SURJECTIVITY OF THE COMPARISON MAP IN BOUNDED COHOMOLOGY FOR HERMITIAN LIE GROUPS

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Abstract. We investigate implications of Gromov's theorem on boundedness of primary characteristic classes for the continuous bounded cohomology of a semisimple Lie group G. We deduce that the comparison map from continuous bounded cohomology to continuous cohomology is surjective for a large class of semisimple Lie groups including all Hermitian groups. Our proof is based on a geometric implementation of the canonical map from the cohomology of the classifying space of G to the continuous group cohomology of G. We obtain this implementation by establishing a variant of Kobayashi-Ono-Hirzebruch duality.

Contents

1. Introduction

This article is concerned with the boundedness problem in continuous cohomology of Lie groups. Given a Lie group G and a class α in the continuous cohomology $H_c^\bullet(G;\mathbb{R})$ of G with real coefficients, one may investigate whether α can be represented by a bounded cocycle. This question may be reformulated in more invariant terms by asking whether α is contained in the image of the *comparison map* $H_{cb}^\bullet(G;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R})$ from continuous bounded cohomology to continuous cohomology [26, Def. 9.2.1]. If this is the case, then the class α is called *bounded*. Dupont [15, Remark 3] asked whether, for a connected semisimple Lie group G without compact factors, *every* continuous cohomology class is bounded in this sense. Despite ongoing efforts for 30 years this question is still open and considered one of the major problems in the theory of continuous bounded cohomology of Lie groups [11, Conjecture 16.1], [27, Conjecture A].

The methods employed so far in order to establish boundedness of specific cohomology classes basically fall into two classes. *Explicit methods* use estimates for explicit cocycles (e.g. those obtained by integration of invariant simplices in the corresponding symmetric space) to obtain boundedness. For example, for simple Lie groups of real rank one boundedness of arbitrary continuous cohomology classes is obtained by estimating integrals over ideal simplices, see

2000 *Mathematics Subject Classification.* 22E41, 53C35, 55N35, 57T15.

The first author was partially supported by SNF grant PP002-102765. The second author was supported by ETH Research Grant TH-01 06-1.

Gromov [17]. Dupont refined this method and proved boundedness of the Kähler class of a semisimple Lie group of Hermitian type in arbitrary rank [15]. Note that, despite being explicit, these methods usually do not provide *sharp* estimates on the L^∞-seminorms of the cohomology classes in question; this requires further effort, see [13] for Kähler classes and [7] for the case of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. For a more recent example of an explicit method see Lafont and Schmidt [23]. In this article we shall apply an *indirect method* for establishing boundedness that does not yield any bounds on cocycles at all but has the advantage that it works for a large class of groups. This method uses Gromov's theorem on boundedness of primary characteristic classes of flat bundles [17, 6] in combination with standard transfer arguments in continuous bounded cohomology [26]. We will explain this method in Section 2.2 below; for now we will show how it may be applied in our case.

To this end, let G be a connected semisimple Lie group without compact factors and with finite center. Then G admits a co-compact lattice Γ, see Borel–Harish-Chandra [3]. The inclusion $\iota_\Gamma : \Gamma \hookrightarrow G$ induces restriction maps $\iota_\Gamma^* \colon H_c^\bullet(G;\mathbb{R}) \to H^\bullet(\Gamma;\mathbb{R})$ in continuous cohomology and $(B\iota_\Gamma)^* \colon H^\bullet(BG;\mathbb{R}) \to H^\bullet(B\Gamma;\mathbb{R})$ on the level of classifying spaces. Note that there is a canonical isomorphism $H^\bullet(\Gamma;\mathbb{R}) \cong H^\bullet(B\Gamma;\mathbb{R})$.

Theorem 1 (Gromov)**.** *Let G be a connected semisimple Lie group without compact factors and with finite center, Γ a co-compact lattice in G, and $\alpha \in H_c^\bullet(G;\mathbb{R})$ a continuous cohomology class. If $\iota_\Gamma^*\alpha \in H^\bullet(\Gamma;\mathbb{R})$ is contained in the image of $(B\iota_\Gamma)^* \colon H^\bullet(BG) \to H^\bullet(B\Gamma;\mathbb{R}) \cong H^\bullet(\Gamma;\mathbb{R})$, then α is bounded.*

It is somewhat folklore that Theorem 1 can be used to show boundedness of certain classes for which explicit methods have failed to establish boundedness so far. However, to our knowledge this approach to boundedness has never been exploited systematically. In the present article we aim to close this gap in the literature. More specifically, for every connected semisimple Lie group without compact factors we explicitly describe those classes in continuous cohomology whose boundedness is a consequence of Theorem 1.

Our main device is a certain *universal map* $\sigma_G \colon H^\bullet(BG;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R})$ that may be characterized as follows.

Proposition 1.1 (Bott)**.** *For every Lie group G there exists a map $\sigma_G \colon H^\bullet(BG;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R})$ such that for all discrete subgroups Γ of G the diagram*

$$\begin{array}{ccc} H^\bullet(BG;\mathbb{R}) & \xrightarrow{\sigma_G} & H_c^\bullet(G;\mathbb{R}) \\ \downarrow{\scriptstyle (B\iota_\Gamma)^*} & & \downarrow{\scriptstyle \iota_\Gamma^*} \\ H^\bullet(B\Gamma;\mathbb{R}) & \xrightarrow{\cong} & H^\bullet(\Gamma;\mathbb{R}) \end{array}$$

commutes. If G admits a co-compact lattice, then σ_G is unique. In fact, it is uniquely determined by the property that it makes the above diagram commute for a single co-compact lattice Γ.

The existence of a natural map $\sigma_H \colon H^\bullet(BH;\mathbb{R}) \to H_c^\bullet(H;\mathbb{R})$ was actually stated by Bott [5] for all sufficiently nice topological groups H. However, the details of his proof never appeared. Fortunately, we only need this result for (not-necessarily connected) Lie groups, and in this case there is a well-known proof based on sheaf cohomology over simplicial spaces. Lacking a good reference we include this proof in Appendix A.

Combining Proposition 1.1 with Theorem 1, we obtain the following boundedness result.

Corollary 1.2. *The image of the universal map $\sigma_G \colon H^\bullet(BG;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R})$ consists of bounded classes.*

Note that the classes in the image of σ_G are exactly those classes to which Gromov's theorem applies. We thus have to determine the image of σ_G explicitly. In order to state our results, we fix the following notation. Let K be a maximal compact subgroup of G. Then (G, K) is a symmetric pair and there exists a dual symmetric pair (G_u, K) with G_u compact (see Section 2.1 for details). Then $\mathcal{X}_u := G_u/K$ is a compact symmetric space, which comes with a canonical K-bundle $p_{G_u} : G_u \to \mathcal{X}_u$. Moreover, there is an isomorphism $H_c^\bullet(G;\mathbb{R}) \cong H^\bullet(\mathcal{X}_u;\mathbb{R})$ (a variant of the van Est isomorphism, see Section 2.1).

Theorem 2. *Under the isomorphism $H_c^\bullet(G;\mathbb{R}) \cong H^\bullet(\mathcal{X}_u;\mathbb{R})$ the image of the universal map σ_G is mapped to the algebra of real-valued characteristic classes of the canonical K-bundle $p_{G_u} : G_u \to \mathcal{X}_u$.*

By the work of Cartan [10] and Borel [1] the cohomology $H^\bullet(\mathcal{X}_u;\mathbb{R})$ of the compact symmetric space $\mathcal{X}_u$ is well-known. This allows us to make Theorem 2 more explicit. In particular, we may describe those groups for which Gromov's theorem yields boundedness of *all* continuous cohomology classes. Recall that a connected semisimple Lie group without compact factors is called *Hermitian* if the corresponding symmetric space $\mathcal{X} = G/K$ admits a G-invariant complex structure. Then, from the classification of real semisimple Lie groups as stated in [18, Table V], we obtain the following result.

