BOUNDED COHOMOLOGY VIA PARTIAL DIFFERENTIAL EQUATIONS, I

TOBIAS HARTNICK AND ANDREAS OTT

ABSTRACT. We present a new technique that employs partial differential equations in order to explicitly construct primitives in the continuous bounded cohomology of Lie groups. As an application, we prove a vanishing theorem for the continuous bounded cohomology of $\mathrm{SL}(2,\mathbb{R})$ in degree four, establishing a special case of a conjecture of Monod.

CONTENTS

1. INTRODUCTION

Ever since Gromov's seminal paper [25], bounded cohomology of discrete groups has proved a useful tool in geometry, topology and group theory. In recent years the scope of bounded cohomology has widened considerably. An important step was taken by Burger and Monod, who extended the theory to the category of locally compact second countable groups under the name of continuous bounded cohomology [14, 38]. Not only did this lead to a breakthrough in the understanding of bounded cohomology of lattices in Lie groups [14], but also triggered a series of discoveries in rigidity theory (e. g. [15, 8, 10, 9, 4, 41, 42, 18, 27, 26]), higher Teichmüller theory (e. g. [11, 13, 12, 1]) and symplectic geometry (e. g. [43, 20]). At the same time, our understanding of the second bounded cohomology has improved. In particular, the approach originally developed for free groups [24] and hyperbolic groups [21] has been extended to larger classes of groups including mapping class groups [2] and acylindrically hyperbolic groups [30, 22]. Moreover, there has been some progress in constructing bounded cohomology classes in higher degree [36, 28, 5, 23].

On the other hand, our knowledge on vanishing results for bounded cohomology in higher degree is still very poor. It was already known to Johnson [32] that the bounded cohomology of an amenable group vanishes in all positive degrees. Here the primitive of a given cocycle

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is obtained by applying an invariant mean. In contrast, for non-amenable groups no general technique for constructing primitives is available so far. In particular, there is not a single non-amenable group whose bounded cohomology is known in all degrees. Actually, the situation is even worse. One may define the *bounded cohomological dimension* of a group Γ to be

$$\mathrm{bcd}(\Gamma) := \sup\{n \mid H^n_b(\Gamma;\mathbb{R}) \neq 0\},$$

where $H^n_b(\Gamma;\mathbb{R})$ is the n-th bounded cohomology of Γ with coefficients in the trivial module $\mathbb{R}$. At present we do not even know whether there exists any group Γ with $\mathrm{bcd}(\Gamma) \notin \{0,\infty\}$.

The few vanishing results we have for the bounded cohomology of non-amenable groups are all based on the vanishing of all cocycles in the respective degree in some resolution. The most far-reaching results in this direction were achieved in [40] by choosing efficient resolutions. However, no such resolutions are known for dealing with the continuous bounded cohomology $H^n_{cb}(H;\mathbb{R})$ of non-amenable connected Lie groups H. For such groups the most efficient resolution that is presently available is the boundary resolution [31, 14]. In this particular resolution cocycles vanish only in degree at most three; in degree greater than three there will inevitably be nonzero cocycles, and one faces the problem of finding primitives. This explains why the few existing vanishing results such as in [14, 16] do not go beyond degree three.

Our goal in this article is to develop a new approach to the construction of primitives in continuous bounded cohomology for real semisimple Lie groups. To demonstrate its effectiveness we settle the following special case of a conjecture due to Monod [39, Problem A].

Theorem 1.1. *Let G be a connected real Lie group that is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Then*

$$H^4_{cb}(G;\mathbb{R}) = 0.$$

Actually, for such G Monod conjectured that $\mathrm{bcd}(G) = 2$, which means that $H^n_{cb}(G;\mathbb{R}) = 0$ for all $n > 2$. In degree $n = 3$ there are no nonzero cocycles in the boundary resolution at all [16], but this is no longer true for $n > 3$. In this sense, Theorem 1.1 is the prototype of a vanishing theorem that requires the construction of primitives. We believe that our method of proof generalizes to arbitrary $n > 3$, and possibly to other Lie groups. This will be addressed in future work.

Monod's conjecture about the bounded cohomology of $\mathrm{SL}_2(\mathbb{R})$ is a special case of a more general conjecture, which would allow to compute the continuous bounded cohomology of arbitrary connected Lie groups. In fact, since continuous bounded cohomology is invariant under division by the amenable radical [14, 38], it is sufficient to compute the continuous bounded cohomology of semsimple Lie groups H without compact factors and with finite center. For such groups it is conjectured [19, 39] that the natural comparison map between the continuous bounded cohomology and the continuous cohomology is an isomorphism. This would imply that $\mathrm{bcd}(H)$ coincides with the dimension of the associated symmetric space, thereby providing examples of groups of arbitrary bounded cohomological dimension. Plenty is known by now about surjectivity of the comparison map [19, 25, 7, 34, 23, 28], while injectivity still remains mysterious in higher degrees. Indeed, injectivity has so far been established only in degree two for arbitrary H [14], and for some rank one groups in degree three [16, 3, 6]. Theorem 1.1 is the first result in degree greater than three. Incidentally, it has an application to the existence of solutions to perturbations of the Spence-Abel functional equation for Rogers' dilogarithm, along the lines suggested in [16]. This will be discussed in the forthcoming article [29].

For the proof of Theorem 1.1 we shall reformulate the problem of constructing bounded primitives in terms of a fixed point problem for the action of G on a certain function space. The main idea is then to describe the fixed points as solutions of a certain system of linear first order partial differential equations. In this way, we obtain primitives by solving the corresponding Cauchy problem. We show that for carefully chosen initial conditions, particular solutions have additional discrete symmetries, which we finally use to deduce boundedness. We will give a more detailed outline of our strategy of proof in Section 2.6 after introducing some notation.

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2. Preliminaries on continuous bounded cohomology

2.1. The boundary action of $\mathrm{PSL}_2(\mathbb{R})$.

The goal of this subsection is to describe a model for the continuous bounded cohomology of $\mathrm{SL}_2(\mathbb{R})$. Since continuous bounded cohomology of connected Lie groups is invariant under local isomorphisms [38, Cor. 7.5.10] we have

$$H^{\bullet}_{cb}(\mathrm{SL}_2(\mathbb{R});\mathbb{R}) \cong H^{\bullet}_{cb}(\mathrm{PSL}_2(\mathbb{R});\mathbb{R}) \cong H^{\bullet}_{cb}(\mathrm{PU}(1,1);\mathbb{R}),$$

where the latter isomorphism is induced by the Cayley transform. We prefer to carry out our computations in the group $G := \mathrm{PU}(1,1)$. The group G acts by fractional linear transformations on the Poincaré disc $\mathbb{D}$ and thereby identifies with the group of orientation-preserving isometries of $\mathbb{D}$, which is an index 2 subgroup in the full isometry group $\widehat{G}$. The actions of G and $\widehat{G}$ extend continuously to the boundary S^1 of $\mathbb{D}$, and the corresponding actions on S^1 will be referred to as the *boundary action* of G respectively $\widehat{G}$. The action of $\widehat{G}$ on S^1 (but not on $\mathbb{D}$) may be identified with the action of $\mathrm{PGU}(1,1)$ by fractional linear transformations. It is well-known that this action is strictly 3-transitive.

Explicitly, elements of $\mathrm{PGU}(1,1)$ can be represented by matrices of the form

$$g_{a,b} := \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ with $|a|^2 - |b|^2 \in \{\pm 1\}$. We denote by $[g_{a,b}]$ the equivalence class of the matrix $g_{a,b}$ in $\mathrm{PGU}(1,1)$ and define $\epsilon := [g_{0,1}]$. Then $\widehat{G} \cong G \rtimes \langle \epsilon \rangle$, where G is given by equivalence classes of matrices of determinant one, and ϵ acts on G via

$$\epsilon g_{a,b} \epsilon^{-1} = g_{\bar{a},\bar{b}}.$$

We will use the following notation concerning subgroups of G: If we define

$$k_\xi := \left[g_{e^{i\xi/2},0}\right], \quad a_s := \left[g_{\cosh(-s/2),\,\sinh(-s/2)}\right], \quad n_t := \left[g_{1+\frac{i}{2}t,\,-\frac{i}{2}t}\right],$$

then the maps $\xi \mapsto k_\xi$, $s \mapsto a_s$ and $t \mapsto n_t$ are one-parameter groups $\mathbb{R} \to G$ whose images we denote by K, A and N, respectively. Our parametrization of these groups is somewhat non-standard, but is convenient for certain computations in local coordinates, see Lemma 3.2 below.

Observe that $K = \mathrm{Stab}_G(0)$ is a maximal compact subgroup of G, that A normalizes N and that $P = AN = \mathrm{Stab}_G(1)$ is a parabolic subgroup of G. Since N is the unipotent radical of P we have $\mathrm{Fix}(N) = \{1\}$, whereas $\mathrm{Fix}(A) = \{\pm 1\}$. Moreover, we obtain an Iwasawa decomposition $G = KAN$, and every elliptic (hyperbolic, parabolic) element in G is conjugate to an element in K (A, N).

2.2. **Cocycles and strict cocycles.** We keep the notation introduced in the last subsection and denote by μ_K the unique K-invariant measure on S^1. Given $n \geq 0$, we shall write $\mathcal{M}((S^1)^{n+1})$ for the space of $\mu_K^{\otimes(n+1)}$-measurable real-valued functions on $(S^1)^n$ and $\mathcal{L}^\infty((S^1)^n)$ for the subspace of bounded functions. The quotients of these spaces obtained by identifying $\mu_K^{\otimes(n+1)}$-almost everywhere coinciding functions will be denoted by $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$ respectively.

We define the homogeneous differential $d : \mathcal{M}((S^1)^{n+1}) \to \mathcal{M}((S^1)^{n+2})$ by

$$df(z_0, \ldots, z_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \, f(z_0, \ldots, \widehat{z_j}, \ldots, z_{n+1}).$$

This induces corresponding differentials on $\mathcal{L}^\infty((S^1)^n)$, $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$. Elements in the kernels of these four differentials are respectively referred to as *strict n-cocycles*, *strict bounded n-cocycles*, *n-cocycles* and *bounded n-cocycles.*

The group $\widehat{G}$ acts diagonally on $(S^1)^{n+1}$, and this action commutes with the action of the symmetric group $\mathfrak{S}_{n+1}$ by permution of the variables. This induces a $\widehat{G} \times \mathfrak{S}_{n+1}$-action on each of the spaces $\mathcal{M}((S^1)^{n+1})$, $\mathcal{L}^\infty((S^1)^{n+1})$, $M((S^1)^{n+1})$ and $L^\infty((S^1)^{n+1})$, and these actions are intertwined by the corresponding homogeneous differentials, in particular they preserve the corresponding cocycles. Given a subgroup $H < \widehat{G}$, an H-invariant (strict, bounded) cocycle is simply called a (strict, bounded) *H-cocycle*, and it is called a (strict, bounded) *H-coboundary* if it is contained in the image of the H-invariants under d.

A major technical inconvenience is caused by the non-surjectivity of the map $\mathcal{M}((S^1)^n)^G \to M((S^1)^n)^G$, which means that a G-cocycle c may not admit an *invariant representative*, i.e., a strict G-cocycle which coincides with c almost everywhere. Fortunately, for bounded function classes existence of invariant representatives follows from [37, Thm. A].

Lemma 2.1 (Invariant representatives)**.** *The maps $\mathcal{L}^\infty((S^1)^{n+1})^G \to L^\infty((S^1)^{n+1})^G$, $n \geq 0$, are surjective. In fact, they admit a family of sections compatible with the homogeneous differentials.*

2.3. **The boundary model for continuous bounded cohomology.** A function class $f \in L^\infty((S^1)^{n+1})$ is called *alternating* provided $\sigma.f = (-1)^\sigma \cdot f$ for all $\sigma \in \mathfrak{S}_{n+1}$, and we denote by $L^\infty_{alt}((S^1)^{n+1}) < L^\infty((S^1)^{n+1})$ the subspace of alternating function classes. Since the actions of $\widehat{G}$ and $\mathfrak{S}_{n+1}$ commute, this subspace is $\widehat{G}$-invariant.

Proposition 2.2 (Boundary model, [38, Thm. 7.5.3])**.** *Given a closed subgroup $H < \widehat{G}$, the continuous bounded cohomology of H is given by the cohomology of the complex $(L^\infty_{\mathrm{alt}}((S^1)^{\bullet+1})^H, d)$, i.e.,*

$$H^n_{cb}(H;\mathbb{R}) \cong H^n\big(L^\infty_{\mathrm{alt}}((S^1)^{\bullet+1})^H, d\big)$$

for all $n \geq 0$.

