. Thus I restrict myself to the classification of a tractable subclass of all Fano threefolds with $b_2 \geq 2$, namely, the primitive ones. My result is the theorem above. In particular, it shows that there are no primitive Fano threefolds missing in the list of [MM81].

As already mentioned, my proof of this theorem follows [MM83], §1 - §6 and §8. We shall proceed in three steps.

First, chapter 2 provides us with all the preliminaries that are necessary for the classification of primitive Fano threefolds with $b_2 \geq 2$. Namely, it introduces some aspects of Mori theory in the context of primitive Fano threefolds. We will learn about Mori's technique of extremal rays of Fano threefolds and its geometric implications. Apart from minor modifications and some supplements, this chapter is a summary of the main results from Mori's paper [Mo].

Second, chapter 3 is a first step towards the classification of primitive Fano threefolds with $b_2 \geq 2$. We will use our knowledge about extremal rays on primitive Fano threefolds from the previous chapter to establish the existence of certain morphisms from any primitive Fano threefold to surfaces or other threefolds. These morphisms will give a first insight into the intrinsic structure of primitive Fano threefolds with $b_2 \geq 2$. In particular, we will see that such Fano threefolds must have $b_2 = 2$ or 3.

Third, in the last two chapters we systematically exploit the results from chapters 2 and 3 in order to classify primitive Fano threefolds with $b_2 \geq 2$. Chapter 4 is devoted to primitive Fano threefolds with $b_2 = 2$, chapter 5 to those with $b_2 = 3$.

There are two appendices. Appendix A contains a list of all types of primitive Fano threefolds with $b_2 \geq 2$, similar to the one in the theorem above, but giving additional technical information relevant to the proofs.

Appendix B provides a summary of basic general results needed frequently in the course of the classification.

The main reference for this thesis is [Ha77] and the first chapter of [De]. In particular, all the standard results used without reference may be found in one of these texts.

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2 Mori theory in the context of Fano threefolds

The purpose of this chapter is to provide a summary of those results from Mori theory that are essential to the classification of primitive Fano threefolds with $\mathrm{b}_2 \geq 2$. We will refer to Mori's paper [Mo] and tailor his results to our needs. This chapter begins with some preliminaries in section 2.1, followed by the main results (statements only) from Mori's theory of extremal rays of Fano threefolds in section 2.2. It concludes with the complete classification of extremal rays of a primitive Fano threefold, in section 2.3. This classification will be the starting point for the subsequent classification of primitive Fano threefolds with $\mathrm{b}_2 \geq 2$.

2.1 Notation and definitions

This section introduces the required formalism to state the results from Mori theory that we need for the classification.

Let X be a variety. We will follow [Mo, Ch. 1, §1] and [De, § 1.3]. By a 1-*cycle* on X we understand an element of the free abelian group $\mathrm{Z}_1(X)$ generated by all the irreducible reduced curves on X. A 1-cycle $\sum n_C C, n_C \in \mathbb{Z}$ is called *effective* if $n_C \geq 0$ for all C. We denote *numerical equivalence of* 1-*cycles* with respect to intersections with Cartier divisors on X by the symbol $\equiv$. Then we set

$$\mathrm{N}_1(X)_{\mathbb{Z}} = \mathrm{Z}_1(X)/\equiv \qquad \text{and} \qquad \mathrm{N}_1(X)_{\mathbb{R}} = \mathrm{N}_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

We endow $\mathrm{N}_1(X)_{\mathbb{R}}$ with its usual real topology.

Likewise, we denote *numerical equivalence of Cartier divisors* with respect to intersections with 1-cycles on X by the symbol $\equiv$, and set

$$\mathrm{N}^1(X)_{\mathbb{Z}} = \mathrm{Ca}(X)/\equiv \qquad \text{and} \qquad \mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{N}^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\mathrm{Ca}(X)$ is the group of Cartier divisors on X.

The vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ is finite-dimensional ([De, § 1.3]), and its dimension is called the *Picard number* $\rho(X)$ of X. In particular, we see that $\rho(X)$ is the rank of $\mathrm{N}^1(X)_{\mathbb{Z}}$.

Via the intersection pairing $(\ ,\ )_X : \mathrm{N}^1(X)_{\mathbb{Z}} \times \mathrm{N}_1(X)_{\mathbb{Z}} \longrightarrow \mathbb{Z}$, which is by definition non-degenerate, $\mathrm{N}_1(X)_{\mathbb{R}}$ is dual to $\mathrm{N}^1(X)_{\mathbb{R}}$.

This shows that the dimension of $\mathrm{N}_1(X)_{\mathbb{R}}$ and the rank of $\mathrm{N}_1(X)_{\mathbb{Z}}$ are both equal to $\rho(X)$.

We define $\mathrm{NE}(X)$ to be the convex cone in $\mathrm{N}_1(X)_{\mathbb{R}}$ generated by the classes of all effective 1-cycles on X.

A half line $R = \mathbb{R}_+ [C]$ in the closure $\overline{\mathrm{NE}}(X)$, where C is an effective 1-cycle on X and $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$, is called an *extremal ray* of X if the following is satisfied:

(i) $(-K_X \cdot C)_X > 0$;

(ii) if Z_1 and Z_2 are classes in $\mathrm{NE}(X)$ such that $Z_1 + Z_2 \in R$, then $Z_1, Z_2 \in R$.

A rational curve l on X is called an *extremal rational curve* if $(-K_X \cdot l)_X \leq \dim(X) + 1$ and the ray $\mathbb{R}_+ [l]$ is an extremal ray of X.

2.2 Extremal rays of Fano threefolds

This section is a summary of the main results of Mori's theory of extremal rays of a Fano threefold, namely, the Cone Theorem and the Contraction Theorem. Moreover, we will collect some useful basic properties of primitive Fano threefolds.

We start with the Cone Theorem, which is Theorem 1.2 in [Mo]. It clarifies the structure of the cone $\mathrm{NE}(X)$ and shows that extremal rays on Fano threefolds always exist. Its importance to our classification will become apparent when we consider the geometric implications of extremal rays below. Note in particular that this theorem implies that $\mathrm{NE}(X)$ is closed in $\mathrm{N}_1(X)_{\mathbb{R}}$.

Theorem 2.1 (Cone Theorem)**.** *Let X be a Fano threefold. Then X contains finitely many extremal rational curves $l_1, \ldots, l_n$ with corresponding extremal rays $R_i = \mathbb{R}_+ [l_i]$ such that*

$$\mathrm{NE}(X) = R_1 + \ldots + R_n.$$

By the theorem, we can make the following definitions: We define the *lenght* of an extremal ray R of X to be the number

$$\mu_R = \min\{(-K_X \cdot C)_X \mid \text{ C is a rational curve on X such that } [C] \in R\}.$$

Moreover we fix, for each extremal ray R of X, an extremal rational curve ℓ_R on X such that $[\ell_R] \in R$ and

$$(-K_X \cdot \ell_R)_X = \mu_R.$$

We digress for a moment and collect some basic properties of Fano threefolds that we will frequently need.

Lemma 2.2. *Let X be a Fano threefold.*

1. $\mathrm{h}^i(\mathcal{O}_X) = 0$ *for all* $i > 0$. *In particular,* $\chi(\mathcal{O}_X) = 1$.

2. $(-K_X \cdot \mathrm{c}_2(X))_X = 24$.

3. *For divisors on X, linear and numerical equivalence are the same, i.e., there is an isomorphism* $\mathrm{N}^1(X)_{\mathbb{Z}} \cong \mathrm{Pic}(X)$.

4. The Picard number $\rho(X)$*, the rank of* $\mathrm{Pic}(X)$ *and the second Betti number* $\mathrm{b}_2(X)$ *are equal.*

5. $\mathrm{Pic}(X)$ *is torsion-free.*

6. For any effective divisor D *on* X*,*

$$(\mathrm{c}_2(X)\cdot D)_X = 6\,\chi(\mathcal{O}_D) + 6\,\chi(\mathcal{O}_D(D)) - 2\,(D^3)_X - ((-K_X)^2\cdot D)_X.$$

Proof. 1. This is an immediate consequence of Kodaira's vanishing theorem [We, VI, 2.4]: $-K_X$ is ample since X is Fano, so $\mathrm{H}^i(X,\mathcal{O}_X) = 0$ for all $i > 0$. Then $\chi(\mathcal{O}_X) = \mathrm{h}^0(\mathcal{O}_X) = 1$.

2. and 3. First of all, we derive a formula for $\chi(\mathcal{O}_X(D))$, where D is any divisor on X. We use intersection theory from [Ha77, A]. By the Riemann-Roch formula [Ha77, A, 4.1],

$$\chi(\mathcal{O}_X(D)) = \deg\left(\mathrm{ch}(\mathcal{O}_X(D))\cdot \mathrm{td}(\mathcal{T}_X)\right)_3. \tag{1}$$

To compute this, note that since X has dimension 3, $\mathrm{A}^i(X) = 0$ for all $i > 3$.

First we turn to the Chern character of $\mathcal{O}_X(D)$. Since $\mathrm{c}_1(\mathcal{O}_X(D)) = D$ by [Ha77, A, 3.C1] and $\mathrm{c}_i(\mathcal{O}_X(D)) = 0$ for $i > 1$ since $\mathcal{O}_X(D)$ has rank 1, we obtain from [Ha77, A, § 4] that

$$\mathrm{ch}(\mathcal{O}_X(D)) = 1 + D + \tfrac{1}{2}\,D^2 + \tfrac{1}{6}\,D^3. \tag{2}$$

Now we compute the Todd class of the tangent sheaf $\mathcal{T}_X$. Since X has dimension 3, $\mathcal{T}_X$ is locally free of rank 3 ([Ha77, p.180]), so $\mathrm{c}_i(X) = \mathrm{c}_i(\mathcal{T}_X) = 0$ for all $i > 3$. Moreover, since $\mathcal{T}_X \cong \Omega_X^*$, $\omega_X \cong \det(\Omega_X)$ and $\mathrm{c}_1(\omega_X) = K_X$, Chern class formalism [Ha77, A, 3.C5] yields $\mathrm{c}_1(X) = -K_X$. Hence we get from [Ha77, A, § 4] that

$$\mathrm{td}(\mathcal{T}_X) = 1 + \tfrac{1}{2}\,(-K_X) + \tfrac{1}{12}\,\big((-K_X)^2 + \mathrm{c}_2(X)\big) + \tfrac{1}{24}\,(-K_X)\cdot \mathrm{c}_2(X). \tag{3}$$

Plugging (2) and (3) into (1) above, we obtain

$$\begin{aligned}\chi(\mathcal{O}_X(D)) \;&=\; \tfrac{1}{24}\,(-K_X\cdot \mathrm{c}_2(X)) + \tfrac{1}{12}\,((-K_X)^2\cdot D) + \tfrac{1}{12}\,(D\cdot \mathrm{c}_2(X)) \\ &\quad + \tfrac{1}{4}\,(-K_X\cdot D^2) + \tfrac{1}{6}\,(D^3).\end{aligned} \tag{4}$$

To verify 2., we choose $D \sim 0$ in (4). Then we obtain $\chi(\mathcal{O}_X) = (1/24)\,(-K_X\cdot \mathrm{c}_2(X))$. By (1) above, $\chi(\mathcal{O}_X) = 1$, and the desired result follows.

In order to prove 3., we have to show that any divisor D on X is linearly equivalent to zero if an only if it is numerically equivalent to zero.

If $D \sim 0$, then $(D\cdot C)_X = 0$ for all curves C on X, i.e., $D \equiv 0$.

Conversely, assume that $D \equiv 0$. Then, using 2. above, we obtain from (4)

$$\chi(\mathcal{O}_X(D)) = 1. \tag{5}$$

On the other hand, since $-K_X$ is ample, $((D-K_X)\cdot C) = (D\cdot C)+(-K_X\cdot C) = (-K_X\cdot C) > 0$ for all curves C on X. Since $\mathrm{NE}(X)$ is closed, Kleiman's criterion [De, 1.27] therefore implies that $D - K_X$ is ample. Thus $\mathrm{h}^i(X, \mathcal{O}_X(D)) = 0$ for all $i > 0$ by Kodaira's vanishing theorem [We, VI, 2.4], and we obtain

$$\chi(\mathcal{O}_X(D)) = \mathrm{h}^0(X, \mathcal{O}_X(D)). \tag{6}$$

By (5) and (6), $\mathrm{h}^0(X, \mathcal{O}_X(D)) = 1$. Hence, by [Ha77, II, 7.7], the complete linear system $|D|$ contains an effective divisor D_0 on X, linearly equivalent to D. Then we can write

$$D_0 = \sum_{i=1}^{k} n_i\, D_i \tag{7}$$

with coefficients $n_i \geq 0$ and prime divisors D_i on X. Since X is projective, there exists an embedding of X into some projective space $\mathbb{P}^N$. Now Bertini's theorem [Ha77, II, 8.18] shows that on cutting X successively with two sufficiently general hyperplanes in $\mathbb{P}^N$, one obtains a curve C_1 in X none of whose components is contained in any of the divisors $D_2, \ldots, D_k$ and which intersects D_1 transversally. We therefore obtain

$$0 = (D\cdot C_1) = (D_0\cdot C_1) = \sum_{i=1}^{k} n_i\,(D_i\cdot C_1) \geq n_1\,(D_1\cdot C_1),$$

where $(D_1\cdot C_1) > 0$. Hence $n_1 = 0$. In the same manner, we see that all the other coefficients n_i in (7) also vanish. This shows that D_0 is linearly equivalent to the zero divisor on X, and we are done.

The second part of the assertion is immediate from this: Since $\sim$ and $\equiv$ are the same,

$$\mathrm{N}^1(X)_{\mathbb{Z}} = (\mathrm{Ca}(X)/\equiv) \cong (\mathrm{Ca}(X)/\sim) = \mathrm{Pic}(X).$$

4. The exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

on X induces an exact sequence of cohomology groups

$$\mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X^*) \longrightarrow \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^2(X, \mathcal{O}_X).$$

Here, $\mathrm{H}^1(X, \mathcal{O}_X)$ and $\mathrm{H}^2(X, \mathcal{O}_X)$ vanish by 1., so the arrow in the middle is an isomorphism. By [Ha77, III, Ex. 4.5], $\mathrm{H}^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$. We conclude $\mathrm{Pic}(X) \cong \mathrm{H}^2(X, \mathbb{Z})$. Combining this with our result in 3., we get $\mathrm{N}^1(X)_{\mathbb{Z}} \cong \mathrm{Pic}(X) \cong \mathrm{H}^2(X, \mathbb{Z})$. Hence these groups have the same rank.

5. Let D be a torsion element of $\mathrm{Pic}(X)$. Then $\alpha\, D \sim 0$ for some nonzero integer α. Hence we get $\alpha\,(D\cdot C) = (\alpha\, D\cdot C) = 0$ and thus $(D\cdot C) = 0$ for all curves C on X, i.e., $D \equiv 0$. By 3., this means $D \sim 0$ and D is zero in $\mathrm{Pic}(X)$.

6. This is just another application of the Riemann-Roch formula. Let D be an effective divisor on X. Using $(-K_X \cdot \mathrm{c}_2(X)) = 24$ from 2., we get from (4) above

$$\begin{aligned}\chi(\mathcal{O}_X(D)) &= 1 + \tfrac{1}{12}\,((-K_X)^2 \cdot D) + \tfrac{1}{12}\,(D \cdot \mathrm{c}_2(X)) + \tfrac{1}{4}\,(-K_X \cdot D^2) + \tfrac{1}{6}\,(D^3),\\ \chi(\mathcal{O}_X(-D)) &= 1 - \tfrac{1}{12}\,((-K_X)^2 \cdot D) - \tfrac{1}{12}\,(D \cdot \mathrm{c}_2(X)) + \tfrac{1}{4}\,(-K_X \cdot D^2) - \tfrac{1}{6}\,(D^3).\end{aligned}$$

We obtain from this

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(-D)) = \tfrac{1}{6}\,((-K_X)^2 \cdot D) + \tfrac{1}{6}\,(D \cdot \mathrm{c}_2(X)) + \tfrac{1}{3}\,(D^3). \tag{8}$$

The standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

By [Ha77, III, Ex. 5.1], we deduce from these sequences

$$\begin{aligned}\chi(\mathcal{O}_X(-D)) &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_D),\\ \chi(\mathcal{O}_X(D)) &= \chi(\mathcal{O}_X) + \chi(\mathcal{O}_D(D)).\end{aligned}$$

We obtain from this

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(-D)) = \chi(\mathcal{O}_D(D)) + \chi(\mathcal{O}_D). \tag{9}$$

Combining (8) and (9) we finally get

$$6\,\chi(\mathcal{O}_D) + 6\,\chi(\mathcal{O}_D(D)) = ((-K_X)^2 \cdot D) + (D \cdot \mathrm{c}_2(X)) + 2\,(D^3),$$

which is the claimed identity. $\square$

Next, we state the Contraction Theorem, which is Theorem 3.1 and 3.2 in [Mo]. It explains the geometric meaning of extremal rays on a Fano threefold.

Theorem 2.3 (Contraction Theorem)**.** *Let X be a Fano threefold, and R an extremal ray of X with associated extremal rational curve ℓ_R. Then there exists a corresponding morphism $f : X \longrightarrow Y$ to a projective variety Y, and with the following properties:*

1. $f_* \mathcal{O}_X \cong \mathcal{O}_Y$.
2. *For any irreducible reduced curve C on X, $[C] \in R$ if and only if $f(C)$ is a point.*

3. There is an exact sequence

$$0 \longrightarrow \operatorname{Pic}(Y) \xrightarrow{f^*} \operatorname{Pic}(X) \xrightarrow{(\,\cdot\,\ell_R)} \mathbb{Z},$$

where $(\,\cdot\,\ell_R)(D) = (D \cdot \ell_R)_X$ *for* $D \in \operatorname{Pic}(X)$.

Moreover, the higher direct images $\mathrm{R}^i f_* \mathcal{O}_X$ *vanish for all* $i > 0$.

Such a morphism f *is unique up to an isomorphism.*

The morphism f of the theorem is called the *Mori contraction* of R and denoted by $\operatorname{cont}_R : X \longrightarrow Y$.

Once we have the classification of extremal rays at our disposal we will be able to show in corollary 2.15 that the third map in the exact sequence of the theorem is actually surjective.

We can use this theorem to deduce two useful properties of Mori contractions.

Corollary 2.4. *Let* X *be a Fano threefold, and* $f : X \longrightarrow Y$ *the contraction corresponding to some extremal ray* R *of* X*. Then*

$$\mathrm{h}^i(X, \mathcal{O}_X) = \mathrm{h}^i(Y, \mathcal{O}_Y)$$

for all $i > 0$.

Proof. By (2.3) above, $\mathrm{R}^i f_* \mathcal{O}_X = 0$ for all $i > 0$. By the Leray spectral sequence ([Ha77, III, Ex. 8.1]), we have isomorphisms $\mathrm{H}^i(X, \mathcal{O}_X) \cong \mathrm{H}^i(Y, f_* \mathcal{O}_X)$ for all $i > 0$. Since f is a Mori contraction, $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ by (2.3) above. Hence $\mathrm{h}^i(X, \mathcal{O}_X) = \mathrm{h}^i(Y, \mathcal{O}_Y)$ for all $i > 0$. □

Corollary 2.5. *Let* X *be a Fano threefold, and* $f : X \longrightarrow Y$ *the contraction corresponding to some extremal ray* R *of* X*. Let* L *be a divisor on* Y*, and* $D = f^* L$*. Then*

$$\chi(\mathcal{O}_D) = 1 - \chi(\mathcal{O}_Y(-L))$$

and

$$\chi(\mathcal{O}_D(D)) = \chi(\mathcal{O}_Y(L)) - 1.$$

Proof. We start with the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

which induces an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

By [Ha77, III, Ex. 5.1], these sequences imply

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) \tag{10}$$

and

$$\chi\left(\mathcal{O}_D(D)\right) = \chi\left(\mathcal{O}_X(D)\right) - \chi\left(\mathcal{O}_X\right). \tag{11}$$

Since X is Fano, $\chi(\mathcal{O}_X) = 1$ by (2.2).

Since $D = f^* L$, we obtain, using the projection formula [Ha77, III, Ex. 8.3],

$$\mathrm{R}^i f_* \mathcal{O}_X(D) \cong \left(\mathrm{R}^i f_*\right) f^* \mathcal{O}_Y(L) \cong \mathcal{O}_Y(L) \otimes \mathrm{R}^i f_* \mathcal{O}_X$$

for all $i \geq 0$. Since $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ and $\mathrm{R}^i f_* \mathcal{O}_X = 0$ for $i > 0$ by (2.3), this implies $f_* \mathcal{O}_X(D) \cong \mathcal{O}_Y(L)$ and $\mathrm{R}^i f_* \mathcal{O}_X(D) = 0$ for all $i > 0$. Hence, by the Leray spectral sequence ([Ha77, III, Ex. 8.1]),

$$\mathrm{H}^i(X, \mathcal{O}_X(D)) \cong \mathrm{H}^i(Y, f_* \mathcal{O}_X(D)) \cong \mathrm{H}^i(Y, \mathcal{O}_Y(L)).$$

Thus we get $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_Y(L))$. Likewise, $\chi(\mathcal{O}_X(-D)) = \chi(\mathcal{O}_Y(-L))$.

By (10) and (11) above, we therefore obtain $\chi(\mathcal{O}_D) = 1 - \chi(\mathcal{O}_Y(-L))$ and $\chi(\mathcal{O}_D(D)) = \chi(\mathcal{O}_Y(L)) - 1$. □

2.3 Classification of extremal rays on primitive Fano threefolds

This section gives a complete classification of extremal rays on a primitive Fano threefold. It is based on Theorem 3.3 in [Mo] and on paragraph 3.3 in [Mi]. We will slightly adapt and present the respective results in theorems 2.7, 2.11 and 2.14 for later use.

Let X be a primitive Fano threefold with $\mathrm{b}_2(X) \geq 2$. By the Contraction Theorem 2.3 there exists, to any extremal ray R of X, a corresponding contraction

$$f = \mathrm{cont}_R : X \longrightarrow Y$$

to a projective variety Y.

We will now classify extremal rays R of X by specifying the following data: the dimension of Y, the morphism f, the length μ of R and its associated extremal rational curve ℓ on X.

Recall from section 2.2 that $\mu = \min\{(-K_X \cdot C) \mid \text{C is a rational curve such that } [C] \in R\}$, and that ℓ is an extremal rational curve on X such that $[\ell] \in R$ and $(-K_X \cdot \ell) = \mu$.

By [Mo, Thm. 3.5], the cases $\dim(Y) = 3$, 2, 1 or 0 are possible. However, the case $\dim(Y) = 0$ cannot occur: For in this case $\mathrm{Pic}(Y) = 0$, so the exact sequence of (2.3) yields an injective homomorphism $\mathrm{Pic}(X) \longrightarrow \mathbb{Z}$. This implies $\mathrm{b}_2(X) \leq 1$ by (2.2 (4)), which is a contradiction. Hence we have to distinguish the cases $\dim(Y) = 3$, 2 or 1.

Case $\dim(\mathbf{Y}) = \mathbf{3}$. By [Mo, Thm. 3.3 and Cor. 3.4.1], there is an irreducible reduced divisor D on X such that $f|_{X-D}$ is an isomorphism and $\dim(f(D)) \leq 1$. Such D is uniquely determined by R and called the *exceptional divisor* of R. Correspondingly, f is called a *divisorial contraction* of X. Note that it is birational. Moreover, f is the blowing-up of Y along the subvariety $f(D)$ with its reduced structure.

By [Mo, Thm. 3.3], there are five different types of extremal rays on X. We will denote them by E_1, E_2, E_3, E_4 and E_5 in such a way that type E_i corresponds to case (3.3.i) in [Mo, Thm. 3.3]. This theorem gives an explicit characterisation of the rays $E_1, \ldots, E_5$, and its statements are summarised in the table given in theorem 2.7 on page 16. However, since we are dealing with primitive Fano threefolds only, we can enhance the statement about rays of type E_1 by means of the following lemma:

Lemma 2.6. *If R is of type E_1, (3.3.1) in [Mo, Thm. 3.3] says that $C = f(D)$ is a nonsingular irreducible curve, Y is nonsingular and $f|_D : D \longrightarrow C$ is a $\mathbb{P}^1$-bundle. Moreover, since X is primitive the following holds:*

1. $C \cong \mathbb{P}^1$
2. $\mathcal{N}^*_{C/Y} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$
3. $D \cong \mathbb{P}^1 \times \mathbb{P}^1$
4. $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)$

Proof. Since X is the blowing-up of Y along C, (B.4) applies. Since $-K_X$ is ample, we obtain the following inequality from (B.4 (4)), which we will need later:

$$(-K_Y \cdot C) + 2 - 2\,\mathrm{g}(C) > 0 \tag{12}$$

Here, $\mathrm{g}(C)$ is the genus of the curve C. Moreover, $\pi = f|_D : D \longrightarrow C$ is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{N}^*_{C/Y})$, $\mathcal{O}_D(-D)$ corresponds to $\mathcal{O}_D(1)$, and

$$-K_X \sim f^*(-K_Y) - D, \tag{13}$$

by (B.4 (2) and (1)).

Twisting the bundle $\mathbb{P}(\mathcal{N}^*_{C/Y})$ by $\mathcal{O}_C(-K_Y)$ gives us a projective space bundle $\pi' : D' = \mathbb{P}(\mathcal{E}) \longrightarrow C$, where

$$\begin{aligned} \mathcal{E} &= \mathcal{N}^*_{C/Y} \otimes \mathcal{O}_C(-K_Y) \\ &\cong \mathcal{N}^*_{C/Y} \otimes \mathcal{O}_C(-K_C) \otimes \textstyle\bigwedge^2 \mathcal{N}_{C/Y} \\ &\cong \mathcal{O}_C(-K_C) \otimes \mathcal{N}_{C/Y}. \end{aligned} \tag{14}$$

Here we used the adjunction formula for $-K_C$ on Y, and formula [Ha77, II, 5.16(b)]. We apply [Ha77, II, 7.9]: There is an isomorphism

$$\varphi : D' \xrightarrow{\cong} D,$$

commuting with the projections π and π' to C. Moreover, using (13), we obtain

$$\begin{aligned} \mathcal{O}_{D'}(1) &\cong \varphi^* \mathcal{O}_D(1) \otimes \pi'^* \mathcal{O}_C(-K_Y) \\ &\cong \varphi^* \mathcal{O}_D(1) \otimes (\varphi^* \circ \pi^*) \mathcal{O}_C(-K_Y) \\ &\cong \varphi^* (\mathcal{O}_D(1) \otimes \pi^* \mathcal{O}_C(-K_Y)) \\ &\cong \varphi^* (\mathcal{O}_D(-D) \otimes f^* \mathcal{O}_Y(-K_Y)) \\ &\cong \varphi^* \mathcal{O}_D(-K_X). \end{aligned} \tag{15}$$

Now since $-K_X$ is ample, so is $\mathcal{O}_D(-K_X)$ by [Ha70, I, 4.1]. Since φ is an isomorphism, this calculation therefore shows that $\mathcal{O}_{D'}(1)$ is ample. Since $D' = \mathbb{P}(\mathcal{E})$, this implies that $\mathcal{E}$ is an ample sheaf on C, by [Ha70, III, 1.1].

In order to find out more about $\mathcal{E}$, we need the following

Claim. $(-K_Y \cdot C) \leq 0$. In particular, C is isomorphic to $\mathbb{P}^1$.

Note that the claim already proves 1. To prove the claim, let us suppose to the contrary, i.e. $(-K_Y \cdot C) > 0$. We will show that $-K_Y$ is ample. For then, Y is Fano and hence X is not primitive since it is the blowing-up of Y along the nonsingular irreducible curve C, which is a contradiction.

By [Ha77, II, 7.5], it suffices to prove that $-m\,K_Y$ is ample, for some $m > 0$. We will do this using criterion [Ha70, I, 4.6], which requires us to verify that $-m\,K_Y$ is generated by its global sections and has positive degree on every irreducible reduced curve on Y.

So let us prove first that $-m\,K_Y$ is generated by its global sections for sufficiently large m:

Since $-K_X$ is ample, $-n\,K_X$ is very ample for sufficiently large n ([Ha77, II, 7.6]). Hence the linear system $|-n\,K_X|$ has no base points ([Ha77, II, 7.1 and 7.8]). Since $f|_{X-D}$ is an isomorphism, it follows from (13) that $-K_X|_{X-D} \cong -K_Y|_{Y-C}$, so

$$\mathrm{H}^0(X - D, -n\,K_X|_{X-D}) \cong \mathrm{H}^0(Y - C, -n\,K_Y|_{Y-C}).$$

This shows that $|-n\,K_Y|$ has no fixed components or no base points outside of C.

Next, we check that C is not contained in the base locus of $|-n\,K_Y|$. The isomorphism

$$f^*\mathcal{O}_Y(-K_Y) \xrightarrow{\cong} \mathcal{O}_X(-K_X + D)$$

from (13) induces an isomorphism

$$f_*\, f^*\, \mathcal{O}_Y(-K_Y) \xrightarrow{\cong} f_*\, \mathcal{O}_X(-K_X + D).$$

Since $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ by (2.3), the projection formula [Ha77, II, Ex. 5.1 (d)] yields $f_* f^* \mathcal{O}_Y(-K_Y) \cong \mathcal{O}_Y(-K_Y) \otimes f_* \mathcal{O}_X \cong \mathcal{O}_Y(-K_Y)$. Hence we arrive at an isomorphism

$$\mathcal{O}_Y(-K_Y) \xrightarrow{\cong} f_* \mathcal{O}_X(-K_X + D).$$

We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(Y, \mathcal{O}_Y(-K_Y)) & \xrightarrow{\cong} & \mathrm{H}^0(Y, f_*\mathcal{O}_X(-K_X + D)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(C, \mathcal{O}_C(-K_Y)) & \xrightarrow{\cong} & \mathrm{H}^0(C, f_*\mathcal{O}_D(-K_X + D)) \end{array} \tag{16}$$

where the vertical maps are the canonical restriction maps. By definition of the direct image functor,

$$\mathrm{H}^0(Y, f_*\mathcal{O}_X(-K_X + D)) = \mathrm{H}^0(X, \mathcal{O}_X(-K_X + D))$$

and

$$\begin{aligned} \mathrm{H}^0(C, f_*\mathcal{O}_D(-K_X + D)) &= \mathrm{H}^0(C, (f|_D)_* \mathcal{O}_D(-K_X + D)) \\ &= \mathrm{H}^0(D, \mathcal{O}_D(-K_X + D)). \end{aligned}$$

Thus diagram (16) takes the following form:

$$\begin{array}{ccc} \mathrm{H}^0(Y, \mathcal{O}_Y(-K_Y)) & \xrightarrow{\cong} & \mathrm{H}^0(X, \mathcal{O}_X(-K_X + D)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(C, \mathcal{O}_C(-K_Y)) & \xrightarrow{\cong} & \mathrm{H}^0(D, \mathcal{O}_D(-K_X + D)) \end{array} \tag{17}$$

Tensoring the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with $\mathcal{O}_X(-K_X + D)$ gives an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_X(-K_X + D) \longrightarrow \mathcal{O}_D(-K_X + D) \longrightarrow 0.$$

Taking cohomology yields an exact sequence

$$\mathrm{H}^0(X, \mathcal{O}_X(-K_X + D)) \longrightarrow \mathrm{H}^0(D, \mathcal{O}_D(-K_X + D)) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X(-K_X)).$$

The first map in this sequence is the canonical restriction map, and therefore equals the vertical map on the right in diagram (17). By Kodaira's vanishing theorem [We, VI, 2.4], $\mathrm{H}^1(X, \mathcal{O}_X(-K_X)) = 0$ since $-K_X$ is ample. Hence we see that the vertical map on the right in diagram (17) is surjective, so we conclude that the canonical restriction map

$$\mathrm{H}^0(Y, \mathcal{O}_Y(-K_Y)) \longrightarrow \mathrm{H}^0(C, \mathcal{O}_C(-K_Y)) \tag{18}$$

is also surjective.
By the Riemann-Roch formula [Ha77, IV, 1.3],

$$h^0(C, \mathcal{O}_C(-K_Y)) \geq \chi(\mathcal{O}_C(-K_Y)) = \deg_C(-K_Y) - (g(C) - 1).$$

By our assumption and [De, (1.3)], $\deg_C(-K_Y) = (-K_Y \cdot C) > 0$. By inequality (12), $(-K_Y \cdot C) > 2\,(g(C) - 1)$. Since $g(C) \geq 0$ always, this shows that

$$h^0(C, \mathcal{O}_C(-K_Y)) > 0.$$

By surjectivity of (18), we conclude from this that $\mathcal{O}_Y(-K_Y)$ and hence also $\mathcal{O}_Y(-n\,K_Y)$ has a global section that does not vanish along C, i.e., C is not contained in the base locus of $|-n\,K_Y|$.