Theorem 3. *Let G be a connected semisimple Lie group G without compact factors and with finite center. Assume that all simple factors of G are either*

(i) *Hermitian;*
(ii) *locally isomorphic to $SO_0(p, q)$ with pq even, or to $Sp(p, q)$ with $p, q \geq 1$;*
(iii) *exceptional with Lie algebra one of the following: $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{g}_{2(2)}$.*

Then the comparison map $H_{cb}^\bullet(G;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R})$ is surjective.

Note that the above list exhausts the vast majority of connected real simple Lie groups that are not complex. In fact, the only real non-complex, non-compact simple factors not covered by Theorem 3 are those locally isomorphic to $SL_n(\mathbb{R})$ for $n \geq 3$, $SU^*(2n)$ for $n \geq 2$, $SO_0(p, q)$ for p and q both odd, and the exceptional ones with Lie algebras $\mathfrak{e}_{6(6)}$ and $\mathfrak{e}_{6(-26)}$. Furthermore, Theorem 3 does not apply to complex semisimple Lie groups either. However, in all these cases our methods still yield partial results, see Section 5.

This article is organized as follows. In Section 2 we fix the notation and collect the necessary background material on continuous (bounded) cohomology of Lie groups. We also explain how to derive Theorem 1 from the original version of Gromov's theorem. Sections 3 and 4 are devoted to the proof of Theorem 2. In Lemma 4.1 we provide a reformulation of this result in terms of commutativity of a certain diagram. We will establish commutativity in two steps. In Section 3 we employ differential-geometric tools to establish a cohomological duality principle that generalizes the classical proportionality principle of Hirzebruch [19] and is reminiscent of a result due to Kobayashi and Ono [22]. This duality is then applied in Section 4 in order to prove commutativity of the diagram from Lemma 4.1. For this purpose it is crucial to describe various canonical maps related to the cohomology of Lie groups as explicitly as possible. Once Lemma 4.1 (and thereby Theorem 2) is established, it is straightforward to compute the range of the universal map σ_G in all cases using classical work of Cartan [10] and Borel [1] in combination with the classification of semisimple real Lie groups [18]. This is carried out in Section 5. We close with a proof of Bott's theorem in Appendix A.

Acknowledgments: We are grateful to M. Burger for pointing out that the cohomology ring of a Hermitian compact symmetric space is generated by characteristic classes and for bringing reference [1] to our attention. While working on the proof of Theorem 3 in the Hermitian case we learned from M. Bucher-Karlsson that a similar theorem should be true for certain even-degree classes in arbitrary semisimple Lie groups. We are indebted to her for this suggestion, which led us to discover Theorem 5, as well as for pointing out the relation of our work to [22]. The second author would like to thank the Department of Mathematics at Rutgers University and the Institut des Hautes Études Scientifiques for their hospitality and excellent working conditions.

2. Preliminaries and notation

Given a topological space X we shall denote by $H^\bullet(X;\mathbb{R})$ its singular cohomology with real coefficients. If X is a manifold, then the singular cohomology of X with real coefficients is canonically isomorphic to the de Rham cohomology of X and we will not distinguish notationally between these two cohomologies.

2.1. Continuous cohomology of Lie groups and its geometric models.

For an arbitrary Hausdorff locally compact group G, the *continuous cohomology* $H_c^\bullet(G;\mathbb{R})$ of G with real coefficients is defined as the cohomology of the complex $(C_c^\bullet(G;\mathbb{R}), d)$, where

$$C_c^n(G;\mathbb{R}) := C(G^{n+1},\mathbb{R})^G, \quad df(g_0,\dots,g_n) := \sum_{i=0}^n (-1)^i f(g_0,\dots,\widehat{g_i},\dots,g_n).$$

Here $C(\cdot,\mathbb{R})$ stands for real-valued continuous functions and $(-)^G$ denotes the functor of G-invariants. The complex $(C_c^\bullet(G;\mathbb{R}), d)$ is called the *homogeneous bar complex* for $H_c^\bullet(G;\mathbb{R})$. It is isomorphic to the *inhomogeneous bar complex* $(C(G^n,\mathbb{R}),\partial)$, where

$$\partial f(g_1,\dots,g_n) := f(g_2,\dots,g_n) + \sum_{j=1}^{n-1}(-1)^j f(g_1,\dots,g_j g_{j+1},\dots,g_n) + (-1)^n f(g_1,\dots,g_{n-1}).$$

If the group $G=\Gamma$ is discrete, the continuity requirement on the cochains becomes vacuous and we will drop the subscript c from the notation. Note that in this case $H^\bullet(\Gamma;\mathbb{R})$ is just the ordinary group cohomology of Γ. In particular, $H^\bullet(\Gamma;\mathbb{R})$ is canonically isomorphic to the singular cohomology $H^\bullet(B\Gamma;\mathbb{R})$ of any given classifying space $B\Gamma$ for Γ. We emphasize that this fails for non-discrete topological groups. However, in favourable cases it is still possible to give a geometric meaning to classes in $H_c^\bullet(G;\mathbb{R})$. We will carry this out in the case where G is a connected semisimple Lie group without compact factors and with finite center. We shall then denote by K a maximal compact subgroup of G and by $\mathcal{X} := G/K$ the associated symmetric space.

Probably the most straightforward geometric interpretation of classes in $H_c^\bullet(G;\mathbb{R})$ is given in terms of harmonic forms on a compact locally symmetric manifold M covered by $\mathcal{X}$. Indeed, by Selberg's refinement [30] of the Borel–Harish-Chandra lemma [3] there exists a co-compact, torsion-free lattice Γ in G. We now fix such a lattice and denote by $M := \Gamma\backslash\mathcal{X}$ the corresponding locally symmetric space. Note that, since Γ is torsion-free, M is smooth. We denote by $\iota_\Gamma : \Gamma \hookrightarrow G$ the inclusion and by $\iota_\Gamma^*\colon H_c^\bullet(G;\mathbb{R}) \to H^\bullet(\Gamma;\mathbb{R})$ the corresponding restriction map.

Lemma 2.1. *The restriction map ι_Γ^* is injective.*

Proof. A left inverse of ι_Γ^* is given by the transfer map, which on the level of cochains is defined by

$$T^n : C(G^{n+1})^\Gamma \to C(G^{n+1})^G, \quad f \mapsto \bar{f},$$

where $\bar{f}$ is given in terms of the G-invariant probability measure μ on G/Γ by

$$\bar{f}(g_0, \ldots, g_n) = \int_{G/\Gamma} f(g_0\dot{g}, \ldots, g_n\dot{g})d\mu(\dot{g}).$$

□

Since the symmetric space $\mathcal{X}$ is contractible, the locally symmetric space $M = \Gamma\backslash\mathcal{X}$ provides a smooth model for $B\Gamma$. In particular, $H^\bullet(\Gamma; \mathbb{R}) \cong H^\bullet(M; \mathbb{R})$. Thus, fixing a Riemannian metric on M we can use Hodge theory to interprete classes in $H_c^\bullet(G; \mathbb{R})$ as harmonic differential forms on M.