2.4. **Even and odd cocycles.** Given $f \in L^\infty_{\mathrm{alt}}((S^1)^{n+1})$, we denote by $f^\pm$ the projections of f onto the ±1-eigenspace of ϵ. Explicitly we have $f^\pm = \frac{1}{2}(f \pm \epsilon.f)$ and $f = f^+ + f^-$. We say that f is *even* if $f = f^+$ and *odd* if $f = f^-$. Since ϵ normalizes G the projections $f \mapsto f^\pm$ preserve the subspace of G-invariants. They also commute with homogeneous differentials, since ϵ does. In particular, every G-invariant alternating (strict) cocycle decomposes uniquely into a sum of a G-invariant alternating (strict) even cocycle and a G-invariant alternating (strict) odd cocycle, and similarly for coboundaries. On the level of cohomology this yields a decomposition

$$H^\bullet_{cb}(G;\mathbb{R}) \cong H^\bullet_{cb}(G;\mathbb{R})_{\mathrm{ev}} \oplus H^\bullet_{cb}(G;\mathbb{R})_{\mathrm{odd}}.$$

The first summand $H^n_{cb}(G;\mathbb{R})_{\mathrm{ev}}$ can be identified with the continuous bounded cohomology $H^n_{cb}(\widehat{G};\mathbb{R})$ of the extended group $\widehat{G}$. Similarly, if we denote by $\mathbb{R}_\epsilon$ the unique non-trivial one-dimensional $\widehat{G}$-module, then $H^\bullet_{cb}(G;\mathbb{R})_{\mathrm{odd}} \cong H^\bullet_{cb}(\widehat{G};\mathbb{R}_\epsilon)$, whence the above decomposition can also be written as

$$H^\bullet_{cb}(G;\mathbb{R}) \cong H^\bullet_{cb}(\widehat{G};\mathbb{R}) \oplus H^\bullet_{cb}(\widehat{G};\mathbb{R}_\epsilon).$$

In particular, the vanishing of $H^\bullet_{cb}(G;\mathbb{R})$ is equivalent to the vanishing of both $H^\bullet_{cb}(\widehat{G};\mathbb{R})$ and $H^\bullet_{cb}(\widehat{G};\mathbb{R}_\epsilon)$. It turns out that the vanishing of $H^\bullet_{cb}(\widehat{G};\mathbb{R}_\epsilon)$ can be deduced using only combinatorial arguments, see Proposition 2.3 below. The vanishing of the first summand is considerably harder and its proof will occupy the rest of this article.

Proposition 2.3. *Every alternating G-invariant bounded 4-cocycle is even. In particular,*

$$H^4_{cb}(G;\mathbb{R})_{\mathrm{odd}} \cong H^4_{cb}(\widehat{G};\mathbb{R}_\epsilon) = 0.$$

The proof of Proposition 2.3 will be given in Appendix A. We remind the reader that non-zero even alternating G-invariant bounded 4-cocycles *do* exist, see [16].

2.5. **Primitives in the boundary model.** Given a bounded G-cocycle $c \in L^\infty((S^1)^5)^G$ and any closed subgroup $H \subset G$, we denote by

$$\mathcal{P}(c)^H := \Big\{p \in M\big((S^1)^4\big)^H \,\Big|\, dp = c\Big\}$$

the space of H-invariant primitives of c and by

$$\mathcal{P}^\infty(c)^H := \Big\{p \in L^\infty\big((S^1)^4\big)^H \,\big|\, dp = c\Big\}$$

the subspace of bounded H-invariant primitives. In view of Proposition 2.2 the proof of Theorem 1.1 amounts to showing that $\mathcal{P}^\infty(c)^G \neq \emptyset$ for any bounded alternating G-cocycle c, and by Proposition 2.3 we may furthermore assume that c is even. Under this assumption we will explicitly construct elements in $\mathcal{P}^\infty(c)^G$. For this purpose, we first define an operator

$$I\colon \mathcal{L}^\infty\big((S^1)^{n+1}\big) \to \mathcal{L}^\infty\big((S^1)^n\big)$$

by

$$I(c)(z_1,\dots,z_n) := \int_{S^1} c(z,z_1\dots,z_n)\,d\mu_K(z). \tag{2.1}$$

It induces an operator $I\colon L^\infty((S^1)^{n+1}) \to L^\infty((S^1)^n)$, which by abuse of notation we denote by the same symbol. By integrating the cocycle equation $dc = 0$, we see that $d(I(c)) = c$ for every cocycle c.

From now on we fix a bounded G-cocycle $c \in L^\infty((S^1)^5)^G$. For the moment we do not need to assume that c is either alternating or even. By K-invariance of the measure μ_K we see from formula (2.1) that the function $I(c)$ is K-invariant, hence a K-invariant primitive of c. It will, however, in general not be G-invariant. In order to obtain G-invariant primitives we amend the operator I in the following way.

We denote by $(S^1)^{(n)} \subset (S^1)^n$ the subset of n-tuples of pairwise distinct points in S^1. Note in particular that the G-action on $(S^1)^{(3)}$ is free and has two open orbits given by positively and negatively oriented triples. We write $C^\infty((S^1)^{(3)})^K$ for the space of K-invariant real-valued smooth functions on $(S^1)^{(3)}$ and consider it as a subspace of $M((S^1)^3)$. We then define an operator

$$P_c\colon C^\infty\big((S^1)^{(3)}\big)^K \to M\big((S^1)^4\big), \quad f \mapsto I(c) + df.$$

A key observation is that all G-invariant bounded primitives of c necessarily lie in the image of the operator P_c. This will allow us to express primitives in terms of smooth (rather than measurable) solutions to differential equations.

Proposition 2.4. *The image of the operator P_c satisfies*

$$\mathcal{P}^\infty(c)^G \subset P_c\big(C^\infty((S^1)^{(3)})^K\big) \subset \mathcal{P}(c)^K.$$

Proof. We have already seen above that $I(c) \in \mathcal{P}^\infty(c)^K$. We conclude that $P_c(f) \in \mathcal{P}(c)^K$ for all $f \in C^\infty((S^1)^{(3)})^K$. It also follows that if $p \in \mathcal{P}^\infty(c)^G$ is any bounded primitive then $d(p - I(c)) = 0$. Hence we must have $p = I(c) + df$ for some *measurable* function f given by $f = I(p - I(c))$. The nontrivial part of the proof is to show that this function f can be chosen to be smooth. In fact, by Lemma 2.1 we may take invariant representatives $\tilde{p}$ and $\tilde{c}$ and set

$$f := I(\tilde{p} - I(\tilde{c})) \in \mathcal{M}((S^1)^3)^K.$$

We claim that this function is smooth on $(S^1)^{(3)}$. Indeed, for every $g \in G$ we have

$$\begin{aligned} f(g.z_1, g.z_2, g.z_3) &= \int_{S^1} \Big(\tilde{p}(z, g.z_1, g.z_2, g.z_3) - I(\tilde{c})(z, g.z_1, g.z_2, g.z_3)\Big) d\mu_K(z) \\ &= \int_{S^1}\int_{S^1} \frac{d(g.\mu_K)}{d\mu_K}(z)\left(\tilde{p}(z,z_1,z_2,z_3) - \frac{d(g.\mu_K)}{d\mu_K}(w)\,\tilde{c}(z,w,z_1,z_2,z_3)\right) d\mu_K(w)\,d\mu_K(z). \end{aligned}$$

Smoothness of the Radon-Nikodym derivative [33, Prop. 8.43] hence implies that the function on G given by $g \mapsto f(g.z_1, g.z_2, g.z_3)$ is smooth. Since G-orbits are open in $(S^1)^{(3)}$ we infer that the function f is smooth. □

2.6. **Strategy of proof.** We briefly outline the strategy for the proof of Theorem 1.1. We shall proceed in three steps.

1. In Section 3, we show that for any function $f \in C^\infty((S^1)^{(3)})^K$ the primitive $P_c(f)$ is G-invariant if and only if f satisfies a certain system of linear first order partial differential equations.
2. In Section 4, we explicitly construct solutions f of this system of differential equations, showing that $\mathcal{P}(c)^G \neq \emptyset$.
3. In Section 5, we prove that there exist particular solutions f with certain additional discrete symmetries. For such functions f we then show boundedness of $P_c(f)$, establishing that $\mathcal{P}^\infty(c)^G \neq \emptyset$.

While the constructions in (1) and (2) work for arbitrary bounded G-cocycles c, the construction in (3) relies on c being alternating and even.

3. Partial differential equations

3.1. **The boundary action in local coordinates.** In order to describe the boundary action of $G = \mathrm{PU}(1,1)$ explicitly, we introduce coordinates as follows. We consider S^1 as a subset of $\mathbb{C}$ and write $z \in S^1$ to denote a complex number z of modulus 1. In addition, it will often be convenient to work with the angular coordinate $\theta \in [0, 2\pi)$ defined by $z = e^{i\theta}$. Correspondingly, on $(S^1)^n$ we will use the two sets of coordinates $(z_0, \dots, z_{n-1})$ and $(\theta_0, \dots, \theta_{n-1})$. Note that, in angular coordinates on S^1, the measure μ_K is given by

$$\int_{S^1} f(z)\, d\mu_K(z) = \fint f(\theta)\, d\theta := \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Convention 3.1. Throughout, all operations on angular coordinates will implicitly be understood modulo 2π. For example, $\theta_2 - \theta_1$ denotes the unique point in the interval $[0, 2\pi)$ that is congruent to $\theta_2 - \theta_1$ modulo 2π.

The G-action on S^1 induces a G-action on the interval $[0, 2\pi)$ by the relation $g.e^{i\theta} = e^{i\, g.\theta}$. Note that in particular $k_\xi.\theta = \theta + \xi$. The next lemma provides formulas for the infinitesimal action of the one-parameter subgroups $\{a_s\}$ and $\{n_t\}$ in angular coordinates.

Lemma 3.2. *The infinitesimal action of $\{a_s\}$ and $\{n_t\}$ in angular coordinates is given by*

$$\frac{d}{ds}(a_s.\eta) = \sin(a_s.\eta), \qquad \frac{d}{dt}(n_t.\eta) = 1 - \cos(n_t.\eta)$$

for $\eta \in [0, 2\pi)$.

Proof. To prove the first formula, we compute

$$\left.\frac{d}{ds}\right|_{s=0} (a_s.\phi) = \frac{1}{i\, e^{i\phi}} \left.\frac{d}{ds}\right|_{s=0} e^{i\, a_s.\phi} = \frac{1}{i\, e^{i\phi}} \frac{d}{ds} \left.\left(\frac{\cosh(-s/2)\, e^{i\phi} + \sinh(-s/2)}{\sinh(-s/2)\, e^{i\phi} + \cosh(-s/2)} \right)\right|_{s=0} = \sin(\phi).$$

Since $\{a_s\}$ is a one-parameter group we further infer that

$$\frac{d}{ds}(a_s.\eta) = \left.\frac{d}{d\sigma}\right|_{\sigma=0} (a_{s+\sigma}.\eta) = \left.\frac{d}{d\sigma}\right|_{\sigma=0} (a_\sigma(a_s.\eta)) = \sin(a_s.\eta).$$

Likewise, for the second formula we compute

$$\frac{d}{dt}\bigg|_{t=0}(n_t.\phi) = \frac{1}{i\,e^{i\phi}}\,\frac{d}{dt}\bigg|_{t=0} e^{i\,n_t.\phi} = \frac{1}{i\,e^{i\phi}}\,\frac{d}{dt}\left(\frac{(1+\frac{i}{2}t)\,e^{i\phi}-\frac{i}{2}t}{\frac{i}{2}t\,e^{i\phi}+1-\frac{i}{2}t}\right)\Bigg|_{t=0} = 1-\cos(\phi)$$

and conclude as above. □

3.2. **Fundamental vector fields.** We denote by $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$ the differential operators that appear as fundamental vector fields for the infinitesimal action of the one-parameter groups $\{k_\xi\}$, $\{a_s\}$ and $\{n_t\}$ on $(S^1)^{(n)}$. By Lemma 3.2, they are given in angular coordinates by

$$\begin{aligned} L_K^{(n)} &= \sum_{j=0}^{n-1} \frac{\partial}{\partial\theta_j} \\ L_A^{(n)} &= \sum_{j=0}^{n-1} \sin(\theta_j)\,\frac{\partial}{\partial\theta_j} \\ L_N^{(n)} &= \sum_{j=0}^{n-1} (1-\cos(\theta_j))\,\frac{\partial}{\partial\theta_j}. \end{aligned}$$

The next lemma is crucial for applications of the operators $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$ in cohomology.