As we have seen above, $|-n\,K_Y|$ has no fixed components or no base points outside of C.

Since C is irreducible, we conclude from these two facts that $|-n\,K_Y|$ can have base points only on C. Hence it can have only finitely many base points. Thus, by Zariski's theorem [Za, Thm. 6.2], $|-n'\,n\,K_Y|$ has no base points for sufficiently large n'. In other words ([Ha77, II, 7.8]), $-m\,K_Y$ is generated by its global sections for sufficiently large m. This is the first condition required by criterion [Ha70, I, 4.6].

Recall that the second condition requires $-m\,K_Y$ to have positive degree on every irreducible reduced curve on Y:

By our assumption, $(-K_Y \cdot C)_Y > 0$. Now consider any irreducible reduced curve Z on Y, distinct from C. It meets C in at most finitely many points. Using $-K_X \sim f^*(-K_Y) - D$ from (13), we therefore obtain

$$(-K_Y \cdot Z)_Y = (f^*(-K_Y) \cdot \tilde{Z})_X = (-K_X \cdot \tilde{Z})_X + (D \cdot \tilde{Z})_X > 0$$

since $-K_X$ is ample and D is effective, where $\tilde{Z}$ denotes the strict transform of Z under the blowing-up f. By [De, (1.3)], it follows that $-K_Y$ and hence also $-m\,K_Y$ has positive degree on every irreducible reduced curve on Y.

Thus the conditions of criterion [Ha70, I, 4.6] are satisfied and we conclude that $-K_Y$ is Fano, which proves the first assertion of the claim.

To prove that $C \cong \mathbb{P}^1$, we combine inequality (12) with the first assertion of the claim and obtain $2\,g(C) - 2 < (-K_Y \cdot C) \leq 0$. Hence $g(C) = 0$, and, by [Ha77, IV, 1.3.5], C is rational. Since C is nonsingular, it is therefore isomorphic to $\mathbb{P}^1$, by [Ha77, I, 6.12].

This completes the proof of the claim.

Now we come back to the sheaf $\mathcal{E} = \mathcal{O}_C(-K_C) \otimes \mathcal{N}_{C/Y}$. Since C has codimension 2 in Y, it is locally free of rank 2 on C ([Ha77, II, 8.17]). Since $C \cong \mathbb{P}^1$ by the claim, $\mathcal{E}$ is decomposable by [Ha77, V, 2.14]. Hence we can write

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

with coefficients $a, b \in \mathbb{Z}$.

Since $\mathcal{E}$ is ample, each of its summands is ample, by [Ha70, III, 1.8]. Hence we must have $a, b > 0$ by [Ha77, II, 7.6.1]. Using formula [Ha77, II, 5.16 (d)], we therefore get

$$\begin{aligned}
\deg_C(\mathcal{E}) &= \deg_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)\right) \\
&= \deg \bigwedge\nolimits^2 \left(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)\right) \\
&= \deg\left(\mathcal{O}_{\mathbb{P}^1}(a+b)\right) \\
&= a+b \\
&\geq 2.
\end{aligned}$$

On the other hand, by (B.2 (3)), (15), (B.4 (4)) and the claim above,

$$\begin{aligned}
\deg_C(\mathcal{E}) &= \mathrm{c}_1(\mathcal{O}_{D'}(1))^2 \\
&= \mathrm{c}_1\left(\varphi^*(\mathcal{O}_D(-K_X)\right)^2 \\
&= ((-K_X|_D)^2)_D \\
&= ((-K_X)^2 \cdot D)_X \\
&= (-K_Y \cdot C) + 2 - \mathrm{g}(C) \\
&\leq 2.
\end{aligned}$$

We conclude from these inequalities that $2 = \deg_C(\mathcal{E}) = a + b$. Since $a, b > 0$, we must therefore have $a = b = 1$. Whence

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

So far, we have proved that $D \cong D' = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$. Twisting by $\mathcal{O}_{\mathbb{P}^1}(-1)$, we see by [Ha77, V, 2.2] that D is isomorphic to the ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1$, which is nothing but $\mathbb{P}^1 \times \mathbb{P}^1$ ([Ha77, V, 2.11.1]). This proves 3.

Now we determine the sheaves $\mathcal{N}^*_{C/Y}$ and $\mathcal{O}_D(D)$. Since $C \cong \mathbb{P}^1$ by the claim, we obtain from (14)

$$\mathcal{N}^*_{C/Y} \cong \mathcal{E}^* \otimes \omega_C^{-1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^1}(2) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

This proves 2. Since $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, we can write $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n)$ with $m, n \in \mathbb{Z}$. Then we obtain, using (B.2 (3)),

$$\begin{aligned}
2\,m\,n &= c_1\left(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m,n)\right)^2 \\
&= c_1\left(\mathcal{O}_D(D)\right)^2 \\
&= c_1\left(\mathcal{O}_D(-1)\right)^2 \\
&= c_1\left(\mathcal{O}_D(1)\right)^2 \\
&= \deg_C\left(\mathcal{N}^*_{C/Y}\right) \\
&= \deg_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) \\
&= 2.
\end{aligned}$$

Since $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$, $\mathcal{O}_D(D)$ has no nontrivial global sections, by [Ha77, II, 7.11]. This implies $m < 0$ or $n < 0$, by [Ha77, II, 7.6.2]. Hence we must have $m = n = -1$, i.e., $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$. □

We summarise our results in the following

Theorem 2.7. *Extremal rays R of X with corresponding contraction $f = \mathrm{cont}_R : X \longrightarrow Y$ to a projective variety Y of dimension 3 satisfy the following:*

f is birational with irreducible reduced exceptional divisor D such that $f|_{X-D}$ is an isomorphism. In particular, D is uniquely determined by R and f is the blowing-up of Y along the subvariety $f(D)$ with its reduced structure. We distinguish five types of extremal rays:

type of R	f *and* D	μ	ℓ
E_1	Y *is nonsingular,* $C = f(D) \cong \mathbb{P}^1$, $\mathcal{N}^*_{C/Y} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$	1	*an exceptional line*
E_2	Y *is nonsingular,* $f(D)$ *is a point,* $D \cong \mathbb{P}^2$, $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$,	2	*a line on* D
E_3	$f(D)$ *is a point,* $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$, $P \times \mathbb{P}^1$ *and* $\mathbb{P}^1 \times Q$ *are numerically equivalent on* X *for all* $P, Q \in \mathbb{P}^1$	1	$P \times \mathbb{P}^1$ *or* $\mathbb{P}^1 \times Q$ *on* D $(P, Q \in \mathbb{P}^1)$
E_4	$f(D)$ *is a point,* D *is isomorphic to an irreducible reduced singular quadric surface in* $\mathbb{P}^3$, $\mathcal{O}_D(D) \cong \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$	1	*a generator of* D
E_5	$f(D)$ *is a point,* $D \cong \mathbb{P}^2$, $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^3}(-2)$	1	*a line on* D

In particular, the divisor $D|_D$ on D is always negative.

We finish the consideration of this case with two useful corollaries which we will need later on.

Corollary 2.8. *Let X be a primitive Fano threefold with an extremal ray R of type E_1, E_2, E_3, E_4 or E_5, and corresponding exceptional divisor D. Then any curve on D can move on D.*

Proof. The claimed assertion is clear at least in case R is of type E_1, E_2, E_3 or E_5. For then, D is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ by (2.7), and the complete linear system of any nonzero effective divisor on these surfaces has positive dimension, by [Ha77, II, 7.8.3].

It remains to check the case R is of type E_4. Then D is isomorphic to an irreducible reduced singular quadric surface in $\mathbb{P}^3$, by (2.7). By the classification of quadric hypersurfaces in [Ha77, I, Ex. 5.12], we may assume that D is defined in $\mathbb{P}^3$ by the polynomial $f = X_0^2 + \ldots + X_r^2$, where $0 \leq r \leq 3$. Since D is irreducible and singular, parts (b) and (c) of this exercise imply $r = 2$. Parts (c) and (d) then show that D is isomorphic to the projective cone over the nonsingular irreducible quadric curve C given by the polynomial $f = X_0^2 + X_1^2 + X_2^2$ in $\mathbb{P}^2$, with vertex a point P in $\mathbb{P}^3$ ([Ha77, I, Ex. 2.10]).

By [Ha77, II, Ex. 6.3 (a)], the corresponding projection $\pi : D - P \longrightarrow C$ induces an isomorphism $\pi^* : \mathrm{Cl}(C) \longrightarrow \mathrm{Cl}(D)$ of Weil divisor class groups. This shows that any curve Z on D is the pull back of some nonzero effective divisor E on C.

Since C is a curve of degree 2 on $\mathbb{P}^2$, $\mathrm{g}(C) = 0$ by [Ha77, V, 1.5.1]. Since C is nonsingular, it follows $C \cong \mathbb{P}^1$, by [Ha77, IV, 1.3.5 and I, 6.12]. This shows that E can move on C, by [Ha77, II, 7.8.3].

Consequently, Z can move in the pull back of the complete linear system of E. $\square$

Corollary 2.9. *Let X be a Fano threefold with an extremal ray R of type E_2, E_3, E_4 or E_5. Then the corresponding exceptional divisor D is mapped to a point by every morphism $g : X \longrightarrow \mathbb{P}^1$, if there is any.*

Proof. Let us consider the restriction $h = g|_D : D \longrightarrow \mathbb{P}^1$. By [Ha77, III, 11.5], it has a Stein factorisation into a morphism with connected fibres from D onto some curve, followed by a finite morphism from this curve onto $\mathbb{P}^1$. Hence, in order to show that h maps D to a point, we may without loss of generality assume that h has connected fibres. By [De, Prop. 1.14], h is uniquely determined by its relative cone of curves, which, in our case, is a nontrivial extremal subcone of $\mathrm{NE}(D)$.

Now, if R is of type E_2, E_4 or E_5, D is isomorphic to $\mathbb{P}^2$ or the the quadric cone in $\mathbb{P}^3$, by (2.7). Hence $\rho(D) = 1$ ([Ha77, II, Ex. 6.5 (c)]). Then $\mathrm{N}_1(D)_{\mathbb{R}}$ has dimension 1 and the only nontrivial extremal subcone of $\mathrm{NE}(D)$ is $\mathrm{NE}(D)$ itself. Thus h maps D to a point.

Likewise, if R is of type E_3, we have an isomorphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\cong} D$, by (2.7). Then $\rho(D) = 2$, i.e. $\mathrm{N}_1(D)_{\mathbb{R}}$ has dimension 2 and $\mathrm{NE}(D)$ has precisely 3 nontrivial extremal subcones: $\mathrm{NE}(D)$ itself and its two edges. Correspondingly, h maps D to a point, or it corresponds, via φ, to the canonical projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ onto the i-th factor. In the latter case we may assume without loss of generality that $\pi_1 = h \circ \varphi$. Fix arbitrary points $P, Q \in \mathbb{P}^1$.

On the one hand,

$$\begin{aligned}
\big(\varphi\,(P \times \mathbb{P}^1) \cdot g^{-1}(P)\big)_X &= \big((\varphi \circ \pi_1^{-1})(P)) \cdot g^{-1}(P)\big)_X \\
&= \big(h^{-1}(P) \cdot g^{-1}(P)\big)_X \\
&= 0.
\end{aligned}$$

On the other hand, we know from (2.7) that $\varphi\,(P \times \mathbb{P}^1)$ and $\varphi\,(\mathbb{P}^1 \times Q)$, considered as curves on X, are numerically equivalent. Thus we get

$$\begin{aligned}
\big(\varphi\,(P \times \mathbb{P}^1) \cdot g^{-1}(P)\big)_X &= \big(\varphi\,(\mathbb{P}^1 \times Q) \cdot g^{-1}(P)\big)_X \\
&= \big(\varphi\,(\mathbb{P}^1 \times Q) \cdot h^{-1}(P)\big)_D \\
&= \big((\varphi \circ \pi_2^{-1})(Q) \cdot (\varphi \circ \pi_1^{-1})(P)\big)_D \\
&= \big(\pi_2^{-1}(Q) \cdot \pi_2^{-1}(P)\big)_{\mathbb{P}^1 \times \mathbb{P}^1} \\
&= 1,
\end{aligned}$$

which is a contradiction. Hence this latter case cannot occur. $\square$

Case $\mathbf{dim(Y) = 2}$. By (3.5.1) of [Mo, Thm. 3.5], Y is nonsingular and $f : X \longrightarrow Y$ is a conic bundle (cf. appendix B). The following result will enable us to characterise this conic bundle more precisely.

Lemma 2.10. *The surface Y is rational.*

Proof. We are going to apply Castelnuovo's criterion [BPV, VI, 2.1].

First, since $f : X \longrightarrow Y$ is a Mori contraction, we obtain for the irregularity $\mathrm{q}(Y)$ of Y $\mathrm{q}(Y) = \mathrm{h}^1(Y, \mathcal{O}_Y) = \mathrm{h}^1(X, \mathcal{O}_X)$, by (2.4). Since X is Fano, $\mathrm{h}^1(X, \mathcal{O}_X) = 0$ by (2.2 (1)). Hence $\mathrm{q}(Y) = 0$.

Now we consider the second plurigenus $P_2(Y) = \mathrm{h}^0(Y, 2\,K_Y)$ of Y and show that it also vanishes.

Since Y is projective, there exists a very ample divisor H on Y. Then it will be enough to prove that $(K_Y \cdot H) < 0$. For if we suppose that $\mathrm{h}^0(Y, 2\,K_Y) \neq 0$, the linear system $|2\,K_Y|$ contains an effective divisor D linearly equivalent to $2\,K_Y$ ([Ha77, II, 7.7 (a)]). But then, $2\,(K_Y \cdot H) = (2\,K_Y \cdot H) = (D \cdot H) \geq 0$ since H is ample, which is a contradiction.

Using the relation $-4\,K_Y \equiv f_*\,(-K_X)^2 + \Delta_f$ from (B.3), we obtain by the projection formula [De, 1.10],

$$\begin{aligned}
-4\,(K_Y \cdot H) &= ((-4\,K_Y) \cdot H) \\
&= \big(f_*\,(-K_X)^2 \cdot H\big) + (\Delta_f \cdot H) \\
&= \big((-K_X)^2 \cdot f^*\,H\big)_X + (\Delta_f \cdot H)_Y\,.
\end{aligned}$$

Ampleness of $-K_X$ implies that $((-K_X)^2 \cdot f^*\,H) > 0$. Ampleness of H implies that $(\Delta_f \cdot H) \geq 0$, since Δ_f is an effective divisor by (B.3 (3)). Hence $(K_Y \cdot H) < 0$.

As a result, $\mathrm{q}(Y) = P_2(Y) = 0$, and the lemma is proved. $\square$

We can distinguish whether the discriminant locus Δ_f of the conic bundle $f : X \longrightarrow Y$ is empty or not.

In the latter case, f has a degenerate fibre, by (B.3 (4)). In this case, we denote the type of the extremal ray R by C_1.

In the first case, all the fibres of f are isomorphic to $\mathbb{P}^1$, by (B.3 (4)). Thus the local-triviality theorem [BPV, I, 10.1] shows that f is a holomorphic projective fibre bundle with fibre $\mathbb{P}^1$. As it is explained in [El, § 3], if the Brauer group $\mathrm{Br}(Y)$ is trivial then the bundle f is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{E})$ associated to some locally free sheaf $\mathcal{E}$ of rank 2 on Y. But since Y is rational by (2.10) above, $\mathrm{Br}(Y) = 0$ (cf. [Gr]). In this case, we denote the type of the extremal ray R by C_2.

We state our results in the following

Theorem 2.11. *Extremal rays R of X with corresponding contraction $f = \mathrm{cont}_R : X \longrightarrow Y$ to a projective variety Y of dimension 2 satisfy the following:*

f is a conic bundle, and Y is a nonsingular rational surface. We distinguish two types of extremal rays:

type of R	f	μ	ℓ
C_1	*Δ_f is a nonzero effective divisor, and f has a degenerate fibre*	*1*	*an irreducible component of a reducible fibre or a reduced part of a multiple fibre*
C_2	*Δ_f is zero, and f is a $\mathbb{P}^1$-bundle associated to some locally free sheaf of rank 2 on Y*	*2*	*a fibre of the $\mathbb{P}^1$-bundle*

Corollary 2.12. *Let X be a Fano threefold and $f : X \longrightarrow Y$ the conic bundle associated to some extremal ray R of type C_1 or C_2 of X, of length μ and with associated extremal rational curve ℓ. Then all fibres of f are numerically equivalent to $(2/\mu)\,\ell$.*

Proof. If R is of type C_1, by (2.11), any degenerate fibre of f is numerically equivalent to $2\,\ell$, and $\mu = 1$. But all fibres of f are numerically equivalent. If R is of type C_2, any fibre is numerically equivalent to ℓ, and $\mu = 2$. □

Case $\mathbf{dim(Y) = 1}$. By (3.5.2) of [Mo, Thm. 3.5], Y is nonsingular and $f : X \longrightarrow Y$ is a *del Pezzo fibration*, i.e. every fibre of f is an irreducible reduced surface which has negative canonical sheaf. The following result is similar to the case $\dim(Y) = 2$ above.

Lemma 2.13. *Y is isomorphic to $\mathbb{P}^1$.*

Proof. Since Y is nonsingular it is enough to show that $\mathrm{g}(Y) = 0$ ([Ha77, IV, 1.3.5 and I, 6.12]). By [Ha77, IV, 1.1], $\mathrm{g}(Y) = \mathrm{h}^1(Y, \mathcal{O}_Y)$. Hence we can argue in complete analogy to the first part of the proof of (2.10) above: Since $f : X \longrightarrow Y$ is a Mori contraction, we obtain $\mathrm{g}(Y) = \mathrm{h}^1(Y, \mathcal{O}_Y) = \mathrm{h}^1(X, \mathcal{O}_X)$, by (2.4). Since X is Fano, $\mathrm{h}^1(X, \mathcal{O}_X) = 0$ by (2.2 (1)). Hence $\mathrm{g}(Y) = 0$. $\square$

We summarise our results in the following

Theorem 2.14. *Extremal rays R of X with corresponding contraction $f = \mathrm{cont}_R : X \longrightarrow Y$ to a projective variety Y of dimension 1 satisfy the following:*

Y is isomorphic to $\mathbb{P}^1$. For all $t \in \mathbb{P}^1$, the fibre X_t of f is an irreducible reduced surface such that $-K_{X_t}$ is ample. We distinguish three types of extremal rays:

type of R	f	μ	ℓ
D_1	*the generic fibre satisfies* $1 \leq (K_{X_t})^2 \leq 6$	*1*	*a line on a fibre*
D_2	*X is embedded in a $\mathbb{P}^3$-bundle P over $\mathbb{P}^1$ such that X_t is isomorphic to an irreducible reduced quadric in $P_t \cong \mathbb{P}^3$, and $(K_{X_t})^2 = 8$, for all $t \in \mathbb{P}^1$*	*2*	*a line on a fibre*
D_3	*X is isomorphic to a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, and $(K_{X_t})^2 = 9$, for all $t \in \mathbb{P}^1$*	*3*	*a line on a fibre*

This completes the classification of extremal rays of X. We conclude this chapter by using it to prove that the exact sequence of theorem 2.3 extends to a short exact sequence. This will have a number of useful consequences.

Corollary 2.15. *Let X be a primitive Fano threefold, and R an extremal ray of X with corresponding contraction $f : X \longrightarrow Y$ and associated extremal rational curve ℓ. Then there exists a split exact sequence*

$$0 \longrightarrow \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{(\,\cdot \ell)} \mathbb{Z} \longrightarrow 0,$$

where $(\,\cdot \ell)(D) = (D \cdot \ell)_X$ for $D \in \mathrm{Pic}(X)$. In particular,

$$\rho(X) = \rho(Y) + 1.$$

Proof. By (2.3), we have this exact sequence, but without the zero on the right. It remains to check that $(\,\cdot \ell)$ is surjective, i.e., we have to find a divisor D on X such that $(D \cdot \ell) = \pm 1$.

If R is of type E_1, E_3, E_4, E_5, C_1 or D_1, then $(-K_X \cdot \ell) = \mu = 1$ by (2.7), (2.11) and (2.14).

If R is of type E_2, the exceptional divisor D on X satisfies $D \cong \mathbb{P}^2$, $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$, and ℓ is a line on D, by (2.7). Hence $(D \cdot \ell) = (D|_D \cdot \ell)_D = (\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(-1)) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))_{\mathbb{P}^2} = -1$.

If R is of type C_2, then, by (2.11), X is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ over Y, associated to some locally free sheaf $\mathcal{E}$ of rank 2 on Y. Moreover, ℓ is a fibre of this bundle. Let $\xi = \mathcal{O}_X(1)$. Then $(\xi \cdot \ell)_X = 1$ by (B.1).

If R is of type D_2 or D_3, we first note that the projective bundles appearing in (2.14) are holomorphic fibre bundles by the local-triviality theorem [BPV, I, 10.1] and hence are projective bundles associated to some locally free sheaf on $\mathbb{P}^1$ by [BPV, V, 4.1]. This ensures that they carry a tautological line bundle.

If R is of type D_2, then, by (2.14), X is embedded in a $\mathbb{P}^3$-bundle P over $\mathbb{P}^1$ such that X_t is an irreducible reduced quadric in $P_t \cong \mathbb{P}^3$ for all $t \in \mathbb{P}^3$. Let E be the divisor on X corresponding to $\mathcal{O}_X(1) = \mathcal{O}_P(1) \otimes \mathcal{O}_X$. Since ℓ is any line in $X_t \subset \mathbb{P}^3$, we obtain $(E \cdot \ell) = (\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(1)) \cdot \ell)_{\mathbb{P}^3} = 1$.

If R is of type D_3, then, by(2.14), X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, and ℓ is a line on $X_t \cong \mathbb{P}^2$. Then $(\mathrm{c}_1(\mathcal{O}_X(1)) \cdot \ell) = (\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}) \cdot \ell)_{\mathbb{P}^2} = 1$.

By [SchSt, 42.12], this short exact sequence splits.

Analogously, we can use the divisor D on X satisfying $(D \cdot \ell) = \pm 1$ to obtain an exact sequence $0 \longrightarrow \mathrm{N}^1(Y)_{\mathbb{Z}} \xrightarrow{f^*} \mathrm{N}^1(X)_{\mathbb{Z}} \xrightarrow{(\,\cdot\,\ell)} \mathbb{Z} \longrightarrow 0$. Hence $\rho(X) = \rho(Y) + 1$. $\square$

Corollary 2.16. *Let X be a primitive Fano threefold with an extremal ray of type D_1, D_2 or D_3. Then $\rho(X) = 2$.*

Proof. By (2.14), $Y \cong \mathbb{P}^1$. Hence $\rho(Y) = 1$, so $\rho(X) = 2$ by (2.15). $\square$

Corollary 2.17. *Let X be a primitive Fano threefold with $\rho(X) = 2$, and let R be an extremal ray of X with corresponding contraction $f : X \longrightarrow Y$. Then* $\mathrm{Pic}(Y)$ *is free of rank* 1*, generated by an ample divisor L on Y.*

Proof. We have an exact sequence as in (2.15) above. Since $\rho(X) = 2$, this shows, by (2.2 (4)), that $\mathrm{Pic}(Y)$ is isomorphic to a subgroup of rank 1 of $\mathrm{Pic}(X)$, by ([SchSt, 51.10]). Moreover, since $\mathrm{Pic}(X)$ is torsion-free by (2.2 (5)), it is free. Hence $\mathrm{Pic}(Y)$ is also free, by [SchSt, 39.6].

By (2.3), Y is projective, so it carries an ample divisor. Since $\mathrm{Pic}(Y) \cong \mathbb{Z}$, we can fix a generator L of $\mathrm{Pic}(Y)$ in such a way that this ample divisor is linearly equivalent to some positive multiple of L. Then, by [Ha77, II, 7.5], L itself must be ample. $\square$

Corollary 2.18. *Let X be a primitive Fano threefold with $\rho(X) = 2$. Then X has no extremal ray of type E_1.*

Proof. Assume to the contrary. Then there is an extremal ray R of type E_1, and $f = \mathrm{cont}_R : X \longrightarrow Y$ is the blowing-up of Y along a nonsingular curve C, by (2.7). We will show that Y is Fano. Then X is not primitive, which is a contradiction.

So let us consider $-K_Y$. Since Y is projective, there exists an ample divisor H on Y corresponding to the embedding of Y into some projective space $\mathbb{P}^N$. Since $\mathrm{Pic}(Y) \cong \mathbb{Z}$ we can write $-K_Y \sim \alpha\, H$, $\alpha \in \mathbb{Z}$. Now Bertini's theorem [Ha77, II, 8.18] shows that on cutting Y successively with two sufficiently general hyperplanes in $\mathbb{P}^N$, one obtains a curve Z in Y which is disjoint from C. In particular, its strict transform $\tilde{Z}$ under the blowing-up f is disjoint from the exceptional divisor D of f. By (B.4 (1)), $-K_X \sim f^*(-K_Y) - D$. Since $f : X - D \longrightarrow Y - C$ is an isomorphism by (2.7), we therefore obtain by the projection formula [De, 1.10]

$$\begin{aligned}
\Big((-K_X) \cdot \tilde{Z}\Big)_X &= \Big(f^*(-K_Y) \cdot \tilde{Z}\Big)_X - \Big(D \cdot \tilde{Z}\Big)_X \\
&= \Big((-K_Y) \cdot f_*(\tilde{Z})\Big)_Y \\
&= ((-K_Y) \cdot Z)_Y \\
&= \alpha\, (H \cdot Z)_Y.
\end{aligned}$$

Here, $((-K_X) \cdot \tilde{Z})_X$ and $(H \cdot Z)_Y$ are positive since $-K_X$ and H are ample. This implies that $\alpha > 0$. Since $-K_Y \sim \alpha\, H$, $-K_Y$ is therefore ample by [Ha77, II, 7.5], i.e. Y is Fano. $\square$

3 A first characterisation of primitive Fano threefolds with $b_2 \geq 2$

The purpose of this chapter is to give a first characterisation of primitive Fano threefolds with $b_2 \geq 2$. Using the classification of extremal rays from section 2.3 we establish the existence of certain contractions on a primitive Fano threefold with $b_2 \geq 2$, depending on the value of b_2. It will turn out that b_2 is always 2 or 3.

A precise statement of these results is given in the following theorem, which will serve as a starting point for the actual classification of primitive Fano threefolds with $b_2 \geq 2$ to be carried out in chapters 4 and 5.

Theorem 3.1. *Let X be a primitive Fano threefold with $b_2(X) \geq 2$. Then the following holds:*

1. $b_2(X) \leq 3$.
2. *If $b_2(X) = 2$, then X has an extremal ray of type C_1 or C_2. Moreover, any contraction corresponding to an extremal ray of X of type C_1 or C_2 is a conic bundle $f : X \longrightarrow \mathbb{P}^2$.*
3. *If $b_2(X) = 3$, then there are the following two cases:*
   - *(i) X has two distinct extremal rays R_1 and R_2, each of type C_1 or C_2. The corresponding contractions are conic bundles $f_1, f_2 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.*
   - *(ii) X has an extremal ray R_1 of type C_1 or C_2 and an extremal ray R_2 of type E_1. The corresponding contractions are a conic bundle $f_1 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, and a birational morphism $f_2 : X \longrightarrow Y$ to a nonsingular projective threefold Y with exceptional divisor $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1,-1)$.*

The remainder of this chapter is devoted to the proof of this theorem.

So let X be a primitive Fano threefold with $b_2(X) \geq 2$. We will proceed in several steps, starting with a review of the types of extremal rays of X.

First, we recall some facts from chapter 2. By the Cone Theorem (2.1), X has a finite number of pairwise distinct extremal rays $R_1, \ldots, R_n$, generating the cone $\mathrm{NE}(X)$. Here, $R_i = \mathbb{R}_+ [\ell_i]$, where $[\ell_i] \in \mathrm{N}_1(X)_{\mathbb{R}}$ is the class of the extremal rational curve $\ell_i = \ell_{R_i}$ associated to R_i. In particular, the classes $[\ell_1], \ldots, [\ell_n]$ generate the cone $\mathrm{NE}(X)$.

Since $\mathrm{b}_2(X) \geq 2$, the vector space $\mathrm{N}_1(X)_{\mathbb{R}}$ has dimension at least 2, by (2.2). Hence the cone $\mathrm{NE}(X)$ has at least 2 distinct extremal rays, i.e., we may assume that X has two distinct extremal rays R_1 and R_2.

Lemma 3.2. *X has an extremal ray of type C_1, C_2, D_1, D_2 or D_3.*

Proof. Assume to the contrary. Then, by the classification of extremal rays in section 2.3, the rays $R_1, \ldots, R_n$ are of type E_1, E_2, E_3, E_4 or E_5. Let $D_1, \ldots, D_n$ denote their corresponding exceptional divisors on X. By (2.7), the D_i are irreducible reduced surfaces. We make the

Claim. $D_1 \cap D_i = \emptyset$ for all $1 < i \leq n$.

To prove the claim, let us assume to the contrary. Then $Z = D_1 \cap D_j$ is nonempty in X for some $1 < j \leq n$. By the dimension theorem [Ha77, I, 7.1], every irreducible component of Z has dimension at least 1. Moreover, Z contains no irreducible component of dimension greater than 1. For otherwise this would imply that $D_1 = D_j$ since D_1 and D_j are irreducible. But R_1 and R_j are distinct rays, so this is impossible, by (2.7) and (2.3). Hence we conclude that Z is a curve on X.

On the one hand, Z is a curve on D_1. Since $D_1|_{D_1}$ is negative by (2.7), we therefore obtain

$$(D_1 \cdot Z)_X = (D_1|_{D_1} \cdot Z)_{D_1} < 0.$$

On the other hand, Z is a curve on D_j. By (2.8), we can move it on D_j out of the intersection $D_1 \cap D_j$ to the effect that non of its components is any longer contained in D_1. Hence

$$(D_1 \cdot Z)_X \geq 0,$$

which is a contradiction. The claim is proved.

Now, since $-K_X$ is ample, $-m\,K_X$ is very ample for sufficiently large m ([Ha77, II, 7.6]), and induces an embedding of X into some projective space $\mathbb{P}^N$. By Bertini's theorem [Ha77, II, 8.18], there are distinct hyperplanes H_1, H_2 in $\mathbb{P}^N$ such that $E_i = H_i \cap X$ are distinct effective divisors on X linearly equivalent to $-m\,K_X$ and such that their intersection $(E_1 \cdot E_2) = E_1 \cap E_2$ is a curve on X.

Since $\mathrm{NE}(X)$ is generated as a cone by the classes of the curves $\ell_1, \ldots, \ell_n$, we can therefore write $(E_1 \cdot E_2) \equiv \sum_{i=1}^n a_i\,\ell_i$ with nonnegative real coefficients a_i.

On the one hand, we obtain

$$((E_1 \cdot E_2) \cdot D_1)_X = a_1\,(\ell_1 \cdot D_1)_X + \sum_{i=2}^n a_i\,(\ell_i \cdot D_1)_X.$$

Here, $(\ell_1 \cdot D_1)_X < 0$ since $D_1|_{D_1}$ is negative and ℓ_1 is a curve on D_1. Moreover, $(\ell_i \cdot D_1)_X = 0$ for all $i > 1$ since ℓ_i is a curve in D_i and D_i is disjoint from D_1 by the claim above. Hence

$$((E_1 \cdot E_2) \cdot D_1)_X \leq 0.$$

On the other hand, since $E_1 \sim E_2 \sim -m\,K_X$, we obtain

$$((E_1 \cdot E_2) \cdot D_1)_X = \left((-m\,K_X)^2 \cdot D_1\right)_X = m^2 \left((-K_X)^2 \cdot D_1\right)_X > 0$$

since $-K_X$ is ample on X.