A different geometric interpretation of classes in $H_c^\bullet(G; \mathbb{R})$ can be given using duality of symmetric spaces. To formulate this duality it is convenient to assume that G is linear, hence embeds into its universal complexification $G^{\mathbb{C}}$. Note that this is not a restriction, since replacing G by a finite quotient will neither affect the continuous cohomology nor boundedness of cohomology classes. Thus assume G to be linear and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ denotes the Killing orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. We further denote by $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$ the *dual Lie algebra* of $\mathfrak{g}$. Then the *compact dual group* G_u of G is defined as the analytic subgroup of $G^{\mathbb{C}}$ associated with the Lie algebra $\mathfrak{g}_u$. It turns out that (G_u, K) is a compact symmetric pair, whence we can define the *compact dual symmetric space* of $\mathcal{X} = G/K$ to be $\mathcal{X}_u := G_u/K$. We can now provide a topological interpretation of $H_c^\bullet(G; \mathbb{R})$ by defining an isomorphism

$$\Psi_G : H^\bullet(\mathcal{X}_u; \mathbb{R}) \cong H_c^\bullet(G; \mathbb{R}). \tag{2.1}$$

This isomorphism is defined by composing the van Est isomorphism [4, Prop. 5.4]

$$\iota_{vE} : \Omega^\bullet(\mathcal{X})^G \to H_c^\bullet(G; \mathbb{R})$$

with an isomorphism

$$\Phi_G : H^\bullet(\mathcal{X}_u; \mathbb{R}) \cong \Omega^\bullet(\mathcal{X})^G \tag{2.2}$$

that will be introduced below.

The van Est isomorphism identifies an invariant differential form on $\mathcal{X}$ with the cohomology class of the cocycle obtained by integrating this form over suitable simplices as described in [16]. All we have to know about this isomorphism for our purposes is the following naturality property. Recall that by a classical lemma of Cartan (see e.g. [18, p. 227]) every invariant form on a symmetric space is closed; thus if Γ is a cocompact, torsion-free lattice in G with associated locally symmetric space $M := \Gamma\backslash\mathcal{X}$, then the inclusion map $\Omega^\bullet(\mathcal{X})^G \to \Omega^\bullet(\mathcal{X})^\Gamma = \Omega^\bullet(M)$ induces a map

$$\pi_! : \Omega^\bullet(\mathcal{X})^G \to H^\bullet(M; \mathbb{R}). \tag{2.3}$$

Then the van Est isomorphism makes the diagram

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{X})^G & \xrightarrow{\pi_!} & H^\bullet(M; \mathbb{R}) \\ {\scriptstyle \iota_{vE}}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_c^\bullet(G; \mathbb{R}) & \xrightarrow{\iota_\Gamma^*} & H^\bullet(\Gamma; \mathbb{R}) \end{array} \tag{2.4}$$

commute.

Let us now define the isomorphism (2.2). By a classical result of Chevalley-Eilenberg [12] the inclusion of G_u-invariant forms induces an isomorphism

$$H^\bullet(\Omega^\bullet(\mathcal{X}_u)^{G_u}, d) \cong H^\bullet(\mathcal{X}_u;\mathbb{R}).$$

But by Cartan's lemma, G_u-invariant forms on $\mathcal{X}_u$ are closed, hence the differential is trivial and every class α in $H^\bullet(\mathcal{X}_u;\mathbb{R})$ can be represented by a unique invariant form $\omega(\alpha)$. Then restriction to the basepoint $o_u = eK$ yields an isomorphism

$$H^\bullet(\mathcal{X}_u;\mathbb{R}) \cong \Omega^\bullet(\mathcal{X}_u)^{G_u} \cong \left(\bigwedge{}^\bullet(i\,\mathfrak{p})^*\right)^K, \quad \alpha \mapsto \omega(\alpha)_{o_u}. \tag{2.5}$$

Combining this with the flip isomorphism

$$\iota^*\colon \left(\bigwedge{}^\bullet(i\,\mathfrak{p})^*\right)^K \xrightarrow{\cong} \left(\bigwedge{}^\bullet\mathfrak{p}^*\right)^K, \quad (\iota^*\alpha)(X_1,\dots,X_n) := \alpha(iX_1,\dots,iX_n), \qquad (X_1,\dots,X_n\in\mathfrak{p}) \tag{2.6}$$

we obtain the desired isomorphism (2.2) as

$$\Phi_G : H^\bullet(\mathcal{X}_u;\mathbb{R}) \cong \left(\bigwedge{}^\bullet\mathfrak{p}^*\right)^K \cong \Omega^\bullet(\mathcal{X})^G, \tag{2.7}$$

where the last isomorphism is given by G-invariant extension.

For later reference we remark at this point that commutativity of the diagram (2.4) may be rephrased in terms of the isomorphism Ψ_G and the homomorphism

$$\Phi_\Gamma := \pi_! \circ \Phi_G\colon H^\bullet(\mathcal{X}_u;\mathbb{R}) \to H^\bullet(M;\mathbb{R}), \tag{2.8}$$

in the following way.

Lemma 2.2. *The diagram*

$$\begin{array}{ccc} H^\bullet(\mathcal{X}_u;\mathbb{R}) & \xrightarrow{\Phi_\Gamma} & H^\bullet(M;\mathbb{R}) \\ {\scriptstyle\Psi_G}\downarrow & & \downarrow{\scriptstyle\cong} \\ H^\bullet_c(G;\mathbb{R}) & \xrightarrow{\iota_\Gamma^*} & H^\bullet(\Gamma;\mathbb{R}). \end{array}$$

commutes.

This clarifies in particular the relation between the above two geometric interpretations of classes in $H^\bullet_c(G;\mathbb{R})$.

A third way to obtain a geometric interpretation for at least some classes in $H^\bullet_c(G;\mathbb{R})$ was already given by Bott's theorem (see Proposition 1.1) in Section 1. Note that the uniqueness assertion in that proposition is an immediate consequence of Lemma 2.1. The existence of σ_G will be proved in Appendix A.

2.2. **Continuous bounded cohomology of Lie groups and Gromov's theorem.** For any Hausdorff locally compact group G the *continuous bounded cohomology* $H^\bullet_{cb}(G;\mathbb{R})$ of G is defined as the cohomology of the complex $(C^\bullet_{cb}(G;\mathbb{R}), d)$ of continuous bounded functions, where

$$C^n_{cb}(G,\mathbb{R}) := C_b(G^{n+1},\mathbb{R})^G, \quad df(g_0,\dots,g_n) = \sum_{i=0}^n (-1)^i f(g_0,\dots,\widehat{g_i},\dots,g_n).$$

In the present generality it was first studied by Burger and Monod [9]. For a discrete group Γ the groups $H^\bullet_b(\Gamma;\mathbb{R}) := H^\bullet_{cb}(\Gamma;\mathbb{R})$ are just the bounded cohomology groups of Trauber (unpublished), which later were popularized through the work of Gromov [17]. Originally introduced as a computational tool for the computation of bounded cohomology of lattices, continuous bounded cohomology of Lie groups has rapidly evolved into a subject in its own right (see [27] for an overview of the literature before 2006 and [8] for some recent applications). Note that the inclusion of complexes

$$(C^\bullet_{cb}(G;\mathbb{R}), d) \hookrightarrow (C^\bullet_c(G;\mathbb{R}), d)$$

induces a *comparison map*

$$c^\bullet_G \colon H^\bullet_{cb}(G;\mathbb{R}) \to H^\bullet_c(G;\mathbb{R}),$$

which is natural in G.

In order to motivate our work, we mention a result of Burger and Monod [9] which states that for any connected Lie group G with amenable radical $R_{am}(G)$ there is an isomorphism

$$H^\bullet_{cb}(G;\mathbb{R}) \cong H^\bullet_{cb}(G/R_{am}(G);\mathbb{R}).$$

In this way the computation of continuous bounded cohomology of connected Lie groups is reduced to the case of semisimple groups with finite center and without compact factors. Concerning such groups the following is conjectured [11, Conjecture 16.1], [27, Conjecture A].

Conjecture. If G is a connected semisimple Lie group without compact factors and with finite center, then the comparison map $H^\bullet_{cb}(G;\mathbb{R}) \to H^\bullet_c(G;\mathbb{R})$ is an isomorphism.

In degree 2 this conjecture was established by Burger and Monod in [9]; beyond that case it is widely open. Note that the surjectivity part of this conjecture is precisely Dupont's question mentioned in Section 1. This explains in particular why attempts to compute the continuous bounded cohomology of Lie groups have revived the interest in this question. In fact, this article arose from our attempts to understand Conjecture 2.2.