Lemma 3.3. *Let $L^{(n)}$ denote one of the operators $L_K^{(n)}$, $L_A^{(n)}$ and $L_N^{(n)}$. Then $L^{(n)}$ commutes with the homogeneous differential in the sense that*

$$d^n \circ L^{(n)} = L^{(n+1)} \circ d^n$$

for every $n > 0$.

Proof. Let $\lambda \in C^\infty([0,2\pi))$ and consider the differential operators

$$L_\lambda^{(n)} := \sum_{j=0}^{n-1} \lambda(\theta_j)\,\frac{\partial}{\partial\theta_j}$$

for every $n > 0$. For any smooth $(n-1)$-cochain q we compute

$$\begin{aligned} \left(L_\lambda^{(n+1)}\big(d^n q\big)\right)(\theta_0,\ldots,\theta_n) &= \sum_{j=0}^{n} \lambda(\theta_j)\,\frac{\partial}{\partial\theta_j}\left(\sum_{\ell=0}^{n}(-1)^\ell\, q\big(\theta_0,\ldots,\widehat{\theta_\ell},\ldots,\theta_n\big)\right) \\ &= \sum_{\ell=0}^{n}(-1)^\ell \sum_{j\neq\ell} \lambda(\theta_j)\,\frac{\partial q}{\partial\theta_j}\big(\theta_0,\ldots,\widehat{\theta_\ell},\ldots,\theta_n\big) \\ &= \sum_{\ell=0}^{n}(-1)^\ell\,\big(L_\lambda^{(n)} q\big)\big(\theta_0,\ldots,\widehat{\theta_\ell},\ldots,\theta_n\big) \\ &= \left(d^n\big(L_\lambda^{(n)} q\big)\right)(\theta_0,\ldots,\theta_n). \end{aligned}$$

□

3.3. **Infinitesimal invariance of primitives.** We are now in a position to characterize G-invariance of primitives $P_c(f)$ in terms of differential equations for the function f. First, let us define functions $c^\sharp$ and $c^\flat$ by

$$c^\sharp(\theta_0,\theta_1,\theta_2) := \fint\fint \cos(\varphi)\, c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi \tag{3.1}$$

and

$$c^\flat(\theta_0,\theta_1,\theta_2) := \fint\fint \sin(\varphi)\, c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi. \tag{3.2}$$

An argument as in the proof of Proposition 2.4 using smoothness of the Radon-Nikodym derivative shows that we may choose the functions $c^\sharp$ and $c^\flat$ to be smooth on $(S^1)^{(3)}$. The next proposition achieves the first step in the agenda outlined in Section 2.6.

Proposition 3.4. *Let $f \in C^\infty((S^1)^{(3)})^K$. Then the primitive $P_c(f) \in \mathcal{P}(c)^K$ is G-invariant if and only if there exist functions $v^\sharp, v^\flat \in C^\infty((S^1)^{(2)})$ such that the triple $(f, v^\sharp, v^\flat)$ satisfies the system of partial differential equations*

$$\begin{cases} L_A^{(3)} f = c^\sharp + dv^\sharp \\ L_N^{(3)} f = c^\flat + dv^\flat. \end{cases} \tag{3.3}$$

Note that all functions appearing in (3.3) are smooth, so all derivatives can be understood classically. The proof of Proposition 3.4 relies on the following lemma.

Lemma 3.5. *The function $I(c)$ is smooth along G-orbits and satisfies*

$$\begin{aligned} L_A^{(4)} I(c) &= -dc^\sharp \\ L_N^{(4)} I(c) &= -dc^\flat, \end{aligned}$$

where $c^\sharp$ and $c^\flat$ are as in (3.1) *and* (3.2).

Proof. An argument as in the proof of Proposition 2.4 shows that the function $I(c)$ can be chosen to be smooth along G-orbits. Fix an invariant representative $\tilde{c}$ of c by Lemma 2.1. Using A-invariance of $\tilde{c}$ and Lemma 3.2 we compute

$$\begin{aligned} L_A^{(4)}(I(c))(\theta_0,\ldots,\theta_3) &= \frac{d}{ds}\bigg|_{s=0} \fint \tilde{c}(\varphi, a_s.\theta_0,\ldots,a_s.\theta_3)\, d\varphi \\ &= \frac{d}{ds}\bigg|_{s=0} \fint \frac{d(a_s.\varphi)}{d\varphi}\, \tilde{c}(\varphi,\theta_0,\ldots,\theta_3)\, d\varphi \\ &= \fint \frac{d}{d\varphi}\, \frac{d(a_s.\varphi)}{ds}\bigg|_{s=0} \tilde{c}(\varphi,\theta_0,\ldots,\theta_3)\, d\varphi \\ &= \fint \frac{d}{d\varphi} \sin(a_s.\varphi)\bigg|_{s=0} \tilde{c}(\varphi,\theta_0,\ldots,\theta_3)\, d\varphi \\ &= \fint \cos(\varphi)\, c(\varphi,\theta_0,\ldots,\theta_3)\, d\varphi. \end{aligned}$$

On the other hand, the cocycle identity

$$\begin{aligned} 0 &= dc(\eta, \varphi, \theta_0, \dots, \theta_3) \\ &= c(\varphi, \theta_0, \dots, \theta_3) - c(\eta, \theta_0, \dots, \theta_3) + \sum_{j=0}^{3} (-1)^j \, c(\eta, \varphi, \theta_0, \dots, \widehat{\theta_j}, \dots, \theta_3) \end{aligned}$$

integrates against $\cos(\varphi)$ to

$$\begin{aligned} 0 &= \fint \fint \cos(\varphi) \, c(\varphi, \theta_0, \dots, \theta_3) \, d\eta \, d\varphi - 0 + \sum_{j=0}^{3} (-1)^j \, c^\sharp(\theta_0, \dots, \widehat{\theta_j}, \dots, \theta_3) \\ &= \fint \cos(\varphi) \, c(\varphi, \theta_0, \dots, \theta_3) \, d\varphi + dc^\sharp(\theta_0, \dots, \theta_3). \end{aligned}$$

This establishes the first identity. Likewise, to prove the second identity we compute

$$\begin{aligned} L_N^{(4)}(I(c))(\theta_0, \dots, \theta_3) &= \frac{d}{dt}\bigg|_{t=0} \fint \tilde{c}(\varphi, n_t.\theta_0, \dots, n_t.\theta_3) \, d\varphi \\ &= \fint \frac{d}{d\varphi} \, \frac{d(n_t.\varphi)}{dt}\bigg|_{t=0} \tilde{c}(\varphi, \theta_0, \dots, \theta_3) \, d\varphi \\ &= \fint \frac{d}{d\varphi} \big(1 - \cos(n_t.\varphi)\big)\bigg|_{t=0} \tilde{c}(\varphi, \theta_0, \dots, \theta_3) \, d\varphi \\ &= \fint \sin(\varphi) \, \tilde{c}(\varphi, \theta_0, \dots, \theta_3) \, d\varphi, \end{aligned}$$

and then integrate the above cocycle identity against $\sin(\varphi)$. □

Proof of Proposition 3.4. First of all, we observe that for $f \in C^\infty((S^1)^{(3)})^K$ we may choose the function

$$P_c(f) = I(c) + df$$

to be smooth along G-orbits. This follows by an argument as in the proof of Proposition 2.4. Hence the primitive $P_c(f)$ is G-invariant if and only if it is infinitesimally G-invariant. Since the subgroups K, A and N generate the group G, this in turn is equivalent to $P_c(f)$ satisfying the system of partial differential equations

$$\begin{cases} L_K^{(4)} P_c(f) = 0 \\ L_A^{(4)} P_c(f) = 0 \\ L_N^{(4)} P_c(f) = 0. \end{cases}$$

The first equation is automatically satisfied since $P_c(f)$ is K-invariant by Proposition 2.4. Writing out the definition of $P_c(f)$ and applying Lemmas 3.3 and 3.5, the remaining two equations are seen to be equivalent to

$$\begin{cases} d\big(L_A^{(3)} f\big) = dc^\sharp \\ d\big(L_N^{(3)} f\big) = dc^\flat. \end{cases}$$

Applying the operator I as in the proof of Proposition 2.4, we conclude that $f \in C^\infty((S^1)^{(3)})^K$ solves this system if and only if there exist *measurable* functions $v^\sharp$ and $v^\flat$ such that the triple

$(f, v^\sharp, v^\flat)$ satisfies system (3.3). However, it is not difficult to see that $v^\sharp$ and $v^\flat$ can be chosen to be smooth. In fact, by (3.3) the functions $dv^\sharp$ and $dv^\flat$ are differences of smooth functions and hence smooth. Now replace $v^\sharp$ and $v^\flat$ by $I(dv^\sharp)$ and $I(dv^\flat)$. □

3.4. **The Frobenius integrability condition.** We now turn to the problem of finding solutions of system (3.3). As we shall see in Proposition 3.7 below, it follows from the classical Frobenius theorem that this system admits a smooth solution $(f, v^\sharp, v^\flat)$ if and only if the functions $v^\sharp$ and $v^\flat$ satisfy a certain integrability condition. In order to state the result we need to introduce the function

$$\check{c}(\phi_1, \phi_2) := \fint \fint \fint \sin(\eta - \varphi)\, c(\eta, \varphi, \psi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi. \tag{3.4}$$

Lemma 3.6. *The function $\check{c}$ is smooth on $(S^1)^{(2)}$ and K-invariant.*

Proof. An argument as in the proof of Proposition 2.4 shows that $\check{c} \in C^\infty((S^1)^{(2)})$. By K-invariance of c and of the measure, for every $\xi \in [0, 2\pi]$ we have

$$\begin{aligned}\check{c}(k_\xi.\phi_1, k_\xi.\phi_2) &= \fint \fint \fint \sin(\eta - \varphi)\, c(\eta, \varphi, \psi, \phi_1 + \xi, \phi_2 + \xi)\, d\eta\, d\varphi\, d\psi \\ &= \fint \fint \fint \sin(\eta - \varphi)\, c(\eta - \xi, \varphi - \xi, \psi - \xi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi \\ &= \fint \fint \fint \sin\big((\eta + \xi) - (\varphi + \xi)\big)\, c(\eta, \varphi, \psi, \phi_1, \phi_2)\, d\eta\, d\varphi\, d\psi = \check{c}(\phi_1, \phi_2).\end{aligned}$$ □

Proposition 3.7 (Integrability condition)**.** *The system* (3.3) *admits a solution* $(f, v^\sharp, v^\flat)$ *if and only if the pair* $(v^\sharp, v^\flat)$ *satisfies the system of partial differential equations*

$$\begin{cases} d\big(L_K^{(2)} v^\sharp + v^\flat\big) = 0 \\ d\big(L_K^{(2)} v^\flat - v^\sharp\big) = 0 \\ d\big(L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat - \check{c}\big) = 0. \end{cases} \tag{3.5}$$

The proof of the proposition relies on the Frobenius theorem and will be deferred to Appendix B. In the sequel, we shall refer to system (3.5) as the *Frobenius system.* It will be enough for our purposes to find a function f satisfying system (3.3) for some pair $(v^\sharp, v^\flat)$. We will hence not attempt to find all solutions of system (3.5). Rather, we will construct a single special solution $(v^\sharp, v^\flat)$.

Proposition 3.8. *Let $r \in C^\infty((0, 2\pi), \mathbb{C})$ be a smooth complex-valued solution of the ordinary differential equation*

$$(1 - e^{-i\phi}) \cdot \frac{dr}{d\phi} = i\, r(\phi) - \check{c}(0, \phi). \tag{3.6}$$

By Convention 3.1, we may define a function $v \in C^\infty((S^1)^{(2)}, \mathbb{C})$ by

$$v(\theta_1, \theta_2) := e^{i\theta_1}\, r(\theta_2 - \theta_1). \tag{3.7}$$

Then the pair $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v))$ is a solution of the Frobenius system (3.5).