This is a contradiction. $\square$

Lemma 3.3. *If X has an extremal ray of type D_1, D_2 or D_3, then it has another one of type C_1 or C_2.*

Proof. Without loss of generality, we may assume that the extremal ray R_1 is of type D_1, D_2 or D_3. By (2.14), there exists a corresponding contraction $f_1 : X \longrightarrow \mathbb{P}^1$. By (2.16), $\rho(X) = 2$.

By the classification of extremal rays in section 2.3, it will be enough to show that R_2 is not of type E_1, E_2, E_3, E_4, E_5, D_1, D_2 or D_3.

First, R_2 is not of type E_1 since $\rho(X) = 2$, by (2.18).

Second, assume that R_2 is of type E_2, E_3, E_4 or E_5. By (2.9), $f_1 : X \longrightarrow \mathbb{P}^1$ maps D_2 to a point. Since all the fibres of f_1 are irreducible by (2.14), it follows that D_2 is a fibre of f_1. Let us consider the extremal rational curve ℓ_2 on D_2.

On the one hand, there is a fibre of f_1 disjoint from D_2. Since all fibres of f_1 are numerically equivalent, this implies

$$(D_2 \cdot \ell_2)_X = 0.$$

On the other hand, $D_2|_{D_2}$ is negative by (2.7), so

$$(D_2 \cdot \ell_2)_X = (D_2|_{D_2} \cdot \ell_2)_{D_2} < 0.$$

This shows that R_2 cannot be of type E_2, E_3, E_4 or E_5.

Lastly, assume that R_2 is of type D_1, D_2 or D_3. Consider the morphism $f = (f_1, f_2) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where each $f_i : X \longrightarrow \mathbb{P}^1$ is the contraction corresponding to the extremal ray R_i of type D_1, D_2 or D_3. It must be finite, for otherwise there would exist an irreducible reduced curve on X which is contained in a fibre of f. Then this curve would be contracted by both f_1 and f_2. Hence, by (2.3), the class of this curve would be contained in $R_1 \cap R_2 = \{0\}$, which is impossible. But this is a contradiction, since X has dimension 3 ([Ha77, II, Ex. 3.22(b)]). $\square$

Proposition 3.4. *X has an extremal ray of type C_1 or C_2.*

Proof. This is an immediate consequence of (3.2) and (3.3). $\square$

By the proposition, we may assume without loss of generality that the extremal ray R_1 is of type C_1 or C_2. Then, by (2.11), its corresponding contraction $f_1 : X \longrightarrow S$ onto a rational nonsingular projective surface S is a conic bundle.

Our next aim is to determine this surface S. Since S is rational, one might ask whether it is a relatively minimal model in the birational equivalence class of rational nonsingular projective surfaces. Indeed, it will be an immediate consequence of the following lemma that the answer is in the affirmative.

Lemma 3.5. *The surface S contains no irreducible reduced curve which has negative self-intersection number on S.*

Proof. Suppose to the contrary. Then S contains an irreducible reduced curve E such that $E^2 < 0$.

Let C be a curve on X such that $f_1(C) = E$. The existence of such a curve can be seen as follows: Since X is projective, we can embed it into some projective space $\mathbb{P}^N$. Fix a fibre of f_1 over some point of E. An application of Bertini's theorem [Ha77, II, 8.18] to both this fibre and the surface $f_1^{-1}(E)$ shows that there exists a hyperplane H in $\mathbb{P}^N$ meeting this fibre transversally and such that the intersection $C = H \cap f_1^{-1}(E)$ is a curve on X. As a consequence, $f_1(C)$ is a closed subset of E which is not a single point. Since E is irreducible, we must therefore have $f_1(C) = E$.

Since $\mathrm{NE}(X)$ is generated as a cone by the classes of the curves $\ell_1, \ldots, \ell_n$, we can write $C \equiv \sum_{i=1}^n a_i\, \ell_i$ with nonnegative real coefficients a_i.

Now $(f_1)_*(C)$ is by construction some positive multiple of E. Since $(E^2)_S < 0$, this implies

$$\sum_{i=1}^n a_i\ \left((f_1)_*(\ell_i) \cdot E\right)_S = \left((f_1)_*(C) \cdot E\right)_S < 0.$$

We conclude $((f_1)_*(\ell_j) \cdot E)_S < 0$ for some j, and since ℓ_1 is contracted by f_1, we may assume without loss of generality that $j = 2$. So let us state explicitly

$$\left((f_1)_*(\ell_2) \cdot E\right)_S < 0. \tag{19}$$

Since E is irreducible, this implies that

$$f_1(\ell_2) = E. \tag{20}$$

In particular, ℓ_2 is a curve on $f_1^{-1}(E)$.

Our next aim is to find out the type of the extremal ray R_2.

Since E is reduced, we obtain by the projection formula [De, 1.10] and (19) above

$$\left(\ell_2 \cdot f_1^{-1}(E)\right)_X = \left(\ell_2 \cdot f_1^*(E)\right)_X = \left((f_1)_*(\ell_2) \cdot E\right)_S < 0. \tag{21}$$

By the classification of extremal rays in section 2.3, this implies that R_2 is of type E_1, E_2, E_3, E_4 or E_5: Assume that R_2 were of type C_1, C_2, D_1, D_2 or D_3. By (2.11) and (2.14), the fibres of f_2 are curves resp. surfaces. Since $f_1^{-1}(E)$ is properly contained in X we could fix an irreducible reduced curve Z which is contained in a fibre of f_2 but not contained in $f_1^{-1}(E)$. Then $(Z \cdot f_1^{-1}(E))_X \geq 0$. Moreover, by (2.3), both classes $[\ell_2]$ and $[Z]$ would lie on $R_2 \setminus \{0\}$, so ℓ_2 would be numerically equivalent to some positive multiple of Z. Hence

$$\left(\ell_2 \cdot f_1^{-1}(E)\right)_X \geq 0,$$

which contradicts (21). So R_2 is of type E_1, E_2, E_3, E_4 or E_5.

The exceptional divisor D_2 of f_2 satisfies the following

Claim. $D_2 = f_1^{-1}(E)$.

To prove this, let us assume to the contrary. By the dimension theorem [Ha77, I, 7.1], every irreducible component of $D_2 \cap f_1^{-1}(E)$ has dimension at least 1. By our assumption, it will follow that $D_2 \cap f_1^{-1}(E)$ is of pure dimension 1 once we know that both D_2 and $f_1^{-1}(E)$ are irreducible.

This is clear for D_2 by (2.7), but to verify that $f_1^{-1}(E)$ is irreducible requires some more justification: Suppose that $f_1^{-1}(E)$ were reducible. By (B.3 (7)), the discriminant locus Δ_{f_1} of the conic bundle f_1 contains E as a connected component. Moreover, $f_1^{-1}(E) = Z_1 \cup Z_2$ with irreducible reduced components Z_1 and Z_2.

The fibre X_s of f_1 over a generic point s of E not contained in the singular locus of E decomposes as $X_s = l_1 \cup l_2$, where each l_i is an irreducible curve on Z_i. In particular,

$$(l_1 \cdot Z_2)_X > 0.$$

On the other hand, the generic fibre $X_{s'}$ of f_1, where s' is a point of S not contained in E, is disjoint from Z_2, so $(X_{s'} \cdot Z_2)_X = 0$. By (2.3), both classes $[l_1]$ and $[X_{s'}]$ lie on $R_1 \setminus \{0\}$, so l_1 is numerically equivalent to some positive multiple of $X_{s'}$. Hence

$$(l_1 \cdot Z_2)_X = 0,$$

which is a contradiction. This shows that $f_1^{-1}(E)$ is irreducible.

Now that we know that $D_2 \cap f_1^{-1}(E)$ is a curve on X, we consider the extremal rational curve ℓ_2. By (20) above, it is contained in $D_2 \cap f_1^{-1}(E)$. By (2.8), we can move ℓ_2 on D_2 out of the curve $D_2 \cap f_1^{-1}(E)$. But this implies that

$$\left(\ell_2 \cdot f_1^{-1}(E)\right)_X \geq 0,$$

which contradicts (21). This proves the claim.

Let us now consider an irreducible component C of a special fibre X_s of f_1 over some point s on E. By the claim above, C is a curve on D_2. Since R_2 is of type E_1, E_2, E_3, E_4 or E_5, $D_2|_{D_2}$ is negative by (2.7), so

$$(C \cdot D_2)_X = (C \cdot D_2|_{D_2})_{D_2} < 0.$$

On the other hand, the generic fibre $X_{s'}$ of f_1 over some point s' on S not contained in E is disjoint from $f_1^{-1}(E)$. By the claim above, we therefore get $(X_{s'} \cdot D_2)_X = (X_{s'} \cdot f_1^{-1}(E))_X = 0$. By (2.3), both classes $[C]$ and $[X_{s'}]$ lie on $R_1 \setminus \{0\}$, so C is numerically equivalent to some positive multiple of $X_{s'}$. This yields

$$(C \cdot D_2)_X = 0,$$

which is a contradiction. □

Corollary 3.6. *The surface S is a relatively minimal model.*

Proof. If this were not the case, S would contain an exceptional curve of the first kind, that is, a nonsingular rational curve with self-intersection number -1 on S ([Ha77, V, 5.4 and 5.7]). This contradicts (3.5). □

To sum up our results, we have shown that S is a nonsingular projective surface which is a relatively minimal model in the birational equivalence class of rational surfaces. Thus, by [Ha77, V, 5.8.2], S is isomorphic to $\mathbb{P}^2$ or the rational ruled surface $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ on $\mathbb{P}^1$, where $e \geq 0$ and $e \neq 1$.

By [Ha77, V, 2.11.3], X_e has a section which is an irreducible reduced curve and which has self-intersection number $-e \leq 0$ on X_e. Hence, by (3.5), e can only take the value 0. In this case, X_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ ([Ha77, V, 2.11.1]). Thus we have proved the following result.

Proposition 3.7. *The surface S is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*

Now we are in a position to complete the proof of the theorem.

By (3.4), there exists a conic bundle $f_1 : X \longrightarrow S$ corresponding to an extremal ray of type C_1 or C_2. By (2.15), $\rho(X) = \rho(S) + 1$. Using (2.2 (4)), we obtain from this the following equation, which provides a link between $\mathrm{b}_2(X)$ and the type of S:

$$\mathrm{b}_2(X) = \rho(S) + 1.$$

By (3.7), S is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the former case, $\rho(S) = 1$ and in the latter case, $\rho(S) = 2$.

This implies that $\mathrm{b}_2(X) = 2$ or 3, which proves the first assertion of the theorem. Moreover, we obtain a converse to this. Namely,

if $\mathrm{b}_2(X) = 2$ then $S \cong \mathbb{P}^2$, and

if $\mathrm{b}_2(X) = 3$ then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Thus we see that in the case $b_2(X) = 2$, any contraction $f : X \longrightarrow Y$ corresponding to an extremal ray of type C_1 or C_2 is a conic bundle with base $Y \cong \mathbb{P}^2$. This proves the second assertion of the theorem. One might wonder why not also to consider the type of the second extremal ray R_2 in order to get more information about X. Actually, this is precisely what we are going to do in chapter 4. We have to postpone it until later, for it will require some supplementary results which go beyond the scope of this chapter.

But in the case $b_2(X) = 3$, things are quite different. Here we have already gathered enough material to give a more detailed characterisation of X by considering both extremal rays R_1 and R_2 of X.

In this case, the considerations above show that any contraction $f : X \longrightarrow Y$ corresponding to an extremal ray of type C_1 or C_2 is a conic bundle with base $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

We can use this to obtain more information about the other extremal ray R_2, as follows.

Proposition 3.8. *If* $b_2(X) = 3$ *then the extremal ray* R_2 *is of type* E_1*,* C_1 *or* C_2*.*

Proof. Since $\rho(X) = b_2(X) = 3$ by (2.2 (3)), (2.16) shows that R_2 cannot be of type D_1, D_2 or D_3. By the classification of extremal rays in section 2.3, it will therefore suffice to show that R_2 is not of type E_2, E_3, E_4 or E_5.

Let us assume to the contrary. Consider the exceptional divisor D_2 corresponding to R_2. Composing our contraction $f_1 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with the canonical projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ onto the i-th factor, we obtain morphisms $g_i = \pi_i \circ f_1 : X \longrightarrow \mathbb{P}^1$, for $i = 1, 2$. By (2.9), both g_1 and g_2 map D_2 to a point. Consequently, the surface D_2 is mapped to a point already by f_1. But all the fibres of f_1 have dimension 1, so this is impossible. $\square$

Accordingly, if R_2 is of type C_1 or C_2 there exists a corresponding contraction which is, as we have remarked above, a conic bundle $f_2 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

If R_2 is of type E_1 then, by (2.7), there exists a corresponding contraction $f_2 : X \longrightarrow Y$ to a smooth projective threefold Y and with exceptional divisor $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1, -1)$. This proves the third assertion of the theorem.

4 Classification of primitive Fano threefolds with $\mathrm{b}_2 = 2$

Our goal in this chapter is to classify primitive Fano threefolds with $\mathrm{b}_2 = 2$. In section 4.1 we establish their main properties, based on our results from the last chapter. A key result is proposition 4.4, which plays a central role in our classification. Specifically, this proposition will enable us to turn information on the types of extremal rays of our Fano threefold into strong restrictions concerning its geometry. After these preparations we work out the actual classification of primitive Fano threefolds with $\mathrm{b}_2 = 2$ in section 4.2.

4.1 Primitive Fano threefolds with $\mathrm{b}_2 = 2$

The purpose of this section is to collect the properties of primitive Fano threefolds with $\mathrm{b}_2 = 2$ that are relevant to their classification.

So let X be a primitive Fano threefold with $\mathrm{b}_2(X) = 2$. Then $\rho(X) = 2$ by (2.2 (4)), so $\mathrm{N}_1(X)_{\mathbb{R}} \cong \mathbb{R}^2$. In particular, the cone $\mathrm{NE}(X)$ has two distinct extremal rays R_1 and R_2.

As we have seen in chapter 2 (2.3 and 2.15), there is, for $i = 1, 2$, a corresponding contraction

$$f_i = \mathrm{cont}_{R_i} : X \longrightarrow Y_i$$

to a projective variety Y_i, with lenght μ_i and associated extremal rational curve ℓ_i, and an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y_i) \xrightarrow{f_i^*} \mathrm{Pic}(X) \xrightarrow{(\,\cdot \ell_i)} \mathbb{Z} \longrightarrow 0. \tag{22}$$

By (2.17), $\mathrm{Pic}(Y_i)$ is free of rank 1, generated by an ample divisor L_i on Y_i. Let us denote by

$$H_i = f_i^* L_i$$

the pull back of L_i to X. It has the following property.

Lemma 4.1. *H_i is a primitive element of* $\mathrm{Pic}(X)$.

Proof. Assume to the contrary. Then $H_i = r\,H$, where $H \in \mathrm{Pic}(X)$ and r is some integer, $r > 1$. Exactness of sequence (22) above then implies $(H \cdot \ell_i) = 0$, so H is contained in $f_i^* \mathrm{Pic}(Y_i)$. But H_i is a generator of $f_i^* \mathrm{Pic}(Y_i)$, so $H_i = r\,H$ is impossible. □

According to (3.1), we may without loss of generality assume that R_1 is of type C_1 or C_2.

Moreover, R_2 is not of type E_1, by (2.18). By the classification of extremal rays in section 2.3, we therefore have the cases R_2 is of type E_2, E_3, E_4, E_5, C_1, C_2, D_1, D_2 or D_3.

Much of our work in this chapter will consist in running through the various cases made up by all possible combinations of the types of R_1 and R_2. For our convenience, we will therefore denote by $(\star - \star\star)$ the case R_1 is of type $\star$ and R_2 is of type $\star\star$.

Since R_1 is of type C_1 or C_2, H_1 has the following property, which we will frequently need.

Lemma 4.2. *The divisor $H_1 = f_1^* L_1$ on X satisfies*

$$H_1^2 \equiv \tfrac{2}{\mu_1}\, \ell_1.$$

Proof. By (3.1), the contraction corresponding to R_1 is a conic bundle $f_1 : X \longrightarrow \mathbb{P}^2$. In particular, L_i corresponds to $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Hence L_1^2 is a point on $\mathbb{P}^2$, so $H_1^2 = f_1^* L_1^2$ is a fibre of f_1. Hence, by (2.12), the assertion follows. $\square$

We will frequently need the following result.

Lemma 4.3. *For $i = 1, 2$, the divisor H_i on X satisfies:*

type of R_i	E_2	E_3 *or* E_4	E_5
$(\mathrm{c}_2(X) \cdot H_i)$	$24/r$	$24/r$	$45/r$

Here, r is the largest integer which divides $-K_X + D_i$ (resp. $-K_X + 2\, D_i$, $2\,(-K_X) + D_i$) in $\mathrm{Pic}(X)$*, where D_i is the exceptional divisor of R_i, if R_i is of type E_3 or E_4 (resp. E_2, E_5) (cf. 2.7).*

type of R_i	C_1	C_2
$(\mathrm{c}_2(X) \cdot H_i)$	$6 + \deg(\Delta_{f_i})$	6

Here, Δ_{f_i} denotes the discriminant locus of the conic bundle $f_i : X \longrightarrow \mathbb{P}^2$ if R_i is of type C_1 (cf. 2.11). In this case, $\deg(\Delta_{f_i}) > 1$.

type of R_i	D_1	D_2	D_3
$(\mathrm{c}_2(X) \cdot H_i)$	$12 - (K_{X_t})^2$	4	3

Here, X_t denotes any fibre of f_i if R_i is of type D_1 (cf. 2.14).

Proof. If R_i is of type E_2, $\mu_i = 2$, $D_i \cong \mathbb{P}^2$, $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{D_i}(-1)$ and ℓ_i is a line on D_i, by (2.7). Then

$$\begin{aligned}((-K_X + 2\,D_i)\cdot \ell_i)_X &= (-K_X\cdot \ell_i)_X + 2\,(D_i\cdot \ell_i)_X\\ &= \mu_i + 2\,(D_i|_{D_i}\cdot \ell_i)_{D_i}\\ &= 2 + 2\,(\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(-1))\cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))_{\mathbb{P}^2}\\ &= 2 + 2\cdot(-1)\\ &= 0.\end{aligned}$$

By exactness of sequence (22), this implies that $-K_X + 2\,D_i$ is contained in $f_i^*\operatorname{Pic}(Y_i) \cong H_i\,\mathbb{Z}$. By definition, r is the largest integer which divides $-K_X+2\,D_i$ in $\operatorname{Pic}(X)$. Since H_i is a primitive element of $\operatorname{Pic}(X)$ by (4.1), we conclude

$$-K_X + 2\,D_i \sim r\,H_i.$$

Then we obtain, using (2.2 (2)),

$$\begin{aligned}(\mathrm{c}_2(X)\cdot H_i) &= (1/r)\,((\mathrm{c}_2(X)\cdot(-K_X)) + 2\,(\mathrm{c}_2(X)\cdot D_i))\\ &= (1/r)\,(24 + 2\,(\mathrm{c}_2(X)\cdot D_i))\,.\end{aligned}$$

To compute the second summand, we use formula (2.2 (6)) which states that

$$(\mathrm{c}_2(X)\cdot D_i) = 6\,\chi(\mathcal{O}_{D_i}) + 6\,\chi(\mathcal{O}_{D_i}(D_i)) - 2\,(D_i^3) - ((-K_X)^2\cdot D_i). \tag{23}$$

Since $D_i \cong \mathbb{P}^2$, $\chi(\mathcal{O}_{D_i}) = \mathrm{h}^0(\mathcal{O}_{D_i}) = 1$, by [Ha77, III, 5.1(b)].
Moreover, $\chi(\mathcal{O}_{D_i}(D_i)) = \chi(\mathcal{O}_{D_i}(-1)) = \mathrm{h}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$.
$(D_i^3) = (D_i|_{D_i})^2_{D_i} = (\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(-1)))^2_{\mathbb{P}^2} = 1$.
$((-K_X)^2\cdot D_i) = (-K_X|_{D_i})^2_{D_i}$. By the adjunction formula, $\mathcal{O}_{D_i}(K_X) \cong \mathcal{O}_{D_i}(K_{D_i} - D_i) \cong \mathcal{O}_{\mathbb{P}^2}(-3+1) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. Hence $((-K_X)^2\cdot D_i) = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(2))^2 = 4$.
Plugging these results into (23) above, we obtain $(\mathrm{c}_2(X)\cdot D_i)_X = 6+0-2-4 = 0$.
Thus our result is

$$(\mathrm{c}_2(X)\cdot H_i) \;\; = 24/r.$$

If R_i is of type E_3 or E_4, $\mu_i = 1$, D_i is isomorphic to an irreducible reduced quadric surface in $\mathbb{P}^3$ and $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{\mathbb{P}^3}(-1)\otimes \mathcal{O}_{D_i}$, by (2.7). To see this in case R_i is of type E_3, we use the Segre embedding $D_i \cong \mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ ([Ha77, I, Ex. 2.15]). Then

$$\begin{aligned}((-K_X + D_i)\cdot \ell_i)_X &= (-K_X\cdot \ell_i)_X + (D_i\cdot \ell_i)_X\\ &= \mu_i + (D_i|_{D_i}\cdot \ell_i)_{D_i}\\ &= 1 + (D_i|_{D_i}\cdot \ell_i)_{D_i}\,.\end{aligned}$$

To compute the second summand, we have to distinguish two cases:

If R_i is of type E_3, $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ and ℓ_i corresponds to $P \times \mathbb{P}^1$ or $\mathbb{P}^1 \times Q$, $P, Q \in \mathbb{P}^1$, by (2.7). Then $(D_i|_{D_i} \cdot \ell_i)_{D_i} = (\mathrm{c}_1(\mathcal{O}(-1,-1)) \cdot (P \times \mathbb{P}^1))_{\mathbb{P}^1\times\mathbb{P}^1} = (\mathrm{c}_1(\mathcal{O}(-1,-1)) \cdot (\mathbb{P}^1 \times Q))_{\mathbb{P}^1\times\mathbb{P}^1} = -1$.

If R_i is of type E_4, D_i is isomorphic to an irreducible reduced singular quadric surface in $\mathbb{P}^3$, $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{D_i}$ and ℓ_i is a generator of D_i, by (2.7). Let Z_0, Z_1, Z_2, Z_3 be homogeneous coordinates on $\mathbb{P}^3$. By ([Ha77, II, Ex. 6.5(b)]), D_i is isomorphic to the cone defined by the equation $Z_0\, Z_1 = Z_2^2$. Moreover, $-D_i|_{D_i}$ corresponds to the divisor on D_i obtained by intersecting D_i with the hyperplane $H_0 = \{Z_0 = 0\}$, and ℓ_i corresponds to the line $Z_1 = Z_2 = 0$ (cf. [Ha77, II, 6.5.2]). This line intersects H_0 transversally in the vertex of the cone D_i, so we obtain $(D_i|_{D_i} \cdot \ell_i)_{D_i} = -(H_0 \cdot \ell_i \cdot D_i)_{\mathbb{P}^3} = -1$.

Now we can finish our calculation:

$$((-K_X + D_i) \cdot \ell_i)_X = 1 + (D_i|_{D_i} \cdot \ell_i)_{D_i} = 1 - 1 = 0.$$

As in case R_2 is of type E_2 above, exactness of sequence (22) implies

$$-K_X + D_i \sim r\, H_i,$$

and we obtain

$$(\mathrm{c}_2(X) \cdot H_i) = (1/r)\,(24 + (\mathrm{c}_2(X) \cdot D_i))\,.$$

To compute the second summand, we use again formula (2.2 (6)):

$$(\mathrm{c}_2(X) \cdot D_i) = 6\,\chi(\mathcal{O}_{D_i}) + 6\,\chi(\mathcal{O}_{D_i}(D_i)) - 2\,(D_i^3) - ((-K_X)^2 \cdot D_i). \qquad (24)$$

Since D_i is a hypersurface in $\mathbb{P}^3$, $\chi(\mathcal{O}_{D_i}) = \mathrm{h}^0(\mathcal{O}_{D_i}) = 1$, by [Ha77, III, Ex. 5.5 (c)]. Moreover, $\chi(\mathcal{O}_{D_i}(D_i)) = \mathrm{h}^0(D_i, \mathcal{O}_{D_i}(-1))$, which is zero since the natural map $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-1)) \longrightarrow \mathrm{H}^0(D_i, \mathcal{O}_{D_i}(-1))$ is surjective, by [Ha77, III, Ex. 5.5 (a)], and $\mathcal{O}_{\mathbb{P}^3}(-1)$ has no nontrivial global sections.
Since $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{D_i}$, we get $(D_i^3)_X = (D_i|_{D_i})^2_{D_i} = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{D_i})^2$ $= (\mathrm{c}_1(\mathcal{O}(-1))^2 \cdot \mathrm{c}_1(\mathcal{O}(2)))_{\mathbb{P}^3} = 2$.
Likewise, $((-K_X)^2 \cdot D_i) = (-K_X|_{D_i})^2_{D_i}$. Since D_i is a hypersurface in $\mathbb{P}^3$, it is a local complete intersection in X, by [Ha77, II, 8.22.2]. Hence we can apply the adjunction formula. We obtain $\omega_X \otimes \mathcal{O}_{D_i} \cong \omega^\circ_{D_i} \otimes \mathcal{O}_{D_i}(-D_i)$, where $\omega^\circ_{D_i}$ is the dualizing sheaf for D_i. Since D_i is a quadric in $\mathbb{P}^3$, $\omega^\circ_{D_i} \cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{\mathbb{P}^3}(-4+2) \otimes \mathcal{O}_{D_i} \cong \mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{D_i}$, by [Ha77, III, 7.11]. Hence we obtain $\mathcal{O}_{D_i}(-K_X) \cong \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{D_i} \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_{D_i}$. Thus we obtain $((-K_X)^2 \cdot D_i)_X = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_{D_i})^2 = (\mathrm{c}_1(\mathcal{O}(1))^2 \cdot \mathrm{c}_1(\mathcal{O}(2)))_{\mathbb{P}^3} = 2$.
Plugging these results into (24) above, we obtain $(\mathrm{c}_2(X) \cdot D_i)_X = 6 + 0 - 4 - 2 = 0$. Thus our result is

$$(\mathrm{c}_2(X) \cdot H_i) \;\; = 24/r.$$

If R_i is of type E_5, $\mu_i = 1$, $D_i \cong \mathbb{P}^2$, $\mathcal{O}_{D_i}(D_i) \cong \mathcal{O}_{D_i}(-2)$ and ℓ_i is a line on D_i, by (2.7). Then

$$\begin{aligned} ((2\,(-K_X) + D_i) \cdot \ell_i)_X &= 2\,((-K_X) \cdot \ell_i)_X + (D_i \cdot \ell_i)_X \\ &= 2\,\mu_i + (D_i|_{D_i} \cdot \ell_i)_{D_i} \\ &= 2 + (\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(-2)) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))_{\mathbb{P}^2} \\ &= 2 - 2 \\ &= 0. \end{aligned}$$

As in case R_2 is of type E_2 above, exactness of sequence (22) implies

$$2\,(-K_X) + D_i \sim r\,H_i,$$

and we obtain

$$(\mathrm{c}_2(X) \cdot H_i) = (1/r)\,(2 \cdot 24 + (\mathrm{c}_2(X) \cdot D_i))\,.$$

To compute the second summand, we use again formula (2.2 (6)):

$$(\mathrm{c}_2(X) \cdot D_i) = 6\,\chi(\mathcal{O}_{D_i}) + 6\,\chi(\mathcal{O}_{D_i}(D_i)) - 2\,(D_i^3) - ((-K_X)^2 \cdot D_i). \tag{25}$$

Since $D_i \cong \mathbb{P}^2$, $\chi(\mathcal{O}_{D_i}) = \mathrm{h}^0(\mathcal{O}_{D_i}) = 1$, by [Ha77, III, 5.1(b)].
Moreover, $\chi(\mathcal{O}_{D_i}(D_i)) = \chi(\mathcal{O}_{D_i}(-2)) = \mathrm{h}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$.
$(D_i^3) = (D_i|_{D_i})^2_{D_i} = (\mathrm{c}_1\,\mathcal{O}_{\mathbb{P}^2}(-2))^2_{\mathbb{P}^2} = 4$.
$((-K_X)^2 \cdot D_i)_X = (-K_X|_{D_i})^2_{D_i}$. By the adjunction formula, $\mathcal{O}_{D_i}(K_X) \cong \mathcal{O}_{D_i}(K_{D_i} - D_i) \cong \mathcal{O}_{\mathbb{P}^2}(-3+2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. Thus we obtain $((-K_X)^2 \cdot D_i)_X = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 1$.
Plugging these results into (25) above, we obtain $(\mathrm{c}_2(X) \cdot D_i)_X = 6+0-8-1 = -3$.
Thus our result is

$$(\mathrm{c}_2(X) \cdot H_i) = (1/r)\,(2 \cdot 24 - 3) = 45/r.$$

If R_i is of type C_1 or C_2, $f_i : X \longrightarrow \mathbb{P}^2$ is a conic bundle, by (2.11). Since L_i is an ample generator of $\mathrm{Pic}(\mathbb{P}^2)$, it is a line on $\mathbb{P}^2$. In particular, H_i is effective, so formula (2.2 (6)) yields

$$(\mathrm{c}_2(X) \cdot H_i) = 6\,\chi(\mathcal{O}_{H_i}) + 6\,\chi(\mathcal{O}_{H_i}(H_i)) - 2\,(H_i^3) - ((-K_X)^2 \cdot H_i). \tag{26}$$

By (2.5) and [Ha77, III, 5.1] we get:
$\chi(\mathcal{O}_{H_i}) = 1 - \chi(\mathcal{O}_{\mathbb{P}^2}(-L_i)) = 1 - \chi(\mathcal{O}_{\mathbb{P}^2}(-1)) = 1 - \mathrm{h}^0(\mathbb{P}^2, \mathcal{O}(-1)) = 1 - 0 = 1$.
$\chi(\mathcal{O}_{H_i}(H_i)) = \chi(\mathcal{O}_{\mathbb{P}^2}(L_i)) - 1 = \chi(\mathcal{O}_{\mathbb{P}^2}(1)) - 1 = \mathrm{h}^0(\mathbb{P}^2, \mathcal{O}(1)) - 1 = \binom{2+1}{2} - 1 = 3 - 1 = 2$ by [Ha77, II, 7.8.3].
Since $L_i^3 = 0$, $H_i^3 = f_i^* L_i^3 = 0$.
By the projection formula [De, 1.10], $((-K_X)^2 \cdot H_i) = ((-K_X)^2 \cdot f_i^* L_i) = ((f_i)_*(K_X^2) \cdot$

$L_i)_{\mathbb{P}^2}$. By (B.3), $(f_i)_*(K_X^2) \equiv -4\, K_{\mathbb{P}^2} - \Delta_{f_i}$. Hence $((-K_X)^2 \cdot H_i)_X = -4\,(K_{\mathbb{P}^2} \cdot L_i)_{\mathbb{P}^2} - (\Delta_{f_i} \cdot L_i)_{\mathbb{P}^2} = -4\,(\mathrm{c}_1(\mathcal{O}(-3)) \cdot \mathrm{c}_1(\mathcal{O}(1)))_{\mathbb{P}^2} - (\Delta_{f_i} \cdot \mathrm{c}_1(\mathcal{O}(1))) = 12 - \deg(\Delta_{f_i})$. Plugging these results into (26) above, we obtain

$$(\mathrm{c}_2(X) \cdot H_i)_X = 6 + 12 - 0 - 12 + \deg(\Delta_{f_i}) = 6 + \deg(\Delta_{f_i}).$$

If R_i is of type C_1, then $\Delta_{f_i} = \mathcal{O}_{\mathbb{P}^2}(a)$ is nonzero, by (2.11), so $a > 0$. If we had $a = 1$, Δ_{f_i} would be a line on $\mathbb{P}^2$. But this contradicts (B.3 (6)), for f_i is a Mori contraction satisfying $\rho(X) = 2 = \rho(\mathbb{P}^2) + 1$. Hence we must have $\deg(\Delta_{f_i}) = a > 1$.

If R_i is of type C_2, then $\Delta_{f_i} = 0$, by (2.11).