Let us now state Gromov's theorem in its original version. Let G denote a connected semisimple Lie group without compact factors and with finite center, and denote by B_*G the Milnor model [24] of the classifying space of G. The important point about this model is that it is functorial in G, and we could equally well work with any other functorial model. If G^δ denotes the discrete group underlying G, then the cohomology classes in $H^\bullet(B_*G^\delta;\mathbb{R})$ are precisely the universal characteristic classes of flat G-bundles, while the elements of $H^\bullet(BG;\mathbb{R})$ are universal characteristic classes of arbitrary G-bundles. By functoriality, the identity map $G^\delta \to G$ induces a map $B_*\iota_\delta : B_*G^\delta \to B_*G$, whence a map

$$H^\bullet(B_*G;\mathbb{R}) \xrightarrow{(B_*\iota_{G^\delta})^*} H^\bullet(B_*G^\delta;\mathbb{R}) \cong H^\bullet(G^\delta;\mathbb{R})$$

in cohomology. Its image consists precisely of those characteristic classes of flat G-bundles which can be extended to characteristic classes of arbitrary G-bundles; these classes are known as *primary characteristic classes* of flat G-bundles. Via the isomorphims $H^\bullet(B_*G^\delta;\mathbb{R}) \cong H^\bullet(G^\delta;\mathbb{R})$ we may consider primary characteristic classes as elements of $H^\bullet(G^\delta;\mathbb{R})$. In particular, we have a well-defined notion of boundedness for such classes. Now Gromov's original theorem takes the following form [17] (see [6] for a stronger version).

Theorem 2.3 (Gromov)**.** *Every primary characteristic class is bounded.*

In order to deduce Theorem 1 from Theorem 2.3 we will need a bounded analog of Lemma 2.1, as follows. If H is a subgroup of G of finite co-volume, then there exists a *bounded transfer map*

$$T^\bullet_b \colon H^\bullet_{cb}(H;\mathbb{R}) \to H^\bullet_{cb}(G;\mathbb{R})$$

that provides a left inverse to the restriction map $H^\bullet_{cb}(G;\mathbb{R}) \to H^\bullet_{cb}(H;\mathbb{R})$, and on the level of cochains is given by the same formula as the usual transfer map defined in the proof of Lemma 2.1 above (see also [26, Proposition 8.6.2]).

Proof of Theorem 1. Let Γ be a torsion-free co-compact lattice in G. The inclusion $\iota_\Gamma : \Gamma \hookrightarrow G$ factors as $\iota_\Gamma = \iota_\delta \circ \iota_\Gamma^\delta$, where $\iota_\Gamma^\delta \colon \Gamma \to G^\delta$ and $\iota_\delta \colon G^\delta \to G$ are the natural maps. By functoriality of B_* it follows from the commuting diagram

$$\begin{array}{ccc}
H_c^\bullet(G;\mathbb{R}) & & H^\bullet(B_*G;\mathbb{R}) \\
\downarrow{\scriptstyle \iota_\delta^*} & & \downarrow{\scriptstyle (B_*\iota_\delta)^*} \\
H^\bullet(G^\delta;\mathbb{R}) & \xleftarrow{\cong} & H^\bullet(B_*G^\delta;\mathbb{R}) \\
\downarrow{\scriptstyle (\iota_\Gamma^\delta)^*} & & \downarrow{\scriptstyle (B_*\iota_\Gamma^\delta)^*} \\
H^\bullet(\Gamma;\mathbb{R}) & \xleftarrow{\cong} & H^\bullet(B_*\Gamma;\mathbb{R})
\end{array}$$

that the image of $(B_*\iota_\Gamma)^*$ consists precisely of the restriction of primary characteristic classes. In particular, using naturality of the comparison map we see that the image of $(B_*\iota_\Gamma)^*$ consists of bounded classes. Now Theorem 1 follows from the commutative diagram

$$\begin{array}{ccccc}
 & & H_c^\bullet(G;\mathbb{R}) & & H^\bullet(B_*G;\mathbb{R}) \\
 & & \downarrow{\scriptstyle \iota_\Gamma^*} & & \downarrow{\scriptstyle (B_*\iota_\Gamma)^*} \\
H_b^\bullet(\Gamma;\mathbb{R}) & \xrightarrow{c_\Gamma^\bullet} & H^\bullet(\Gamma;\mathbb{R}) & \xleftarrow{\cong} & H^\bullet(B_*G^\delta;\mathbb{R}) \\
\downarrow{\scriptstyle T_b^\bullet} & & \downarrow{\scriptstyle T^\bullet} & & \\
H_{cb}^\bullet(G;\mathbb{R}) & \xrightarrow{c_G^\bullet} & H_c^\bullet(G;\mathbb{R}) & &
\end{array}$$

□

3. Cohomological duality for symmetric spaces

3.1. **Preliminaries.** Since the cohomology of non-compact symmetric spaces is always trivial while that of compact symmetric spaces is always non-trivial, there is no obvious way to detect duality of symmetric spaces on the level of cohomology. However, as pointed out by Hirzebruch [19], one obtains a meaningful notion of cohomological duality if one compares the cohomology of a compact symmetric space $\mathcal{X}_u$ to the cohomology of a compact *locally* symmetric space M that is covered by its non-compact dual symmetric space $\mathcal{X}$. In the sequel we will fix such a locally symmetric space $M = \Gamma\backslash\mathcal{X}$ and use the notation introduced in the last section.

Our goal in this section is to compare real-valued characteristic classes of the canonical K-bundles $G_u \to \mathcal{X}_u = G_u/K$ and $\Gamma\backslash G \to M$, where the latter bundle is induced by the projection $G \to \mathcal{X}$. Let $f_{G_u} \colon G_u \to BK$ and $f_{\Gamma\backslash G} \colon \Gamma\backslash G \to BK$ be the classifying maps of these bundles. For every $c \in H^\bullet(BK;\mathbb{R})$ we then denote by $c(G_u) := f_{G_u}^* c$ and $c(\Gamma\backslash G) := f_{\Gamma\backslash G}^* c$ the corresponding real-valued characteristic classes.

We shall apply Chern-Weil theory in order to compare these classes [14, 21]. Let us briefly recall the basic facts. For every principal K-bundle $P \to X$ over a closed manifold X and every connection 1-form $A \in \Omega^1(P, \mathfrak{k})$ on P, we denote by $F_A \in \Omega^2(P, \mathfrak{k})$ its curvature 2-form. It is given by $F_A = dA + \frac{1}{2}[A \wedge A]$; moreover it is horizontal and thus descends to a 2-form $F_A \in \Omega^2\big(X, P(\mathfrak{k})\big)$ on X with values in the adjoint bundle $P(\mathfrak{k}) := P \times_K \mathfrak{k}$. Recall that K acts on the space $S^k(\mathfrak{k}^*)$ of symmetric k-multilinear functions on $\mathfrak{k}$ via the diagonal adjoint action. We denote by $I^k(\mathfrak{k}^*) \subset S^k(\mathfrak{k}^*)$ the subset of K-invariants. Given a K-invariant symmetric function $f \in I^k(\mathfrak{k}^*)$ and a connection 1-form $A \in \Omega^1(P, \mathfrak{k})$, we obtain a well-defined closed

$2k$-form

$$f(F_A, \dots, F_A) \in \Omega^{2k}(X)$$

on X which then defines a class in the de Rham cohomology $H^{2k}(X;\mathbb{R})$. The Chern-Weil theorem then asserts the following [14, Thm. 8.1].

Lemma 3.1. *Let K be a compact Lie group.*

(i) *There are no characteristic classes of principal K-bundles in odd degree, i.e. we have $H^{2k+1}(BK;\mathbb{R}) = \{0\}$ for all $k \geq 0$.*

(ii) *For every characteristic class $c \in H^{2k}(BK;\mathbb{R})$ there exists a unique $f \in I^k(\mathfrak{k}^*)$ such that the following holds. For every principal K-bundle $P \to X$ over a compact manifold X and every connection 1-form $A \in \Omega^1(P, \mathfrak{k})$ we have*

$$c(P) = [f(F_A, \dots, F_A)] \in H^{2k}(X;\mathbb{R}).$$

3.2. **An explicit version of Kobayashi-Ono-Hirzebruch duality.** The goal of this section is to establish the following result.