Proof. We remind the reader that throughout the proof we adhere to Convention 3.1. First observe that if v is a solution of the system

$$\begin{cases} L_K^{(2)} v = i\, v \\ e^{-i\theta_1}\dfrac{\partial v}{\partial \theta_1} + e^{-i\theta_2}\dfrac{\partial v}{\partial \theta_2} = \check{c}, \end{cases} \tag{3.8}$$

then $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v))$ is a solution of (3.5). Indeed, taking real and imaginary parts of the first equation in (3.8) we obtain

$$L_K^{(2)} v^\sharp = -v^\flat, \quad L_K^{(2)} v^\flat = v^\sharp,$$

while taking the real part of the second equation yields

$$L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat = \check{c}.$$

Next consider the transformation

$$u(\theta_1, \theta_2) := e^{-i\theta_1}\, v(\theta_1, \theta_2).$$

Then the first equation in (3.8) is equivalent to

$$L_K^{(2)} u = 0, \tag{3.9}$$

and the second equation is equivalent to

$$\begin{aligned} \check{c}(\theta_1, \theta_2) &= e^{-i\theta_1}\frac{\partial}{\partial \theta_1}\Big(e^{i\theta_1}\, u(\theta_1, \theta_2)\Big) + e^{-i\theta_2}\frac{\partial}{\partial \theta_2}\Big(e^{i\theta_1}\, u(\theta_1, \theta_2)\Big) \\ &= i\, u(\theta_1, \theta_2) + \frac{\partial u}{\partial \theta_1} + e^{i(\theta_1 - \theta_2)}\frac{\partial u}{\partial \theta_2} \\ &= i\, u(0, \theta_2 - \theta_1) + \big(1 - e^{i(\theta_1 - \theta_2)}\big)\frac{\partial u}{\partial \theta_1}. \end{aligned} \tag{3.10}$$

Here we used that $\partial_{\theta_1} u = -\partial_{\theta_2} u$ by (3.9). Let now r be a solution of equation (3.6) and set $v(\theta_1, \theta_2) := e^{i\theta_1}\, r(\theta_2 - \theta_1)$. Then $u(\theta_1, \theta_2) = r(\theta_2 - \theta_1)$ obviously satisfies (3.9). By K-invariance of $\check{c}$ from Lemma 3.6 we have $\check{c}(\theta_1, \theta_2) = \check{c}(0, \theta_2 - \theta_1)$. It follows that u solves equation (3.10). □

4. CONSTRUCTION OF PRIMITIVES

4.1. **Solving the Frobenius system.** The first step in the construction of the primitive $P_c(f)$ is to solve the differential equation (3.6) for the function r. As we have seen in Propositions 3.8 and 3.7, the function r then gives rise to a special solution $(v^\sharp, v^\flat)$ of the Frobenius system (3.5), which in turn determines the inhomogeneities in system (3.3) in such a way that this system admits a solution f.

The complex ordinary differential equation (3.6) can be solved by applying the method of variation of constants. Its general solution $r \in C^\infty((0, 2\pi), \mathbb{C})$ is given by

$$r(\phi) = \big(1 - e^{i\phi}\big) \cdot \left(C_0 - \frac{1}{2}\int_\pi^\phi \frac{\check{c}(0, \zeta)}{1 - \cos(\zeta)}\, d\zeta\right),$$

where C_0 is an arbitrary complex constant. Note that different choices of C_0 lead to cohomologous cochains v. We may therefore assume $C_0 = 0$, obtaining

$$r(\phi) = -\frac{1}{2}\big(1 - e^{i\phi}\big) \cdot \int_{\pi}^{\phi} \frac{\check{c}(0,\zeta)}{1 - \cos(\zeta)}\, d\zeta. \tag{4.1}$$

We shall henceforth be working with the function r defined by this formula. A crucial observation is the following lemma.

Lemma 4.1 (Boundedness of the inhomogeneity)**.** *The function r is bounded. In particular, the inhomogeneities in system* (3.3) *are bounded.*

Proof. Observe that $\|\check{c}\|_\infty \leq \|c\|_\infty$ and $\sqrt{1-\cos(\phi)} \cdot |\cot(\phi/2)| = \sqrt{2} \cdot |\cos(\phi/2)| \leq \sqrt{2}$. Hence for all $\phi \in (0, 2\pi)$ we have an estimate

$$\begin{aligned}
|r(\phi)| &\leq \frac{1}{2} \cdot \|\check{c}\|_\infty \cdot |1 - e^{i\phi}| \cdot \left| \int_{\pi}^{\phi} \frac{1}{1 - \cos(\zeta)}\, d\zeta \right| \\
&= \frac{\sqrt{2}}{2} \cdot \|\check{c}\|_\infty \cdot \sqrt{1 - \cos(\phi)} \cdot |\cot(\phi/2)| \\
&\leq \|c\|_\infty.
\end{aligned}$$

□

4.2. **Cauchy initial value problem.** Recall that solutions to a first order linear partial differential equation may be constructed explicitly by integration along its characteristic curves, with initial values prescribed on some non-characteristic hypersurface [17, Ch. 3]. We shall now apply this principle in order to explicitly construct solutions of the system (3.3).

Let the function r be given by formula (4.1). By Proposition 3.8 and Lemma 4.1 this determines a bounded solution $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v))$ of the Frobenius system (3.5) by

$$v(\theta_1, \theta_2) := e^{i\theta_1}\, r(\theta_2 - \theta_1). \tag{4.2}$$

We shall henceforth keep the function v defined that way. By Proposition 3.7 the system (3.3) then admits a solution $f \in C^\infty((S^1)^{(3)})$ satisfying the system of equations

$$L_K^{(3)} f = 0 \tag{4.3a}$$

$$L_A^{(3)} f = c^\sharp + dv^\sharp \tag{4.3b}$$

$$L_N^{(3)} f = c^\flat + dv^\flat. \tag{4.3c}$$

We know from Section 3.2 that the characteristic curves for these equations are precisely the orbits for the actions of the one-parameter groups $K = \{k_\xi\}$, $A = \{a_s\}$ and $N = \{n_t\}$ on $(S^1)^{(3)}$, respectively. In order to construct the function f we may therefore proceed as follows.

(1) We use equation (4.3a) in order to construct f with initial values $f_0 := f|_{H_0}$ prescribed on the hypersurface

$$H_0 := \big\{(z_0, z_1, z_2) \in (S^1)^{(3)} \,\big|\, z_0 := 1\big\}.$$

Of course, equation (4.3a) just says that f is constant along the K-orbits in $(S^1)^{(3)}$. The hypersurface H_0 is non-characteristic for the equation (4.3a) since it intersects transversally with the K-orbits in $(S^1)^{(3)}$. Note that in order for f to be compatible with the remaining equations (4.3b) and (4.3c), the hypersurface H_0 has to be invariant under the actions of A and N. This, however, is indeed the case since the point 1 remains fixed under these two actions.

(2) We use equation (4.3c) in order to construct f_0 with initial values $f_1 := f_0|_{H_1}$ prescribed on the union of A-orbits

$$H_1 := \{a_s.(1,i,-i) \mid -\infty < s < \infty\} \cup \{a_s.(1,-i,i) \mid -\infty < s < \infty\} \subset H_0.$$

Note that H_1 is non-characteristic for the equation (4.3c) since it intersects transversally with the N-orbits in H_0. Moreover, in order for f_0 to be compatible with the remaining equation (4.3b), the curve H_1 has to be invariant under the action of A, which is obviously the case.

(3) We use equation (4.3b) in order to construct f_1 with initial values $f_2 := f_1|_{H_2}$ prescribed on the set of *base points*

$$H_2 := \{(1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3})\} \subset H_1.$$

Note that equation (4.3b), when restricted to the curve H_1, becomes an ordinary differential equation which can be solved directly.

In particular, we see that solutions of system (3.3) are uniquely determined by the initial values of f_2 on the set of base points H_2 and hence form a 2-parameter family. We will work out the details of (1) in Section 4.3, while the details of (2) and (3) will be worked out in Section 4.4.

4.3. **Reduction of variables.** In angular coordinates, the hypersurface H_0 introduced in the previous subsection is given by

$$H_0 = \{(\theta_0, \theta_1, \theta_2) \in [0, 2\pi)^3 \mid \theta_0 = 0, \theta_1 \neq \theta_2, \theta_1 \neq 0 \neq \theta_2\}.$$

The canonical projection $(\theta_0, \theta_1, \theta_2) \mapsto (\theta_1, \theta_2)$ identifies H_0 with the open subset $\Omega := (0, 2\pi)^2 \setminus \Delta$ of the square, where $\Delta \subset (0, 2\pi)^2$ denotes the diagonal in the open square. The coordinates in Ω will be denoted by (ϕ_1, ϕ_2). Moreover, we write $\Omega_\pm := \{(\phi_1, \phi_2) \mid \phi_1 \lesseqgtr \phi_2\}$ for the open subsets corresponding to the two G-orbits in $(S^1)^{(3)}$ consisting of positively and negatively oriented triples. The restriction $f_0 := f|_{H_0}$ is then given by $f_0(\phi_1, \phi_2) = f(0, \phi_1, \phi_2)$. Note that by K-invariance the function f is recovered from f_0 by

$$f(\theta_0, \theta_1, \theta_2) = f_0(\theta_1 - \theta_0, \theta_2 - \theta_0). \tag{4.4}$$

Since the hypersurface H_0 is invariant under the actions of A and N, the system (3.3) restricts to the system

$$\begin{cases} L_A^{(2)} f_0 = c_0^\sharp + (dv^\sharp)_0 \\ L_N^{(2)} f_0 = c_0^\flat + (dv^\flat)_0. \end{cases} \tag{4.5}$$

Here we denote by $c_0^\sharp$, $c_0^\flat$, $(dv^\sharp)_0$ and $(dv^\flat)_0$ the respective restrictions of the functions $c^\sharp$, $c^\flat$, $dv^\sharp$ and $dv^\flat$, i. e.,

$$c_0^\sharp(\phi_1, \phi_2) := c^\sharp(0, \phi_1, \phi_2), \quad c_0^\flat(\phi_1, \phi_2) := c^\flat(0, \phi_1, \phi_2) \tag{4.6}$$

and

$$(dv^\sharp)_0(\phi_1, \phi_2) := dv^\sharp(0, \phi_1, \phi_2), \quad (dv^\flat)_0(\phi_1, \phi_2) := dv^\flat(0, \phi_1, \phi_2). \tag{4.7}$$

Note that these functions are smooth on Ω.

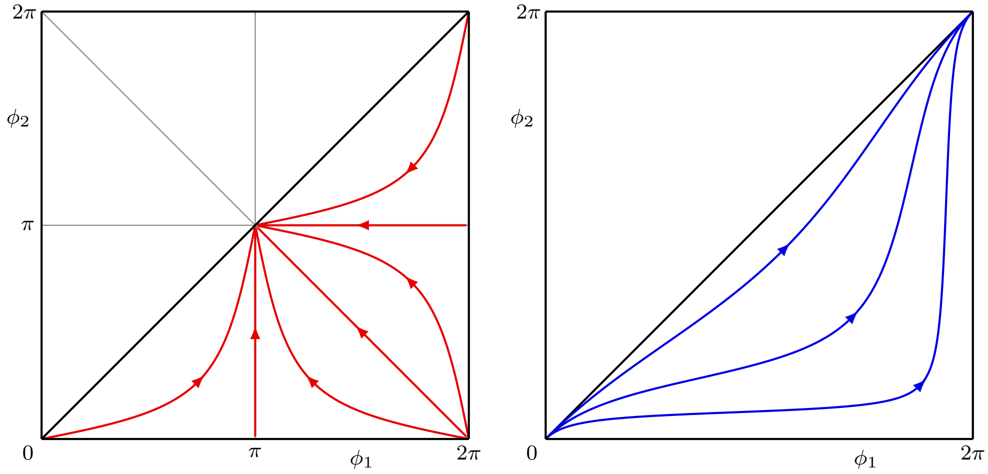

Figure 1. A-orbits (left) and N-orbits (right) in the domain Ω_-

4.4. **Method of characteristics.** We now construct the function f_0 by integrating equations (4.3b) and (4.3c) along their characteristic curves. Recall that the characteristics for these equations are precisely the orbits for the actions of the one-parameter groups $A = \{a_s\}$ and $N = \{n_t\}$ on Ω (see Figure 1). It will be convenient to abbreviate the inhomogeneities appearing on the right-hand side of system (4.5) by

$$F_c^\sharp := c_0^\sharp + (dv^\sharp)_0 \quad \text{and} \quad F_c^\flat := c_0^\flat + (dv^\flat)_0. \tag{4.8}$$

If we prescribe the value of f_0 on a single point in each of the orbits Ω_+ and Ω_-, the function f_0 will be uniquely determined by the relations

$$f_0(a_S.\phi_1, a_S.\phi_2) - f_0(\phi_1, \phi_2) = \int_0^S F_c^\sharp(a_s.\phi_1, a_s.\phi_2)\, ds$$

and

$$f_0(n_T.\phi_1, n_T.\phi_2) - f_0(\phi_1, \phi_2) = \int_0^T F_c^\flat(n_t.\phi_1, n_t.\phi_2)\, dt.$$