If R_i is of type D_1, D_2 or D_3, $f_i : X \longrightarrow \mathbb{P}^1$ by (2.14). Since L_i is an ample generator of $\mathrm{Pic}(\mathbb{P}^1)$, it is a point on $\mathbb{P}^1$. In particular, H_i is effective, so formula (2.2 (6)) yields

$$(\mathrm{c}_2(X) \cdot H_i) = 6\,\chi(\mathcal{O}_{H_i}) + 6\,\chi(\mathcal{O}_{H_i}(H_i)) - 2\,(H_i^3) - ((-K_X)^2 \cdot H_i). \qquad (27)$$

As in case R_i of type C_1 or C_2 above, we get by (2.5) and [Ha77, III, 5.1]:
$\chi(\mathcal{O}_{H_i}) = 1 - \chi(\mathcal{O}_{\mathbb{P}^1}(-L_i)) = 1 - \chi(\mathcal{O}_{\mathbb{P}^1}(-1)) = 1 - \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}(-1)) = 1 - 0 = 1$.
$\chi(\mathcal{O}_{H_i}(H_i)) = \chi(\mathcal{O}_{\mathbb{P}^1}(L_i)) - 1 = \chi(\mathcal{O}_{\mathbb{P}^1}(1)) - 1 = \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}(1)) - 1 = \binom{1+1}{1} - 1 = 2 - 1 = 1$ by [Ha77, II, 7.8.3].
Since $L_i^3 = 0$, $H_i^3 = f_i^* L_i^3 = 0$.
By the adjunction formula, $((-K_X)^2 \cdot H_i)_X = (K_X|_{H_i})^2_{H_i} = (K_{H_i} - H_i|_{H_i})^2_{H_i}$. Since L_i is a point on $\mathbb{P}^1$, $H_i|_{H_i} = 0$. Moreover, $H_i = f_i^* L_i$ is a fibre of f_i. Hence, by (2.14), $((-K_X)^2 \cdot H_i)_X = K_{X_t}^2$, where X_t, $t \in \mathbb{P}^1$ is a fibre of f_i.
Plugging these results into (27) above, we obtain:

If R_i is of type D_1, then $(\mathrm{c}_2(X) \cdot H_i)_X = 6 + 6 - 0 - K_{X_t}^2 = 12 - K_{X_t}^2$.

If R_i is of type D_2, then $K_{X_t}^2 = 8$ by (2.14), and thus $(\mathrm{c}_2(X) \cdot H_i)_X = 6 + 6 - 0 - 8 = 4$.

If R_i is of type D_3, then $K_{X_t}^2 = 9$ by (2.14), and thus $(\mathrm{c}_2(X) \cdot H_i)_X = 6 + 6 - 0 - 9 = 3$. $\square$

Now we come to the promised proposition which clarifies the structure of $\mathrm{Pic}(X)$ and therefore plays a crucial role in our classification.

Proposition 4.4. *$\{H_1, H_2\}$ is a $\mathbb{Z}$-basis of $\mathrm{Pic}(X) \cong \mathrm{N}^1(X)_{\mathbb{Z}}$, and $\{\ell_1, \ell_2\}$ is the corresponding dual basis of $\mathrm{N}_1(X)_{\mathbb{Z}}$. In particular,*

$$-K_X \sim \mu_2\, H_1 + \mu_1\, H_2.$$

Proof. First, we check that H_1 and H_2 are linearly independent over $\mathbb{R}$. So let

$$0 = \lambda_1\, H_1 + \lambda_2\, H_2$$

with coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$. By exactness of sequence (22), $(H_1 \cdot \ell_1) = (f_1^* L_1 \cdot \ell_1) = 0$. Hence we obtain

$$0 = \lambda_1 \, (H_1 \cdot \ell_1) + \lambda_2 \, (H_2 \cdot \ell_1) = \lambda_2 \, (H_2 \cdot \ell_1).$$

Here, $(H_2 \cdot \ell_1)$ is nonzero. To see this, assume to the contrary. Recall from the beginning of this section that ℓ_1 and ℓ_2 generate $\mathrm{N}_1(X)_{\mathbb{R}}$. Hence, by exactness of sequence (22), the functional $(H_2, \ )$ would be identically zero on $\mathrm{N}_1(X)_{\mathbb{R}}$. Since the intersection pairing $(\ , \ )_X : \mathrm{N}^1(X)_{\mathbb{R}} \times \mathrm{N}_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$ is nondegenerate, the class of H_2 in $\mathrm{N}^1(X)_{\mathbb{Z}}$ would therefore be zero. But $\mathrm{N}^1(X)_{\mathbb{Z}} \cong \mathrm{Pic}(X)$ by (2.2 (3)), so we would have $H_2 \sim 0$, which is impossible by exactness of (22). We conclude $\lambda_2 = 0$. Likewise, $\lambda_1 = 0$.

Now we prove that H_1 and H_2 generate $\mathrm{Pic}(X)$. This will require considerable effort.

As we have just seen, $H_1 \, \mathbb{Z} \oplus H_2 \, \mathbb{Z}$ is a free subgroup of rank 2 of $\mathrm{Pic}(X)$. Since $\mathrm{Pic}(X)$ is free of rank 2 by (2.2), the quotient

$$\mathrm{Pic}(X) \, / \, (H_1 \, \mathbb{Z} \oplus H_2 \, \mathbb{Z}) \tag{28}$$

is a finite abelian group ([SchSt, 39.8]). Let a denote its order.

We claim that there is an isomorphism of groups

$$\mathrm{Pic}(X) \, / \, (H_1 \, \mathbb{Z} \oplus H_2 \, \mathbb{Z}) \cong \mathbb{Z} \, / \, (H_2 \cdot \ell_1) \, \mathbb{Z}. \tag{29}$$

To see this, consider the epimorphism

$$\mathrm{Pic}(X) \xrightarrow{(\ \cdot \ell_1)} \mathbb{Z} \xrightarrow{\nu_1} \mathbb{Z} \, / \, (H_2 \cdot \ell_1) \, \mathbb{Z},$$

which is obtained by composing the third map in sequence (22) with the canonical epimorphism. We check that it has kernel $H_1 \, \mathbb{Z} \oplus H_2 \, \mathbb{Z}$. This will give us the desired isomorphism. If H is in this kernel, $(H \cdot \ell_1)$ is contained in $(H_2 \cdot \ell_1) \, \mathbb{Z}$. Then $(H \cdot \ell_1) = (\alpha_2 \, H_2 \cdot \ell_1)$ for some $\alpha_2 \in \mathbb{Z}$. By exactness of sequence (22), this implies that H and $\alpha_2 \, H_2$ differ by some element of $f_1^* \ \mathrm{Pic}(Y_1) = H_1 \, \mathbb{Z}$. So $H = \alpha_1 \, H_1 + \alpha_2 \, H_2$, $\alpha_1 \in \mathbb{Z}$. Conversely, if this holds, then $(H \cdot \ell_1) = \alpha_2 \, (H_2 \cdot \ell_1)$, which is mapped to zero by ν_1.

Likewise, there is an isomorphism

$$\mathrm{Pic}(X) \, / \, (H_1 \, \mathbb{Z} \oplus H_2 \, \mathbb{Z}) \cong \mathbb{Z} \, / \, (H_1 \cdot \ell_2) \, \mathbb{Z}. \tag{30}$$

Since L_i is ample, it is nef and hence also $H_i = f_i^* L_i$ is nef, by [De, 1.25]. Thus $(H_1 \cdot \ell_2)$ and $(H_2 \cdot \ell_1)$ are both nonnegative and we deduce from (29) and (30) that

$$(H_1 \cdot \ell_2) = a = (H_2 \cdot \ell_1). \tag{31}$$

By exactness of sequences (22),

$$(H_1 \cdot \ell_1) = 0 = (H_2 \cdot \ell_2). \tag{32}$$

Recall from the above that H_1 and H_2 are linearly independent over $\mathbb{R}$. Hence their classes form a basis of the 2-dimensional vector space $N^1(X)_{\mathbb{R}} \cong \mathrm{Pic}(X) \otimes \mathbb{R}$. Thus we get a relation in $\mathrm{Pic}(X) \otimes \mathbb{R}$

$$-K_X \sim \beta_1 \, H_1 + \beta_2 \, H_2$$

with real coefficients β_1, β_2. Using (31) and (32), we obtain from this

$$\begin{aligned} \mu_1 &= (-K_X \cdot \ell_1) \\ &= \beta_1 \, (H_1 \cdot \ell_1) + \beta_2 \, (H_2 \cdot \ell_1) \\ &= \beta_2 \cdot a. \end{aligned}$$

Likewise, we get $\mu_2 = \beta_1 \cdot a$. Hence the relation becomes

$$a \, (-K_X) \sim \mu_2 \, H_1 + \mu_1 \, H_2. \tag{33}$$

Now we see from (28), (31), (32) and (33) that in order to complete the proof of the proposition it will be enough to prove that $a = 1$.

The key will be the equality

$$24 \, a = a \, (-K_X \cdot \mathrm{c}_2(X)) = \mu_2 \, (H_1 \cdot \mathrm{c}_2(X)) + \mu_1 \, (H_2 \cdot \mathrm{c}_2(X)) \, , \tag{34}$$

which follows from (33) and (2.2 (2)).

We will combine it with a suitable estimation of $(\mathrm{c}_2(X) \cdot H_i)$. This estimation, depending on the type of R_i, is obtained as follows:

If R_i is of type C_1, then $f_i : X \longrightarrow \mathbb{P}^2$ is a conic bundle, by (3.1). By (4.3),

$$(\mathrm{c}_2(X) \cdot H_i) = 6 + \deg(\Delta_{f_i}),$$

where Δ_{f_i} is the discriminant locus of f_i. By (2.11), Δ_{f_i} is not empty. Hence $\deg(\Delta_{f_i}) > 0$. By (B.3), $\Delta_{f_i} \equiv -4 \, K_{\mathbb{P}^2} - (f_i)_*(K_X^2)$. So we get by the projection formula [De, 1.10]

$$\begin{aligned} \deg(\Delta_{f_i}) &= -4 \, \deg(K_{\mathbb{P}^2}) - \deg\big((f_i)_*(K_X^2)\big) \\ &= -4 \, \deg(\mathcal{O}_{\mathbb{P}^2}(-3)) - \big((f_i)_*(K_X^2) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))\big)_{\mathbb{P}^2} \\ &= 12 - \big((-K_X)^2 \cdot f_i^* \, \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))\big)_X \, . \end{aligned}$$

Since $-K_X$ is ample, $((-K_X)^2 \cdot f_i^* \, \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1)))_X > 0$. Hence $\deg(\Delta_{f_i}) < 12$. We obtain the estimation

$$7 \leq \deg(\Delta_{f_i}) \leq 17. \tag{35}$$

If R_i is of type D_1, then

$$(c_2(X) \cdot H_i) = 12 - (K_{X_t})^2,$$

where X_t is the generic fibre of f_i, by (4.3) above. By (2.14), $1 \leq (K_{X_t})^2 \leq 6$. Hence we get the estimation

$$6 \leq (c_2(X) \cdot H_i) \leq 11. \tag{36}$$

Since R_2 is of type E_2, E_3, E_4, E_5, C_1, C_2, D_1, D_2 or D_3, combining the results of (4.3), (35) and (36) we obtain the following basic estimation:

$$(c_2(X) \cdot H_2) \leq 24 \text{ or } = 45 \tag{37}$$

We are now going to apply this to equation (34) above in order to determine the value of a. We have to distinguish the following cases.

Case $(\mathbf{C_1} - \mathbf{E_3}, \mathbf{E_4}, \mathbf{E_5}, \mathbf{C_1}, \mathbf{D_1})$. By (2.7), (2.11) and (2.14), $\mu_1 = \mu_2 = 1$. So equation (34) takes the following form:

$$24\,a = (H_1 \cdot c_2(X)) + (H_2 \cdot c_2(X))$$

By (35), $7 \leq (H_1 \cdot c_2(X)) \leq 17$. Now we use (37): If $(c_2(X) \cdot H_2) = 45$, we obtain

$$52 = 7 + 45 \leq 24\,a \leq 17 + 45 = 62,$$

which is a contradiction. Hence we must have $(c_2(X) \cdot H_2) \leq 24$. Then we get

$$24\,a \leq 17 + 24 = 41,$$

which implies that $a = 1$.

Case $(\mathbf{C_1} - \mathbf{E_2}, \mathbf{C_2}, \mathbf{D_2})$. By (2.7), (2.11) and (2.14), $\mu_1 = 1$ and $\mu_2 = 2$. So equation (34) takes the following form:

$$24\,a = 2\ (H_1 \cdot c_2(X)) + (H_2 \cdot c_2(X))$$

We claim that a is odd. To check this, assume to the contrary. By (33), $a\,(-K_X) \sim 2\,H_1 + H_2$. Then H_2 would be divisible by 2 in $\mathrm{Pic}(X)$, which is not possible since H_2 is a primitive element of $\mathrm{Pic}(X)$, by(4.1).

By (35), $(H_1 \cdot c_2(X)) \leq 17$. By (4.3), $(c_2(X) \cdot H_2) \leq 24$. Hence we obtain

$$24\,a \leq 2 \cdot 17 + 24 = 58.$$

Since a is odd, it follows that $a = 1$.

Case $(\mathbf{C_1} - \mathbf{D_3})$. By (2.11) and (2.14), $\mu_1 = 1$ and $\mu_2 = 3$. So equation (34) takes the following form:

$$24\,a = 3\ (H_1 \cdot c_2(X)) + (H_2 \cdot c_2(X))$$

We claim that a is odd. To check this, we proceed as follows: By (33), $a\,(-K_X) \sim 3\,H_1 + H_2$. By (31) above, we obtain

$$\begin{aligned} a^3\,(-K_X)^3 &= 3^3\,(H_1^3) + 3 \cdot 3^2\,(H_1^2 \cdot H_2) + 3 \cdot 3\,(H_1 \cdot H_2^2) + (H_2^3) \\ &= 3^3 \cdot 2\,(\ell_1 \cdot H_2) \\ &= 3^3 \cdot 2\,a. \end{aligned}$$

Here we used that $H_1^3 = f_1^* L_1^3 = 0$, and that $f_2 : X \longrightarrow \mathbb{P}^1$ by (2.14) since R_2 is of type D_3, so $L_2 = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1}(1))$ and hence $H_2^2 = f_2^* L_2^2 = 0$. Moreover, we used that $H_1^2 \equiv (2/\mu_1)\,\ell_1 = 2\,\ell_1$, by (4.2). Carrying on, we obtain

$$a^2\,(-K_X)^3 = 3^3 \cdot 2.$$

This shows that a cannot be even.

By (35), $(H_1 \cdot \mathrm{c}_2(X)) \leq 17$. By (4.3), $(\mathrm{c}_2(X) \cdot H_2) = 3$. Hence we obtain

$$24\,a \leq 3 \cdot 17 + 3 = 54.$$

Since a is odd, it follows that $a = 1$.

Case $(\mathbf{C_2} - \mathbf{E_3}, \mathbf{E_4}, \mathbf{E_5}, \mathbf{C_1}, \mathbf{D_1})$. By (2.7), (2.11) and (2.14), $\mu_1 = 2$ and $\mu_2 = 1$. So equation (34) takes the following form:

$$24\,a = (H_1 \cdot \mathrm{c}_2(X)) + 2\ (H_2 \cdot \mathrm{c}_2(X))$$

By (4.3), $(H_1 \cdot \mathrm{c}_2(X)) = 6$. Similarly to case $(C_1 - E_2, C_2, D_2)$ above, we see that a is odd. Now we use (37):

If $(\mathrm{c}_2(X) \cdot H_2) \leq 24$, we obtain $24\,a \leq 6 + 2 \cdot 24$, i.e.,

$$12\,a \leq 3 + 24 = 27.$$

Since a is odd, it follows that $a = 1$.

If $(\mathrm{c}_2(X) \cdot H_2) = 45$, we obtain $24\,a = 6 + 2 \cdot 45$, i.e.,

$$12\,a = 3 + 45 = 48,$$

which is a contradiction.

Case $(\mathbf{C_2} - \mathbf{E_2}, \mathbf{C_2}, \mathbf{D_2})$. By (2.7), (2.11) and (2.14), $\mu_1 = \mu_2 = 2$. So equation (34) takes the following form:

$$24\,a = 2\ (H_1 \cdot \mathrm{c}_2(X)) + 2\ (H_2 \cdot \mathrm{c}_2(X))$$

By (4.3), $(H_1 \cdot \mathrm{c}_2(X)) = 6$. Now we use (4.3):

If R_2 is of type E_2, then $(H_2 \cdot \mathrm{c}_2(X)) = 24/r$, and we get

$$12\,a = 6 + \tfrac{24}{r}.$$

The left hand side is an integral multiple of 12, so $24/r$ is an odd multiple of 6, i.e., $r = 4$. Hence $a = 1$. If R_2 is of type C_2 or D_2, then $(H_2 \cdot \mathrm{c}_2(X)) \leq 6$, and we get

$$12\,a \leq 6 + 6 = 12,$$

which implies that $a = 1$.

Case $(\mathbf{C_2} - \mathbf{D_3})$**.** By (2.11) and (2.14), $\mu_1 = 2$ and $\mu_2 = 3$. So equation (34) takes the following form:

$$24\,a = 3\;(H_1 \cdot \mathrm{c}_2(X)) + 2\;(H_2 \cdot \mathrm{c}_2(X))$$

By (4.3), $(H_1 \cdot \mathrm{c}_2(X)) = 6$ and $(H_2 \cdot \mathrm{c}_2(X)) = 3$. Hence we get

$$24\,a = 3 \cdot 6 + 2 \cdot 3 = 24,$$

which implies that $a = 1$. □

4.2 Classification of primitive Fano threefolds with $\mathrm{b}_2 = 2$

Now we are in a position to start with the actual classification of primitive Fano threefolds X with $\mathrm{b}_2(X) = 2$. We will run through all possible configurations of the types of R_1 and R_2, each time working out their geometric implications by means of proposition 4.4.

As explained in the beginning of the previous section, if we take R_1 to be of type C_1 or C_2 we need to consider the cases that R_2 is of type E_2, E_3, E_4, E_5, C_1, C_2, D_1, D_2 or D_3.

This will give us precisely nine types of primitive Fano threefolds with $\mathrm{b}_2 = 2$, which make up the first part of the Theorem in chapter 1. A more detailed list can be found in appendix A.

4.2.1 Case R_2 is of type E_2, E_3, E_4 or E_5

We consider the case that R_2 is of type E_2, E_3, E_4 or E_5. By (3.1), we are in the following situation:

$$\begin{array}{ccc} X & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\big\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Here, $f_1 : X \longrightarrow \mathbb{P}^2$ is a conic bundle, corresponding to the ray R_1 of type C_1 or C_2, and $f_2 : X \longrightarrow Y$ is a contraction, corresponding to the ray R_2 of type E_2, E_3, E_4 or E_5, to a projective variety Y of dimension 3, and with exceptional divisor D.

We will determine the structure of the conic bundle f_1 by means of the divisor D.

Lemma 4.5. *Extremal rays of X and the morphism $f_1|_D : D \longrightarrow \mathbb{P}^2$ satisfy the following:*

If R_1 is of type C_1, then R_2 is of type E_3 or E_4 and $f_1|_D$ is a double covering.

If R_1 is of type C_2, then R_2 is of type E_2 or E_5 and $f_1|_D$ is an isomorphism.

Proof. First of all, we prove that $f_1|_D$ is finite and surjective. So let us assume to the contrary. Then there exists an irreducible reduced curve C_1 on D which is contained in a fibre of f_1. Since $D|_D$ is negative by (2.7), we therefore have

$$(D \cdot C_1)_X = (D|_D \cdot C_1)_D < 0.$$

Since D is properly contained in X, there exists an irreducible reduced curve C_2 in a fibre of f_1 which is not contained in D. Hence $(D \cdot C_2)_X \geq 0$. By (2.3), both classes $[C_1]$ and $[C_2]$ lie on $R_1 \setminus \{0\}$, so C_1 is numerically equivalent to some positive multiple of C_2 and we get

$$(D \cdot C_1)_X \geq 0.$$

This is a contradiction, so $f_1|_D$ must be finite. In particular, its image is a closed subvariety of $\mathbb{P}^2$ of dimension 2, so $f_1|_D$ is surjective.

By (2.12), the fibres of f_1 are all numerically equivalent to $(2/\mu_1)\,\ell_1$. Since D is reduced by (2.7), we therefore obtain

$$\deg(f_1|_D) = \tfrac{2}{\mu_1}\,(D \cdot \ell_1)_X. \tag{38}$$

We are now going to compute $(D \cdot \ell_1)_X$.

By (4.4), $-K_X \sim \mu_2\, H_1 + \mu_1\, H_2$. Moreover, $H_1^2 \equiv (2/\mu_1)\,\ell_1$ by (4.2). Hence we get

$$\begin{aligned} H_1^2 &\equiv \tfrac{1}{\mu_2^2}\,(-K_X - \mu_1\, H_2)^2 \\ &\equiv \tfrac{1}{\mu_2^2}\left((-K_X)^2 - 2\mu_1\,(-K_X)\cdot H_2 + \mu_1^2\, H_2^2\right). \end{aligned} \tag{39}$$

Since R_2 is of type E_2, E_3, E_4 or E_5, $f_2(D)$ is a point on Y by (2.7). Now since L_2 is ample on Y, $n\,L_2$ is very ample for sufficiently large n ([Ha77, II, 7.6]) and therefore obtained as a hyperplane section with respect to the corresponding embedding of Y into some projective space. By Bertini's theorem [Ha77, II, 8.18], we then see that L_2 is linearly equivalent to a divisor on Y which does not contain the point $f_2(D)$. Since $H_2 = f_2^* L_2$, we conclude that

$$H_2 \cdot D \equiv 0.$$

Thus, using $H_1^2 \equiv (2/\mu_1)\,\ell_1$ and (39), we obtain

$$\tfrac{2}{\mu_1}\,(\ell_1 \cdot D)_X = (H_1^2 \cdot D)_X = \tfrac{1}{\mu_2^2}\,((-K_X)^2 \cdot D)_X = \tfrac{1}{\mu_2^2}\,(K_X|_D)_D^2.$$

As we have already remarked in the proof of (4.3), we can apply the adjunction formula to D even in the case R_2 is of type E_4. We obtain $\mathcal{O}_D(K_X) \cong \omega_D^\circ \otimes \mathcal{O}_D(-D)$, where ω_D° is the dualizing sheaf for D. Thus we have

$$\tfrac{2}{\mu_1}\,(\ell_1 \cdot D)_X = \tfrac{1}{\mu_2^2}\,\mathrm{c}_1\left(\omega_D^\circ \otimes \mathcal{O}_D(-D)\right)^2 . \tag{40}$$

To carry on, we have to distinguish the type of R_1.

If R_1 is of type C_1, then $\mu_1 = 1$. We will consider equation (40) for each type of R_2, using (2.7):

If R_2 is of type E_2, then $\mu_2 = 2$, $D \cong \mathbb{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$. Hence we get

$$2\,(\ell_1 \cdot D) = \tfrac{1}{4}\,\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(-3+1))^2 = 1.$$

If R_2 is of type E_3, then $\mu_2 = 1$, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1,-1)$. Hence we get

$$2\,(\ell_1 \cdot D) = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2+1,-2+1))^2 = 2.$$

If R_2 is of type E_4, then $\mu_2 = 1$, D is isomorphic to an irreducible reduced singular quadric surface in $\mathbb{P}^3$ and $\mathcal{O}_D(D) \cong \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$. By [Ha77, III, 7.11], $\omega_D^\circ \cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(D) \otimes \mathcal{O}_D \cong \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^3}(-4+2) \cong \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^3}(-2)$. Hence we obtain

$$\begin{aligned} 2\,(\ell_1 \cdot D) &= \mathrm{c}_1\left(\mathcal{O}_{\mathbb{P}^3}(-2+1) \otimes \mathcal{O}_D\right)^2 \\ &= \left(\mathrm{c}_1(\mathcal{O}(-1))^2 \cdot \mathrm{c}_1(\mathcal{O}(D))\right)_{\mathbb{P}^3} \\ &= \left(\mathrm{c}_1(\mathcal{O}(-1))^2 \cdot \mathrm{c}_1(\mathcal{O}(2))\right)_{\mathbb{P}^3} \\ &= 2. \end{aligned}$$

If R_2 is of type E_5, then $\mu_2 = 1$, $D \cong \mathbb{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_D(-2)$. Hence we get

$$2\,(\ell_1 \cdot D) = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(-3+2))^2 = 1.$$

Since $(\ell_1 \cdot D)$ is an integer, these equations show that R_2 can only be of type E_3 or E_4. Moreover, in this case, $(\ell_1 \cdot D) = 1$, so $\deg(f_1|_D) = 2$ by (38) above. This proves the first assertion.

If R_1 is of type C_2, then $\mu_1 = 2$ by (2.11). Assume that R_2 were of type E_3 or E_4. Then we would have $\mu_2 = 1$ by (2.7) and we would obtain from (4.4) above that $-K_X \sim H_1 + 2\,H_2$. Hence, by (2.2 (2)) and (4.3),

$$\begin{aligned} 24 &= (\mathrm{c}_2(X) \cdot (-K_X)) \\ &= (\mathrm{c}_2(X) \cdot H_1) + 2\,(\mathrm{c}_2(X) \cdot H_2) \\ &= 6 + \tfrac{48}{r} \end{aligned}$$

for some integer r, which is a contradiction. Thus R_2 is of type E_2 or E_5. Now we can refer to our calculations above which show that equation (40) takes the following forms: If R_2 is of type E_2, then $(\ell_1 \cdot D) = 1$, and if R_2 is of type E_5, then $(\ell_1 \cdot D) = 1$.

Thus, in this case, $f_1|_D$ has degree 1 by (38) above. Hence it is bijective and therefore an isomorphism, by (B.5). $\square$

By (4.5), the following configurations of types of extremal rays of X are possible.

Case ($\mathbf{C_1 - E_3}$ or $\mathbf{E_4}$). By (B.3 (2)), the conic bundle structure of X gives us a natural embedding

$$\begin{array}{ccc} X & \xrightarrow{\iota} & P = \mathbb{P}\left((f_1)_*\mathcal{O}_X(-K_X)\right) \\ & & \Big\downarrow \pi \\ & & \mathbb{P}^2 \end{array} \tag{41}$$

which establishes X as a divisor on some $\mathbb{P}^2$-bundle P over $\mathbb{P}^2$. We are now going to exploit this fact in order to obtain a characterisation of X. First, we have to determine this $\mathbb{P}^2$-bundle, i.e., we have to compute the direct image $(f_1)_*\mathcal{O}_X(-K_X)$ on $\mathbb{P}^2$.

We begin with the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

which yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X - D) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_D(-K_X) \longrightarrow 0.$$

By [Ha77, III, § 8], this induces an exact sequence

$$\begin{aligned} 0 \longrightarrow (f_1)_*\mathcal{O}_X(-K_X - D) \longrightarrow (f_1)_*\mathcal{O}_X(-K_X) \longrightarrow (f_1)_*\mathcal{O}_D(-K_X) \\ \longrightarrow \mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X - D). \end{aligned}$$

We claim that $\mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X-D)$ vanishes: Since f_1 is flat by (B.3 (1)), this follows essentially from the semicontinuity theorem [Ha77, III, 12.9], as follows.

Let X_t denote the fibre of f_1 over $t \in \mathbb{P}^2$, considered as a closed subscheme of X. Then we have to verify that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 1$ for all $t \in \mathbb{P}^2$.

For generic t, $X_t \cong \mathbb{P}^1$ by (B.3 (4)). Since $f_1|_D$ has degree 2 by (4.5), D is reduced and $X_t \equiv 2\,\ell_1$ by (2.12),

$$\deg_{X_t}((-K_X-D)|_{X_t}) = ((-K_X-D)\cdot X_t)_X = 2\,(-K_X\cdot\ell_1)_X-(D\cdot X_t)_X = 2-2 = 0.$$

Hence we conclude that $\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t}$.

For special $t \in \Delta_{f_1}$, $X_t = l_1 + l_2$ is degenerate, where $l_i \cong \mathbb{P}^1$, by (B.3 (4)). Moreover, $l_i \equiv \ell_1$ by (2.11) and, as we have seen in the proof of (4.5) above, $(D \cdot \ell_1)_X = 1$. We obtain

$$\deg_{l_i}((-K_X - D)|_{l_i}) = ((-K_X - D) \cdot l_i)_X = (-K_X \cdot \ell_1)_X - (D \cdot \ell_1)_X = 1 - 1 = 0.$$

Hence we conclude that $-K_X - D$ is trivial along both l_1 and l_2, which implies $\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t}$.

Summing up, we have proved that

$$\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t} \tag{42}$$

for all $t \in \mathbb{P}^2$.

Since f_1 is a conic bundle X_t is connected, by (B.3 (4)). Hence

$$\mathrm{h}^0(X_t, \mathcal{O}_{X_t}) = 1 \tag{43}$$

for all $t \in \mathbb{P}^2$. Moreover, since f_1 is flat it follows from [Ha77, III, 9.10] in combination with [Ha77, III, Ex. 5.3] that $\chi(\mathcal{O}_{X_t}) = \mathrm{h}^0(X_t, \mathcal{O}_{X_t}) - \mathrm{h}^1(X_t, \mathcal{O}_{X_t})$ is independent of t. Hence we conclude that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t})$ is independent of t. Thus we may compute its value using the generic fibre $X_t \cong \mathbb{P}^1$. But then it is immediate that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}) = 0$, by [Ha77, III, 5.1].

By (42) above, this implies that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 0$ for all $t \in \mathbb{P}^2$. Therefore, $\mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X - D) = 0$.

Hence we have an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X(-K_X - D) \longrightarrow (f_1)_*\mathcal{O}_X(-K_X) \longrightarrow (f_1)_*\mathcal{O}_D(-K_X) \longrightarrow 0. \tag{44}$$

We are now going to work out the direct images appearing in this sequence.

Note that, no matter whether R_2 is of type E_3 or E_4, D is isomorphic to an irreducible reduced quadric surface in $\mathbb{P}^3$, by (2.7). To see this in case R_2 is of type E_3 we use the Segre embedding $D \cong \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ ([Ha77, I, Ex. 2.15]). We will write, for $n \in \mathbb{Z}$, $\mathcal{O}_D(n) = \mathcal{O}_{\mathbb{P}^3}(n) \otimes \mathcal{O}_D$. Then $\mathcal{O}_D(-1,-1)$ corresponds to $\mathcal{O}_D(-1)$ if R_2 is of type E_3. Thus in both cases we have

$$\mathcal{O}_D(D) \cong \mathcal{O}_D(-1). \tag{45}$$

Recall from (4.5) that $f_1|_D : D \longrightarrow \mathbb{P}^2$ is a double covering. For our computations we will need the following

Claim. For all $n \in \mathbb{Z}$, $(f_1|_D)^* \,\mathcal{O}_{\mathbb{P}^2}(n) \cong \mathcal{O}_D(n)$.

It will be enough to prove this for $n = 1$. If R_2 is of type E_4, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ and we can write $(f_1|_D)^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1) \cong \mathcal{O}_D(a, b)$, $a, b \in \mathbb{Z}$. By the projection formula [De, 1.10],

$$2\,a\,b = \mathrm{c}_1(\mathcal{O}_D(a,b))^2 = \deg(f_1|_D) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 2 \cdot 1 = 2.$$

We conclude $a\,b = 1$. If we had $a, b < 0$ then $\mathcal{O}_D(a,b)$ would have no nontrivial global sections which is impossible since $\mathcal{O}_D(a,b)$ is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$. Thus we must have $a, b > 0$, whence $a = b = 1$. By the above, $\mathcal{O}_D(1,1)$ corresponds to $\mathcal{O}_D(1)$.

If R_2 is of type E_3, the hyperplane section homomorphism $\mathrm{Pic}(\mathbb{P}^3) \longrightarrow \mathrm{Pic}(D)$ is an isomorphism, by [Ha77, II, Ex. 6.2 and 6.5 (c)]. Hence we can write $(f_1|_D)^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_D(a)$, $a \in \mathbb{Z}$. By the projection formula,

$$\mathrm{c}_1(\mathcal{O}_D(a))^2 = \deg(f_1|_D) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 2 \cdot 1 = 2.$$

Since D is a quadric surface in $\mathbb{P}^3$,

$$\begin{aligned} \mathrm{c}_1(\mathcal{O}_D(a))^2 &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(a) \otimes \mathcal{O}_D)^2 \\ &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(a))^2 \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(D)) \\ &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(a))^2 \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^3}(2)) \\ &= 2\,a^2. \end{aligned}$$

Combining both equations, we conclude $a^2 = 1$. By [Ha77, III, Ex. 5.5], the natural map $\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a)) \longrightarrow \mathrm{H}^0(D, \mathcal{O}_D(a))$ is surjective. This shows that if $a < 0$ then $\mathcal{O}_D(a)$ would have no nontrivial global sections which is impossible since $\mathcal{O}_D(a)$ is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$. Thus we must have $a > 0$, whence $a = 1$. The claim is proved.