Proposition 3.2 (Kobayashi-Ono)**.** *Let $k \geq 0$, and let $c \in H^{2k}(BK;\mathbb{R})$ be a characteristic class. Then the corresponding characteristic classes of the canonical K-bundles $G_u \to \mathcal{X}_u$ and $\Gamma\backslash G \to M$ are related by*

$$\Phi_\Gamma\big(c(G_u)\big) = (-1)^k \cdot c(\Gamma\backslash G),$$

where $\Phi_\Gamma \colon H^{2k}(\mathcal{X}_u;\mathbb{R}) \to H^{2k}(M;\mathbb{R})$ is the homomorphism (2.8).

After finishing a first version of this article, we learned from M. Bucher-Karlsson that the existence of a map $\Phi : H^{2k}(\mathcal{X}_u;\mathbb{R}) \to H^{2k}(M;\mathbb{R})$ with the property that $\Phi\big(c(G_u)\big) = (-1)^k \cdot c(\Gamma\backslash G)$ follows from more general results of Kobayashi and Ono [22]. However, for our purposes it is crucial to have an explicit description of this map on the level of cocycles, i.e., we need to be able to identify Φ with Φ_Γ. In order to keep the present article self-contained at this important point we decided to include a proof of Proposition 3.2 that is taylored to our needs. We would also like to point out that the main idea underlying this type of duality results already appears in the proof of Hirzebruch's proportionality principle [19].

Proof. We will prove the claimed relation by unraveling the definition of the homomorphism Φ_Γ given in Section 2.1, using the results from Chern-Weil theory discussed in the previous subsection.

First of all, we introduce appropriate connection 1-forms on the bundles $\Gamma\backslash G \to M$ and $G_u \to \mathcal{X}_u$. Let us denote by $\theta_G \in \Omega^1(G;\mathfrak{g})$ and $\theta_{G_u} \in \Omega^1(G_u;\mathfrak{g}_u)$ the Maurer-Cartan forms on G and G_u, respectively, and denote by

$$\pi_{\mathfrak{k}} \colon \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to \mathfrak{k}, \quad \pi_{\mathfrak{k}}^u \colon \mathfrak{g}_u = \mathfrak{k} \oplus i\,\mathfrak{p} \to \mathfrak{k}$$

the canonical projections. Then

$$\tilde{A} := \pi_{\mathfrak{k}} \circ \theta_G \in \Omega^1(G;\mathfrak{k})$$

defines a connection 1-form on the bundle $G \to \mathcal{X}$. This form is invariant under the action of the auxiliary lattice Γ in G and hence descends to a connection 1-form $A \in \Omega^1(\Gamma\backslash G;\mathfrak{k})$ on the bundle $\Gamma\backslash G \to M$. Likewise,

$$A_u := \pi_{\mathfrak{k}}^u \circ \theta_{G_u} \in \Omega^1(G_u;\mathfrak{k})$$

defines a connection 1-form on the bundle $G_u \to \mathcal{X}_u$.

Now let $c \in H^{2k}(BK;\mathbb{R})$ be a characteristic class and denote by $f \in I^k(\mathfrak{k}^*)$ the corresponding k-multilinear invariant function on $\mathfrak{k}$ as in Lemma 3.1. Then the characteristic class

$$c(G_u) \in H^{2k}(\mathcal{X}_u;\mathbb{R})$$

is represented as a class in de Rham cohomology by the closed form

$$f(F_{A_u},\dots,F_{A_u})\in\Omega^{2k}(\mathcal{X}_u).$$

Since the Maurer-Cartan form $\theta_{G_u}\in\Omega^1(G_u;\mathfrak{k})$ is G_u-invariant, the connection 1-form $A_u\in\Omega^1(G_u;\mathfrak{k})$ and hence also the form $f(F_{A_u},\dots,F_{A_u})\in\Omega^{2k}(\mathcal{X}_u)$ on $\mathcal{X}_u$ are both G_u-invariant. Thus the class $[f(F_{A_u},\dots,F_{A_u})]$ is mapped under (2.5) to the K-invariant $2k$-linear form

$$\big(f(F_{A_u},\dots,F_{A_u})\big)_{o_u}\in\Big(\bigwedge{}^{2k}(i\,\mathfrak{p})^*\Big)^K$$

on $i\,\mathfrak{p}$. In order to determine the image of this form under the isomorphism (2.6) we use the following claim.

Claim. $\iota^*\Big(\big(f(F_{A_u},\dots,F_{A_u})\big)_{o_u}\Big)=(-1)^k\cdot\big(f(F_{\tilde A},\dots,F_{\tilde A})\big)_o.$

We will prove the claim by a direct calculation. First, we note that for $iX_1,\dots,iX_{2k}\in i\,\mathfrak{p}\cong T_{o_u}\mathcal{X}_u$ we have

$$\begin{aligned}&\big(f(F_{A_u},\dots,F_{A_u})\big)_{o_u}(iX_1,\dots,iX_{2k})\\&\qquad=\frac{1}{(2k)!}\sum_{\sigma\in\mathfrak{S}_{2k}}(-1)^\sigma f\big((F_{A_u})_e(iX_{\sigma(1)},iX_{\sigma(2)}),\dots,(F_{A_u})_e(iX_{\sigma(2k-1)},iX_{\sigma(2k)})\big),\end{aligned}$$

where $e\in G_u$ is the unit element, and the summation is taken over all permutations σ of $(1,2,\dots,2k)$. On the left-hand side of this identity we consider F_{A_u} as a form on $\mathcal{X}_u$ whereas on the right-hand side we regard it as a form on G_u. Likewise, for $X_1,\dots,X_{2k}\in\mathfrak{p}\cong T_o\mathcal{X}$ we have

$$\begin{aligned}&(f(F_{\tilde A},\dots,F_{\tilde A}))_o(X_1,\dots,X_{2k})\\&\qquad=\frac{1}{(2k)!}\sum_{\sigma\in\mathfrak{S}_{2k}}(-1)^\sigma f\big((F_{\tilde A})_e(X_{\sigma(1)},X_{\sigma(2)}),\dots,(F_{\tilde A})_e(X_{\sigma(2k-1)},X_{\sigma(2k)})\big),\end{aligned}$$

where $e\in G$ is the unit element of G. Again, on the left-hand side of this identity $F_{\tilde A}$ is regarded as a form on $\mathcal{X}$ whereas on the right-hand side it is considered as a form on G. Since f is k-linear we see from these formulas and the definition of the isomorphism (2.6) that it will be enough to establish the relation

$$\big(F_{A_u}\big)_e(iX,iY)=-\big(F_{\tilde A}\big)_e(X,Y)$$

for $X,Y\in\mathfrak{p}$. To this end, we recall that the Maurer-Cartan form on G_u satisfies the identities

$$(\theta_{G_u})_{o_u}(iX)=iX,\ X\in\mathfrak{p}\quad\text{and}\quad d\theta_{G_u}+\frac{1}{2}\big[\theta_{G_u}\wedge\theta_{G_u}\big]=0.$$

Then we obtain

$$\begin{aligned}
\big(F_{A_u}\big)_e(iX,iY) &= \Big(\big(dA_{G_u}\big)_e + \frac{1}{2}\big[A_{G_u}\wedge A_{G_u}\big]_e\Big)(iX,iY)\\
&= \Big(\pi^u_{\mathfrak{k}}\circ\big(d\theta_{G_u}\big)_e + \frac{1}{2}\big[\pi^u_{\mathfrak{k}}\circ d\theta_{G_u}\wedge \pi^u_{\mathfrak{k}}\circ d\theta_{G_u}\big]_e\Big)(iX,iY)\\
&= \frac{1}{2}\big(-\pi^u_{\mathfrak{k}}\circ\big[\theta_{G_u}\wedge\theta_{G_u}\big]_e + \big[\pi^u_{\mathfrak{k}}\circ d\theta_{G_u}\wedge \pi^u_{\mathfrak{k}}\circ d\theta_{G_u}\big]_e\big)(iX,iY)\\
&= -\frac{1}{2}\big[\theta_{G_u}\wedge\theta_{G_u}\big]_e(iX,iY)\\
&= -\frac{1}{2}\Big(\big[(\theta_{G_u})_e(iX),(\theta_{G_u})_e(iY)\big] - \big[(\theta_{G_u})_e(iY),(\theta_{G_u})_e(iX)\big]\Big)\\
&= -[iX,iY]\\
&= [X,Y].
\end{aligned}$$

A similar computation shows that

$$\big(F_{\tilde{A}}\big)_e(X,Y) = -[X,Y].$$

This proves the claim.