More precisely, let us denote by

$$\Delta^{\mathrm{op}} := \big\{(\phi, 2\pi - \phi) \,\big|\, \phi \in (0, 2\pi) \setminus \{\pi\}\big\} \subset \Omega$$

the antidiagonal in Ω, which corresponds to the hypersurface H_1 introduced in Section 4.2. Note that it has two connected components. In order to compute the function f_0 we first introduce new coordinates on Ω that are adapted to the N-orbits. For every point $(\phi_1, \phi_2) \in \Omega$ we define $\Phi(\phi_1, \phi_2) \in (0, \pi) \cup (\pi, 2\pi)$ in such a way that $(\Phi(\phi_1, \phi_2), 2\pi - \Phi(\phi_1, \phi_2))$ is the point of intersection of the antidiagonal with the unique N-orbit passing through the point (ϕ_1, ϕ_2). We then define $T(\phi_1, \phi_2) \in (-\infty, \infty)$ by the relation

$$(\phi_1, \phi_2) = n_{T(\phi_1,\phi_2)}.\big(\Phi(\phi_1, \phi_2), 2\pi - \Phi(\phi_1, \phi_2)\big).$$

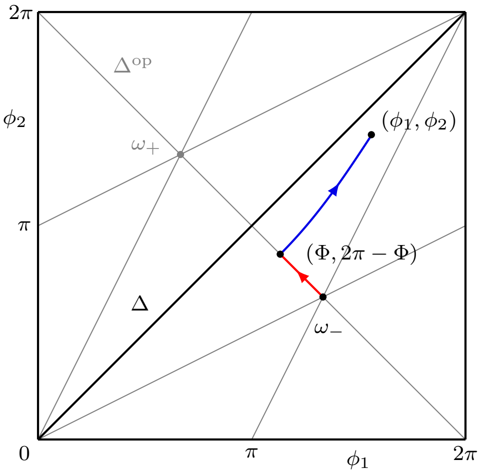

FIGURE 2. A path traveling along A- and N-orbits from the basepoint ω_- to some point (ϕ_1, ϕ_2) in Ω_-

For later reference we note that

$$T(\phi_1, \phi_2) = -\frac{1}{2}\left(\cot\left(\frac{\phi_1}{2}\right) + \cot\left(\frac{\phi_2}{2}\right)\right). \tag{4.9}$$

Here we used that the action of the one-parameter subgroup $\{n_t\}$ is given by the formula $n_t.\phi = 2\cot(-t + \operatorname{arccot}(\phi/2))$. Integrating the second equation in (4.5) along the N-orbits in Ω, with initial values f_1 prescribed on the antidiagonal Δ^{op}, we then obtain

$$\begin{aligned} f_0(\phi_1, \phi_2) = f_1\big((\Phi(\phi_1, \phi_2), 2\pi - \Phi(\phi_1, \phi_2)\big) \\ + \int_0^{T(\phi_1,\phi_2)} F_c^{\flat}\big(n_t.\Phi(\phi_1, \phi_2), n_t.(2\pi - \Phi(\phi_1, \phi_2))\big)\, dt \end{aligned} \tag{4.10}$$

for every $(\phi_1, \phi_2) \in \Omega$. It remains to compute the function f_1 along the antidiagonal. Let

$$\omega_+ := (2\pi/3, 4\pi/3) \quad \text{and} \quad \omega_- := (4\pi/3, 2\pi/3)$$

be the points in Ω corresponding to the base points in H_2 introduced in Section 4.2. Note that ω_+ and ω_- coincide with the barycenters of the triangles enclosing the domains Ω_+ and Ω_- (see Figure 2). Define a new coordinate $S(\phi) \in (-\infty, \infty)$ on each component of the antidiagonal Δ^{op} by the relation

$$(\phi, 2\pi - \phi) = a_{S(\phi)}.\omega_{\pm},$$

depending on whether the point $(\phi, 2\pi - \phi)$ lies in Ω_+ or Ω_-. Integrating the first equation in (4.5) along the A-orbits in Δ^{op}, with initial values f_2 prescribed on the base points $\{\omega_+, \omega_-\}$, we get

$$f_1(\phi, 2\pi - \phi) = f_2(\omega_{\pm}) + \int_0^{S(\phi)} F_c^{\sharp}(a_s.\omega_{\pm})\, ds \tag{4.11}$$

for every $\phi \in (0, \pi) \cup (\pi, 2\pi)$.

4.5. **Explicit primitives.** Combining the results from the previous subsections, we are now in a position to give the following explicit characterization of primitives.

Proposition 4.2 (Explicit primitives)**.** *Let $v^\sharp$ and $v^\flat$ be the real and imaginary parts of the function v defined by formula* (4.2)*, where r is as in* (4.1)*. Then the following hold.*

(i) Let $f \in C^\infty((S^1)^{(3)})^K$. The primitive $P_c(f) \in \mathcal{P}(c)^K$ is G-invariant if and only if the function f solves system (3.3)*.*

(ii) There is a one-to-one correspondence between solutions $f \in C^\infty((S^1)^{(3)})^K$ of system (3.3) *and solutions $f_0 \in C^\infty(\Omega)$ of system* (4.5) *via the relation*

$$f(\theta_0,\theta_1,\theta_2) = f_0(\theta_1-\theta_0,\theta_2-\theta_0).$$

(iii) Every pair $(f_0(\omega_+), f_0(\omega_-)) \in \mathbb{R}^2$ of initial values uniquely determines a smooth solution $f_0 \in C^\infty(\Omega)$ of system (4.5) *by the formula*

$$\begin{aligned}f_0(\phi_1,\phi_2) = f_0(\omega_\pm) &+ \int_0^{S(\Phi(\phi_1,\phi_2))} F_c^\sharp(a_s.\omega_\pm)\,ds \\ &+ \int_0^{T(\phi_1,\phi_2)} F_c^\flat\big(n_t.\Phi(\phi_1,\phi_2), n_t.(2\pi-\Phi(\phi_1,\phi_2))\big)\,dt,\end{aligned} \tag{4.12}$$

where the functions $F_c^\sharp$ and $F_c^\flat$ are as in (4.8)*. Conversely, any smooth solution of system* (4.5) *arises in this way.*

Proof. Let $f \in C^\infty((S^1)^{(3)})^K$. Assertion (i) holds by Proposition 3.4, while (ii) was proved in Section 4.3. Finally, our considerations in Section 4.4 show that the function f_0 satisfies (4.5) if and only if it is given by formulas (4.10) and (4.11), in terms of integration along the unique path in Ω starting at $\omega_\pm$ and traveling to (ϕ_1,ϕ_2) along A- and N-orbits via the point $(\Phi(\phi_1,\phi_2), 2\pi-\Phi(\phi_1,\phi_2)$ on the antidiagonal (see Figure 2), with initial values prescribed at $\omega_\pm$. This proves (iii). □

The proposition achieves the second step in the agenda outlined in Section 2.6. In particular, it shows that solutions f_0 of system (4.5) form a 2-parameter family.

5. Boundedness of primitives

5.1. **Symmetries.** Our construction of primitives in the previous section was valid for arbitrary G-cocycles $c \in L^\infty((S^1)^5)$. However, for the proof of boundedness of primitives, which we shall discuss in this section, it turns out to be essential that c be alternating and even. As we have seen in Subsection 2.5, this is not a loss of generality. The following proposition discusses symmetries of the various functions appearing in the construction of primitives resulting from these additional assumptions.

Proposition 5.1 (Symmetries)**.** *Assume that the cocycle c is alternating and even. Then the inhomogeneities on the right-hand sides of systems* (3.3) *and* (4.5) *have the following properties.*

(i) The functions $c^\sharp + dv^\sharp$ and $c^\flat + dv^\flat$ are alternating.

(ii) The function $F_c^\sharp = c_0^\sharp + (dv^\sharp)_0$ is antisymmetric about the antidiagonal Δ^{op} in Ω. In particular, it vanishes along the antidiagonal.

(iii) The function $F_c^\flat = c_0^\flat + (dv^\flat)_0$ is symmetric about the antidiagonal Δ^{op} in Ω.

Proof. We begin with the following observation. The function $\check{c}$ defined in (3.4) is alternating since c is assumed to be alternating. By Lemma 3.6, the function $\check{c}$ is K-invariant. Hence

$$\check{c}(0,\zeta) = \check{c}(-\zeta,0) = -\check{c}(0,-\zeta).$$

By Convention 3.1, substituting ζ by $2\pi-\zeta$ we infer from this that

$$\int_{\pi}^{\phi} \frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta = -\int_{\pi}^{2\pi-\phi} \frac{\check{c}(0,-\zeta)}{1-\cos(-\zeta)}\,d\zeta = \int_{\pi}^{-\phi} \frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta \tag{5.1}$$

for every $\phi \in (0,2\pi)$. Recall moreover from Section 4.2 that $(v^\sharp, v^\flat) := (\mathrm{Re}(v), \mathrm{Im}(v))$, where

$$v(\theta_1,\theta_2) = e^{i\theta_1}\, r(\theta_2-\theta_1)$$

and r is as in (4.1). Let us prove (i). Since c is alternating, it is immediate from (3.1) and (3.2) that $c^\sharp$ and $c^\flat$ are alternating. By (4.1) and (5.1) we have

$$\begin{aligned} v(\theta_1,\theta_2) &= e^{i\theta_1}\, r(\theta_2-\theta_1) \\ &= -\frac{1}{2}\left(e^{i\theta_1}-e^{i\theta_2}\right)\cdot \int_{\pi}^{\theta_2-\theta_1} \frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta \\ &= \frac{1}{2}\left(e^{i\theta_2}-e^{i\theta_1}\right)\cdot \int_{\pi}^{\theta_1-\theta_2} \frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta = -v(\theta_2,\theta_1). \end{aligned} \tag{5.2}$$

It follows that dv, and hence $dv^\sharp$ and $dv^\flat$ are alternating. This proves (i). For the proof of (ii) and (iii) we have to show that

$$F_c^\sharp(\phi_1,\phi_2) = -F_c^\sharp(-\phi_2,-\phi_1), \quad F_c^\flat(\phi_1,\phi_2) = F_c^\flat(-\phi_2,-\phi_1). \tag{5.3}$$

To this end, we first note that by (4.6) and evenness of c we have

$$\begin{aligned} c_0^\sharp(-\phi_2,-\phi_1) &= \fint\fint \cos(\varphi)\, c(\eta,\varphi,0,-\phi_2,-\phi_1)\,d\eta\,d\varphi \\ &= \fint\fint \cos(\varphi)\, c(-\eta,-\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi \\ &= \fint\fint \cos(-\varphi)\, c(\eta,\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi \\ &= -\fint\fint \cos(\varphi)\, c(\eta,\varphi,0,\phi_1,\phi_2)\,d\eta\,d\varphi = -c_0^\sharp(\phi_1,\phi_2), \end{aligned} \tag{5.4}$$

and similarly

$$\begin{aligned} c_0^\flat(-\phi_2,-\phi_1) &= \fint\fint \sin(\varphi)\, c(\eta,\varphi,0,-\phi_2,-\phi_1)\,d\eta\,d\varphi \\ &= \fint\fint \sin(-\varphi)\, c(\eta,\varphi,0,\phi_2,\phi_1)\,d\eta\,d\varphi \\ &= \fint\fint \sin(\varphi)\, c(\eta,\varphi,0,\phi_1,\phi_2)\,d\eta\,d\varphi = c_0^\flat(\phi_1,\phi_2). \end{aligned} \tag{5.5}$$

Applying (5.1) as in the proof of (i) above, we obtain

$$\begin{aligned} v(-\theta_1,-\theta_2) &= -\frac{1}{2}\left(e^{-i\theta_1}-e^{-i\theta_2}\right)\cdot\int_{\pi}^{-\theta_2+\theta_1}\frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta \\ &= -\frac{1}{2}\left(e^{-i\theta_1}-e^{-i\theta_2}\right)\cdot\int_{\pi}^{\theta_2-\theta_1}\frac{\check{c}(0,\zeta)}{1-\cos(\zeta)}\,d\zeta = \overline{v(\theta_1,\theta_2)}. \end{aligned}$$

Combining this with (5.2) we arrive at

$$v(\theta_1,\theta_2) = -\overline{v(-\theta_2,-\theta_1)}.$$

Consider the function $(dv)_0(\phi_1,\phi_2) := dv(0,\phi_1,\phi_2)$. The previous two identities imply that

$$\begin{aligned} (dv)_0(\phi_1,\phi_2) &= v(\phi_1,\phi_2) - v(0,\phi_2) + v(0,\phi_1) \\ &= -\overline{\left(v(-\phi_2,-\phi_1) - v(0,-\phi_1) + v(0,-\phi_2)\right)} = -\overline{(dv)_0(-\phi_2,-\phi_1)}. \end{aligned}$$