Now we turn to the computation of the first sheaf in sequence (44). Combining (42) and (43) above, we obtain that $\mathrm{h}^0(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 1$ for all $t \in \mathbb{P}^2$. Since f_1 is flat, it follows from [Ha77, III, 12.9] that $(f_1)_*\mathcal{O}_X(-K_X - D)$ is locally free of rank 1 on $\mathbb{P}^2$. Hence we can write $(f_1)_*\mathcal{O}_X(-K_X - D) \cong \mathcal{O}_{\mathbb{P}^2}(k)$, $k \in \mathbb{Z}$. Then

$$\mathcal{O}_X(-K_X - D) \cong f_1^*\,(f_1)_*\mathcal{O}_X(-K_X - D) \cong f_1^*\,\mathcal{O}_{\mathbb{P}^2}(k).$$

To figure out the value of k, note that by the claim above and the adjunction formula

$$\mathcal{O}_D(k) \cong (f_1|_D)^*\,\mathcal{O}_{\mathbb{P}^2}(k) \cong \mathcal{O}_D(-K_X - D) \cong (\omega_D^\circ)^{-1},$$

where ω_D° is the dualizing sheaf for D. By [Ha77, III, 7.11],

$$\omega_D^\circ \cong \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(D) \otimes \mathcal{O}_D \cong \mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_D \cong \mathcal{O}_D(-2). \tag{46}$$

Hence we obtain $\mathcal{O}_D(k) \cong \mathcal{O}_D(2)$, so $k = 2$. Whence

$$(f_1)_*\,\mathcal{O}_X(-K_X - D) \cong \mathcal{O}_{\mathbb{P}^2}(2). \tag{47}$$

In order to compute the third sheaf in sequence (44), note that these calculations further show that $\mathcal{O}_D(-K_X - D) \cong \mathcal{O}_D(2)$. Combining this with $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1)$ from (45) we obtain

$$\mathcal{O}_D(-K_X) \cong \mathcal{O}_D(1).$$

Using the claim above and the projection formula [Ha77, II, Ex. 5.1 (d)], we therefore get

$$\begin{aligned}(f_1)_* \mathcal{O}_D(-K_X) &\cong (f_1)_*\mathcal{O}_D(1)\\ &\cong (f_1)_* \left(\mathcal{O}_D \otimes (f_1)^* \mathcal{O}_{\mathbb{P}^2}(1)\right)\\ &\cong (f_1)_*\mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^2}(1)\\ &\cong (f_1|_D)_*\mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}^2}(1).\end{aligned}$$

Since $f_1|_D : D \longrightarrow \mathbb{P}^2$ is a double covering by (4.5), it is cyclic. Hence, by ([BPV, I, 17.2]), $(f_1|_D)_* \mathcal{O}_D \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$, where $b \in \mathbb{Z}$ such that $\omega_D^\circ \cong (f_1|_D)^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(b))$. Since $\omega_D^\circ \cong \mathcal{O}_D(-2)$ by (46) this implies, by the claim above,

$$\mathcal{O}_D(-2) \cong (f_1|_D)^* \left(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(b)\right) \cong (f_1|_D)^* \mathcal{O}_{\mathbb{P}^2}(b-3) \cong \mathcal{O}_D(b-3).$$

We conclude $b = 1$, so $(f_1|_D)_* \mathcal{O}_D \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and we obtain

$$\begin{aligned}(f_1)_* \mathcal{O}_D(-K_X) &\cong (\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(1)\\ &\cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1).\end{aligned} \tag{48}$$

Plugging in the results of (47) and (48), sequence (44) takes the following form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(2) \longrightarrow (f_1)_*\mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0.$$

By [Ha77, III, Ex. 6.1], this sequence splits. For

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O}(2)) &\cong \mathrm{Ext}^1(\mathcal{O}, (\mathcal{O} \oplus \mathcal{O}(-1)) \otimes \mathcal{O}(2))\\ &\cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(2) \oplus \mathcal{O}(1))\\ &\cong \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(2) \oplus \mathcal{O}(1))\end{aligned}$$

by [Ha77, III, 6.3 and 6.7], and this cohomology group vanishes because the natural exact sequence on $\mathbb{P}^2$

$$0 \longrightarrow \mathcal{O}(2) \longrightarrow \mathcal{O}(2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

induces an exact sequence in cohomology

$$\mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(2)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(2) \oplus \mathcal{O}(1)) \longrightarrow \mathrm{H}^1(\mathbb{P}^2, \mathcal{O}(1))$$

whose first and third term vanishes by [Ha77, III, 5.1 (b)]. Hence we conclude

$$(f_1)_* \mathcal{O}_X(-K_X) \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2).$$

Embedding (41) now takes the form

$$\begin{array}{ccc} X & \xrightarrow{\iota} & P = \mathbb{P}(\mathcal{E}) \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^2 \end{array}$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.

In particular, X is a divisor on P, and since $\operatorname{Pic}(P) \cong \pi^* \operatorname{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}$ by [Ha77, II, Ex. 7.9] we can write

$$\mathcal{O}_P(X) \cong \mathcal{O}_P(n) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(a) \tag{49}$$

where $n, a \in \mathbb{Z}$. We are now going to figure out these coefficients. First, by the adjunction formula,

$$\mathcal{O}_X(-K_X) \cong \mathcal{O}_P(-K_P) \otimes \mathcal{O}_P(-X) \otimes \mathcal{O}_X. \tag{50}$$

By (B.2 (1)) we obtain

$$\begin{aligned} \mathcal{O}_P(-K_P) &\cong \mathcal{O}_P(3) \otimes \pi^* \left(\omega_{\mathbb{P}^2} \otimes \det \mathcal{E}\right)^{-1} \\ &\cong \mathcal{O}_P(3) \otimes \pi^* \left(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(3)\right) \\ &\cong \mathcal{O}_P(3), \end{aligned} \tag{51}$$

using

$$\begin{aligned} \det \mathcal{E} &\cong \textstyle\bigwedge^3 \left(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)\right) \\ &\cong \mathcal{O} \otimes \mathcal{O}(1) \otimes \mathcal{O}(2) \\ &\cong \mathcal{O}(3). \end{aligned}$$

By (B.3 (2)), the embedding $\iota : X \longrightarrow P$ is induced by the exact sequence

$$f_1^* \, (f_1)_* \, \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow 0$$

on X, which implies

$$\mathcal{O}_X(-K_X) \cong \iota^* \, \mathcal{O}_P(1) \cong \mathcal{O}_P(1) \otimes \mathcal{O}_X \tag{52}$$

by [Ha77, II, 7.12]. Plugging in this, (51) and (49), relation (50) becomes

$$\mathcal{O}_P(1) \otimes \mathcal{O}_X \cong \mathcal{O}_P(3-n) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(-a) \otimes \mathcal{O}_X.$$

Comparing coefficients we obtain $n = 2$ and $a = 0$, whence

$$\mathcal{O}_P(X) \cong \mathcal{O}_P(2). \tag{53}$$

This is the desired explicit description of X as a divisor on the bundle $P = \mathbb{P}(\mathcal{E})$.

In particular, we can use it to compute $(-K_X)^3$. In our case here, formula (B.2 (4)) applies and we get

$$\begin{aligned}(-K_X)^3 &= 2\ \mathrm{c}_1(\mathcal{E})^2 - 2\ \mathrm{c}_2(\mathcal{E}) + 4\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{O}) + 6\ \mathrm{c}_1(\mathcal{E}) \cdot K_{\mathbb{P}^2} \\ &\quad + 9\ \mathrm{c}_1(\mathcal{O}) \cdot K_{\mathbb{P}^2} + 6\, K_{\mathbb{P}^2}^2 + 3\ \mathrm{c}_1(\mathcal{O})^2.\end{aligned}$$

Since $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$, by [Ha77, A, 3.C3 and C5]

$$\begin{aligned}&\mathrm{c}_1(\mathcal{E}) = \mathrm{c}_1(\mathcal{O}) + \mathrm{c}_1(\mathcal{O}(1)) + \mathrm{c}_1(\mathcal{O}(2)) = \mathrm{c}_1(\mathcal{O}(3)),\\ &\mathrm{c}_2(\mathcal{E}) = \mathrm{c}_1(\mathcal{O}(1)) \cdot \mathrm{c}_1(\mathcal{O}(2)) = 2,\\ &\mathrm{c}_1(\mathcal{O}) = 0.\end{aligned}$$

Thus $\mathrm{c}_1(\mathcal{E})^2 = 9$. Moreover, $K_{\mathbb{P}^2} = \mathcal{O}(-3)$, so $\mathrm{c}_1(\mathcal{E}) \cdot K_{\mathbb{P}^2} = -9$ and $K_{\mathbb{P}^2}^2 = 9$. Plugging these results into our formula above, we obtain

$$\begin{aligned}(-K_X)^3 &= 2 \cdot 9 - 2 \cdot 2 + 4 \cdot 0 + 6 \cdot (-9) + 9 \cdot 0 + 6 \cdot 9 + 3 \cdot 0 \\ &= 14.\end{aligned}$$

We are now going to work out a more explicit description of X as a double covering of the bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$.

Recall from the above that we have an embedding

$$\begin{tikzcd} X \arrow[r, "\iota"] & P = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) \arrow[d, "\pi"] \\ & \mathbb{P}^2 \end{tikzcd}$$

such that X is a divisor on P linearly equivalent to $\mathcal{O}_P(2)$. By [Ha77, II, 7.12 and Ex. 7.8], there is a section $\sigma : \mathbb{P}^2 \longrightarrow P$ of the bundle P corresponding to the natural exact sequence

$$0 \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{O} \longrightarrow 0 \tag{54}$$

on $\mathbb{P}^2$ and satisfying $\sigma^* \mathcal{O}_P(1) \cong \mathcal{O}_{\mathbb{P}^2}$. Let $S = \sigma(\mathbb{P}^2)$ denote its image. Then $\mathcal{O}_P(1) \otimes \mathcal{O}_S \cong (\sigma \circ \pi|_S)^* \mathcal{O}_P(1) \cong (\pi|_S)^* \sigma^* \mathcal{O}_P(1) \cong (\pi|_S)^* \mathcal{O}_{\mathbb{P}^2} \cong \mathcal{O}_S$, i.e., $\mathcal{O}_P(1)$ is trivial along S.

We can use this to show that $X \cap S = \emptyset$. Namely, if we assume that $X \cap S$ were not empty, it would have dimension at least 1, by [Ha77, I, 7.1], for S is a surface and P has dimension 4. Since P is projective ([Ha77, II, 7.10]), $X \cap S$ would therefore contain a curve C. Now we know from (52) that $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(1)$, and, as we have just seen, $\mathcal{O}_S(1) \cong \mathcal{O}_S$. Hence

$$(-K_X \cdot C)_X = (c_1(\mathcal{O}_X(1)) \cdot C)_X = (c_1(\mathcal{O}_S(1)) \cdot C)_S = (c_1(\mathcal{O}_S) \cdot C)_S = 0.$$

On the other hand,

$$(-K_X \cdot C)_X > 0$$

since $-K_X$ is ample, which is a contradiction.

Now let us consider the blowing-up $\tilde{\pi} : \tilde{P} \longrightarrow P$ of P along the section S.

To go on, we will need a local description of this blowing-up: As it is explained in [Ha77, II, § 7], $\tilde{P}$ is obtained as the **Proj** of the sheaf of graded algebras $\mathcal{S} = \bigoplus_{d\geq 0} \mathcal{I}^d$ associated to the ideal sheaf $\mathcal{I}$ of S on P. Let U be any affine open subset of $\mathbb{P}^2$. Then $W = \pi^{-1}(U)$ is an open subset of P and $\tilde{\pi}^{-1}(W) \cong \operatorname{Proj} \mathcal{S}(W)$. To obtain a local description of W, fix sections $\xi_0 \in \mathcal{O}(U)$, $\xi_1 \in \mathcal{O}(1)(U)$ and $\xi_2 \in \mathcal{O}(2)(U)$ which are trivializing over U. They generate the symmetric algebra $\bigoplus_{d\geq 0} \mathrm{S}^d(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$ locally over U, so $W = \pi^{-1}(U) \cong \operatorname{Proj}\, \mathcal{O}_U[\xi_0, \xi_1, \xi_2] \cong \mathbb{P}^2_U$. Since S is the section of P corresponding to sequence (54) above, its restriction to $W \cong \mathbb{P}^2_U$ is given as the common zero locus of $\pi^*\xi_1$ and $\pi^*\xi_2$, i.e., locally over W the ideal sheaf $\mathcal{I}$ is generated by $\pi^*\xi_1$ and $\pi^*\xi_2$. Using homogeneous coordinates with respect to ξ_0, ξ_1, ξ_2 on the relative projective space $\mathbb{P}^2_U$, we may therefore identify $S \cap W$ with the single point $(1, 0, 0)$ on $\mathbb{P}^2_U$. This way of thinking of S locally as a point on the relative projective space $\mathbb{P}^2_U$ will automatically lead us to an alternative description of the blowing up of P along S.

Namely, it follows from the construction above in combination with [Ha77, II, 7.12.1] that blowing up P along S is, locally over W, the same as blowing up $\mathbb{P}^2_U$ in the point $(1, 0, 0)$. Arguing in complete analogy with [Ha77, V, 2.11.4] we then see that $\mathbb{P}^2_U$ blown up in $(1, 0, 0)$ is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^1_U$, where $\mathbb{P}^1_U$ is identified with the set of points in $\mathbb{P}^2_U$ whose first homogeneous coordinate vanishes. Moreover, the exceptional divisor of this blowing up is a section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$.

Since $\mathbb{P}^1_U$ is identified with the $\mathbb{P}^1$-bundle $V_7'|_U = \mathbb{P}(\mathcal{O}_U(1) \oplus \mathcal{O}_U(2))$ over U, we can express our result by saying that $\tilde{P}|_W$ is isomorphic to the bundle $\mathbb{P}(\mathcal{O}_{V_7'|_U} \oplus \mathcal{O}_{V_7'|_U}(1))$ over V_7'.

So far, we have only given local descriptions of globally existing objects. Hence our construction above globalises and we obtain an isomorphism

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\cong} & N' = \mathbb{P}\left(\mathcal{O}_{V_7'} \oplus \mathcal{O}_{V_7'}(1)\right) \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \nu'} \\ P = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) & & V_7' = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(2)) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle p'} \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{array} \tag{55}$$

under which the exceptional divisor of the blowing up of P along its section S corresponds to a section T' of $\nu' : N' \longrightarrow V_7'$.

Note that since X is a divisor on P disjoint from S, X is embedded into N' as a divisor disjoint from T'.

We can still simplify this situation by twisting the bundle $V_7' = \mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(2))$ by $\mathcal{O}(-1)$. Then we get a bundle $p : V_7 = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1)) \longrightarrow \mathbb{P}^2$. By [Ha77, II, 7.9], there is an isomorphism $\varphi : V_7' \longrightarrow V_7$, commuting with the projections p' and p, under which $\mathcal{O}_{V_7'}(1) \cong (\varphi^*)^{-1}\,\mathcal{O}_{V_7}(1) \otimes p'^*\,\mathcal{O}(1)$. Thus the bundle

$$\nu' : N' = \mathbb{P}(\mathcal{O}_{V_7'} \oplus \mathcal{O}_{V_7'}(1)) \longrightarrow V_7'$$

is isomorphic to the bundle

$$\nu : N = \mathbb{P}(\mathcal{O}_{V_7} \oplus \mathcal{O}_{V_7}(1) \otimes p^*\,\mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow V_7. \tag{56}$$

By (B.2 (1)),

$$\begin{aligned} \omega_{V_7} &\cong \mathcal{O}_{V_7}(-2) \otimes p^*\left(\omega_{\mathbb{P}^2} \otimes \det(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))\right) \\ &\cong \mathcal{O}_{V_7}(-2) \otimes p^*\left(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(1)\right) \\ &\cong \mathcal{O}_{V_7}(-2) \otimes p^*\,\mathcal{O}_{\mathbb{P}^2}(-2). \end{aligned}$$

Thus we conclude $\mathcal{O}_{V_7}(1) \otimes p^*\,\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{V_7}(-\frac{1}{2}K_{V_7})$. Using this, bundle (56) can be written as

$$\nu : N = \mathbb{P}\left(\mathcal{O}_{V_7} \oplus \mathcal{O}_{V_7}(-\tfrac{1}{2}K_{V_7})\right) \longrightarrow V_7.$$

Moreover, we write T for the image of the section T' under φ.

Modifying diagram (55) appropriately, we can now summarize our results: There is an isomorphism

$$\begin{tikzcd} \tilde{P} \arrow[r, "\cong"] \arrow[d, "\tilde{\pi}"] & N = \mathbb{P}\left(\mathcal{O}_{V_7} \oplus \mathcal{O}_{V_7}(-\frac{1}{2}K_{V_7})\right) \arrow[d, "\nu"] \\ P = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) \arrow[d, "\pi"] & V_7 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \arrow[d, "p"] \\ \mathbb{P}^2 & \mathbb{P}^2 \end{tikzcd} \tag{57}$$

under which the exceptional divisor of the blowing up of P along its section S corresponds to a section T of the bundle $\nu : N \longrightarrow V_7$. Moreover, X is embedded into N as a divisor disjoint from T.

Now we claim that the induced morphism $\nu|_X : X \longrightarrow V_7$ is a double covering: To see this, we first check that $\nu|_X$ is finite and surjective: Consider any fibre F of ν. Let $t = (p \circ \nu)(F)$ be its projection down to $\mathbb{P}^2$. Since $F \cong \mathbb{P}^1$ contains a point of the section T, it cannot be contained in X, for X is disjoint from T. This shows that F intersects X in at most finitely many points, i.e., $\nu|_X$ is finite. In particular, its image is a closed subvariety of V_7 of dimension 3, so $\nu|_X$

is surjective. Next, unravelling the construction of isomorphism (57) we see that this isomorphism takes F to a line G in the fibre $P_t = \pi^{-1}(t) \cong \mathbb{P}^2$ of the bundle P. But we know from (53) that $\mathcal{O}_P(X) \cong \mathcal{O}_P(2)$ on P. Thus X restricts to $\mathcal{O}(2)$ on the fibre $P_t \cong \mathbb{P}^2$. Hence

$$(X \cdot F)_N = (X \cdot G)_P = (X|_{P_t} \cdot G)_{P_t} = (\mathrm{c}_1(\mathcal{O}(2)) \cdot G)_{\mathbb{P}^2} = 2. \tag{58}$$

This shows that $\nu|_X$ has degree 2.

Since ν is a double covering it is cyclic. Hence, by [BPV, I, 17.1 (iii)], its branch locus B is the divisor on V_7 determined by

$$\mathcal{O}_X(K_X) \cong \nu^* \left(\mathcal{O}_{V_7}(K_{V_7}) \otimes \mathcal{O}_{V_7}(\tfrac{1}{2}\, B)\right) \otimes \mathcal{O}_X. \tag{59}$$

In order to compute the left hand side of this relation, we need to describe X as a divisor on N. Since $\operatorname{Pic}(N) \cong \nu^* \operatorname{Pic}(V_7) \oplus \mathbb{Z}$ by [Ha77, II, Ex. 7.9], we can write

$$X \sim \nu^* L + n\, \xi$$

where $L \in \operatorname{Pic}(V_7)$, $n \in \mathbb{Z}$ and $\xi = \mathcal{O}_N(1))$. Since X is disjoint from the section T of ν, it follows that $L \sim 0$. Then, intersecting with any fibre F of ν we obtain, by (B.1) and (58), $2 = (X \cdot F)_N = n\,(\xi \cdot F)_N = n$. Hence

$$\mathcal{O}_N(X) \cong \mathcal{O}_N(2).$$

Moreover, by (B.2 (1)),

$$\begin{aligned} \omega_N &\cong \mathcal{O}_N(-2) \otimes \nu^* \left(\mathcal{O}_{V_7}(K_{V_7}) \otimes \det\left(\mathcal{O}_{V_7} \oplus \mathcal{O}_{V_7}(-\tfrac{1}{2} K_{V_7})\right)\right) \\ &\cong \mathcal{O}_N(-2) \otimes \nu^* \mathcal{O}_{V_7}(\tfrac{1}{2} K_{V_7}). \end{aligned}$$

Therefore, by the adjunction formula,

$$\begin{aligned} \mathcal{O}_X(K_X) &\cong \omega_N \otimes \mathcal{O}_N(X) \otimes \mathcal{O}_X \\ &\cong \mathcal{O}_N(-2) \otimes \nu^* \mathcal{O}_{V_7}(\tfrac{1}{2} K_{V_7}) \otimes \mathcal{O}_N(2) \otimes \mathcal{O}_X \\ &\cong \nu^* \mathcal{O}_{V_7}(\tfrac{1}{2} K_{V_7}) \otimes \mathcal{O}_X. \end{aligned}$$

Comparing with (59) we see that $B \sim -K_{V_7}$.

We can now state our final result: X is isomorphic to a double covering of $V_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ whose branch locus is a member of $|-K_{V_7}|$. This is type no. 3 in table A.1.

Case ($\mathbf{C_2 - E_2}$ or $\mathbf{E_5}$). By (2.11), the conic bundle f_1 is isomorphic to the projective space bundle associated to some locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^2$. Hence we can write

$$f_1 : X \cong \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^2.$$

We are now going to determine this sheaf $\mathcal{E}$ by means of the divisor D.

By (4.5) above, $f_1|_D : D \longrightarrow \mathbb{P}^2$ is an isomorphism. The standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

By [Ha77, III, § 8], this induces an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X \longrightarrow (f_1)_*\mathcal{O}_X(D) \longrightarrow (f_1)_*\mathcal{O}_D(D) \longrightarrow \mathrm{R}^1(f_1)_*\mathcal{O}_X.$$

Since $\mathrm{R}^1(f_1)_*\mathcal{O}_X = 0$ by (2.3), we arrive at an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X \longrightarrow (f_1)_*\mathcal{O}_X(D) \longrightarrow (f_1)_*\mathcal{O}_D(D) \longrightarrow 0. \tag{60}$$

We are now going to work out the direct images appearing in this sequence.

By [Ha77, II, 7.11], it is immediate that

$$(f_1)_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^2}. \tag{61}$$

Since $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{P}(\mathcal{E})) \cong f_1^* \,\mathrm{Pic}(\mathbb{P}^2) \oplus \mathbb{Z}$ by [Ha77, II, Ex. 7.9], we can write

$$\mathcal{O}_X(D) \cong f_1^*\mathcal{L} \otimes \mathcal{O}_X(n)$$

where $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^2)$ and $n \in \mathbb{Z}$. Intersecting D with any fibre F of f_1 we obtain, by (B.1),

$$1 = (D \cdot F) = (\mathrm{c}_1(\mathcal{O}_X(D)) \cdot F) = (f_1^* \,\mathrm{c}_1(\mathcal{L}) \cdot F) + n\,(\mathrm{c}_1(\mathcal{O}_X(1)) \cdot F) = n.$$

Hence the projection formula [Ha77, II, Ex. 5.1 (d)] together with [Ha77, II, 7.11] yields

$$\begin{aligned} (f_1)_*\mathcal{O}_X(D) &\cong (f_1)_*(f_1^*\mathcal{L} \otimes \mathcal{O}_X(1)) \\ &\cong \mathcal{L} \otimes (f_1)_*\mathcal{O}_X(1) \\ &\cong \mathcal{L} \otimes \mathcal{E}. \end{aligned} \tag{62}$$

When computing the direct image $(f_1)_*\mathcal{O}_D(D)$ we have to bear in mind that this sheaf depends on the type of R_2. Namely, by (2.7), $\mathcal{O}_D(D) \cong \mathcal{O}_D(e)$, where $e = -1$ if R_2 is of type E_2 and $e = -2$ if R_2 is of type E_5. Since $f_1|_D$ is an isomorphism we therefore obtain

$$\begin{aligned} (f_1)_*\mathcal{O}_D(D) &\cong (f_1|_D)_*\mathcal{O}_D(D) \\ &\cong (f_1|_D)_*\mathcal{O}_D(e) \\ &\cong \mathcal{O}_{\mathbb{P}^2}(e). \end{aligned} \tag{63}$$

Plugging in the results of (61), (62) and (63), sequence (60) takes the following form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{L} \otimes \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(e) \longrightarrow 0$$

By [Ha77, III, Ex. 6.1], this sequence splits. For

$$\mathrm{Ext}^1(\mathcal{O}(e), \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-e)) \cong H^1(\mathbb{P}^2, \mathcal{O}(-e))$$

by [Ha77, III, 6.7 and 6.3], and this cohomology group vanishes by the Kodaira vanishing theorem [We, VI, 2.4] because $\mathcal{O}(-e) \otimes \omega_{\mathbb{P}^2}^{-1} \cong \mathcal{O}(-e) \otimes \mathcal{O}(3) \cong \mathcal{O}(3-e)$ is ample on $\mathbb{P}^2$ since $e \leq 0$. Hence we conclude that

$$\mathcal{L} \otimes \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(e).$$

Then we obtain, by [Ha77, II, 7.9],

$$\begin{aligned} X &\cong \mathbb{P}(\mathcal{E}) \\ &\cong \mathbb{P}(\mathcal{L} \otimes \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-e)) \\ &\cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-e)\right). \end{aligned} \tag{64}$$

Now it remains to compute $(-K_X)^3$. In our case here, formula (B.2 (2)) applies and we get

$$(-K_X)^3 = 2\ \mathrm{c}_1(\mathcal{O} \oplus \mathcal{O}(-e))^2 - 8\ \mathrm{c}_2(\mathcal{O} \oplus \mathcal{O}(-e)) + 6\,(K_{\mathbb{P}^2})^2.$$

By [Ha77, A, 3.C3 and C5],

$$\begin{aligned} \mathrm{c}_1(\mathcal{O} \oplus \mathcal{O}(-e)) &= \mathrm{c}_1(\mathcal{O}) + \mathrm{c}_1(\mathcal{O}(-e)) \\ &= \mathrm{c}_1(\mathcal{O}(-e)), \\ \mathrm{c}_2(\mathcal{O} \oplus \mathcal{O}(-e)) &= \mathrm{c}_1(\mathcal{O}) \cdot \mathrm{c}_1(\mathcal{O}(-e)) \\ &= 0. \end{aligned}$$

Thus $\mathrm{c}_1(\mathcal{O} \oplus \mathcal{O}(-e))^2 = e^2$. Moreover, $(K_{\mathbb{P}^2})^2 = \mathrm{c}_1(\mathcal{O}(-3))^2 = 9$. Plugging these results into the formula above, we obtain

$$(-K_X)^3 = 2 \cdot e^2 - 0 + 6 \cdot 9 = 2\,e^2 + 54. \tag{65}$$

Resubstituting the values of e into (64) and (65) we finally get:

If R_2 is of type E_2, then $e = -1$ and

$$X \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\right),$$
$$(-K_X)^3 = 2 \cdot (-1)^2 + 54 = 56.$$

This is type no. 8 in table A.1.

If R_2 is of type E_5, then $e = -2$ and

$$X \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\right),$$
$$(-K_X)^3 = 2 \cdot (-2)^2 + 54 = 62.$$

This is type no. 9 in table A.1.

4.2.2 Case R_2 is of type C_1 or C_2

We consider the case that R_2 is of type C_1 or C_2. By (3.1), we are in the following situation:

$$\begin{tikzcd} X \arrow[r, "f_2"] \arrow[d, "f_1"'] & \mathbb{P}^2 \\ \mathbb{P}^2 & \end{tikzcd}$$

Here, $f_1, f_2 : X \longrightarrow \mathbb{P}^2$ are conic bundles corresponding to distinct rays R_1 and R_2 of type C_1 or C_2.

We will describe X as a covering space of its image under the morphism

$$f = (f_1, f_2) : X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

We need a couple of lemmas.

Lemma 4.6. *The morphism f is finite.*

Proof. Assume to the contrary. Then there exists an irreducible reduced curve on X which is contained in a fibre of f. Then this curve is contracted by both f_1 and f_2. Hence, by (2.3), the class of this curve is contained in $R_1 \cap R_2 = \{0\}$, which is a contradiction. $\square$

Since f is finite, $f(X)$ is a closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ of dimension 3. In particular, it is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$. The morphism f induces a finite morphism

$$\tilde{f} : X \longrightarrow f(X).$$

We are going to consider the divisor

$$f_*(X) = \deg(\tilde{f}) \cdot f(X) \tag{66}$$

on $\mathbb{P}^2 \times \mathbb{P}^2$ (cf. [Ha77, A, § 1]).

Note that in our case here, $L_i = \mathcal{O}(1)$ on the i-th factor of $\mathbb{P}^2 \times \mathbb{P}^2$. Hence $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$ is generated by the divisors

$$M_i = \pi_i^* L_i,$$

$i = 1, 2$, where $\pi_i : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ denotes the canonical projection onto the i-th factor. In particular, $f^* M_i = (f^* \circ \pi_i^*)\, L_i = f_i^*\, L_i = H_i$.

Lemma 4.7. *$f_*(X)$ is linearly equivalent to the divisor*

$$\tfrac{2}{\mu_2} M_1 + \tfrac{2}{\mu_1} M_2.$$

Proof. Since $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$ is generated by the divisors M_i, we can write

$$f_*(X) \sim a_1\, M_1 + a_2\, M_2$$

with coefficients $a_1, a_2 \in \mathbb{Z}$. Fix a curve $C_1 = (\text{point} \times \text{line})$ on $\mathbb{P}^2 \times \mathbb{P}^2$. By construction, $M_1 \sim (\text{line} \times \mathbb{P}^2)$ and $M_2 \sim (\mathbb{P}^2 \times \text{line})$, whence $(M_1 \cdot C_1) = 0$ and $(M_2 \cdot C_1) = 1$. We obtain

$$(f_*(X) \cdot C_1)_{\mathbb{P}^2 \times \mathbb{P}^2} = a_2.$$

Since $M_1^2 \cdot M_2 \equiv (\text{point} \times \mathbb{P}^2) \cdot (\mathbb{P}^2 \times \text{line}) \equiv (\text{point} \times \text{line})$, we have $M_1^2 \cdot M_2 \equiv C_1$. By (4.2), $H_1^2 \equiv (2/\mu_1)\, \ell_1$. Hence, by the projection formula [Ha77, A, 1.A4] and (4.4), we obtain

$$\begin{aligned}
(f_*(X) \cdot C_1)_{\mathbb{P}^2 \times \mathbb{P}^2} &= \big(f_*(X) \cdot M_1^2 \cdot M_2\big)_{\mathbb{P}^2 \times \mathbb{P}^2} \\
&= \big(X \cdot (f^* M_1)^2 \cdot f^* M_2\big)_X \\
&= \big(H_1^2 \cdot H_2\big)_X \\
&= \tfrac{2}{\mu_1}\, (\ell_1 \cdot H_2)_X \\
&= \tfrac{2}{\mu_1}.
\end{aligned}$$

We conclude $a_2 = 2/\mu_1$. Similarly, using a curve $C_2 = (\text{line} \times \text{point})$ we obtain $a_1 = 2/\mu_2$. □

Lemma 4.8. *If the morphism $\tilde{f} : X \longrightarrow f(X)$ has degree 1, then it is an isomorphism.*

Proof. We have seen above that $Y = f(X)$ is a subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$ of dimension 3. By [Ha77, II, Ex. 3.8], $\tilde{f} : X \longrightarrow Y$ factors into $\nu \circ g$, where $\nu : \tilde{Y} \longrightarrow Y$ is the normalisation of Y and $g : X \longrightarrow \tilde{Y}$ is a surjective morphism with connected fibres.

Since $\tilde{f}$ has degree 1, g is bijective und must therefore be an isomorphism, by (B.5). This allows us to assume without loss of generality that $\tilde{Y} = X$ and $\tilde{f} : X \longrightarrow Y$ is the normalisation of Y.