Continuing with the proof of the proposition, we note that since the Maurer-Cartan form $\theta_G\in\Omega^1(G;\mathfrak{k})$ is G-invariant, it follows that the connection 1-form $\tilde{A}\in\Omega^1(G;\mathfrak{k})$ and hence also the form $f(F_{\tilde{A}},\dots,F_{\tilde{A}})\in\Omega^{2k}(\mathcal{X})$ on $\mathcal{X}$ are both G-invariant. Hence the $2k$-linear form $\big(f(F_{\tilde{A}},\dots,F_{\tilde{A}})\big)_o$ on $\mathfrak{p}$ is mapped under the second isomorphism in (2.7) to the form

$$f(F_{\tilde{A}},\dots,F_{\tilde{A}})\in\Omega^{2k}(\mathcal{X})^G.$$

This form is in particular Γ-invariant, so it now finally follows from Lemma 3.1 that it gets mapped under (2.8) to the characteristic class $c(\Gamma\backslash G)$. ☐

3.3. **A proportionality principle.** In his seminal article [19], Hirzebruch proved a duality principle for characteristic numbers of certain locally symmetric spaces. Generalizations of this proportionality principle were obtained by Kamber and Tondeur [20]. In this section we explain how Proposition 3.2 may be used to reproduce some of these results.

For any closed oriented manifold X of dimension m, we denote by

$$\langle\,\cdot\,,[X]\rangle\colon H^m(X;\mathbb{R})\to\mathbb{R}$$

the pairing of classes of top degree in the singular cohomology of X with the fundamental class $[X]\in H_m(X;\mathbb{R})$ in the singular homology of X. Then we have the following proportionality principle.

Corollary 3.3. *Set $m:=\dim(M)=\dim(\mathcal{X}_u)$, and fix an orientation of $\mathcal{X}_u$ and M. Then there exists a real number $a(\Gamma)\neq 0$ such that for any collection $c_1,\dots,c_r\in H^\bullet(BK;\mathbb{R})$ of characteristic classes satisfying*

$$\deg(c_1)+\dots+\deg(c_r)=m$$

we have

$$\big\langle c_1(G_u)\cup\dots\cup c_r(G_u),[\mathcal{X}_u]\big\rangle = a(\Gamma)\cdot\big\langle c_1(\Gamma\backslash G)\cup\dots\cup c_r(\Gamma\backslash G),[M]\big\rangle.$$

Proof. By Lemma 3.1 (i) we may without loss of generality assume m to be even. Since $H^m(\mathcal{X}_u;\mathbb{R})\cong\mathbb{R}$ there exists a real number a' such that the linear functionals

$$\big\langle\,\cdot\,,[\mathcal{X}_u]\big\rangle,\big\langle\Phi_\Gamma(\,\cdot\,),[M]\big\rangle\colon H^m(\mathcal{X}_u;\mathbb{R})\to\mathbb{R}$$

are related by

$$\big\langle\,\cdot\,,[\mathcal{X}_u]\big\rangle = a'\cdot\big\langle\Phi_\Gamma(\,\cdot\,),[M]\big\rangle.$$

Then Proposition 3.2 yields

$$\begin{aligned}\big\langle c_1(G_u)\cup\cdots\cup c_r(G_u),[\mathcal{X}_u]\big\rangle &= a'\cdot\big\langle\Phi_\Gamma\big(c_1(G_u)\cup\cdots\cup c_r(G_u)\big),[M]\big)\big\rangle\\ &= (-1)^{m/2}\cdot a'\cdot\big\langle c_1(\Gamma\backslash G)\cup\cdots\cup c_r(\Gamma\backslash G),[M]\big\rangle.\end{aligned}$$

This shows in particular that $a'\neq 0$. Now define $a(\Gamma):=(-1)^{m/2}\cdot a'$. □

4. A geometric implementation of the universal map

The goal of this section is to prove Theorem 2. To this end it will be convenient to reformulate the assertion of this theorem in terms of commutativity of a certain diagram. We will describe this reformulation in Section 4.1, and prove commutativity of this diagram in Section 4.2.

4.1. **Definition of the geometric map and reformulation of Theorem 2.** Our main tool in the proof of Theorem 2 is a certain geometric map $T_G : H^\bullet(B_*G;\mathbb{R})\to H_c^\bullet(G;\mathbb{R})$ which we shall now define. Recall from Section 2.2 that B_*G denotes a functorial model of the classifying space of G, which for definiteness we may choose to be the Milnor model. We will keep the notation introduced in Section 3. In particular, for any subgroup H of G we denote by $\iota_H\colon H\to G$ the corresponding inclusion.

Since K is a maximal compact subgroup of G, the inclusion $\iota_K\colon K\to G$ is a homotopy equivalence by the polar decomposition of G. Now if $EG\to BG$ is a universal G-bundle, then $BK := EG/K$ is a classifying space for K; the polar decomposition of G then shows that BK deformation retracts onto BG. This implies in particular that ι_K induces an isomorphism

$$(B_*\iota_K)^*\colon H^\bullet(B_*G;\mathbb{R})\to H^\bullet(B_*K;\mathbb{R}).$$

The main ingredient in the construction of T_G is the classifying map $f_{G_u}\colon \mathcal{X}_u\to B_*K$ of the canonical K-bundle $p_{G_u} : G_u\to\mathcal{X}_u$. This map gives rise to a map

$$f^*_{G_u} : H^\bullet(B_*K;\mathbb{R})\to H^\bullet(\mathcal{X}_u;\mathbb{R})$$

whose image is given by the algebra of real-valued characteristic classes of the bundle p_{G_u}. Recall the isomorphism $\Psi_G\colon H^\bullet(\mathcal{X}_u;\mathbb{R})\to H_c^\bullet(G;\mathbb{R})$ from Section 2.1. We define the geometric map by

$$T_G := \Psi_G\circ f^*_{G_u}\circ(B_*\iota_K)^*\colon H^\bullet(B_*G;\mathbb{R})\to H_c^\bullet(G;\mathbb{R}). \tag{4.1}$$

Next we observe that Theorem 2 states that the geometric map T_G and the universal map σ_G have the same image. We will prove here the stronger statement that these two maps actually agree up to a sign. In odd degrees this follows trivially from $H^{2k+1}(B_*K;\mathbb{R})=\{0\}$ (see Lemma 3.1); in even degrees we claim that

$$\sigma_G = (-1)^k\cdot T_G\colon H^{2k}(B_*G;\mathbb{R})\to H_c^{2k}(G;\mathbb{R}). \tag{4.2}$$

In view of the uniqueness assertion of Proposition 1.1, in order to prove Theorem 2 it will therefore suffice to prove the following lemma.