Recall from (4.7) that $(dv^\sharp)_0$ and $(dv^\flat)_0$ are the real and imaginary parts of $(dv)_0$. Hence we conclude that

$$(dv^\sharp)_0(\phi_1,\phi_2) = -(dv^\sharp)_0(-\phi_2,-\phi_1), \quad (dv^\flat)_0(\phi_1,\phi_2) = (dv^\flat)_0(-\phi_2,-\phi_1). \tag{5.6}$$

The identities (5.3) now follow from (5.4), (5.5) and (5.6), which proves (ii) and (iii). □

Next we consider symmetries of the solutions of system (4.5). We introduce some notation first. The $\mathfrak{S}_3$-action on $(S^1)^{(3)}$ commutes with the K-action, whence it descends to an action on Ω. To describe this action explicitly, we denote by s_1 and s_2 the Coxeter generators of $\mathfrak{S}_3$ that act on $(S^1)^{(3)}$ by swapping coordinates in the pairs (θ_0,θ_1) and (θ_1,θ_2), respectively. Then, with respect to the coordinates (ϕ_1,ϕ_2) on Ω, the actions of s_1 and s_2 are given by

$$s_1.(\phi_1,\phi_2) = (-\phi_1,\phi_2-\phi_1), \quad s_2.(\phi_1,\phi_2) = (\phi_2,\phi_1).$$

A function $h_0 \in C^\infty(\Omega)$ will be called *alternating under the action of* $\mathfrak{S}_3$ if $s.h_0 = (-1)^s\, h_0$ for all $s \in \mathfrak{S}_3$. Thus a function $h_0 \in C^\infty(\Omega)$ is alternating under the action of $\mathfrak{S}_3$ if and only if the function $h \in C^\infty((S^1)^{(3)})^K$ defined by

$$h(\theta_0,\theta_1,\theta_2) = h_0(\theta_1-\theta_0,\theta_2-\theta_0)$$

is alternating in the usual sense.

Proposition 5.2 (Alternating solutions)**.** *Assume that the cocycle c is alternating and even. A solution $f_0 \in C^\infty(\Omega)$ of system (4.5) is alternating under the action of $\mathfrak{S}_3$ if and only if*

$$f_0(\omega_+) = -f_0(\omega_-). \tag{5.7}$$

In this case the primitive $P_c(f) \in \mathcal{P}(c)^G$, where f is defined by (4.4), is alternating.

Proof. First of all, we observe that $\mathfrak{S}_3$ acts on the base points $\{\omega_+,\omega_-\}$ by

$$s_1.\omega_\pm = \omega_\mp, \quad s_2.\omega_\pm = \omega_\mp. \tag{5.8}$$

Hence, if f_0 is alternating under the action of $\mathfrak{S}_3$ it follows that

$$f_0(\omega_+) = (-1)^{s_1}\, f_0(s_1.\omega_+) = -f_0(\omega_-).$$

Conversely, let f_0 be a solution of system (4.5) that satisfies (5.7). We will prove that f_0 coincides with its antisymmetrization under the action of $\mathfrak{S}_3$. By Proposition 4.2 (ii), the function f_0 corresponds to a solution $f \in C^\infty((S^1)^{(3)})^K$ of system (3.3) via

$$f(\theta_0, \theta_1, \theta_2) = f_0(\theta_1 - \theta_0, \theta_2 - \theta_0).$$

Let now

$$\hat{f} := \frac{1}{6} \cdot \sum_{s \in \mathfrak{S}_3} (-1)^s \, s.f$$

be the antisymmetrization of f. Then $\hat{f} \in C^\infty((S^1)^{(3)})^K$, and we further claim that $\hat{f}$ solves system (3.3) as well. To see this, observe that in system (3.3) the operators $L_A^{(3)}$ and $L_N^{(3)}$ are symmetric, while by Proposition 5.1 (i) the inhomogeneities $c^\sharp + dv^\sharp$ and $c^\flat + dv^\flat$ are alternating. Now by K-invariance, the function $\hat{f}$ gives rise to a function $\hat{f}_0 \in C^\infty(\Omega)$ via

$$\hat{f}(\theta_0, \theta_1, \theta_2) = \hat{f}_0(\theta_1 - \theta_0, \theta_2 - \theta_0).$$

Then Proposition 4.2 (ii) implies that $\hat{f}_0$ solves system (4.5). Moreover, we have

$$\hat{f}_0 = \frac{1}{6} \cdot \sum_{s \in \mathfrak{S}_3} (-1)^s \, s.f_0,$$

whence $\hat{f}_0$ is alternating under the action of $\mathfrak{S}_3$. It follows from (5.7) and (5.8) that $\hat{f}_0(\omega_\pm) = f_0(\omega_\pm)$. The uniqueness statement in Proposition 4.2 (iii) implies that $\hat{f}_0$ coincides with f_0. □

The proposition shows that solutions f_0 of system (4.5) that are alternating under the action of $\mathfrak{S}_3$ form a 1-parameter family.

5.2. **Boundedness.** In order to complete the proof of Theorem 1.1 it remains to show that among the G-invariant primitives we constructed in Section 4, there actually exist bounded ones. This is the content of the next proposition, which crucially relies on the symmetries unveiled in the previous subsection.

Proposition 5.3 (Boundedness). *Assume that the cocycle c is alternating and even. Let $f_0 \in C^\infty(\Omega)$ be a solution of system* (4.5) *that is alternating under the action of $\mathfrak{S}_3$. Then the corresponding primitive $P_c(f) \in \mathcal{P}(c)^G$, where f is defined by* (4.4)*, is bounded.*

The proof of the proposition relies on the following three basic observations.

Lemma 5.4. *Let the function f_0 be defined by formula* (4.12)*. If f_0 is bounded along the line segments*

$$(0, 2\pi/3) \cup (2\pi/3, 2\pi) \ni \xi \mapsto (2\pi/3, \xi)$$

and

$$(0, 4\pi/3) \cup (4\pi/3, 2\pi) \ni \xi \mapsto (4\pi/3, \xi),$$

and f is given by formula (4.4)*, then the corresponding primitive $P_c(f) \in \mathcal{P}(c)^G$ is bounded.*

Proof. By Proposition 4.2, $P_c(f) = I(c) + df$ is G-invariant. By 3-transitivity of the G-action on S^1 and since $I(c)$ is bounded, we therefore deduce that $P_c(f)$ is bounded if and only if the function

$$z \mapsto df\big(1, e^{2\pi i/3}, e^{4\pi i/3}, z\big)$$

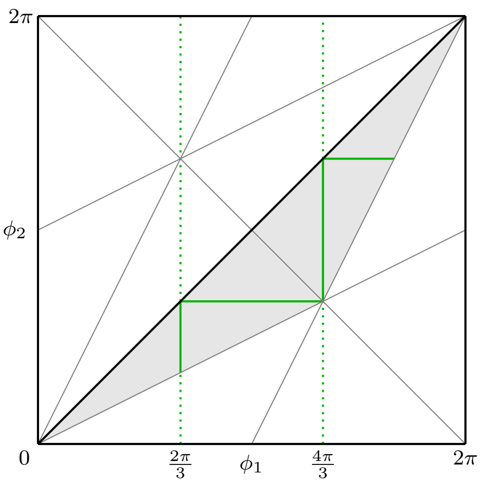

FIGURE 3. A fundamental domain for the $\mathfrak{S}_3$-action on Ω (shaded), and the images of the line segments $\xi \mapsto (2\pi/3, \xi)$ and $\xi \mapsto (4\pi/3, \xi)$ therein under the $\mathfrak{S}_3$-action

is bounded. Writing $z = e^{i\xi}$, we may express this function as

$$\begin{aligned}\xi \mapsto\ & f(2\pi/3, 4\pi/3, \xi) - f(0, 4\pi/3, \xi) + f(0, 2\pi/3, \xi) - f(0, 2\pi/3, 4\pi/3)\\ &= f_0(2\pi/3, \xi - 2\pi/3) - f_0(4\pi/3, \xi) + f_0(2\pi/3, \xi) - f_0(2\pi/3, 4\pi/3).\end{aligned}$$

The lemma follows. ☐

Lemma 5.5. *Let C be a compact subset of the open square $(0, 2\pi)^2$. If the function f_0 defined by formula (4.12) is bounded along the antidiagonal Δ^{op} in Ω, then it is bounded on the subset $C \cap \Omega$ of Ω.*

Proof. By Lemma 4.1 the function $F_c^\flat = c_0^\flat + (dv^\flat)_0$ is bounded. Moreover, by assumption we have $|f_0|_{\Delta^{\mathrm{op}}}| \leq M$ for some constant $M > 0$. Hence we obtain from formula (4.12) the estimate

$$\left|f_0(\phi_1, \phi_2)\right| \leq M + \left\|F_c^\flat\right\|_\infty \cdot \left|T(\phi_1, \phi_2)\right|$$

for all $(\phi_1, \phi_2) \in \Omega$. It remains to show that the function T is bounded on $C \cap \Omega$. By compactness of C it will be enough to prove that the function $T\colon \Omega \to \mathbb{R}$ extends to a continuous function on the open square $(0, 2\pi)^2$. This, however, is immediate from formula (4.9). ☐

Lemma 5.6. *Assume that the cocycle c is alternating. Then the function f_0 defined by formula (4.12) is locally constant along the antidiagonal Δ^{op} in Ω.*

Proof. Since c is alternating, the inhomogeneity $F_c^\sharp$ vanishes along the antidiagonal Δ^{op} by Proposition 5.1 (ii). The lemma now follows from formula (4.12). ☐

Example 5.7. Assume that the cocycle c is alternating. Consider the special solution f_0 determined by the initial values $f_0(\omega_\pm) = 0$. It is alternating under the action of $\mathfrak{S}_3$ by Proposition 5.2. Moreover, by Lemma 5.6 it vanishes along the antidiagonal Δ^{op}. Since under the action of $\mathfrak{S}_3$ the components of Δ^{op} get identified with the medians of the triangles enclosing the domains Ω_+ and Ω_-, we further infer that f_0 also vanishes along these medians. Moreover, by Proposition 5.1 (iii) the function $F_c^\flat$ is symmetric about the antidiagonal. Thus we see from formula (4.12) that the special solution f_0 is antisymmetric with respect to the antidiagonal, and hence antisymmetric with respect to all medians.

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. By Proposition 4.2 (iii) the function f_0 is given by formula (4.12). Hence by Lemma 5.4 it suffices to show that f_0 is bounded along the line segments $\xi \mapsto (2\pi/3, \xi)$ and $\xi \mapsto (4\pi/3, \xi)$. Since f_0 is alternating under the action of $\mathfrak{S}_3$, it suffices to prove that f_0 is bounded along the images of these line segments in any fundamental domain for the $\mathfrak{S}_3$-action on Ω. In fact, we may choose the fundamental domain in such a way that the image line segments lie inside a compact subset of $(0, 2\pi)^2$ (see Figure 3). By Lemma 5.6 and Lemma 5.5, the function f_0 is then bounded on these line segments. □

Theorem 1.1 follows by combining Propositions 4.2, 5.2 and 5.3.

Appendix A. Vanishing of odd cocycles

The goal of this appendix is to prove Proposition 2.3, which states that every bounded alternating G-invariant 4-cocycle is necessarily even. Our strategy here is inspired by [16] in that we consider Fourier transforms of cocycles and study the conditions imposed on them by G-invariance. Given $n \in \mathbb{N}_0$, we denote by $c(\mathbb{Z}^{n+1})$ the space of complex-valued sequences indexed by $\mathbb{Z}^{n+1}$ and by $\ell^2(\mathbb{Z}^{n+1})$ the subspace of square-summable sequences. We denote by $e_0, \ldots, e_n$ the standard basis of $\mathbb{Z}^{n+1}$ and use the multi-index notation

$$\underline{k} := (k_0, \ldots, k_n) := \sum_{j=0}^{n} k_j e_j.$$

We then write $c_{alt}(\mathbb{Z}^{n+1})$ and $\ell^2_{\mathrm{alt}}(\mathbb{Z}^{n+1})$, respectively, for the corresponding subspaces of alternating sequences and define two linear operators $A^{(n)}_{\pm} : \ell^2_{alt}(\mathbb{Z}^{n+1}) \to c_{alt}(\mathbb{Z}^{n+1})$ by

$$\begin{aligned} (A^{(n)}_{+}F)(\underline{k}) &:= \sum_{j=0}^{n} (k_j + 1) \cdot F(\underline{k} + e_j), \\ (A^{(n)}_{-}F)(\underline{k}) &:= \sum_{j=0}^{n} (k_j - 1) \cdot F(\underline{k} - e_j). \end{aligned}$$

Definition A.1. An element $C \in \ell^2_{\mathrm{alt}}(\mathbb{Z}^{n+1})$ is called a *combinatorial n-cocycle* if the following hold:

(i) $C(k_0, \ldots, k_n) = 0$ unless $k_j = 0$ for precisely one $j \in \{0, \ldots, n\}$.
(ii) $C(k_0, \ldots, k_n) = 0$ unless $k_0 + \cdots + k_n = 0$.
(iii) $C \in \ker(A^{(n)}_{+}) \cap \ker(A^{(n)}_{-})$.