We will use duality theory to show that $\tilde{f}$ is an isomorphism. Namely, by [Re, Prop. 2.11],

$$\omega_X^\circ \cong \mathcal{H}om_{\mathcal{O}_Y}\left(\tilde{f}_* \mathcal{O}_X, \omega_Y^\circ\right) \tag{67}$$

where ω_X° and ω_Y° denote the dualizing sheaves for X resp. Y. By [Ha77, III, 7.12], $\omega_X^\circ \cong \omega_X$. Since $\tilde{f}$ has degree 1, $Y = f(X) = f_*(X)$ is a Cartier divisor on $\mathbb{P}^2 \times \mathbb{P}^2$. In particular, Y is a local complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$. Hence we obtain (cf. [Ha77, III, 7.11]), using $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(Y) \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2/\mu_2, 2/\mu_1)$ from (4.7),

$$\begin{aligned}
\omega_Y^\circ &\cong \omega_{\mathbb{P}^2 \times \mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(Y) \otimes \mathcal{O}_Y \\
&\cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, -3) \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}\left(\tfrac{2}{\mu_2}, \tfrac{2}{\mu_1}\right) \otimes \mathcal{O}_Y \\
&\cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}\left(\tfrac{2}{\mu_2} - 3, \tfrac{2}{\mu_1} - 3\right) \otimes \mathcal{O}_Y.
\end{aligned} \tag{68}$$

In particular, ω_Y° is an invertible sheaf on Y.

Since Y is a local complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$, it satisfies the S_2-condition (cf. [Re, 2.2]). Hence, by [Re, Prop. 2.3], (67) implies the subadjunction formula

$$\tilde{f}^* K_Y^\circ \sim K_X + C. \tag{69}$$

Here, K_X and K_Y° denote the divisors corresponding to the dualizing sheaves ω_X and ω_Y°, and C is the divisor corresponding to the conductor ideal sheaf of the normalisation f (cf. [Re, 2.1]). We can rewrite (68) as

$$K_Y^\circ \sim \left(\tfrac{2}{\mu_2} - 3\right) M_1|_Y + \left(\tfrac{2}{\mu_1} - 3\right) M_2|_Y.$$

Hence we get

$$\begin{aligned}\tilde{f}^* K_Y^\circ &\sim \left(\tfrac{2}{\mu_2} - 3\right) \tilde{f}^* M_1 + \left(\tfrac{2}{\mu_1} - 3\right) \tilde{f}^* M_2 \\ &\sim \left(\tfrac{2}{\mu_2} - 3\right) H_1 + \left(\tfrac{2}{\mu_1} - 3\right) H_2. \end{aligned} \tag{70}$$

Moreover, by (4.4)

$$K_X \sim -\mu_2\, H_1 - \mu_1\, H_2. \tag{71}$$

Since R_i is of type C_1 or C_2, μ_i only takes the values 1 or 2, by (2.11). Hence $2/\mu_i - 3 = -\mu_i$, for $i = 1, 2$. Thus, comparing (70) and (71) we see that $\tilde{f}^* K_Y^\circ \sim K_X$, so C in (69) must be zero. But this means that the normalisation $\tilde{f} : X \longrightarrow Y$ actually is an isomorphism. □

Lemma 4.9. $(-K_X)^3 = 6\,(\mu_1^2 + \mu_2^2)$

Proof. By (4.4), $-K_X \sim \mu_2\, H_1 + \mu_1\, H_2$. Since L_i is a line on $\mathbb{P}^2$, $H_i^3 = (f_i^*\, L_i)^3 = f_i^*(L_i^3) = 0$. By (4.2), $H_i^2 \equiv (2/\mu_i)\, \ell_i$. Hence, by (4.4),

$$\begin{aligned}(-K_X)^3 &= (\mu_2\, H_1 + \mu_1\, H_2)^3 \\ &= \mu_2^3\, H_1^3 + 3\,\mu_2^2\,\mu_1\, H_1^2 \cdot H_2 + 3\,\mu_2\,\mu_1^2\, H_1 \cdot H_2^2 + \mu_1^3\, H_2^3 \\ &= 3\,\mu_2^2\,\mu_1 \cdot \tfrac{2}{\mu_1}\,(H_2 \cdot \ell_1) + 3\,\mu_2\,\mu_1^2 \cdot \tfrac{2}{\mu_2}\,(H_1 \cdot \ell_2) \\ &= 6\left(\mu_1^2 + \mu_2^2\right).\end{aligned}$$

□

We now come to the actual classification. Since R_1 and R_2 are of type C_1 or C_2, the possible types of extremal rays of X are as follows.

Case ($\mathbf{C_1 - C_1}$). In this case, $\mu_1 = \mu_2 = 1$ by (2.11). Hence, by (66) and (4.7),

$$\deg(\tilde{f}) \cdot f(X) = f_*(X) \sim 2\,M_1 + 2\,M_2.$$

Since M_1 and M_2 form a basis of $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$, this implies that $\deg(\tilde{f})$ is either 1 or 2.

In the former case, $f(X) \sim 2\,M_1 + 2\,M_2$ and $\tilde{f} : X \longrightarrow f(X)$ is an isomorphism, by (4.8). Hence X is isomorphic to a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$.

In the latter case, $W_6 = f(X) \sim M_1 + M_2$ is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$ and $\tilde{f} : X \longrightarrow W_6$ is a double covering.

Claim. W_6 is nonsingular.

Suppose not and aim for contradiction. First, note that both $\pi_1|_{W_6}$ and $\pi_2|_{W_6}$ are equidimensional of dimension 1. To see this, recall that $W_6 = f(X)$ and $f = (f_1, f_2) : X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. Then, since f is finite, any fibre of f_1 is mapped to a curve by f_2, and vice versa.

W_6 is defined by a nonzero polynomial $H(z, w)$, bihomogeneous of bidegree $(1, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, where $z = (z_0, z_1, z_2)$ resp. $w = (w_0, w_1, w_2)$ denote homogeneous coordinates on the factors of $\mathbb{P}^2 \times \mathbb{P}^2$. We may without loss of generality assume that W_6 is singular at a point (x_0, y_0) in the affine chart $\{z_0 \neq 0 \text{ and } w_0 \neq 0\}$ of $\mathbb{P}^2 \times \mathbb{P}^2$ with local coordinates $x = (x_1, x_2), x_i = z_i/z_0$ and $y = (y_1, y_2), y_i = w_i/w_0$. In this chart, W_6 is defined by the polynomial $\tilde{H}(x, y)$, obtained by dehomogenizing $H(z, w)$. By [Ha77, I, § 5], the Jacobian matrix $\left(\partial\tilde{H}/\partial x, \partial\tilde{H}/\partial y\right)$ is zero at (x_0, y_0). Since $\tilde{H}(x, y)$ is linear in x and y separatedly, and $\tilde{H}(x_0, y_0) = 0$, this means that each $\tilde{H}(x, y_0)$ and $\tilde{H}(x_0, y)$, considered as a polynomial in x resp. y only, is the zero polynomial. Hence $W_6 \cap \pi_1^{-1}(x_0)$ and $W_6 \cap \pi_2^{-1}(y_0)$ locally have dimension 2, which is a contradiction. This proves the claim.

Since $\tilde{f}$ is a double covering, it is cyclic. Hence, by [BPV, I,17.1 (iii)], its branch locus B is the divisor on W_6 determined by

$$K_X \sim \tilde{f}^* \left(K_{W_6} + \tfrac{1}{2}\,B\right). \tag{72}$$

By the adjunction formula,

$$\begin{aligned} K_{W_6} &\sim (K_{\mathbb{P}^2 \times \mathbb{P}^2} + W_6)\,|_{W_6} \\ &\sim (-3\,M_1 - 3\,M_2 + M_1 + M_2)\,|_{W_6} \\ &\sim -2\,(M_1 + M_2)\,|_{W_6}, \end{aligned}$$

whence

$$\tilde{f}^* K_{W_6} \sim -2\,\tilde{f}^*\,(M_1 + M_2)\,|_{W_6} \sim -2\,H_1 - 2\,H_2.$$

On the other hand, by (4.4),

$$K_X \sim -H_1 - H_2.$$

Hence relation (72) implies that $B \sim -K_{W_6}$, i.e., $\tilde{f}$ is branched along a divisor on W_6 which is a member of $|-K_{W_6}|$.

In both cases, formula (4.9) yields

$$(-K_X)^3 = 6\left(\mu_1^2 + \mu_2^2\right) = 6 \cdot (1+1) = 12.$$

This is type no. 2 in table A.1.

Case $(\mathbf{C_1} - \mathbf{C_2})$. In this case, $\mu_1 = 1$ and $\mu_2 = 2$ by (2.11). Hence, by (66) and (4.7),

$$\deg(\tilde{f}) \cdot f(X) = f_*(X) \sim M_1 + 2\, M_2.$$

This shows that $\deg(f)$ must be 1. Thus $f(X) \sim M_1 + 2\, M_2$ and $\tilde{f} : X \longrightarrow f(X)$ is an isomorphism, by (4.8). Hence X is isomorphic to a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$.

Formula (4.9) yields

$$(-K_X)^3 = 6\left(\mu_1^2 + \mu_2^2\right) = 6 \cdot (1+4) = 30.$$

This is type no. 5 in table A.1.

Case $(\mathbf{C_2} - \mathbf{C_1})$. This case is equivalent to case $(C_1 - C_2)$.

Case $(\mathbf{C_2} - \mathbf{C_2})$. In this case, $\mu_1 = \mu_2 = 2$ by (2.11). Hence, by (66) and (4.7),

$$\deg(\tilde{f}) \cdot f(X) = f_*(X) \sim M_1 + M_2.$$

This implies $\deg(f) = 1$. Thus $f(X) \sim M_1 + M_2$ and $\tilde{f} : X \longrightarrow f(X)$ is an isomorphism, by (4.8). Hence X is isomorphic to a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$.

Formula (4.9) yields

$$(-K_X)^3 = 6\left(\mu_1^2 + \mu_2^2\right) = 6 \cdot (4+4) = 48.$$

This is type no. 6 in table A.1.

4.2.3 Case R_2 is of type D_1, D_2 or D_3

We consider the case that R_2 is of type D_1, D_2 or D_3. By (3.1), we are in the following situation:

$$\begin{array}{ccc} X & \xrightarrow{f_2} & \mathbb{P}^1 \\ {\scriptstyle f_1}\big\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Here, $f_1 : X \longrightarrow \mathbb{P}^2$ is a conic bundle corresponding to the ray R_1 of type C_1 or C_2, and $f_2 : X \longrightarrow \mathbb{P}^1$ is a contraction corresponding to the ray R_2 of type D_1, D_2 or D_3.

We will describe X as a covering space of $\mathbb{P}^2 \times \mathbb{P}^1$ by means of the morphism

$$f = (f_1, f_2) : X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1.$$

We need a couple of lemmas.

Lemma 4.10. *Extremal rays of X and the morphism $f : X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ satisfy the following:*

If R_1 is of type C_1 then R_2 is of type D_1 or D_2 and f is a double covering.

If R_1 is of type C_2 then R_2 is of type D_3 and f is an isomorphism.

Proof. First, we prove that f is finite and surjective. So let us assume to the contrary. Then there exists an irreducible reduced curve on X which is contained in a fibre of f. This curve is contracted by both f_1 and f_2. Hence, by (2.3), the class of this curve is contained in $R_1 \cap R_2 = \{0\}$, which is a contradiction. In particular, the image of f is a closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ of dimension 3, so f is surjective.

By (2.12), the fibres of f_1 are all numerically equivalent to $(2/\mu_1)\,\ell_1$. Since L_2 is a point on $\mathbb{P}^1$ $H_2 = f_2^*(L_2)$ is a fibre of f_2. In particular, it is reduced. We therefore obtain, by (4.4),

$$\deg(f) = \tfrac{2}{\mu_1}\,(H_2 \cdot \ell_1)_X = \tfrac{2}{\mu_1}. \tag{73}$$

If R_1 is of type C_1, then $\mu_1 = 1$ by (2.11). Assume that R_2 were of type D_3. Then $\mu_2 = 3$ by (2.14) and we would obtain from (4.4) $-K_X \sim 3\,H_1 + H_2$. Hence, by (2.2 (2)),

$$24 = (\mathrm{c}_2(X) \cdot (-K_X)) = 3\,(\mathrm{c}_2(X) \cdot H_1) + (\mathrm{c}_2(X) \cdot H_2).$$

Here, $(\mathrm{c}_2(X) \cdot H_1) > 7$ and $(\mathrm{c}_2(X) \cdot H_2) = 3$, by (4.3). Thus we would have

$$24 > 3 \cdot 7 + 3 = 24,$$

which is a contradiction. Hence R_2 must be of type D_1 or D_2. By (73), f has degree 2.

If R_1 is of type C_2, then $\mu_1 = 2$ by (2.11). By (73), f has degree 1, so it is bijective and therefore an isomorphism, by (B.5). Then X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ and it is immediate from (2.14) that R_2 is of type D_3. □

Lemma 4.11. $(-K_X)^3 = 6\,\mu_2^2$

Proof. By (4.4), $-K_X \sim \mu_2\, H_1 + \mu_1\, H_2$. Since L_1 is a line on $\mathbb{P}^2$, $H_1^3 = (f_1^*\, L_1)^3 = f_1^*(L_1^3) = 0$. Since L_2 is a point on $\mathbb{P}^1$, $H_2^2 = 0$. By (4.2), $H_1^2 \equiv (2/\mu_1)\,\ell_1$. Hence, by (4.4),

$$\begin{aligned}
(-K_X)^3 &= (\mu_2\, H_1 + \mu_1\, H_2)^3 \\
&= \mu_2^3\, H_1^3 + 3\,\mu_2^2\,\mu_1\, H_1^2 \cdot H_2 + 3\,\mu_2\,\mu_1^2\, H_1 \cdot H_2^2 + \mu_1^3\, H_2^3 \\
&= 3\,\mu_2^2\,\mu_1 \cdot \tfrac{2}{\mu_1}\,(H_2 \cdot \ell_1)_X \\
&= 6\,\mu_2^2.
\end{aligned}$$

□

We now come to the actual classification. Recall that L_i corresponds to $\mathcal{O}(1)$ on the i-th factor of $\mathbb{P}^2 \times \mathbb{P}^1$. Hence $\operatorname{Pic}(\mathbb{P}^2 \times \mathbb{P}^1)$ is generated by the divisors $M_i = \pi_i^* L_i$, $i = 1, 2$, where π_i denotes the canonical projection onto the i-th factor of $\mathbb{P}^2 \times \mathbb{P}^1$. In particular, $f^* M_i = (f^* \circ \pi_i^*)\, L_i = f_i^*\, L_i = H_i$.

By (4.10), we obtain the following possibilities for types of extremal rays of X.

Case $(\mathbf{C_1 - D_1})$. In this case, $\mu_1 = \mu_2 = 1$ by (2.11) and (2.14). By (4.10), f is a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$.

Hence it is cyclic and, by [BPV, I, 17.1 (iii)], its branch locus B is the divisor on $\mathbb{P}^2 \times \mathbb{P}^1$ determined by

$$K_X \sim f^* \left(K_{\mathbb{P}^2 \times \mathbb{P}^1} + \tfrac{1}{2}\, B \right). \tag{74}$$

Since $\operatorname{Pic}(\mathbb{P}^2 \times \mathbb{P}^1)$ is generated by M_1 and M_2, we can write

$$B \sim b_1\, M_1 + b_2\, M_2$$

with coefficients $b_1, b_2 \in \mathbb{Z}$. Then (74) takes the form

$$\begin{aligned}
K_X &\sim f^* \left(-3\, M_1 - 2\, M_2 + \tfrac{b_1}{2}\, M_1 + \tfrac{b_2}{2}\, M_2 \right) \\
&\sim \left(\tfrac{b_1}{2} - 3 \right) f^* M_1 + \left(\tfrac{b_2}{2} - 2 \right) f^* M_2 \\
&\sim \left(\tfrac{b_1}{2} - 3 \right) H_1 + \left(\tfrac{b_2}{2} - 2 \right) H_2.
\end{aligned}$$

On the other hand, by (4.4),

$$K_X \sim -H_1 - H_2.$$

Since H_1 and H_2 form a basis of $\mathrm{Pic}(X)$ by (4.4), comparing coefficients yields $b_1 = 4$ and $b_2 = 2$. Thus the double covering f is branched along a divisor on $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree $(4, 2)$.

Formula (4.11) yields

$$(-K_X)^3 = 6\,\mu_2^2 = 6 \cdot 1 = 6.$$

This is type no. 1 in table A.1.

Case ($\mathbf{C_1 - D_2}$). In this case, $\mu_1 = 1$ and $\mu_2 = 2$ by (2.11) and (2.14). By (4.10), f is a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$.

To determine its branch locus, we proceed as in case (C1 - D1) above. Namely, the branch divisor is given by

$$B \sim b_1\, M_1 + b_2\, M_2,$$

with coefficients $b_1, b_2 \in \mathbb{Z}$ satisfying

$$K_X \quad \sim \quad \left(\tfrac{b_1}{2} - 3\right)\, H_1 + \left(\tfrac{b_2}{2} - 2\right)\, H_2.$$

On the other hand, by (4.4),

$$K_X \sim -2\, H_1 - H_2.$$

Comparing coefficients yields $b_1 = b_2 = 2$. Thus the double covering f is branched along a divisor on $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree $(2, 2)$.

Formula (4.11) yields

$$(-K_X)^3 = 6\,\mu_2^2 = 6 \cdot 4 = 24.$$

This is type no. 4 in table A.1.

Case ($\mathbf{C_2 - D_3}$). In this case, $\mu_1 = 1$ and $\mu_2 = 3$ by (2.11) and (2.14). By (4.10) above, f is an isomorphism, i.e., X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$.

Formula (4.11) yields

$$(-K_X)^3 = 6\,\mu_2^2 = 6 \cdot 9 = 54.$$

This is type no. 7 in table A.1.

5 Classification of primitive Fano threefolds with $\mathrm{b}_2 = 3$

In this chapter we will refine the characterisation of primitive Fano threefolds with $\mathrm{b}_2 = 3$ given in chapter 3 to a complete classification.

So let X be a primitive Fano threefold with $\mathrm{b}_2(X) = 3$. According to (3.1), X has two distinct extremal rays R_1 and R_2, where R_1 is of type C_1 or C_2 and R_2 is of type C_1, C_2 or E_1. We will run through all possible configurations of the types of R_1 and R_2, each time working out their geometric implications. As in chapter 4, we will denote by $(\star - \star\star)$ the case that R_1 is of type $\star$ and that R_2 is of type $\star\star$.

This will give us precisely four types of primitive Fano threefolds with $\mathrm{b}_2 = 3$, which make up the second part of the Theorem in chapter 1. A more detailed list can be found in appendix A.

5.1 Case R_2 is of type E_1

We consider the case that R_2 is of type E_1. By (3.1), we are in the following situation:

$$\begin{array}{ccc} X & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\big\downarrow & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

Here, $f_1 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a conic bundle corresponding to the ray R_1 of type C_1 or C_2, and $f_2 : X \longrightarrow Y$ is a contraction, corresponding to the ray R_2 of type E_1, to a nonsingular projective threefold Y and with exceptional divisor $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1, -1)$.

We will determine the structure of the conic bundle f_1 by means of the divisor D.

Lemma 5.1. *The morphism $f_1|_D : D \cong \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ satisfies the following:*

If R_1 is of type C_1, then $f_1|_D$ is a double covering.

If R_1 is of type C_2, then $f_1|_D$ is an isomorphism.

Proof. First, we prove that $f_1|_D$ is finite and surjective. So let us assume to the contrary. Then there exists an irreducible reduced curve C_1 on D which is contained in a fibre of f_1. Since $D|_D$ is negative by (2.7), we therefore have

$$(D \cdot C_1)_X = (D|_D \cdot C_1)_D < 0.$$

Since D is properly contained in X, there exists an irreducible reduced curve C_2 in a fibre of f_1 which is not contained in D. Hence $(D \cdot C_2)_X \geq 0$. By (2.3), both classes $[C_1]$ and $[C_2]$ lie on $R_1 \setminus \{0\}$, so C_1 is numerically equivalent to some positive multiple of C_2, and we get

$$(D \cdot C_1)_X \geq 0.$$

This is a contradiction, so $f_1|_D$ must be finite. In particular, its image is a closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ of dimension 2, so $f_1|_D$ is surjective.

Since the fibres of f_1 are all numerically equivalent to $(2/\mu_1)\,\ell_1$ by (2.12) and D is reduced by (2.7), we obtain

$$\deg f_1|_D = \tfrac{2}{\mu_1}(D \cdot \ell_1). \tag{75}$$

In order to compute $(D \cdot \ell_1)$ we will express the divisor D in terms of data corresponding to the conic bundle f_1. The exact sequence of (2.15)

$$0 \longrightarrow \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{f_1^*} \operatorname{Pic}(X) \xrightarrow{(\,\cdot \ell_1)} \mathbb{Z} \longrightarrow 0 \tag{76}$$

splits, so

$$\operatorname{Pic}(X) \cong f_1^* \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \oplus \mathbb{Z}E,$$

where $E \in \operatorname{Pic}(X)$ such that $(E \cdot \ell_1) = 1$. Hence we can write

$$D \;\sim\; f_1^* L + a\,E, \tag{77}$$

where $L \in \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $a \in \mathbb{Z}$. Then we get

$$(D^3) = (f_1^* L + a\,E)^3 = f_1^* L^3 + 3\,a\,(f_1^*(L^2) \cdot E) + 3\,a^2 (f_1^* L \cdot E^2) + a^3\,E^3.$$

Since L is a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$, $f_1^* L^3 = 0$ and we can write $\mathcal{O}(L) \cong \mathcal{O}(k,l)$, $k, l \in \mathbb{Z}$. Then L^2 corresponds to $2\,kl$ times a point on $\mathbb{P}^1 \times \mathbb{P}^1$, so $f_1^*(L^2) \equiv 2\,kl\,F$, where F is a fibre of f_1. By (2.12), $F \equiv (2/\mu_1)\,\ell_1$. So we obtain $f_1^*(L^2) \equiv (4\,kl/\mu_1)\,\ell_1$, whence

$$\begin{aligned}(D^3) &= 3\,a \cdot \tfrac{4\,kl}{\mu_1}(E \cdot \ell_1) + 3\,a^2\left(f_1^* L \cdot E^2\right) + \alpha^3\,E^3 \\ &= 6\,a \cdot \tfrac{2\,kl}{\mu_1} + 3\,a^2\left(f_1^* L \cdot E^2\right) + a^3\,E^3.\end{aligned}$$

On the other hand, recall from (2.7) that f_2 is the blowing-up of Y along a curve $C \cong \mathbb{P}^1$ with exceptional divisor D and $\mathcal{N}^*_{C/Y} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Hence, by (B.4 (3)),

$$(D^3) = \deg_C(\mathcal{N}^*_{C/Y}) = \deg_{\mathbb{P}^1}\left(\textstyle\bigwedge^2 (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))\right) = \deg_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2)) = 2.$$

Combining both results, we obtain

$$2 = 6\,a \cdot \tfrac{2\,kl}{\mu_1} + 3\,a^2 \left(f_1^*L \cdot E^2\right) + a^3\,E^3.$$

This shows that a is nonzero and divides 2. Since μ_1 is either 1 or 2 by (2.11), we further see that a cannot be 2, for then the right hand side would be divisible by 4. By (77) and exactness of sequence (76), $(D \cdot \ell_1) = (f_1^*L \cdot \ell_1) + a\,(E \cdot \ell_1) = a$, so a is positive. Hence we must have $a = 1$, and we obtain from (75)

$$\deg(f_1|_D) = \tfrac{2}{\mu_1}\,(D \cdot \ell_1) = \tfrac{2}{\mu_1}\,a = \tfrac{2}{\mu_1}.$$

If R_1 is of type C_1 then $\mu_1 = 1$ by (2.11), so $\deg(f_1|_D) = 2$.

If R_1 is of type C_2 then $\mu_1 = 2$ by (2.11), so $(f_1|_D)$ has degree 1. Hence it is bijective and therefore an isomorphism, by (B.5). □

Case $(\mathbf{C_1} - \mathbf{E_1})$. By (B.3 (2)), the conic bundle structure of X gives us a natural embedding

$$\begin{array}{ccc} X & \xrightarrow{\iota'} & P' = \mathbb{P}\left((f_1)_*\mathcal{O}_X(-K_X)\right) \\ & & \Big\downarrow \pi' \\ & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \tag{78}$$

which establishes X as a divisor on some $\mathbb{P}^2$-bundle P' over $\mathbb{P}^1 \times \mathbb{P}^1$. We are now going to exploit this fact in order to obtain a characterisation of X. First, we have to determine this $\mathbb{P}^2$-bundle, i.e., we have to compute the direct image $(f_1)_*\mathcal{O}_X(-K_X)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

We begin with the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

which yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X - D) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_D(-K_X) \longrightarrow 0.$$

By [Ha77, III, § 8], this induces an exact sequence

$$\begin{aligned} 0 \longrightarrow (f_1)_*\mathcal{O}_X(-K_X - D) \longrightarrow (f_1)_*\mathcal{O}_X(-K_X) \longrightarrow (f_1)_*\mathcal{O}_D(-K_X) \\ \longrightarrow \mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X - D). \end{aligned}$$

We claim that $\mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X - D)$ vanishes: Since f_1 is flat by (B.3 (1)), this follows essentially from the semicontinuity theorem [Ha77, III, 12.9], as follows.

Let X_t denote the fibre of f_1 over $t \in \mathbb{P}^1 \times \mathbb{P}^1$, considered as a closed subscheme of X. Then we have to verify that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 1$ for all $t \in \mathbb{P}^1 \times \mathbb{P}^1$.

For generic t, $X_t \cong \mathbb{P}^1$ by (B.3 (4)). Since $f_1|_D$ has degree 2 by (5.1), D is reduced and $X_t \equiv 2\,\ell_1$ by (2.12),

$$\deg_{X_t}((-K_X - D)|_{X_t}) = ((-K_X - D)\cdot X_t)_X = 2\,(-K_X \cdot \ell_1)_X - (D\cdot X_t)_X = 2-2 = 0.$$

Hence we conclude that $\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t}$.

For special $t \in \Delta_{f_1}$, $X_t = l_1 + l_2$ is degenerate, where $l_i \cong \mathbb{P}^1$, by (B.3 (4)). Moreover, $l_i \equiv \ell_1$ by (2.11) and, as we have seen in the proof of (5.1), $(D\cdot \ell_1)_X = 1$. We obtain

$$\deg_{l_i}((-K_X - D)|_{l_i}) = ((-K_X - D)\cdot l_i)_X = (-K_X \cdot \ell_1)_X - (D \cdot \ell_1)_X = 1 - 1 = 0.$$

Hence we conclude that $-K_X - D$ is trivial along both l_1 and l_2, which implies $\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t}$.

Summing up, we have proved that

$$\mathcal{O}_{X_t}(-K_X - D) \cong \mathcal{O}_{X_t} \tag{79}$$

for all $t \in \mathbb{P}^2$.

Since f_1 is a conic bundle X_t is connected, by (B.3 (4)). Hence

$$\mathrm{h}^0(X_t, \mathcal{O}_{X_t}) = 1 \tag{80}$$

for all $t \in \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, since f_1 is flat it follows from [Ha77, III, 9.10] in combination with [Ha77, III, Ex. 5.3] that $\chi(\mathcal{O}_{X_t}) = \mathrm{h}^0(X_t, \mathcal{O}_{X_t}) - \mathrm{h}^1(X_t, \mathcal{O}_{X_t})$ is independent of t. Hence we conclude that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t})$ is independent of t. Thus we may compute its value using the generic fibre $X_t \cong \mathbb{P}^1$. But then it is immediate that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}) = 0$, by [Ha77, III, 5.1].

By (79) above, this implies that $\mathrm{h}^1(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 0$ for all $t \in \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, $\mathrm{R}^1(f_1)_*\mathcal{O}_X(-K_X - D) = 0$.

Hence we have an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X(-K_X - D) \longrightarrow (f_1)_*\mathcal{O}_X(-K_X) \longrightarrow (f_1)_*\mathcal{O}_D(-K_X) \longrightarrow 0. \tag{81}$$

We are now going to work out the direct images appearing in this sequence.

This will require us to know more about the morphism

$$f_1|_D : D \cong \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

In fact, in our case here we can give an explicit description of $f_1|_D$ as follows: By (5.1), $f_1|_D$ is a double covering. In particular, it is cyclic. By [BPV, I, §17], it is completely determined by the invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying

$$\omega_D \cong (f_1|_D)^* \left(\omega_{\mathbb{P}^1\times\mathbb{P}^1} \otimes \mathcal{L}\right).$$

Writing $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$, $a,b \in \mathbb{Z}$ we obtain

$$\mathcal{O}_D(-2,-2) \cong (f_1|_D)^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a-2,b-2).$$

Since $\mathcal{O}_D(-2,-2)$ has no nontrivial global sections, we must have $a-2, b-2 < 0$. By the projection formula [De, 1.10],

$$\mathrm{c}_1(\mathcal{O}_D(-2,-2))^2 = \deg(f_1|_D)\cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a-2,b-2))^2,$$

which implies $2\cdot(-2)^2 = 2\cdot 2\,(a-2)(b-2)$. Hence we conclude $a=0$ and $b=1$ or vice versa. For symmetry reasons, we may assume without loss of generality that

$$\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1).$$

We are now going to prove that $f_1|_D$ equals the morphism

$$id \times p : D \cong \mathbb{P}^1\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1\times\mathbb{P}^1,$$

where $p:\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the double covering defined by $\mathcal{O}_{\mathbb{P}^1}(1)$ (cf. [BPV, I, §17]).

By [Ha77, II, 6.9], we have the following formula:

$$(id\times p)^*\,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\alpha,\beta) \cong \mathcal{O}_D(\alpha, 2\,\beta) \text{ for } \alpha,\beta\in\mathbb{Z}. \tag{82}$$

The morphism $id\times p$ is a double covering. Hence, by [BPV, I, §17], it is determined by the invertible sheaf $\mathcal{L}'$ on $\mathbb{P}^1\times\mathbb{P}^1$ satisfying

$$\omega_{\mathbb{P}^1\times\mathbb{P}^1} \cong (id\times p)^*(\omega_{\mathbb{P}^1\times\mathbb{P}^1}\otimes\mathcal{L}').$$

Writing $\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a',b')$, $a',b'\in\mathbb{Z}$, we obtain by (82)

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2) &\cong (id\times p)^*\,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a'-2,b'-2) \\ &\cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a'-2, 2\,(b'-2)). \end{aligned}$$

We conclude $a'=0$ and $b'=1$. Hence we have $\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \cong \mathcal{L}$, and this implies that

$$f_1|_D = id\times p, \tag{83}$$

which is the desired explicit description of $f_1|_D$.