Lemma 4.1. *The diagram*

$$\begin{array}{ccc} H^{2k}(B_*G;\mathbb{R}) & \xrightarrow{(-1)^k\cdot T_G} & H_c^{2k}(G;\mathbb{R}) \\ \downarrow{\scriptstyle (B_*\iota_\Gamma)^*} & & \downarrow{\scriptstyle \iota_\Gamma^*} \\ H^{2k}(B_*\Gamma;\mathbb{R}) & \xrightarrow{\cong} & H^{2k}(\Gamma;\mathbb{R}) \end{array}$$

commutes.

4.2. **Proof of Lemma 4.1.** Firstly, by Proposition 3.2 the triangle

$$\begin{array}{ccccc} & & H^{2k}(B_*K;\mathbb{R}) & & \\ & \swarrow{\scriptstyle (-1)^k\cdot f_{G_u}^*} & & \searrow{\scriptstyle f_{\Gamma\backslash G}^*} & \\ H^{2k}(\mathcal{X}_u;\mathbb{R}) & & \xrightarrow{\Phi_\Gamma} & & H^{2k}(M;\mathbb{R}) \end{array} \tag{4.3}$$

commutes. Secondly, we may attach to this diagram the commuting square obtained in Lemma 2.2, obtaining a commutative diagram

$$\begin{array}{ccccc} H^{2k}(B_*K;\mathbb{R}) & \xrightarrow{(-1)^k\cdot f_{G_u}^*} & H^{2k}(\mathcal{X}_u;\mathbb{R}) & \xrightarrow{\Psi_G} & H_c^{2k}(G;\mathbb{R}) \\ \downarrow{\scriptstyle f_{\Gamma\backslash G}^*} & \swarrow{\scriptstyle \Phi_\Gamma} & & & \downarrow{\scriptstyle \iota_\Gamma^*} \\ H^{2k}(M;\mathbb{R}) & & \xrightarrow{\cong} & & H^{2k}(\Gamma;\mathbb{R}). \end{array} \tag{4.4}$$

Observe that Lemma 4.1 will be proven once we show commutativity of the diagram

$$\begin{array}{ccc} H^{2k}(B_*G;\mathbb{R}) & \xrightarrow{(B_*\iota_K)^*} & H^{2k}(B_*K;\mathbb{R}) \\ \downarrow{\scriptstyle (B_*\iota_\Gamma)^*} & & \downarrow{\scriptstyle f_{\Gamma\backslash G}^*} \\ H^{2k}(B_*\Gamma;\mathbb{R}) & \xrightarrow{f_{\mathcal{X}}^*} & H^{2k}(M;\mathbb{R}). \end{array} \tag{4.5}$$

Indeed, attaching diagram (4.5) to diagram (4.4) from the left, we obtain the commutative diagram

$$\begin{array}{ccccccc} H^{2k}(B_*G;\mathbb{R}) & \xrightarrow{(B_*\iota_K)^*} & H^{2k}(B_*K;\mathbb{R}) & \xrightarrow{(-1)^k\cdot f_{G_u}^*} & H^{2k}(\mathcal{X}_u;\mathbb{R}) & \xrightarrow{\Psi_G} & H_c^{2k}(G;\mathbb{R}) \\ \downarrow{\scriptstyle (B_*\iota_\Gamma)^*} & & & & & & \downarrow{\scriptstyle \iota_\Gamma^*} \\ H^{2k}(B_*\Gamma;\mathbb{R}) & & & \xrightarrow{\cong} & & & H^{2k}(\Gamma;\mathbb{R}) \end{array}$$

By (4.1) the upper row coincides with $(-1)^k \cdot T_G$, so we have arrived at the commutative diagram in Lemma 4.1. Thus it remains to prove the following lemma.

Lemma 4.2. *The diagram* (4.5) *commutes.*

Proof. It suffices to prove that the square

$$\begin{array}{ccc} B_*K & \xrightarrow{B_*\iota_K} & B_*G \\ \uparrow{\scriptstyle f_{\Gamma\backslash G}} & & \uparrow{\scriptstyle B_*\iota_\Gamma} \\ M & \xrightarrow{f_{\mathcal{X}}} & B_*\Gamma \end{array}$$

commutes up to homotopy. So we have to show that there is a homotopy

$$B_*\iota_K \circ f_{\Gamma\backslash G} \simeq B_*\iota_\Gamma \circ f_\mathcal{X}$$

between maps from M to B_*G. Since homotopy classes of maps from M to B_*G are in one-to-one correspondence with isomorphism classes of G-bundles over M this is equivalent to the existence of an isomorphism of G-bundles

$$(B_*\iota_K \circ f_{\Gamma\backslash G})^*EG \cong (B_*\iota_\Gamma \circ f_\mathcal{X})^*EG \tag{4.6}$$

over M. Now we see from the pullback diagrams

$$\begin{CD} (\Gamma\backslash G)\times_K G @>>> E_*K\times_K G @>>> E_*G \\ @VVV @VVV @VVV \\ M @>{f_{\Gamma\backslash G}}>> B_*K @>{B_*\iota_K}>> B_*G \end{CD}$$

and

$$\begin{CD} \Gamma\backslash(\mathcal{X}\times G) @>>> E_*\Gamma\times_\Gamma G @>>> E_*G \\ @VVV @VVV @VVV \\ M @>{f_\mathcal{X}}>> B_*\Gamma @>{B_*\iota_\Gamma}>> B_*G \end{CD}$$

that

$$(B_*\iota_K \circ f_{\Gamma\backslash G})^*EG \cong (\Gamma\backslash G)\times_K G \quad \text{and} \quad (B_*\iota_\Gamma \circ f_\mathcal{X})^*EG \cong \Gamma\backslash(\mathcal{X}\times G)$$

as G-bundles over M. Note that in the second diagram the quotient $\Gamma\backslash(\mathcal{X}\times G)$ is taken with respect to the diagonal action induced by the standard left actions of Γ on $\mathcal{X} = G/K$ and G. Thus (4.6) is a consequence of the following claim.

Claim. *The bundles $(\Gamma\backslash G)\times_K G$ and $\Gamma\backslash(\mathcal{X}\times G)$ are isomorphic as G-bundles over M.*

To prove the claim we write down the isomorphism explicitly. Let us use the notation

$$\pi_1\colon P_1 := (\Gamma\backslash G)\times_K G \to M, \quad [\Gamma g_1, g_2] \mapsto \Gamma g_1 K$$

and

$$\pi_2\colon P_2 := \Gamma\backslash((G/K)\times G) \to M, \quad [g_1K, g_2] \mapsto \Gamma g_1 K$$

for the two bundles. Then the map

$$\varphi\colon G\times G \to G\times G, \quad (g_1, g_2) \mapsto (g_1, g_1g_2).$$

descends to a morphism of G-bundles $P_1 \to P_2$. Similarly, the inverse

$$\varphi^{-1}\colon G\times G \to G\times G, \quad (g_1, g_2) \mapsto (g_1, g_1^{-1}g_2)$$

descends to a morphism $P_2 \to P_1$. This proves the claim and finishes the proof of the lemma. □

This completes the proof of Lemma 4.1 and thereby the proof of Theorem 2.

5. Applications

Let (G_u, K) be a compact symmetric pair. Then the real cohomology of the compact symmetric space $\mathcal{X}_u = G_u/K$ was described by H. Cartan [10] (see also [1] for a different approach, which generalizes to the integral cohomology).