According to [16, Sec. 3.1] the Fourier transform $\widehat{c}$ of a G-invariant alternating bounded n-cocycle c is a combinatorial n-cocycle. Observe that c is even if and only if $\widehat{c}$ is *even* in the sense that $\widehat{c}(-\underline{k}) = \widehat{c}(\underline{k})$. Proposition 2.3 will therefore be a consequence of the following combinatorial result.

Proposition A.2. *Every combinatorial* 4*-cocycle is even.*

For the proof of Proposition A.2 we need two preparatory lemmas. Given a combinatorial n-cocycle C we denote by $\mathrm{supp}(C) := \{\underline{k} \in \mathbb{Z}^{n+1} \mid C(\underline{k}) \neq 0\}$ the *support* of C.

Lemma A.3. *Let C be a combinatorial* 4*-cocycle and $(k_0, k_1, k_2, k_3, k_4) \in \mathrm{supp}(C)$. Then there exists $\sigma \in \mathfrak{S}_5$ such that*

$$k_{\sigma(0)} < k_{\sigma(1)} < k_{\sigma(2)} = 0 < k_{\sigma(3)} < k_{\sigma(4)}.$$

Proof. Since C is alternating and vanishes unless precisely one of its entries is 0, it suffices to show that C vanishes on those $\underline{k} \in \mathbb{Z}^5$ which satisfy either

$$k_0 > k_1 = 0 > k_2 > k_3 > k_4 \tag{A.1}$$

or $k_0 < k_1 = 0 < k_2 < k_3 < k_4$. We are going to show $C(k_0, 0, k_2, k_3, k_4) = 0$ whenever $\underline{k}$ satisfies (A.1) and leave the second, analogous case to the reader. Our proof will be by induction on k_0.

If $k_0 \leq 5$ then the condition $k_2 + k_3 + k_4 = -k_0$ cannot be satisfied for $\underline{k}$ satisfying (A.1), hence $C(\underline{k}) = 0$. Otherwise we use $C \in \ker(A_+^{(4)})$ to deduce that

$$\begin{aligned} 0 &= (A_+^{(4)}C)(k_0 - 1, 0, k_2, k_3, k_4) \\ &= k_0 \cdot C(k_0, 0, k_2, k_3, k_4) + 0 + (k_2 + 1) \cdot C(k_0 - 1, 0, k_2 + 1, k_3, k_4) \\ &\quad + (k_3 + 1) \cdot C(k_0 - 1, 0, k_2, k_3 + 1, k_4) + (k_2 + 1) \cdot C(k_0 - 1, 0, k_2, k_3, k_4 + 1). \end{aligned}$$

The third summand on the right-hand side vanishes by antisymmetry if $k_2 + 1 = 0$, and by the induction hypothesis otherwise. Similarly, the last two summands vanish. Now the assumption $k_0 \neq 0$ implies $C(k_0, 0, k_2, k_3, k_4) = 0$. □

Lemma A.4. *Let C, D be combinatorial* 4*-cocycles such that*

$$C(-n - 1, -n, 0, n, n + 1) = D(-n - 1, -n, 0, n, n + 1)$$

holds for all $n > 0$. Then $C = D$.

Proof. Since the space of combinatorial cocycles is linear, we may assume $D = 0$ and hence

$$\forall n > 0 : \; C(-n - 1, -n, 0, n, n + 1) = 0.$$

We have to show that $C = 0$. Using that C is alternating and Lemma A.3, it suffices to prove that $C(k_0, k_1, 0, k_3, k_4) = 0$ whenever $k_0 < k_1 < 0 < k_3 < k_4$. Since $C \in \ker(A_+^{(4)})$ we have

$$\begin{aligned} 0 &= (A_+^{(4)}C)(k_0, k_1, 0, k_3, k_4 - 1) \\ &= (k_0 + 1) \cdot C(k_0 + 1, k_1, 0, k_3, k_4 - 1) + (k_1 + 1) \cdot C(k_0, k_1 + 1, 0, k_3, k_4 - 1) + 0 \\ &\quad + (k_3 + 1) \cdot C(k_0, k_1, 0, k_3 + 1, k_4 - 1) + k_4 \cdot C(k_0, k_1, 0, k_3, k_4). \end{aligned}$$

We may rewrite this as

$$\begin{aligned} C(k_0, k_1, 0, k_3, k_4) &= -\frac{k_0 + 1}{k_4} \cdot C(k_0 + 1, k_1, 0, k_3, k_4 - 1) \\ &\quad - \frac{k_1 + 1}{k_4} \cdot C(k_0, k_1 + 1, 0, k_3, k_4 - 1) \end{aligned} \tag{A.2}$$

$$- \frac{k_3+1}{k_4} \cdot C(k_0, k_1, 0, k_3+1, k_4-1).$$

Now we iterate this recursion. In each step we get a sum of terms of the form $C(k_0, k_1, 0, k_3, k_4)$ where the distance between k_3 and k_4 is smaller than in the previous step. We can thus run the iteration until we arrive at terms of the form $C(k_0, k_1, 0, n, n+1)$ with $0 < n$. We may furthermore assume that $k_0 < k_1 < 0$ since C is alternating. It then remains to show that

$$\forall k_0 < k_1 < 0 < n : \; C(k_0, k_1, 0, n, n+1) = 0 \tag{A.3}$$

We prove this by a double induction on n and $|k_0|$.

If $n = 1$ then the condition $k_0 + k_1 = -2n - 1$ forces $(k_0, k_1) = (-2, -1)$ and we are done by hypothesis. Now assume that $n > 1$ is arbitrary. The condition $k_0 + k_1 = -2n - 1$ forces $|k_0| \geq n + 1$. If $|k_0| = n + 1$ then $(k_0, k_1) = (-n - 1, -n)$ and we are again done by hypothesis. It thus remains to show (A.3) for $n > 1$ and $|k_0| > n + 1$, where we assume

$$C(k_0', k_1, 0, n', n'+1) = 0$$

if either $n' < n$ or $n = n'$ and $|k_0|' < |k_0|$.

Now since $C \in \ker(A_-^{(4)})$ we have

$$\begin{aligned}
0 &= (A_-^{(4)} C)(k_0+1, k_1, 0, n, n+1) \\
&= k_0 \cdot C(k_0, k_1, 0, n, n+1) + (k_1 - 1) \cdot C(k_0+1, k_1-1, 0, n, n+1) + 0 \\
&\quad + (n-1) \cdot C(k_0+1, k_1, 0, n-1, n+1) + n \cdot C(k_0+1, k_1, 0, n, n).
\end{aligned}$$

The second of the five summands on the right-hand side vanishes by induction hypothesis of the inner induction on k_0, while the last summand vanishes by antisymmetry. Since $k_0 < -n - 1 < -2$ we have $k_0 \neq 0$ and thus

$$C(k_0, k_1, 0, n, n+1) = -\frac{n-1}{k_0} \cdot C(k_0+1, k_1, 0, n-1, n+1).$$

To deal with the expression on the right-hand side we again apply (A.2) with $k_3 = n - 1$ and $k_4 = n + 1$. We thereby find

$$\begin{aligned}
C(k_0, k_1, 0, n, n+1) &= \frac{(n-1) \cdot (k_0+1)}{k_0 \cdot (n+1)} C(k_0+1, k_1, 0, n-1, n) \\
&\quad + \frac{(n-1) \cdot (k_1+1)}{k_0 \cdot (n+1)} C(k_0, k_1+1, 0, n-1, n) \\
&\quad + \frac{(n-1) \cdot n}{k_0 \cdot (n+1)} C(k_0, k_1, 0, n, n).
\end{aligned}$$

Here the first two summands vanish by the induction hypothesis of the outer induction on n, and the last summand vanishes by antisymmetry. This shows that $C(k_0, k_1, 0, n, n+1) = 0$ and finishes the proof of the lemma. □

Proof of Proposition A.2. Given a combinatorial 4-cocycle C we define a function $D : \mathbb{Z}^5 \to \mathbb{C}$ by $D(k_0, \ldots, k_4) := C(-k_0, \ldots, -k_4)$. We claim that D is a combinatorial cocycle. Since C is alternating and in ℓ^2, D is alternating and in ℓ^2 as well. Conditions (i) and (ii) are obvious. Since $C \in \ker(A_-^{(4)})$ we have

$$\begin{aligned}(A_+^{(4)}D)(\underline{k}) &= \sum_{j=0}^{4}(k_j+1)\cdot D(\underline{k}+e_j) \;=\; -\sum_{j=0}^{4}(-k_j-1)\cdot C(-(\underline{k}+e_j))\\ &= -\sum_{j=0}^{4}(-k_j-1)\cdot C(-\underline{k}-e_j) \;=\; -(A_-^{(4)}C)(-k_0,\dots,-k_4) \;=\; 0,\end{aligned}$$

which shows that $D \in \ker(A_+^{(4)})$. Dually, $C \in \ker(A_+^{(4)})$ implies $D \in \ker(A_-^{(4)})$, which finishes the proof that D is a combinatorial cocycle.

On the other hand, antisymmetry of C yields

$$D(-n-1,-n,0,n,n+1) = C(n+1,n,0,-n,-n-1) = C(-n-1,-n,0,n,n+1).$$

Now Lemma A.4 implies that $C(\underline{k}) = D(\underline{k}) = C(-\underline{k})$, which means that C is even. □

This finishes the proof of Proposition 2.3.

Appendix B. The Frobenius integrability condition

The goal of this appendix is to prove Proposition 3.7. We have to show that the system

$$\text{(B.1)}\qquad \begin{cases} L_K^{(3)}f = 0\\ L_A^{(3)}f = c^\sharp + dv^\sharp\\ L_N^{(3)}f = c^\flat + dv^\flat \end{cases}$$

admits a solution $(f, v^\sharp, v^\flat)$ if and only if the pair $(v^\sharp, v^\flat)$ satisfies the Frobenius system (3.5). Here we consider f and $v^\sharp$, $v^\flat$ as smooth functions on the domains $D := [0,2\pi)^{(3)}$ and $[0,2\pi)^{(2)}$, respectively. It will be convenient to replace system (B.1) by the the equivalent system

$$\text{(B.2)}\qquad \begin{cases} L_K^{(3)}f = 0\\ L_A^{(3)}f = c^\sharp + dv^\sharp\\ L_N^{(3)}f - L_K^{(3)}f = c^\flat + dv^\flat. \end{cases}$$

Consider the product $D \times \mathbb{R}$. We denote the coordinates on D by $(\theta_0, \theta_1, \theta_2)$ and the coordinate on $\mathbb{R}$ by θ_3. The graph $\Gamma_f := \{((\theta_0,\theta_1,\theta_2), f(\theta_0,\theta_1,\theta_2))\}$ of the function f is a 3-dimensional submanifold of $D \times \mathbb{R}$. Define vector fields X, Y, Z on $D \times \mathbb{R}$ by

$$\begin{aligned} X &:= L_K^{(3)},\\ Y &:= L_A^{(3)} + \left(c^\sharp + dv^\sharp\right)\partial_{\theta_3},\\ Z &:= L_N^{(3)} - L_K^{(3)} + \left(c^\flat + dv^\flat\right)\partial_{\theta_3}. \end{aligned}$$

Since G acts strictly 3-transitively on D, it follows that these vector fields span a distribution E of constant rank 3 on $D \times \mathbb{R}$. Then a triple $(f, v^\sharp, v^\flat)$ is a solution of system (B.2) if and only if the graph Γ_f is an integral manifold for E. Hence the Frobenius theorem (see e. g. [35, Ch. 11.]) implies that system (B.2) admits a solution $(f, v^\sharp, v^\flat)$ if and only if the distribution E is integrable, i. e., the vector fields X, Y, Z form an involutive system. Note that

$$\left[L_K^{(3)}, L_A^{(3)}\right] = L_K^{(3)} - L_N^{(3)}, \quad \left[L_K^{(3)}, L_N^{(3)} - L_K^{(3)}\right] = L_A^{(3)}, \quad \left[L_A^{(3)}, L_N^{(3)} - L_K^{(3)}\right] = L_K^{(3)}.$$

Hence the vector fields X, Y, Z form an involutive system if and only if

$$[X, Y] = -Z, \quad [X, Z] = Y, \quad [Y, Z] = X. \tag{B.3}$$

We shall now make these conditions explicit. We start with two preliminary lemmas.