Now we come back to the sheaves in sequence (81). To compute the first sheaf in this sequence we combine (79) and (80) to obtain $\mathrm{h}^0(X_t, \mathcal{O}_{X_t}(-K_X - D)) = 1$ for all $t\in\mathbb{P}^2$. Since f_1 is flat, it follows from [Ha77, III, 12.9] that $(f_1)_*\mathcal{O}_X(-K_X-D)$ is locally free of rank 1 on $\mathbb{P}^1\times\mathbb{P}^1$. Hence we can write $(f_1)_*\mathcal{O}_X(-K_X-D) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)$ with $k,l\in\mathbb{Z}$. Then

$$\mathcal{O}_X(-K_X-D) \cong f_1^*\,(f_1)_*\,\mathcal{O}_X(-K_X-D) \cong f_1^*\,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l).$$

Hence, by the adjunction formula, (83) and (82),

$$\begin{aligned}
\mathcal{O}_D(2,2) &\cong \mathcal{O}_D(-K_D) \\
&\cong \mathcal{O}_D(-K_X - D) \\
&\cong (f_1|_D)^* \, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l) \\
&\cong (id \times p)^* \, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l) \\
&\cong \mathcal{O}_D(k, 2\,l).
\end{aligned}$$

This implies $k = 2$ and $l = 1$, so

$$(f_1)_*\mathcal{O}_X(-K_X - D) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1). \tag{84}$$

Next, we compute $(f_1)_*\mathcal{O}_D(-K_X)$. The calculation above in particular shows that $\mathcal{O}_D(-K_X - D) \cong \mathcal{O}_D(2,2)$. Combining this with $\mathcal{O}_D(D) \cong \mathcal{O}_D(-1,-1)$, we obtain $\mathcal{O}_D(-K_X) \cong \mathcal{O}_D(1,1)$. Then, by (83), the projection formula [De, 1.10] and (82),

$$\begin{aligned}
(f_1)_*\mathcal{O}_D(-K_X) &\cong (f_1)_* \, \mathcal{O}_D(1,1) \\
&\cong (id \times p)_* \, (\mathcal{O}_D(1,2) \otimes \mathcal{O}_D(0,-1)) \\
&\cong (id \times p)_* \, ((id \times p)^* \, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \otimes \mathcal{O}_D(0,-1)) \\
&\cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \otimes (id \times p)_* \, \mathcal{O}_D(0,-1).
\end{aligned}$$

Claim. $(id \times p)_* \, \mathcal{O}_D(0,-1) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)^{\oplus 2}$

To prove this, we first compute $p_*\mathcal{O}_{\mathbb{P}^1}(1)$: By [Ha77, IV, Ex. 2.6], $p_*\mathcal{O}_{\mathbb{P}^1}(1)$ is locally free of rank 2 on $\mathbb{P}^1$ since p has degree 2. Hence, by [Ha77, V, 2.14], we can write $p_*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, $a, b \in \mathbb{Z}$. Then

$$\det p_*\mathcal{O}_{\mathbb{P}^1}(1) \cong \det(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathcal{O}_{\mathbb{P}^1}(a+b).$$

On the other hand, $p_* \, \mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, which follows from [BPV, I, 17.2] since p was defined by $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, by [Ha77, IV, Ex. 2.6 (a)],

$$\begin{aligned}
\det p_*\mathcal{O}_{\mathbb{P}^1}(1) &\cong (\det p_* \, \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \\
&\cong \det\,(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \\
&\cong \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \\
&\cong \mathcal{O}_{\mathbb{P}^1}.
\end{aligned}$$

We conclude $a+b = 0$, and hence $p_*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$. Then, by [Ha77, II, 6.9] and the projection formula [Ha77, II, Ex. 5.1 (d)],

$$\begin{aligned}
p_* \, \mathcal{O}_{\mathbb{P}^1}(-1) &\cong p_* \, (\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) \\
&\cong p_* \, (\mathcal{O}_{\mathbb{P}^1}(1) \otimes p^* \, \mathcal{O}_{\mathbb{P}^1}(-1)) \\
&\cong p_*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \\
&\cong (\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \\
&\cong \mathcal{O}_{\mathbb{P}^1}(a-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1).
\end{aligned}$$

Hence we may assume without loss of generality that $a \geq 0$. Then, using [Ha77, II, 7.8.3], we get

$$\begin{aligned} 0 &= \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &= \mathrm{h}^0(\mathbb{P}^1, p_* \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &= \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)) \\ &= \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a-1)) + \mathrm{h}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a-1)) \\ &= \tbinom{a-1+1}{1} + 0 \\ &= a. \end{aligned}$$

Thus $p_* \mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and we obtain

$$\begin{aligned} (id \times p)_* \mathcal{O}_D(0,-1) &\cong \mathcal{O}_{\mathbb{P}^1} \boxtimes p_* \mathcal{O}_{\mathbb{P}^1}(-1) \\ &\cong \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \\ &\cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,-1)^{\oplus 2}. \end{aligned}$$

This proves the claim.

Then, by the claim,

$$\begin{aligned} (f_1)_* \mathcal{O}_D(-K_X) &\cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,-1)^{\oplus 2} \\ &\cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)^{\oplus 2}. \end{aligned}$$

Plugging in this and (84), sequence (81) takes the following form:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1) \longrightarrow (f_1)_* \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)^{\oplus 2} \longrightarrow 0.$$

By [Ha77, III, Ex. 6.1], this sequence splits. For

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}(1,0)^{\oplus 2}, \mathcal{O}(2,1)) &\cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-1,0)^{\oplus 2} \otimes \mathcal{O}(2,1)) \\ &\cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(1,1)^{\oplus 2}) \\ &\cong \mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)^{\oplus 2}) \end{aligned}$$

by [Ha77, III, 6.3 and 6.7], and this cohomology group vanishes because the natural exact sequence on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}(1,1) \longrightarrow \mathcal{O}(1,1)^{\oplus 2} \longrightarrow \mathcal{O}(1,1) \longrightarrow 0$$

induces an exact sequence in cohomology

$$\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \longrightarrow \mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)^{\oplus 2}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$$

whose first and third term is zero by the Kodaira vanishing theorem [We, VI, 2.4] since $\mathcal{O}(1,1) \otimes \omega_{\mathbb{P}^1 \times \mathbb{P}^1}^{-1} \cong \mathcal{O}(1,1) \otimes \mathcal{O}(2,2) \cong \mathcal{O}(3,3)$ is ample. Hence we conclude

$$(f_1)_* \mathcal{O}_X(-K_X) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)^{\oplus 2}.$$

The embedding (78) now takes the form

$$\begin{array}{ccc} X & \xrightarrow{\iota'} & P' = \mathbb{P}(\mathcal{E}') \\ & & \big\downarrow{\scriptstyle \pi'} \\ & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \tag{85}$$

where $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)^{\oplus 2}$.

In particular, X is a divisor on P', and since $\operatorname{Pic}(P') \cong \pi'^* \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \oplus \mathbb{Z}$ ([Ha77, II, Ex. 7.9]) we can write

$$\mathcal{O}_{P'}(X) \cong \mathcal{O}_{P'}(n) \otimes \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b) \tag{86}$$

where $n, a, b \in \mathbb{Z}$. We are now going to figure out these coefficients. By the adjunction formula,

$$\mathcal{O}_X(-K_X) \cong \mathcal{O}_{P'}(-K_{P'}) \otimes \mathcal{O}_{P'}(-X) \otimes \mathcal{O}_X. \tag{87}$$

By (B.2 (1)), we obtain

$$\begin{aligned} \mathcal{O}_{P'}(-K_{P'}) &\cong \mathcal{O}_{P'}(3) \otimes \pi'^* \left(\omega_{\mathbb{P}^1\times\mathbb{P}^1} \otimes \det \mathcal{E}'\right)^{-1} \\ &\cong \mathcal{O}_{P'}(3) \otimes \pi'^* \left(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-4,-1)\right) \\ &\cong \mathcal{O}_{P'}(3) \otimes \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1) \end{aligned} \tag{88}$$

using

$$\begin{aligned} \det \mathcal{E}' &\cong \textstyle\bigwedge^3 \left(\mathcal{O}(2,1) \oplus \mathcal{O}(1,0)^{\oplus 2}\right) \\ &\cong \mathcal{O}(2,1) \otimes \mathcal{O}(1,0) \otimes \mathcal{O}(1,0) \\ &\cong \mathcal{O}(4,1). \end{aligned}$$

By (B.3 (2)), the embedding $\iota' : X \longrightarrow P'$ is induced by the exact sequence

$$f_1^* (f_1)_* \mathcal{O}_X(-K_X) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow 0$$

on X, which implies

$$\mathcal{O}_X(-K_X) \cong \iota'^* \mathcal{O}_{P'}(1) \cong \mathcal{O}_{P'}(1) \otimes \mathcal{O}_X$$

by [Ha77, II, 7.12]. Plugging this, (88) and (86) into relation (87), we get

$$\mathcal{O}_{P'}(1) \otimes \mathcal{O}_X \cong \mathcal{O}_{P'}(3-n) \otimes \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2-a, 1-b) \otimes \mathcal{O}_X.$$

Comparing coefficients we obtain $n = 2$, $a = -2$ and $b = 1$, whence

$$\mathcal{O}_{P'}(X) \cong \mathcal{O}_{P'}(2) \otimes \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1). \tag{89}$$

This is the desired explicit description of X as a divisor on the bundle $P' = \mathbb{P}(\mathcal{E}')$.

However, we can improve this description a little by twisting the bundle $\mathcal{E}'$ by $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-1)$. Let

$$\begin{aligned}\mathcal{E} &= \mathcal{E}' \otimes \mathcal{M} \\ &\cong \big(\mathcal{O}(2,1) \oplus \mathcal{O}(1,0)^{\oplus 2}\big) \otimes \mathcal{O}(-2,-1) \\ &\cong \mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}\end{aligned}$$

with associated $\mathbb{P}^2$-bundle $\pi : P = \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Then, by [Ha77, II, 7.9], there is an isomorphism

$$\varphi : P \xrightarrow{\cong} P',$$

commuting with the projections π and π' to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, by (86)

$$\begin{aligned}\mathcal{O}_P(\varphi^{-1}(X)) &\cong \varphi^* \mathcal{O}_{P'}(X) \\ &\cong \varphi^* \mathcal{O}_{P'}(2) \otimes \varphi^* \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1) \\ &\cong (\varphi^* \mathcal{O}_{P'}(1))^{\otimes 2} \otimes \varphi^* \pi'^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1) \\ &\cong \big(\mathcal{O}_P(1) \otimes \pi^* \mathcal{M}^{-1}\big)^{\otimes 2} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1) \\ &\cong \mathcal{O}_P(2) \otimes \pi^* \mathcal{M}^{-2} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1) \\ &\cong \mathcal{O}_P(2) \otimes \pi^* (\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,2) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,1)) \\ &\cong \mathcal{O}_P(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3).\end{aligned}$$

For our convenience, we simply identify X with $\varphi^{-1}(X)$ on P. We arrive at the following final result: There is an embedding

$$\begin{array}{ccc} X & \xrightarrow{\iota = \varphi^{-1}\circ\iota'} & P = \mathbb{P}\left(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}\right) \\ & & \big\downarrow \pi \\ & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

such that

$$\mathcal{O}_P(X) \cong \mathcal{O}_P(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3).$$

It remains to compute $(-K_X)^3$. In our case here, formula (B.2 (4)) applies and we get

$$\begin{aligned}(-K_X)^3 = {} & 2\ \mathrm{c}_1(\mathcal{E})^2 - 2\ \mathrm{c}_2(\mathcal{E}) + 4\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{F}) + 6\ \mathrm{c}_1(\mathcal{E}) \cdot K_{\mathbb{P}^1\times\mathbb{P}^1} \\ & + 9\ \mathrm{c}_1(\mathcal{F}) \cdot K_{\mathbb{P}^1\times\mathbb{P}^1} + 6\, K^2_{\mathbb{P}^1\times\mathbb{P}^1} + 3\ \mathrm{c}_1(\mathcal{F})^2,\end{aligned}$$

where $\mathcal{F} \cong \mathcal{O}(2,3)$. Since $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}$, by [Ha77, A, 3.C3 and C5]

$$\begin{aligned}&\mathrm{c}_1(\mathcal{E}) = \mathrm{c}_1(\mathcal{O}) + 2\ \mathrm{c}_1(\mathcal{O}(-1,-1)) = 2\ \mathrm{c}_1(\mathcal{O}(-1,-1)), \\ &\mathrm{c}_2(\mathcal{E}) = \mathrm{c}_1(\mathcal{O}(-1,-1))^2 = 2.\end{aligned}$$

Thus $c_1(\mathcal{E})^2 = 8$. Moreover, $K_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}(-2,-2)$, so $c_1(\mathcal{E})\cdot c_1(\mathcal{F}) = -10$, $c_1(\mathcal{E})\cdot K_{\mathbb{P}^1\times\mathbb{P}^1} = 8$, $c_1(\mathcal{F})\cdot K_{\mathbb{P}^1\times\mathbb{P}^1} = -10$, $K^2_{\mathbb{P}^1\times\mathbb{P}^1} = 8$ and $c_1(\mathcal{F})^2 = 12$. Plugging these results into our formula above, we obtain

$$\begin{aligned}(-K_X)^3 &= 2\cdot 8 - 2\cdot 2 + 4\cdot(-10) + 6\cdot 8 + 9\cdot(-10) + 6\cdot 8 + 3\cdot 12\\ &= 14.\end{aligned}$$

This is type no. 2 in table A.2.

Case $(\mathbf{C_2} - \mathbf{E_1})$. By (2.11), the conic bundle f_1 is isomorphic to the projective space bundle associated to some locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^1\times\mathbb{P}^1$. Hence we can write

$$f_1 : X \cong \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^1\times\mathbb{P}^1.$$

We are now going to determine this sheaf $\mathcal{E}$ by means of the divisor D.

By (5.1) above, $f_1|_D : D \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$ is an isomorphism. The standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

By [Ha77, III, § 8], this induces an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X \longrightarrow (f_1)_*\mathcal{O}_X(D) \longrightarrow (f_1)_*\mathcal{O}_D(D) \longrightarrow \mathrm{R}^1(f_1)_*\mathcal{O}_X.$$

Since $\mathrm{R}^1(f_1)_*\mathcal{O}_X = 0$ by (2.3), we arrive at an exact sequence

$$0 \longrightarrow (f_1)_*\mathcal{O}_X \longrightarrow (f_1)_*\mathcal{O}_X(D) \longrightarrow (f_1)_*\mathcal{O}_D(D) \longrightarrow 0. \tag{90}$$

We are now going to work out the direct images appearing in this sequence.

By [Ha77, II, 7.11], it is immediate that

$$(f_1)_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}. \tag{91}$$

Since $\mathrm{Pic}(X) \cong f_1^* \,\mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1) \oplus \mathbb{Z}$ by [Ha77, II, Ex. 7.9], we can write

$$\mathcal{O}_X(D) \cong f_1^*\mathcal{L} \otimes \mathcal{O}_X(n)$$

where $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1)$ and $n \in \mathbb{Z}$. Intersecting D with any fibre F of f_1 we obtain, by (B.1),

$$1 = (D\cdot F) = (c_1(\mathcal{O}_X(D))\cdot F) = (f_1^*\, c_1(\mathcal{L})\cdot F) + n\,(c_1(\mathcal{O}_X(1))\cdot F) = n.$$

Hence the projection formula [Ha77, II, Ex. 5.1 (d)] together with [Ha77, II, 7.11] yields

$$\begin{aligned}(f_1)_*\mathcal{O}_X(D) &\cong (f_1)_*(f_1^*\mathcal{L}\otimes\mathcal{O}_X(1))\\ &\cong \mathcal{L}\otimes(f_1)_*\mathcal{O}_X(1)\\ &\cong \mathcal{L}\otimes\mathcal{E}. \qquad (92)\end{aligned}$$

Lastly, since $f_1|_D$ is an isomorphism we obtain

$$\begin{aligned}(f_1)_*\mathcal{O}_D(D) &\cong (f_1|_D)_*\mathcal{O}_D(D)\\ &\cong (f_1|_D)_*\mathcal{O}_D(-1,-1)\\ &\cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1). \qquad (93)\end{aligned}$$

Plugging in the results of (91), (92) and (93), sequence (90) takes the following form:

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\longrightarrow\mathcal{L}\otimes\mathcal{E}\longrightarrow\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\longrightarrow 0$$

By [Ha77, III, Ex. 6.1], this sequence splits. For

$$\mathrm{Ext}^1(\mathcal{O}(-1,-1),\mathcal{O})\cong\mathrm{Ext}^1(\mathcal{O},\mathcal{O}(1,1))\cong\mathrm{H}^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,1))$$

by [Ha77, III, 6.7 and 6.3], and this cohomology group vanishes by the Kodaira vanishing theorem [We, VI, 2.4] because $\mathcal{O}(1,1)\otimes\omega^{-1}_{\mathbb{P}^1\times\mathbb{P}^1}\cong\mathcal{O}(1,1)\otimes\mathcal{O}(2,2)\cong\mathcal{O}(3,3)$ is ample. Hence we conclude that

$$\mathcal{L}\otimes\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1).$$

Then we obtain, by [Ha77, II, 7.9],

$$\begin{aligned}X &\cong \mathbb{P}(\mathcal{E})\\ &\cong \mathbb{P}(\mathcal{L}\otimes\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))\\ &\cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\right).\end{aligned}$$

Now it remains to compute $(-K_X)^3$. In our case here, formula (B.2 (2)) applies and we get

$$(-K_X)^3 = 2\ \mathrm{c}_1(\mathcal{O}\oplus\mathcal{O}(1,1))^2 - 8\ \mathrm{c}_2(\mathcal{O}\oplus\mathcal{O}(1,1)) + 6\,(K_{\mathbb{P}^1\times\mathbb{P}^1})^2.$$

By [Ha77, A, 3.C3 and C5],

$$\begin{aligned}\mathrm{c}_1(\mathcal{O}\oplus\mathcal{O}(1,1)) &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})+\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))\\ &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)),\\ \mathrm{c}_2(\mathcal{O}\oplus\mathcal{O}(1,1)) &= \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})\cdot\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))\\ &= 0.\end{aligned}$$

Thus $\mathrm{c}_1(\mathcal{O}\oplus\mathcal{O}(1,1))^2 = 2$. Moreover, $(K_{\mathbb{P}^1\times\mathbb{P}^1})^2 = \mathrm{c}_1(\mathcal{O}(-2,-2))^2 = 8$. Plugging these results into the formula above, we obtain

$$(-K_X)^3 = 2\cdot 2 - 0 + 6\cdot 8 = 52.$$

This is type no. 4 in table A.2.

5.2 Case R_2 is of type C_1 or C_2

We consider the case that R_2 is of type C_1 or C_2. By (3.1), we are in the following situation:

$$\begin{array}{ccc} X & \xrightarrow{f_2} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle f_1}\big\downarrow & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

Here, $f_1, f_2 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ are conic bundles corresponding to distinct rays R_1 and R_2 of type C_1 or C_2.

Before we begin, note that we may assume without loss of generality that $((P_1 \times \mathbb{P}^1) \cdot f_2(C_1))_{\mathbb{P}^1\times\mathbb{P}^1} > 0$ for some irreducible reduced curve C_1 in a fibre of the conic bundle f_1 and some point $P_1 \in \mathbb{P}^1$. For, since $R_1 \cap R_2 = \{0\}$, it follows by (2.3) that C_1 is not contracted by f_2. Hence $f_2(C_1)$ is a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ and will have positive intersection with at least one of the divisors $P_1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times Q_1$, where $P_1, Q_1 \in \mathbb{P}^1$.

We will describe X as a covering space of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by means of the morphism

$$\begin{array}{rcl} f = (f_1, \pi_1 \circ f_2) : X & \xrightarrow{(f_1,f_2)} & (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \\ & & \big\downarrow {\scriptstyle (p_1, \pi_1 \circ p_2)} \\ & & (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^1 \end{array} \tag{94}$$

where $p_i : (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ denote the canonical projections onto the i-th factor. We need a couple of lemmas.

Lemma 5.2. *The morphism f is finite and surjective.*

Proof. First, we prove that f is finite. Let us assume to the contrary. Then there exists an irreducible reduced curve C_2 on X which is contained in a fibre of f. Hence $\pi_1 \circ f_2$ maps C_2 to some point P_2 on $\mathbb{P}^1$, and we deduce from this $(f_2^* (P_2 \times \mathbb{P}^1) \cdot C_2)_X = 0$. Moreover, C_2 is contracted by f_1, and so is the curve C_1 introduced above. Thus, by (2.3), both classes $[C_1]$ and $[C_2]$ lie on $R_1 \setminus \{0\}$. Hence C_1 is numerically equivalent to some positive multiple of C_2 and we get

$$\left(f_2^* \left(P_2 \times \mathbb{P}^1\right) \cdot C_1\right)_X = 0.$$

On the other hand, since $((P_1 \times \mathbb{P}^1) \cdot f_2(C_1))_{\mathbb{P}^1\times\mathbb{P}^1} > 0$ by the above, we obtain by the projection formula [De, 1.10]

$$\begin{aligned} \left(f_2^* \left(P_2 \times \mathbb{P}^1\right) \cdot C_1\right)_X &= ((P_2 \times \mathbb{P}^1) \cdot f_2(C_1))_{\mathbb{P}^1\times\mathbb{P}^1} \\ &= ((P_1 \times \mathbb{P}^1) \cdot f_2(C_1))_{\mathbb{P}^1\times\mathbb{P}^1} \\ &> 0, \end{aligned}$$

which is a contradiction. Hence f must be finite. In particular, its image is a closed subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of dimension 3, so f is surjective. $\square$

We are now going to characterize f more closely. First, we take a look at the discriminant loci Δ_{f_i} of the conic bundles $f_1, f_2 : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. By (B.3 (3)), we can write

$$\Delta_{f_1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_2, a_1) \text{ and } \Delta_{f_2} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_4, a_3),$$

with coefficients $a_i \in \mathbb{Z}$. The somewhat mysterious ordering of these coefficients is necessary to obtain the result of lemma 5.5 below. Since Δ_{f_i} is effective, $a_i \geq 0$ for all i. If R_1 is of type C_1 we can say even more:

Lemma 5.3. *If R_1 is of type C_1 then $a_1, a_2 > 0$.*

Proof. To prove this, assume without loss of generality that $a_1 = 0$. Since Δ_{f_1} is nonzero by (2.11), we have $a_2 > 0$. By (B.3 (2)), $\Delta_{f_1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_2, 0)$ is a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with only ordinary double points. It follows that Δ_{f_1} is the disjoint union of a_2 copies of $\mathbb{P}^1$. But this contradicts (B.3 (6)), for f_1 is a Mori contraction satisfying $\rho(X) = 3 = \rho(\mathbb{P}^1 \times \mathbb{P}^1) + 1$. $\square$

According to diagram (94) above, we introduce the following divisors: On $\mathbb{P}^1 \times \mathbb{P}^1$, let $L_1 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0)$, $L_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)$ and $L_3 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0)$. On $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we define divisors

$$M_1 = p_1^* L_1, M_2 = p_1^* L_2 \text{ and } M_3 = p_2^* L_3.$$

Their pull backs to X will be denoted by

$$H_i = f^* M_i.$$

Next, we determine the degree of f. Note that the fibres of f_1 are all numerically equivalent to $(2/\mu_1)\, \ell_1$, by (2.12). Moreover, H_3 is a reduced surface in X. Writing $a = (H_3 \cdot \ell_1)$, we therefore obtain, by diagram (94) above,

$$\deg(f) = \tfrac{2}{\mu_1}\, (H_3 \cdot \ell_1) = \tfrac{2}{\mu_1}\, a. \tag{95}$$

In particular, $a > 0$. By (2.15), there is an exact sequence

$$0 \longrightarrow \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{f_1^*} \operatorname{Pic}(X) \xrightarrow{(\,\cdot \ell_1)} \mathbb{Z} \longrightarrow 0.$$

Thus the computation

$$((a\,(-K_X) - \mu_1\, H_3) \cdot \ell_1) = a\,(-K_X \cdot \ell_1) - \mu_1\,(H_3 \cdot \ell_1) = a\,\mu_1 - \mu_1\, a = 0$$

shows that $a\,(-K_X) - \mu_1\, H_3$ is contained in $f_1^* \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. Hence, by definition of H_1 and H_2, we can write

$$a\,(-K_X) \sim c_1\, H_1 + c_2\, H_2 + \mu_1\, H_3. \tag{96}$$

The following two lemmas will help us to compute the constants a, c_1 and c_2.

Lemma 5.4. *The coefficients a, c_1 and c_2 satisfy the following:*

1. $a\,\mu_1\,(8-a_3) = 4\,c_1\,c_2$
2. $a\,(8-a_1) = 4\,c_2$
3. $a\,(8-a_2) = 4\,c_1$

In particular, $0 \le a_i \le 7$ for $i = 1,2,3$. Moreover, if R_1 is of type C_2 then $a_1 = a_2 = 0$; if R_2 is of type C_2 then $a_3 = 0$.

Proof. 1. By (96), $a\,(-K_X) \sim c_1\,H_1 + c_2\,H_2 + \mu_1\,H_3$. Since $H_i^2 = f^*\,M_i^2 = 0$ for all i, we get

$$a^2\,(-K_X)^2 \equiv 2\,c_1\,c_2\,H_1 \cdot H_2 + 2\,c_1\,\mu_1\,H_1 \cdot H_3 + 2\,c_2\,\mu_1\,H_2 \cdot H_3. \qquad (97)$$

By (B.3), $(f_2)_*(K_X^2) \equiv -4\,K_{\mathbb{P}^1\times\mathbb{P}^1} - \Delta_{f_2}$. Recall that $\Delta_{f_2} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a_4, a_3)$. Hence, by the projection formula [De, 1.10],

$$\begin{aligned}
\big((-K_X)^2 \cdot H_3\big) &= \big((-K_X)^2 \cdot f_2^*\,L_3\big) \\
&= ((-4\,K_{\mathbb{P}^1\times\mathbb{P}^1} - \Delta_{f_2}) \cdot L_3)_{\mathbb{P}^1\times\mathbb{P}^1} \\
&= ((-4\,\mathrm{c}_1(\mathcal{O}(-2,-2)) - \mathrm{c}_1(\mathcal{O}(a_4,a_3)) \cdot \mathrm{c}_1(\mathcal{O}(1,0)))_{\mathbb{P}^1\times\mathbb{P}^1} \\
&= 8 - a_3.
\end{aligned}$$

Since $-K_X$ is ample, this must be positive, i.e., $a_3 \le 7$. Since Δ_{f_2} is effective, $a_3 \ge 0$.

Since $L_1 \cdot L_2$ is a point on $\mathbb{P}^1\times\mathbb{P}^1$, $H_1 \cdot H_2 = f_1^*\,L_1 \cdot L_2$ is numerically equivalent to a fibre of f_1. Hence, by (2.12), $H_1 \cdot H_2 \equiv (2/\mu_1)\,\ell_1$, and we obtain

$$(H_1 \cdot H_2 \cdot H_3) = \tfrac{2}{\mu_1}\,(H_3 \cdot \ell_1) = \tfrac{2}{\mu_1}\,a.$$

Intersecting (97) with H_3, we therefore get $a^2\,(8-a_3) = 2\,c_1\,c_2\,(2/\mu_1)\,a$, which proves 1. If R_2 is of type C_2, then $\Delta_{f_2} = 0$ by (2.11), so $a_3 = 0$.

Assertions 2. and 3. are proved analogously. □

Lemma 5.5. *For $i = 1,2,3$, the divisor H_i satisfies*

$$(\mathrm{c}_2(X) \cdot H_i) = 4 + a_i.$$

Proof. Without loss of generality, we may consider only H_1. It is an effective divisor on X, so formula (2.2 (6)) yields

$$(\mathrm{c}_2(X) \cdot H_1) = 6\,\chi(\mathcal{O}_{H_1}) + 6\,\chi(\mathcal{O}_{H_1}(H_1)) - 2\,(H_1^3) - ((-K_X)^2 \cdot H_1). \qquad (98)$$

By definition, $H_1 = f^*\,p_1^*\,L_1 = f_1^*\,L_1$. Hence we obtain, by (2.5) and Kodaira's vanishing theorem [We, VI, 2.4]:
$\chi(\mathcal{O}_{H_1}) = 1 - \chi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-L_1)) = 1 - \chi(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)) = 1 -$

$\mathrm{h}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)) = 1 - 0 = 1$.
$\chi(\mathcal{O}_{H_1}(H_1)) = \chi(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(L_1)) - 1 = \chi(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)) - 1 = \mathrm{h}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)) - 1 = \binom{1+1}{1} - 1 = 2 - 1 = 1$ by [Ha77, II, 7.8.3].
Since $L_1^3 = 0$, $H_1^3 = f_1^* L_1^3 = 0$.
We have already seen in the proof of lemma (5.4) above that $((-K_X)^2 \cdot H_1) = 8 - a_1$.

Plugging these results into (98) above, we obtain

$$(\mathrm{c}_2(X) \cdot H_1) = 6 + 6 - 0 - 8 + a_1 = 4 + a_1.$$

□

We now come to the actual classification. Since R_1 and R_2 are of type C_1 or C_2, the possible types of extremal rays of X are as follows.

Case ($\mathbf{C_1 - C_1}$ or $\mathbf{C_2}$). In this case, $\mu_1 = 1$ by (2.11). First, we determine the degree of f. Using (5.5) and (2.2 (2)), relation (96) above yields

$$24\,a = c_1\,(4 + a_1) + c_2\,(4 + a_2) + \mu_1\,(4 + a_3). \tag{99}$$

We can rearrange the equations from (5.4) to get

$$\begin{aligned} a\,(4 + a_1) &= 4\,(3\,a - c_2), \\ a\,(4 + a_2) &= 4\,(3\,a - c_1), \\ a\,(4 + a_3) &= 4\,(3\,a - c_1\,c_2). \end{aligned}$$

Combining this with (99) we obtain

$$\begin{aligned} 24\,a^2 &= 4\,c_1\,(3\,a - c_2) + 4\,c_2\,(3\,a - c_1) + 4\,(3\,a - c_1\,c_2) \\ &= 12\,\left(a\,(c_1 + c_2 + 1) - c_1\,c_2\right), \end{aligned}$$

whence

$$2\,a^2 + c_1\,c_2 = a\,(c_1 + c_2 + 1). \tag{100}$$

By (5.4), $c_1, c_2 \geq 1$, so $c_1 + c_2 \leq c_1\,c_2 + 1$. Hence we get

$$\begin{aligned} 2\,a^2 + c_1\,c_2 &\leq a\,c_1\,c_2 + 2a, \text{ so} \\ 2\,a\,(a-1) &\leq c_1\,c_2\,(a-1). \end{aligned}$$

We claim that $a = 1$. For otherwise $a > 1$, and we obtain from this $8\,a \leq 4\,c_1\,c_2$. By (5.4 (1)), $4\,c_1\,c_2 \leq a\,(8 - a_3)$ and $a_3 \geq 0$, so we conclude that $a_3 = 0$. But then, again by (5.4 (1)),

$$c_1\,c_2 = 2\,a \tag{101}$$

and we obtain from (100)

$$2\,a + 1 = c_1 + c_2.$$

Combining these equations we get $c_1 + c_2 = c_1\,c_2 + 1$. Without loss of generality, this implies $c_1 = 1$. Then (101) yields $c_2 = 2\,a$, so we conclude from (5.4 (2)) that $a_1 = 0$. But this contradicts (5.3) above, so we must have $a = 1$. Thus, by (95), f has degree 2, i.e., it is a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Hence it is cyclic and, by [BPV, I, 17.1 (iii)], its branch locus B is the divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ determined by

$$K_X \sim f^* \left(K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1} + \tfrac{1}{2}\, B \right). \tag{102}$$

Since $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ is generated by M_1, M_2 and M_3 we can write

$$B \sim b_1\, M_1 + b_2\, M_2 + b_2\, M_2$$

with coefficients $b_1, b_2, b_3 \in \mathbb{Z}$. Moreover,

$$K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1} \sim -2\, M_1 - 2\, M_2 - 2\, M_3.$$

Then (102) above takes the form

$$\begin{aligned} K_X &\sim f^* \left(-2\, M_1 - 2\, M_2 - 2\, M_2 + \tfrac{b_1}{2}\, M_1 + \tfrac{b_2}{2}\, M_2 + \tfrac{b_3}{2}\, M_3 \right) \\ &\sim \left(\tfrac{b_1}{2} - 2 \right) f^* M_1 + \left(\tfrac{b_2}{2} - 2 \right) f^* M_2 + \left(\tfrac{b_2}{2} - 2 \right) f^* M_2 \\ &\sim \left(\tfrac{b_1}{2} - 2 \right) H_1 + \left(\tfrac{b_2}{2} - 2 \right) H_2 + \left(\tfrac{b_2}{2} - 2 \right) H_2. \end{aligned}$$

On the other hand, since $a_1, a_2 > 0$ by (5.3), equations (5.4 (2) and (3)) then imply that $4\,c_i < 8$, so we obtain $c_1 = c_2 = 1$. Hence, by (96),

$$K_X \sim -H_1 - H_2 - H_2.$$

Comparing coefficients yields $b_1 = b_2 = b_3 = 2$. Thus the double covering f is branched along a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of tridegree $(2, 2, 2)$.

As in the proof of (5.4), we calculate

$$(-K_X)^3 = 6\,(H_1 \cdot H_2 \cdot H_3) = 6 \cdot \tfrac{2}{\mu_1}\, a = 12.$$

This is type no. 1 in table A.2.