Theorem 4 (H. Cartan)**.** *Let (G_u, K) be a compact symmetric pair and $\mathcal{X}_u = G_u/K$. Then there exist graded subalgebras $H^\bullet_{ev}, H^\bullet_{odd} \subset H^\bullet(\mathcal{X}_u;\mathbb{R})$ with the following properties.*

(i) *$H^\bullet(\mathcal{X}_u;\mathbb{R}) = H^\bullet_{ev} \otimes H^\bullet_{odd}$ as graded algebras.*
(ii) *$H^\bullet_{ev}$ is the algebra of real characteristic classes of the canonical K-bundle $p_{G_u} : G_u \to \mathcal{X}_u$; it is concentrated in even degree.*
(iii) *Under the map $p^*_{G_u} : H^\bullet(\mathcal{X}_u;\mathbb{R}) \to H^\bullet(G_u;\mathbb{R})$ the subalgebra $H^\bullet_{odd}$ is mapped isomorphically onto $p^*_{G_u}(H^\bullet(\mathcal{X}_u;\mathbb{R}))$; the latter, and hence also $H^\bullet_{odd}$, is generated by odd degree classes.*

The decomposition in (i) induces via the isomorphism $H^\bullet(\mathcal{X}_u;\mathbb{R}) \cong H^\bullet_c(G;\mathbb{R})$ a corresponding decomposition

$$H^\bullet_c(G;\mathbb{R}) = H^\bullet_{ev}(G) \otimes H^\bullet_{odd}(G), \tag{5.1}$$

Then Theorem 2 takes the following form.

Corollary 5.1. *The primary chacteristic classes of flat G-bundles are precisely the classes in the image of $H^\bullet_{ev}(G)$ under the natural restriction map $\iota^*_\delta : H^\bullet_c(G;\mathbb{R}) \to H^\bullet(G^\delta;\mathbb{R})$.*

Combining this with Theorem 1 we obtain the following main result.

Theorem 5. *Every class in $H^\bullet_{ev}(G)$ is bounded.*

For every semisimple Lie group G the algebra $H^\bullet_{ev}(G)$ is known [25], hence Theorem 5 yields an explicit boundedness result for each such G. Of particular interest is the case where $H^\bullet_c(G;\mathbb{R}) = H^\bullet_{ev}(G)$, since in this case Theorem 5 gives a positive answer to Dupont's question. We recall the following result of Cartan [10].

Proposition 5.2 (H. Cartan)**.** *Let G be a semisimple Lie group without compact factors and with finite center. Then $H^\bullet_c(G;\mathbb{R}) = H^\bullet_{ev}(G)$ if and only if*

$$\mathrm{rk}_{\mathbb{R}}(G_u) = \mathrm{rk}_{\mathbb{R}}(K). \tag{5.2}$$

Note that condition (5.2) is satisfied if and only if it is satisfied by every simple factor. We may therefore assume from now on that G is simple. If G happens to be a complex simple Lie group, then $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ and thus $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{k}$. This implies $G_u = K \times K$, hence condition (5.2) can never be satisfied. The non-complex, non-compact simple real Lie groups together with their dual compact symmetric pairs are listed in [18, Table V]. We may then read off from this list those pairs that satisfy condition (5.2).

Proposition 5.3. *Let G be a non-complex, non-compact simple real Lie group with finite center. Then exactly one of the following three cases holds.*

(i) *G is Hermitian.*
(ii) *G is not Hermitian, but locally isomorphic to $SO_0(p, q)$ with pq even or $Sp(p, q)$ with $p, q \geq 1$ or exceptional with Lie algebra one of the following: $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{g}_{2(2)}$.*
(iii) *G is locally isomorphic to $SL_n(\mathbb{R})$ for $n \geq 3$, $SU^*(2n)$ for $n \geq 2$, $SO_0(p, q)$ for p and q both odd or exceptional with Lie algebras $\mathfrak{e}_{6(6)}$ or $\mathfrak{e}_{6(-26)}$.*

In cases (i) and (ii) condition (5.2) *is satisfied. In case (iii) condition* (5.2) *is not satisfied.*

In view of the preceeding remarks, Theorem 3 now follows from Theorem 5, Proposition 5.2 and Proposition 5.3.

Appendix A. On a theorem of Bott

The purpose of this appendix is to provide a short proof of the following result of Bott (which is precisely the existence part of Proposition 1.1).

Proposition A.1 (Bott)**.** *Let G be Lie group. Then there exists a map $\sigma_G : H^\bullet(BG;\mathbb{R}) \to H^\bullet_c(G;\mathbb{R})$ such that for every discrete subgroup Γ of G with inclusion map $\iota_\Gamma : \Gamma \to G$ the diagram*

$$\begin{array}{ccc} H^\bullet(BG;\mathbb{R}) & \xrightarrow{\sigma_G} & H^\bullet_c(G;\mathbb{R}) \\ {\scriptstyle (B\iota_\Gamma)^*}\downarrow & & \downarrow{\scriptstyle \iota_\Gamma^*} \\ H^\bullet(B\Gamma;\mathbb{R}) & \xrightarrow{\cong} & H^\bullet(\Gamma;\mathbb{R}) \end{array}$$

commutes.

Bott suggested to use the theory of spaces with two topologies (see [28]) for a proof of Proposition A.1. Here we present a short proof using simplicial sheaf cohomology [14, 31]. The advantage of this proof is that it carries over to the case of Lie groupoids, and in fact we learned that the argument given below is folklore in the groupoid community.

Every Hausdorff topological group G may be considered as a topological category (with one object) and we denote by $G_\bullet$ the nerve of this category (see e.g. [14, p. 76], where this nerve is denoted $NG(\bullet)$). By definition $G_\bullet$ is a simplicial space, i.e., a simplicial object in the category of topological spaces. If we assume that G is a (possibly disconnected, not necessarily second countable) Lie group G, then the fat geometric realization $B_*G := \|G_\bullet\|$ of $G_\bullet$ is a functorial model for the classifying space of G (see e.g. [14, pp. 77-78]), which is closely related to Segal's model [29]. On each of the spaces G_n we now consider the sheaf $\underline{\mathbb{R}}$ of locally constant real valued functions and the sheaf C^0 of continuous real valued functions. Since these sheaves are compatible with the face and degeneracy maps, we obtain sheaves $\underline{\mathbb{R}}$ and C^0 over the simplicial space $G_\bullet$ (see [31, Sec. 3] for the notion of a sheaf over a simplicial space). We can thus form the corresponding sheaf cohomology groups. Then we have the following lemma.

Lemma A.2. (i) *$H^\bullet(G_\bullet;\underline{\mathbb{R}}) \cong H^\bullet(B_*G;\mathbb{R})$.*
(ii) *$H^\bullet(G_\bullet;C^0) \cong H^\bullet_c(G;\mathbb{R})$.*
(iii) *Under these isomorphisms the inclusion of simplicial sheaves $i : \underline{\mathbb{R}} \hookrightarrow C^0$ induces a natural $\mathbb{R}$-algebra homomorphism*

$$i^* : H^\bullet(B_*G;\mathbb{R}) \to H^\bullet_c(G;\mathbb{R}),$$

which for discrete G coincides with the canonical isomorphism.

Proof. (i) The sheaf $\underline{\mathbb{R}}$ admits a flabby resolution

$$\underline{\mathbb{R}} \to C^0 \xrightarrow{d} C^1 \to \dots$$

by the sheaves C^q of singular real q-cochains (i.e., for $U_n \subset G_n$ the group $C^q(U_n)$ consists of singular real q-cochains in U_n). Hence $H^\bullet(G_\bullet;\underline{\mathbb{R}})$ is the cohomology of the total complex associated to the double complex $\{C^q(G_n)\}$. Then (i) is a consequence of [14, Prop. 5.15].
(ii) The sheaves C^0 on G_n are flabby, hence acyclic, and thus the double complex computing $H^\bullet(G_\bullet;C^0)$ collapses to the inhomogeneous bar complex for $H^\bullet_c(G;\mathbb{R})$.
(iii) Naturality follows from naturality of the nerve construction. For discrete G the sheaves $\underline{\mathbb{R}}$ and C^0 coincide, hence the map becomes tautological. ☐

Now Proposition A.1 is an immediate consequence of assertion (iii) of Lemma A.2. However, it seems impossible to obtain the geometric implementation T_G of σ_G (see Section 4) from the above sheaf theoretic definition without appealing to the universal property.

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