Lemma B.1. *The functions* $c^\sharp$ *and* $c^\flat$ *defined in* (3.1) *and* (3.2) *satisfy*

$$L_K^{(3)} c^\sharp = -c^\flat, \quad L_K^{(3)} c^\flat = c^\sharp.$$

Proof. More generally, we prove that for any function $\lambda \in C^\infty((0, 2\pi))$, the function

$$c_\lambda(\theta_0, \theta_1, \theta_2) := \fint \fint \lambda(\varphi)\, c(\eta, \varphi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi$$

satisfies $L_K^{(3)} c_\lambda = c_{\lambda'}$. Indeed, by K-invariance of the cocycle c and the measure, we have

$$\begin{aligned} L_K^{(3)} c_\lambda(\theta_0, \theta_1, \theta_2) &= \frac{d}{d\xi}\bigg|_{\xi=0} \fint \fint \lambda(\varphi)\, c(\eta, \varphi, \theta_0 + \xi, \theta_1 + \xi, \theta_2 + \xi)\, d\eta\, d\varphi \\ &= \frac{d}{d\xi}\bigg|_{\xi=0} \fint \fint \lambda(\varphi)\, c(\eta - \xi, \varphi - \xi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi \\ &= \fint \fint \frac{d}{d\xi}\bigg|_{\xi=0} \lambda(\varphi + \xi)\, c(\eta, \varphi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi \\ &= \fint \fint \lambda'(\varphi)\, c(\eta, \varphi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi. \end{aligned}$$

□

Lemma B.2. *The function* $\check{c}$ *defined in* (3.4) *satisfies*

$$L_K^{(3)} c^\sharp - L_N^{(3)} c^\sharp + L_A^{(3)} c^\flat = -d\check{c}. \tag{B.4}$$

Proof. Let us consider the left-hand side of (B.4). In a first step, using G-invariance of c and K-invariance of the measure, we compute

$$\begin{aligned} L_K^{(3)} c^\sharp(\theta_0, \theta_1, \theta_2) &= \frac{d}{d\xi}\bigg|_{\xi=0} \fint \fint \cos(\varphi)\, c(\eta, \varphi, \theta_0 + \xi, \theta_1 + \xi, \theta_2 + \xi)\, d\eta\, d\varphi \\ &= \fint \fint \frac{d}{d\xi} \cos(\varphi + \xi)\bigg|_{\xi=0} c(\eta, \varphi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi \end{aligned}$$

and

$$\begin{aligned} L_N^{(3)} c^\sharp(\theta_0, \theta_1, \theta_2) &= \frac{d}{dt}\bigg|_{t=0} \fint \fint \cos(\varphi)\, c(\eta, \varphi, n_t.\theta_0, n_t.\theta_1, n_t.\theta_2)\, d\eta\, d\varphi \\ &= \fint \fint \frac{d}{dt} \left(\cos(n_t.\varphi) \frac{d(n_t.\varphi)}{d\varphi} \frac{d(n_t.\eta)}{d\eta} \right)\bigg|_{t=0} c(\eta, \varphi, \theta_0, \theta_1, \theta_2)\, d\eta\, d\varphi, \end{aligned}$$

and similarly

$$\begin{aligned} L_A^{(3)} c^\flat(\theta_0,\theta_1,\theta_2) &= \frac{d}{ds}\bigg|_{s=0} \fint\fint \sin(\varphi)\, c(\eta,\varphi,a_s.\theta_0,a_s.\theta_1,a_s.\theta_2)\, d\eta\, d\varphi \\ &= \fint\fint \frac{d}{ds}\left(\sin(a_s.\varphi)\,\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\right)\bigg|_{s=0} c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\, d\eta\, d\varphi. \end{aligned}$$

Second, using Lemma 3.2 we compute the derivatives appearing in the above formulas. Firstly, we have $\frac{d}{d\xi}(\cos(\varphi+\xi))\Big|_{\xi=0} = -\sin(\varphi)$. Moreover,

$$\begin{aligned} &\frac{d}{dt}\left(\cos(n_t.\varphi)\,\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\right)\bigg|_{t=0} \\ &= -\sin(n_t.\varphi)\,\frac{d(n_t.\varphi)}{dt}\,\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0} + \cos(n_t.\varphi)\,\frac{d}{d\varphi}\frac{d(n_t.\varphi)}{dt}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0} \\ &\qquad + \cos(n_t.\varphi)\,\frac{d(n_t.\varphi)}{d\varphi}\frac{d}{d\eta}\frac{d(n_t.\eta)}{dt}\bigg|_{t=0} \\ &= -\sin(n_t.\varphi)\,\big(1-\cos(n_t.\varphi)\big)\,\frac{d(n_t.\varphi)}{d\varphi}\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0} \\ &\qquad + \cos(n_t.\varphi)\,\frac{d}{d\varphi}\big(1-\cos(n_t.\varphi)\big)\,\frac{d(n_t.\eta)}{d\eta}\bigg|_{t=0} \\ &\qquad\qquad + \cos(n_t.\varphi)\,\frac{d(n_t.\varphi)}{d\varphi}\frac{d}{d\eta}\big(1-\cos(n_t.\eta)\big)\bigg|_{t=0} \\ &= -\sin(\varphi)\,(1-\cos(\varphi)) + \cos(\varphi)\sin(\varphi) + \cos(\varphi)\sin(\eta) \\ &= 2\sin(\varphi)\cos(\varphi) + \cos(\varphi)\sin(\eta) - \sin(\varphi). \end{aligned}$$

Lastly,

$$\begin{aligned} &\frac{d}{ds}\left(\sin(a_s.\varphi)\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\right)\bigg|_{s=0} \\ &= \cos(a_s.\varphi)\,\frac{d(a_s.\varphi)}{ds}\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} + \sin(a_s.\varphi)\,\frac{d}{d\varphi}\frac{d(a_s.\varphi)}{ds}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} \\ &\qquad + \sin(a_s.\varphi)\,\frac{d(a_s.\varphi)}{d\varphi}\frac{d}{d\eta}\frac{d(a_s.\eta)}{ds}\bigg|_{s=0} \\ &= \cos(a_s.\varphi)\sin(a_s.\varphi)\,\frac{d(a_s.\varphi)}{d\varphi}\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} + \sin(a_s.\varphi)\frac{d}{d\varphi}\sin(a_s.\varphi)\,\frac{d(a_s.\eta)}{d\eta}\bigg|_{s=0} \\ &\qquad + \sin(a_s.\varphi)\,\frac{d(a_s.\varphi)}{d\varphi}\,\frac{d}{d\eta}\sin(a_s.\eta)\bigg|_{s=0} \\ &= 2\sin(\varphi)\cos(\varphi) + \sin(\varphi)\cos(\eta). \end{aligned}$$

Summing up, we obtain

$$\begin{aligned}&L_K^{(3)}c^\sharp(\theta_0,\theta_1,\theta_2)-L_N^{(3)}c^\sharp(\theta_0,\theta_1,\theta_2)+L_A^{(3)}c^\flat(\theta_0,\theta_1,\theta_2)\\ &=\fint\fint\big(\sin(\varphi)\cos(\eta)-\cos(\varphi)\sin(\eta)\big)\,c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi\\ &=-\fint\fint\sin(\eta-\varphi)\,c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi.\end{aligned}\tag{B.5}$$

Now we turn to the computation of the right-hand side of (B.4). The cocycle identity for c yields

$$\begin{aligned}0=dc(\eta,\varphi,\psi,\theta_0,\theta_1,\theta_2)=c(\varphi,\psi,\theta_0,\theta_1,\theta_2)-c(\eta,\psi,\theta_0,\theta_1,\theta_2)+c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\\ -\sum_{j=0}^{2}(-1)^j\,c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta_j},\dots,\theta_2).\end{aligned}$$

We multiply this identity by $\sin(\eta-\varphi)$ and integrate over the variables η,φ,ψ. Integrating the first term, we get

$$\begin{aligned}&\fint\fint\fint\sin(\eta-\varphi)\,c(\varphi,\psi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi\,d\psi\\ &=\fint\fint\left(\fint\sin(\eta-\varphi)\,d\eta\right)c(\varphi,\psi,\theta_0,\theta_1,\theta_2)\,d\varphi\,d\psi\;=\;0.\end{aligned}$$

Likewise, the integral of the second term vanishes. We are thus left with

$$\begin{aligned}&\fint\fint\sin(\eta-\varphi)\,c(\eta,\varphi,\theta_0,\theta_1,\theta_2)\,d\eta\,d\varphi\\ &=\fint\fint\fint\sin(\eta-\varphi)\left(\sum_{j=0}^{2}(-1)^j\,c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta_j},\dots,\theta_2)\right)d\eta\,d\varphi\,d\psi\\ &=\sum_{j=0}^{2}(-1)^j\fint\fint\fint\sin(\eta-\varphi)\,c(\eta,\varphi,\psi,\theta_0,\dots,\widehat{\theta_j},\dots,\theta_2)\,d\eta\,d\varphi\,d\psi=d\check{c}(\theta_0,\theta_1,\theta_2).\end{aligned}$$

Comparing this with (B.5) above, formula (B.4) follows. □

We are now in a position to finish the proof of Proposition 3.7 by spelling out the integrability conditions (B.3). Consider the first identity in (B.3). We have

$$[X,Y]=\big[L_K^{(3)},L_A^{(3)}\big]+\big(L_K^{(3)}(c^\sharp+dv^\sharp)\big)\partial_{\theta_3}=L_K^{(3)}-L_N^{(3)}+\big(L_K^{(3)}(c^\sharp+dv^\sharp)\big)\partial_{\theta_3}.$$

Recall from Lemma B.1 and Lemma 3.3 that

$$L_K^{(3)}c^\sharp=-c^\flat,\quad L_K^{(3)}dv^\sharp=dL_K^{(2)}v^\sharp.$$

Thus

$$[X,Y]=L_K^{(3)}-L_N^{(3)}+\big(-c^\flat+dL_K^{(2)}v^\sharp\big)\partial_{\theta_3}.$$

Comparing this to $-Z$ we find

$$[X,Y]=-Z\quad\iff\quad d\big(L_K^{(2)}v^\sharp+v^\flat\big)=0.\tag{B.6}$$

Likewise, for the second identity in (B.3) we have

$$[X, Z] = Y \quad \Longleftrightarrow \quad d\big(L_K^{(2)} v^\flat - v^\sharp\big) = 0. \tag{B.7}$$

Finally, observe that

$$\begin{aligned}[Y, Z] &= \big[L_A^{(3)}, L_N^{(3)} - L_K^{(3)}\big] + \big[L_A^{(3)}, (c^\flat + dv^\flat)\partial_{\theta_3}\big] - \big[L_N^{(3)} - L_K^{(3)}, (c^\sharp + dv^\sharp)\partial_{\theta_3}\big] \\ &= L_K^{(3)} + \Big(\big(L_K^{(2)} - L_N^{(2)}\big)dv^\sharp + L_A^{(3)} dv^\flat + \big(L_K^{(3)} - L_N^{(3)}\big)c^\sharp + L_A^{(3)} c^\flat\Big)\partial_{\theta_3}.\end{aligned}$$

By Lemma B.2 and Lemma 3.3, this becomes

$$[Y, Z] = L_K^{(3)} + \Big(d\big(L_K^{(2)} - L_N^{(2)}\big)v^\sharp + dL_A^{(2)} v^\flat - d\check{c}\Big)\partial_{\theta_3}.$$

We deduce that

$$[Y, Z] = X \quad \Longleftrightarrow \quad d\big(L_K^{(2)} v^\sharp - L_N^{(2)} v^\sharp + L_A^{(2)} v^\flat - \check{c}\big) = 0. \tag{B.8}$$

Combining (B.6), (B.7) and (B.8), Proposition 3.7 now follows from (B.3).

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Mathematics Department, Technion, Haifa, 32000, Israel
E-mail address: `tobias.hartnick@gmail.com`

Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany
E-mail address: `aott@mathi.uni-heidelberg.de`