Case ($\mathbf{C_2} - \mathbf{C_1}$ or $\mathbf{C_2}$). In this case, $\mu_1 = 2$ and $a_1 = a_2 = 0$ by (2.11). Combining the equations from (5.4), we get

$$a^2\,(8 - a_1)\,(8 - a_2) = 4\,a\,\mu_1\,(8 - a_3)$$

which in our case here becomes $8^2\,a^2 = 8\,a\,(8 - a_3) \leq 8^2\,a$. Since a is positive, this implies $a = 1$. Hence, by (95), f has degree 1, so it is bijective. By (B.5), it is therefore an isomorphism, i.e., X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Moreover, by (5.4 (2) and (3)), $c_1 = c_2 = 2$. Thus we obtain from (96)

$$-K_X \sim 2\,H_1 + 2\,H_2 + 2\,H_3.$$

As in the proof of (5.4), we calculate

$$(-K_X)^3 = 8 \cdot 6\,(H_1 \cdot H_2 \cdot H_3) = 48 \cdot \tfrac{2}{\mu_1}\,a = 48.$$

This is type no. 3 in table A.2.

A Tables

The following tables are a summary of our results in chapters 4 and 5.

Table A.1. *Primitive Fano threefolds X with $\mathrm{b}_2(X) = 2$ are classified as follows.*

no.	$(-K_X)^3$	*type of X*	*type of R_1 and R_2*	*page*
1	*6*	*a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ whose branch locus is a divisor of bidegree* $(2,4)$	$(C_1 - D_1)$	*60*
2	*12*	*a double covering of W_6 (see no. 6) whose branch locus is a member of $\|-K_{W_6}\|$, or a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree* $(2,2)$	$(C_1 - C_1)$	*57*
3	*14*	*a double covering of V_7 (see no. 8) whose branch locus is a member of* $\|-K_{V_7}\|$	$(C_1 - E_3$ *or* $E_4)$	*43*
4	*24*	*a double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ whose branch locus is a divisor of bidegree* $(2,2)$	$(C_1 - D_2)$	*61*
5	*30*	*a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree* $(1,2)$	$(C_1 - C_2)$	*58*
6	*48*	*W_6, a nonsingular divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree* $(1,1)$	$(C_2 - C_2)$	*58*
7	*54*	$\mathbb{P}^2 \times \mathbb{P}^1$	$(C_2 - D_3)$	*61*
8	*56*	$V_7 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	$(C_2 - E_2)$	*51*
9	*62*	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	$(C_2 - E_5)$	*51*

Table A.2. *Primitive Fano threefolds X with $\mathrm{b}_2(X) = 3$ are classified as follows.*

no.	$(-K_X)^3$	*type of X*	*type of R_1 and R_2*	*page*
1	*12*	*a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree* $(2,2,2)$	$(C_1 - C_1$ *or* $C_2)$	*76*
2	*14*	*a nonsingular member of $\|\mathcal{O}_P(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,3)\|$ on the $\mathbb{P}^2$-bundle* $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)^{\oplus 2}) \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1$	$(C_1 - E_1)$	*64*
3	*48*	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(C_2 - C_1$ *or* $C_2)$	*77*
4	*52*	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$	$(C_2 - E_1)$	*71*

B Supplementary results

This appendix collects basic general results mainly concerning projective space bundles, conic bundles and blow-ups of threefolds along curves.

Let Y be a nonsingular variety of dimension n and $\mathcal{E}$ a locally free sheaf of rank r on Y. Following [Ha77, II, § 7], there exists an associated projective space bundle

$$\pi : X = \mathbb{P}(\mathcal{E}) \longrightarrow Y$$

where X is a nonsingular variety of dimension $n + r - 1$, equipped with a tautological line bundle $\mathcal{O}_X(1)$. Let $\xi \in \mathrm{A}^1(X)$ denote the class of the divisor corresponding to $\mathcal{O}_X(1)$.

Lemma B.1. *Assume that $\mathcal{E}$ is of rank $r = 2$. Then any fibre F of the $\mathbb{P}^1$-bundle X satisfies $(\xi \cdot F)_X = 1$.*

Proof. By construction, $\mathcal{O}_X(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on each fibre $F \cong \mathbb{P}^1$ of $\pi : X \longrightarrow Y$ ([Ha77, II, § 7]). We get $(\xi \cdot F)_X = \deg_F(\mathcal{O}_X(\xi)|_F) = \deg_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)) = 1$. □

Proposition B.2. *Let $\pi : X = \mathbb{P}(\mathcal{E}) \longrightarrow Y$ be the projective space bundle associated to some locally free sheaf $\mathcal{E}$ of rank r on a nonsingular variety Y of dimension n.*

1. *The canonical sheaf of X is given by*

$$\omega_X \cong \mathcal{O}_X(-r) \otimes \pi^* \left(\omega_Y \otimes \det \mathcal{E}\right).$$

2. *If Y is of dimension $n = 2$ and $\mathcal{E}$ of rank $r = 2$, then*

$$(-K_X)^3 = 2\ \mathrm{c}_1(\mathcal{E})^2 - 8\ \mathrm{c}_2(\mathcal{E}) + 6\,(K_Y)^2.$$

3. *If Y is of dimension $n = 1$ and $\mathcal{E}$ of rank $r = 2$, then*

$$\mathrm{c}_1(\mathcal{O}_X(1))^2 = \deg_Y(\mathcal{E}).$$

4. *If Y is of dimension $n = 2$, $\mathcal{E}$ of rank $r = 3$ and D a nonsingular divisor on X such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(2) \otimes \pi^* \mathcal{F}$ for some invertible sheaf $\mathcal{F}$ on Y, then*

$$\begin{aligned}(-K_D)^3 \;=\;& 2\ \mathrm{c}_1(\mathcal{E})^2 - 2\ \mathrm{c}_2(\mathcal{E}) + 4\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{F}) + 6\ \mathrm{c}_1(\mathcal{E}) \cdot K_Y \\ &+ 9\ \mathrm{c}_1(\mathcal{F}) \cdot K_Y + 6\,K_Y^2 + 3\ \mathrm{c}_1(\mathcal{F})^2.\end{aligned}$$

Proof. 1. By [Ha77, II, 8.11], there is an exact sequence

$$0 \longrightarrow \pi^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

on X, but without the zero on the left. However, since the projection π is locally trivial, it is a submersion and thus the pull back π^* on the level of one-forms is injective. Hence the sequence is exact on the left as well. In our situation, the sheaves in this sequence are locally free of rank $n, n+r-1, r-1$ respectively ([Ha77, II, 8.13 and 8.15]). Taking exterior powers therefore yields, by [Ha77, II, Ex. 5.16 (d,e)],

$$\begin{aligned} \omega_X &\cong \bigwedge^{n+r-1} \Omega_X \\ &\cong \bigwedge^{n} \pi^*\Omega_Y \otimes \bigwedge^{r-1} \Omega_{X/Y} \\ &\cong \pi^* \left(\bigwedge^{n} \Omega_Y\right) \otimes \bigwedge^{r-1} \Omega_{X/Y} \\ &\cong \pi^*\omega_Y \otimes \omega_{X/Y}. \end{aligned}$$

Plugging in $\omega_{X/Y} \cong \mathcal{O}_X(-r) \otimes \pi^* \det \mathcal{E}$, which is proved in [Ha77, III, Ex. 8.4 (b)], the assertion follows.

2. We are now going to apply the formula just proved to compute $(-K_X)^3$, using intersection theory as in [Ha77, A]. For our convenience, let us write $\mathcal{L} = \omega_Y \otimes \det \mathcal{E}$. Then

$$\begin{aligned} (-K_X)^3 &= (2\,\xi - \mathrm{c}_1(\pi^*\mathcal{L}))^3 \\ &= 8\,\xi^3 - 12\,\xi^2 \cdot \pi^*\,\mathrm{c}_1(\mathcal{L}) + 6\,\xi \cdot \pi^*\,\mathrm{c}_1(\mathcal{L})^2 - \pi^*\,\mathrm{c}_1(\mathcal{L})^3. \end{aligned} \tag{103}$$

Here the last summand is zero since $\mathrm{c}_1(\mathcal{L})^3 \in \mathrm{A}^3(Y)$ and $\mathrm{A}^3(Y) = 0$ because Y is of dimension 2. To compute the powers of ξ, we proceed as follows: By [Ha77, A, § 3], the divisor ξ satisfies the relation

$$\pi^*\,\mathrm{c}_0(\mathcal{E}) \cdot \xi^2 - \pi^*\,\mathrm{c}_1(\mathcal{E}) \cdot \xi + \pi^*\,\mathrm{c}_2(\mathcal{E}) = 0$$

in $\mathrm{A}^2(X)$, where $\mathrm{c}_i(\mathcal{E}) \in \mathrm{A}^i(Y)$. Since $\mathrm{c}_0(\mathcal{E}) = 1$, this becomes

$$\xi^2 = \pi^*\,\mathrm{c}_1(\mathcal{E}) \cdot \xi - \pi^*\,\mathrm{c}_2(\mathcal{E}). \tag{104}$$

Using this, we obtain

$$\begin{aligned} \xi^3 &= \pi^*\,\mathrm{c}_1(\mathcal{E}) \cdot \xi^2 - \pi^*\,\mathrm{c}_2(\mathcal{E}) \cdot \xi \\ &= (\pi^*\,\mathrm{c}_1(\mathcal{E}))^2 \cdot \xi - \pi^*\,\mathrm{c}_1(\mathcal{E}) \cdot \pi^*\,\mathrm{c}_2(\mathcal{E}) - \pi^*\,\mathrm{c}_2(\mathcal{E}) \cdot \xi \\ &= \pi^*(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) \cdot \xi - \pi^*(\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_2(\mathcal{E})) \\ &= \pi^*(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) \cdot \xi \end{aligned} \tag{105}$$

since $\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_2(\mathcal{E}) \in \mathrm{A}^3(Y) = 0$. Plugging (104) and (105) into (103), we get

$$\begin{aligned}
(-K_X)^3 &= 8\left[\pi^*(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) \cdot \xi\right] \\
&\quad - 12\left[\pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi - \pi^* \mathrm{c}_2(\mathcal{E})\right] \cdot \pi^* \mathrm{c}_1(\mathcal{L}) \\
&\quad + 6\,\xi \cdot \pi^* \mathrm{c}_1(\mathcal{L})^2 \\
&= 8\,\pi^*(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) \cdot \xi \\
&\quad - 12\,\pi^*(\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{L})) \cdot \xi - 12\,\pi^*(\mathrm{c}_2(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{L})) \\
&\quad + 6\,\pi^* \mathrm{c}_1(\mathcal{L})^2 \cdot \xi
\end{aligned}$$

Here, $\mathrm{c}_2(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{L}) \in \mathrm{A}^3(Y) = 0$. Now, by (B.1) above, for any cycle $Z \in \mathrm{A}^2(Y)$,

$$\pi^* Z \cdot \xi = \deg Z$$

where the degree is the natural group homomorphism $\mathrm{A}^2(Y) \longrightarrow \mathbb{Z}$. For our convenience we will suppress this degree homomorphism and simply write Z instead of $\deg Z$. Then we obtain

$$(-K_X)^3 = 8\,(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) - 12\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{L}) + 6\ \mathrm{c}_1(\mathcal{L})^2.$$

The Chern class formalism of [Ha77, A, 3.C4 and 3.C5], applied to $\mathcal{L} = \omega_Y \otimes \det \mathcal{E}$, yields

$$\begin{aligned}
\mathrm{c}_1(\mathcal{L}) &= \mathrm{c}_1(\omega_Y) + \mathrm{c}_1(\det \mathcal{E}) \\
&= K_Y + \mathrm{c}_1(\mathcal{E}).
\end{aligned}$$

Hence we get

$$\begin{aligned}
(-K_X)^3 &= 8\,(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})) - 12\ \mathrm{c}_1(\mathcal{E}) \cdot [K_Y + \mathrm{c}_1(\mathcal{E})] \\
&\quad + 6\ [K_Y + \mathrm{c}_1(\mathcal{E})]^2 \\
&= 8\ \mathrm{c}_1(\mathcal{E})^2 - 8\ \mathrm{c}_2(\mathcal{E}) - 12\ \mathrm{c}_1(\mathcal{E}) \cdot K_Y \\
&\quad - 12\ \mathrm{c}_1(\mathcal{E})^2 + 6\,(K_Y)^2 + 12\,K_Y \cdot \mathrm{c}_1(\mathcal{E}) + 6\ \mathrm{c}_1(\mathcal{E})^2 \\
&= 2\ \mathrm{c}_1(\mathcal{E})^2 - 8\ \mathrm{c}_2(\mathcal{E}) + 6\,(K_Y)^2.
\end{aligned}$$

3. We keep the notation of 2. From (104) we get

$$\xi^2 = \pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi - \pi^* \mathrm{c}_2(\mathcal{E}) = \pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi \tag{106}$$

since $\mathrm{c}_2(\mathcal{E}) \in \mathrm{A}^2(Y)$ and $\mathrm{A}^2(Y) = 0$ since Y has dimension 1. But this is just $\mathrm{c}_1(\mathcal{O}_X(1))^2 = \deg_Y(\mathcal{E})$.

4. By the adjunction formula,

$$(-K_D)^3_D = ((-K_X - D)|_D)^3_D = (-K_X - D)^3_X \cdot D. \tag{107}$$

By 1. above,

$$K_X \sim -3\,\xi + \pi^* \mathrm{c}_1(\omega_Y \otimes \det \mathcal{E}),$$

and, by our assumption,

$$D \sim 2\,\xi + \pi^* \mathrm{c}_1(\mathcal{F}).$$

Now we plug this into (107). Writing $a = -1$, $b = 2$ and $\mathcal{G} = \omega_Y \otimes \det \mathcal{E} \otimes \mathcal{F}$ to make equations more tractable, we obtain

$$\begin{aligned}
(-K_D)^3 &= -\,(a\,\xi + \pi^* \mathrm{c}_1(\omega_Y \otimes \det \mathcal{E} \otimes \mathcal{F}))^3 \cdot (b\,\xi + \pi^* \mathrm{c}_1(\mathcal{F})) \\
&= -\,(a\,\xi + \pi^* \mathrm{c}_1(\mathcal{G}))^3 \cdot (b\,\xi + \pi^* \mathrm{c}_1(\mathcal{F})) \\
&= -\,\big(a^3\,\xi^3 + 3\,a^2\xi^2 \cdot \pi^* \mathrm{c}_1(\mathcal{G}) + 3\,a\xi \cdot \pi^* \mathrm{c}_1(\mathcal{G})^2 + \pi^* \mathrm{c}_1(\mathcal{G})^3\big) \\
&\quad\ \cdot (b\,\xi + \pi^* \mathrm{c}_1(\mathcal{F})) \\
&= -a^3 b\,\xi^4 + \big(3a^2 b\,\pi^* \mathrm{c}_1(\mathcal{G}) + a^3\,\pi^* \mathrm{c}_1(\mathcal{F})\big) \cdot \xi^3 \\
&\quad - \big(3ab\,\pi^* \mathrm{c}_1(\mathcal{G})^2 + 3a^2\,\pi^*\,(\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{F}))\big) \cdot \xi^2 \\
&\quad - 3a\,\pi^*\,\big(\mathrm{c}_1(\mathcal{G})^2 \cdot \mathrm{c}_1(\mathcal{F})\big) \cdot \xi \\
&= -a^3 b\,\xi^4 - \big(3a^2 b\,\pi^* \mathrm{c}_1(\mathcal{G}) + a^3\,\pi^* \mathrm{c}_1(\mathcal{F})\big) \cdot \xi^3 \\
&\quad - \big(3ab\,\pi^* \mathrm{c}_1(\mathcal{G})^2 + 3a^2\,\pi^*\,(\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{F}))\big) \cdot \xi^2. \qquad (108)
\end{aligned}$$

Here we used $\mathrm{c}_1(\mathcal{G})^3, \mathrm{c}_1(\mathcal{G})^2 \cdot \mathrm{c}_1(\mathcal{F}) \in \mathrm{A}^3(Y) = 0$. To compute the powers of ξ, we proceed similarly as in 2. above: By [Ha77, A, § 3], the divisor ξ satisfies the relation

$$\pi^* \mathrm{c}_0(\mathcal{E}) \cdot \xi^3 - \pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi^2 + \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi - \pi^* \mathrm{c}_3(\mathcal{E}) = 0$$

in $\mathrm{A}^3(X)$, where $\mathrm{c}_i(\mathcal{E}) \in \mathrm{A}^i(Y)$. Since $\mathrm{c}_0(\mathcal{E}) = 1$ and $\mathrm{c}_3(\mathcal{E}) \in \mathrm{A}^3(Y) = 0$, this becomes

$$\xi^3 = \pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi^2 - \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi. \qquad (109)$$

Using this, we obtain

$$\begin{aligned}
\xi^4 &= \pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi^3 - \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi^2 \\
&= \pi^* \mathrm{c}_1(\mathcal{E})^2 \cdot \xi^2 - \pi^*\,(\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_2(\mathcal{E})) \cdot \xi - \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi^2 \\
&= \pi^*\,\big(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})\big) \cdot \xi^2 - \pi^*\,(\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_2(\mathcal{E})) \cdot \xi \\
&= \pi^*\,\big(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})\big) \cdot \xi^2 \qquad (110)
\end{aligned}$$

since $\mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_2(\mathcal{E}) \in \mathrm{A}^3(Y) = 0$. Plugging (109) and (110) into (108), we get

$$\begin{aligned}
(-K_D)^3 &= -a^3 b\,\big[\pi^*\,\big(\mathrm{c}_1(\mathcal{E})^2 - \mathrm{c}_2(\mathcal{E})\big) \cdot \xi^2\big] \\
&\quad - \big(3a^2 b\,\pi^* \mathrm{c}_1(\mathcal{G}) + a^3\,\pi^* \mathrm{c}_1(\mathcal{F})\big) \cdot \big[\pi^* \mathrm{c}_1(\mathcal{E}) \cdot \xi^2 - \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi\big] \\
&\quad - \big(3ab\,\pi^* \mathrm{c}_1(\mathcal{G})^2 + 3a^2\,\pi^*\,(\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{F}))\,\big) \cdot \xi^2 \\
&= -\big[a^3 b\,\pi^* \mathrm{c}_1(\mathcal{E})^2 - a^3 b\ \mathrm{c}_2(\mathcal{E}) \\
&\quad + 3a^2 b\,\pi^*\,(\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{E})) + a^3\,\pi^*\,(\mathrm{c}_1(\mathcal{F}) \cdot \mathrm{c}_1(\mathcal{E})) \\
&\quad + 3ab\,\pi^* \mathrm{c}_1(\mathcal{G})^2 + 3a^2\,\pi^*\,(\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{F}))\,\big] \cdot \xi^2 \\
&\quad - \big[3a^2 b\,\pi^* \mathrm{c}_1(\mathcal{G}) + a^3\,\pi^* \mathrm{c}_1(\mathcal{F})\big] \cdot \pi^* \mathrm{c}_2(\mathcal{E}) \cdot \xi. \qquad (111)
\end{aligned}$$

Here, the second summand is zero since both $\mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_2(\mathcal{E})$ and $\mathrm{c}_1(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{E})$ lie in $\mathrm{A}^3(Y) = 0$.

Now since $\mathcal{O}_X(1)$ restricts to $\mathcal{O}_F(1)$ on every fibre $F \cong \mathbb{P}^2$ of the $\mathbb{P}^2$-bundle X, we have

$$(\xi^2 \cdot F)_X = (\xi|_F)^2_F = \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 1.$$

This shows that for any cycle $Z \in \mathrm{A}^2(Y)$,

$$\pi^* Z \cdot \xi = \deg Z$$

where the degree is the natural group homomorphism $\mathrm{A}^2(Y) \longrightarrow \mathbb{Z}$. For our convenience we will suppress this degree homomorphism and simply write Z instead of $\deg Z$. Then (111) implies

$$\begin{aligned}(-K_D)^3 &= -a^3 b\ \mathrm{c}_1(\mathcal{E})^2 + a^3 b\ \mathrm{c}_2(\mathcal{E}) - 3a^2 b\ \mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{E}) - a^3\ \mathrm{c}_1(\mathcal{F}) \cdot \mathrm{c}_1(\mathcal{E}) \\ &\quad - 3ab\ \mathrm{c}_1(\mathcal{G})^2 - 3a^2\ \mathrm{c}_1(\mathcal{G}) \cdot \mathrm{c}_1(\mathcal{F}).\end{aligned}$$

As in 2. above, Chern class formalism applied to $\mathcal{G} = \omega_Y \otimes \det \mathcal{E} \otimes \mathcal{F}$ yields

$$\mathrm{c}_1(\mathcal{G}) = K_Y + \mathrm{c}_1(\mathcal{F}) + \mathrm{c}_1(\mathcal{E}).$$

Hence we obtain

$$\begin{aligned}(-K_D)^3 &= -a^3 b\ \mathrm{c}_1(\mathcal{E})^2 + a^3 b\ \mathrm{c}_2(\mathcal{E}) - 3a^2 b\ [K_Y + \mathrm{c}_1(\mathcal{F}) + \mathrm{c}_1(\mathcal{E})] \cdot \mathrm{c}_1(\mathcal{E}) \\ &\quad - a^3\ \mathrm{c}_1(\mathcal{F}) \cdot \mathrm{c}_1(\mathcal{E}) - 3ab\ [K_Y + \mathrm{c}_1(\mathcal{F}) + \mathrm{c}_1(\mathcal{E})]^2 \\ &\quad - 3a^2\ [K_Y + \mathrm{c}_1(\mathcal{F}) + \mathrm{c}_1(\mathcal{E})] \cdot \mathrm{c}_1(\mathcal{F}) \\ &= -(a^3 b + 3a^2 b + 3ab)\ \mathrm{c}_1(\mathcal{E})^2 + a^3 b\ \mathrm{c}_2(\mathcal{E}) \\ &\quad - (a^3 + 3a^2 b + 6ab + 3a^2)\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{F}) \\ &\quad - (3a^2 b + 6ab)\ \mathrm{c}_1(\mathcal{E}) \cdot K_Y - (6ab + 3a^2)\ \mathrm{c}_1(\mathcal{F}) \cdot K_Y \\ &\quad - 3ab\, K_Y^2 - (3ab + 3a^2)\ \mathrm{c}_1(\mathcal{F})^2.\end{aligned}$$

Resubstituting $a = -1$ and $b = 2$, we finally obtain

$$\begin{aligned}(-K_D)^3 &= 2\ \mathrm{c}_1(\mathcal{E})^2 - 2\ \mathrm{c}_2(\mathcal{E}) + 4\ \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{c}_1(\mathcal{F}) + 6\ \mathrm{c}_1(\mathcal{E}) \cdot K_Y \\ &\quad + 9\ \mathrm{c}_1(\mathcal{F}) \cdot K_Y + 6\, K_Y^2 + 3\ \mathrm{c}_1(\mathcal{F})^2.\end{aligned}$$

□

A morphism $f : X \longrightarrow S$ from a nonsingular threefold X onto a nonsingular projective surface S is a *conic bundle* if every fibre of f is isomorphic to a conic, i.e., a scheme of zeroes of a nonzero homogeneous form of degree 2 on $\mathbb{P}^2$. The set

$$\Delta_f = \{s \in S \mid f^{-1}(s) \text{ is singular}\}$$

is called the *discriminant locus* of f.

Proposition B.3. *Let $f : X \longrightarrow S$ be a conic bundle as in the definition above.*

1. *f is a flat morphism.*

2. *The sheaf $f_* \omega_X^{-1}$ is locally free of rank 3, and the natural map*

$$X \longrightarrow \mathbb{P}(f_* \omega_X^{-1})$$

corresponding to the exact sequence $f^* f_* \left(\omega_X^{-1}\right) \longrightarrow \omega_X^{-1} \longrightarrow 0$ is an embedding.

3. *If Δ_f is not empty then it is a curve with only ordinary double points.*

4. *The fibres X_s, $s \in S$, of f satisfy the following:*

   If $s \in S - \Delta_f$ then X_s is isomorphic to $\mathbb{P}^1$.

   If s is a regular point of Δ_f then X_s is reducible and decomposes as $l_1 + l_2$, where each l_i is isomorphic to $\mathbb{P}^1$ and l_1 and l_2 intersect transversally in one point.

   If s is a double point of Δ_f then X_s is non-reduced and isomorphic to a double line on $\mathbb{P}^2$.

5. $-4\, K_S \equiv f_*(K_X^2) + \Delta_f$

6. *If $\rho(X) = \rho(S) + 1$ then Δ_f has no nonsingular rational connected component.*

7. *Let C be an irreducible reduced curve on S. If $f^{-1}(C)$ is reducible then Δ_f contains C as a connected component. Moreover, $f_1^{-1}(C) = Z_1 \cup Z_2$ with irreducible reduced components Z_1 and Z_2.*

Proof. 1. and 2. This is proved in [Be, I, Prop. 1.2 (i) and (ii)].

3. and 4. This is proved in [Be, I, Prop. 1.2 (iii)]. Namely, it is shown:
If $s \in S - \Delta_f$, then X_s is isomorphic to a nonsingular conic in $\mathbb{P}^2$. Hence, by [Ha77, V, 1.5.1 and IV, 1.3.5], it is isomorphic to $\mathbb{P}^1$.
If s is a regular point of Δ_f, then X_s is isomorphic to the scheme of zeroes of $X_0^2 + X_1^2$ on $\mathbb{P}^2$. Hence it is reducible and decomposes as $l_1 \cup l_2$, where each l_i is isomorphic to $\mathbb{P}^1$ and l_1 and l_2 intersect transversally in one point.
If s is a singular point of Δ_f, then X_s is isomorphic to the scheme of zeroes of X_0^2 on $\mathbb{P}^2$. Hence it is a double line on $\mathbb{P}^2$ and non-reduced.

5. This formula is proved in [Mi, 4.11].

6. Assume to the contrary. Let C be a nonsingular rational connected component of Δ_f. As it is explained in [Mi, 4.2], it follows from [Be, I, Prop. 1.5] that $f^{-1}(C)$ is reducible with two components. On the other hand, since $\rho(X) = \rho(S) + 1$, $f^{-1}(C)$ is irreducible, by [Mi, 4.5]. This is a contradiction. $\square$

We collect some properties of the blowing-up of a nonsingular threefold along a nonsingular curve.

Proposition B.4. *Let $f : X \longrightarrow Y$ be the blowing-up of a nonsingular threefold Y along a nonsingular curve C on Y, with exceptional divisor D. Then the following holds:*

1. $-K_X \sim f^*(-K_Y) - D$.
2. *The induced morphism $f|_D : D \longrightarrow C$ is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{N}^*_{\mathcal{C}/\mathcal{Y}})$ over C, and $\mathcal{O}_D(-D) \cong \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{\mathcal{C}/\mathcal{Y}})}(1)$.*
3. $(D^3)_X = \deg_C(\mathcal{N}^*_{C/Y})$.
4. $((-K_X)^2 \cdot D)_X = (-K_Y \cdot C)_Y + 2 - 2\,\mathrm{g}(C)$.

Proof. 1. This is immediate from [Ha77, II, Ex.8.5 (b)] since C is of codimension 2 in Y.

2. This follows from [Ha77, II, 8.24 (b,c)], using $\mathcal{O}_D(-D) \cong \mathcal{O}_X(-D) \otimes \mathcal{O}_D \cong \mathcal{I}_D \otimes \mathcal{O}_X/\mathcal{I}_D \cong \mathcal{I}_D/\mathcal{I}_D^2$.

3. By 2. above, $(D^3)_X = (D|_D \cdot D|_D)_D = ((-D|_D) \cdot (-D|_D))_D = \mathrm{c}_1(\mathcal{O}_D(1))^2$. Since $D \cong \mathbb{P}(\mathcal{N}^*_{C/Y})$, $\mathrm{c}_1(\mathcal{O}_D(1))^2 = \deg_C(\mathcal{N}^*_{C/Y})$, by (B.2 (3)).

4. We apply the formalism of the proof of (B.2 (3)) to the $\mathbb{P}^1$-bundle $\pi = f|_D : D = \mathbb{P}(\mathcal{N}^*_{C/Y}) \longrightarrow C$. In particular, we write $\xi = \mathcal{O}_D(1)$. Then

$$((-K_X)^2 \cdot D)_X = ((-K_X|_D)^2)_D$$

By the adjunction formula, (B.2 (1)) and 2. above,

$$\begin{aligned}
\mathcal{O}_D(-K_X) &\cong \mathcal{O}_D(-K_D) \otimes \mathcal{O}_D(D) \\
&\cong \mathcal{O}_D(2) \otimes \pi^* \left(\omega_C \otimes \det \mathcal{N}^*_{C/Y}\right)^{-1} \otimes \mathcal{O}_D(-1) \\
&\cong \mathcal{O}_D(1) \otimes \pi^* \mathcal{L},
\end{aligned}$$

where we write $\mathcal{L} = \omega_C^{-1} \otimes \det \mathcal{N}_{C/Y}$ for our convenience. Then we get

$$\begin{aligned}
\left((-K_X)^2 \cdot D\right)_X &= (\xi + \mathrm{c}_1(\pi^*\mathcal{L}))^2 \\
&= \xi^2 + 2\,\xi \cdot \pi^* \mathrm{c}_1(\mathcal{L}) + \pi^* \mathrm{c}_1(\mathcal{L})^2 \\
&= \xi^2 + 2\,\pi^* \mathrm{c}_1(\mathcal{L}) \cdot \xi
\end{aligned}$$

since $\mathrm{c}_1(\mathcal{L})^2 \in \mathrm{A}^2(C)$, and $\mathrm{A}^2(C) = 0$ because C is a curve. As we have seen in the proof of (B.2 (3)),

$$\xi^2 = \pi^* \mathrm{c}_1\left(\mathcal{N}^*_{C/Y}\right) \cdot \xi.$$

Hence we get

$$\begin{aligned}
((-K_X)^2 \cdot D)_X &= \pi^* \mathrm{c}_1(\mathcal{N}^*_{C/Y}) \cdot \xi + 2\,\pi^* \mathrm{c}_1(\mathcal{L}) \cdot \xi \\
&= \pi^* \left(\mathrm{c}_1(\mathcal{N}^*_{C/Y}) + 2\,\mathrm{c}_1(\mathcal{L})\right) \cdot \xi.
\end{aligned}$$

By (B.1) above, for any cycle $Z \in \mathrm{A}^2(Y)$,

$$\pi^* Z \cdot \xi = \deg Z$$

where the degree is the natural group homomorphism $\mathrm{A}^2(Y) \longrightarrow \mathbb{Z}$. For our convenience we will suppress this degree homomorphism and simply write Z instead of $\deg Z$. By [Ha77, A, 3.C3, C4, C5],

$$\begin{aligned}
\mathrm{c}_1(\mathcal{L}) &= \mathrm{c}_1(\omega_C^{-1}) + \mathrm{c}_1(\det \mathcal{N}_{C/Y}) \\
&= \mathrm{c}_1\left(\mathcal{N}_{C/Y} \otimes \omega_C^{-1}\right) \\
&= \mathrm{c}_1\left(\mathcal{N}_{C/Y}\right) - \mathrm{c}_1(\omega_C).
\end{aligned}$$

By the adjunction formula, $\mathcal{N}_{C/Y} \otimes \omega_C^{-1} \cong \mathcal{O}_C(C) \otimes \omega_C^{-1} \cong \mathcal{O}_C(-K_Y)$. So we obtain

$$\begin{aligned}
((-K_X)^2 \cdot D)_X &= \mathrm{c}_1(\mathcal{N}^*_{C/Y}) + 2\ \mathrm{c}_1(\mathcal{L}) \\
&= \mathrm{c}_1(\mathcal{N}^*_{C/Y}) + \mathrm{c}_1(\mathcal{N}_{C/Y}) - \mathrm{c}_1(\omega_C) + \mathrm{c}_1(\mathcal{O}_C(-K_Y)) \\
&= \mathrm{c}_1(\mathcal{O}_C(-K_Y)) - \mathrm{c}_1(\omega_C) \\
&= \deg_C(-K_Y) - \deg_C(K_C) \\
&= (-K_Y \cdot C) + 2 - 2\ \mathrm{g}(C).
\end{aligned}$$

Here we used [Ha77, IV, 1.3.3]. □

Lemma B.5. *Let $f : X \longrightarrow Y$ be a morphism from a nonsingular variety X to a normal variety Y. If f is bijective then it is an isomorphism.*

Proof. By generic smoothness [Ha77, III, 10.7] there is a nonempty nonsingular open subset $V \subseteq Y$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \longrightarrow V$ is smooth. By the inverse function theorem ([GH, 0.2, p.19]), $f|_{f^{-1}(V)}$ is an isomorphism, i.e., $f : X \longrightarrow Y$ is birational. Then $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ by Zariski's main theorem [Ha77, III, 11.4], so f is an isomorphism. □

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