RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Themen der Geometrie
Heidelberg, 03.07.2015

Exercise sheet
Hyperbolic Surfaces

Exercise 1. On models of hyperbolic plane:

1. Prove that the cross ratio is invariant by the action of $PSL(2, \mathbb{C})$.
2. Prove that the cross ratio of 4 points is real if and only if the 4 points lie on a generalized circle (circle o a straight line).
3. Prove that balls for the hyperbolic metric in the upper half model H are Euclidean discs (possibly with a different center).
4. Prove the following formula for the distance d_H between points of the upper half plane:
$$\cosh d_H(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2\ \mathrm{Im}(z_1)\ \mathrm{Im}(z_2)}$$
5. Consider a Riemannian metric on an open subset of $\mathbb{R}^2$ of the form $g(x, y)(dx^2 + dy^2)$. Prove that its curvature is[1]
$$K = -\frac{1}{2g}\left(\frac{\partial^2 \log g}{\partial x^2} + \frac{\partial^2 \log g}{\partial y^2}\right)$$
6. Using the previous formula, compute the curvature of the hyperbolic metric on the upper half plane and on the disc, respectively
$$\frac{1}{y^2}(dx^2 + dy^2), \qquad \frac{4}{(1 - x^2 - y^2)^2}(dx^2 + dy^2)$$
.

Exercise 2. On isometries of hyperbolic plane:

1. Prove that $\mathrm{Isom}^+(H)$ acts transitively on the unit tangent bundle of H.
2. Prove that $\mathrm{Isom}^+(H)$ acts transitively on pairs of equidistant points.
3. Prove that $\mathrm{Isom}^+(H)$ acts transitively on triples of cyclically ordered points of ∂H.
4. There are two ways to define a topology on $\mathrm{Isom}^+(H)$. One way is to identify it with $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm \mathrm{Id}$, and identify matrices with points of $\mathbb{R}^4$. The other way is embed $\mathrm{Isom}^+(H)$ in the space of continuous maps from H to H, and put on the space of continuous maps the compact-open topology[2]. Prove that the two topologies on $\mathrm{Isom}^+(H)$ coincide.
5. Given $x \in H$ and $K \subset H$ compact, consider the set $A \in \mathrm{Isom}^+(H) \mid A(x) \in K$. Prove that this set is compact.

Exercise 3. On the definition of topological manifolds:

1. Find a topological space locally homeomorphic to $\mathbb{R}^2$, but not Hausdorff.
2. Find a topological space that is Hausdorff and locally homeomorphic to $\mathbb{R}^2$, but not second countable[1].
3. Prove that the topological product of 2 manifolds is a manifold.
4. Prove that spheres are manifolds.

Exercise 4. On topology of surfaces:

1. Prove that the 3 definitions of the torus given during the course give homeomorphic surfaces.
2. Prove that, if $g = 0$ and $n \geq 3$, or if $g = 1$ and $n \geq 1$, or if $g \geq 2$, then the surface $S_{g,n}$ can be decomposed in $2g - 2 + n$ pairs of pants by $3g - 3 + n$ simple closed curves.
3. The surface S_g is homeomorphic to the quotient of a $(4g)$-gon by the edge identification given during the course $(aba^{-1}b^{-1}cdc^{-1}d^{-1}\dots)$
4. Prove that, if $g = 0$ and $n \geq 3$, or if $g > 0$ and $n \geq 1$, every surface $S_{g,n}$ admits a topological ideal triangulation.

Exercise 5. On atlases and structures on manifolds:

1. Find a function that is C^k but not C^{k+1}, or C^∞ but not C^ω.
2. Prove that all surfaces $S_{g,n}$ admit a structure of oriented C^ω manifold (or do it for C^∞, if you don't like C^ω).
3. Prove that all surfaces $S_{g,n}$ admit a complex structure.

Exercise 6. On coverings:

1. Prove that every covering map is surjective and a local homeomorphism.
2. Prove that the group of Deck transformations acts by homeomorphisms.
3. Prove that the group of Deck transformations is a discrete group for the compact-open topology[2].
4. Prove that the group of Deck transformations acts properly discontinuously and freely.

Exercise 7. On hyperbolic surfaces:

1. Complete the proof of the existence of the developing map, more precisely, prove that the definition does not depend on the choice of charts (U_i, ϕ_i), and that it does not depend on the choice of subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$.

2. Prove that the systole is a positive number.

3. Prove that a compact hyperbolic surface has a shortest simple closed curve (this implies the previous one).

Exercise 8. Prove that a sequence in $(\mathbb{R}_{>0})^{\mathcal{C}}$ converges to a limit point for the product topology if and only if it converges point-wise to the same limit point.

[1] This exercise requires some knowledge that is not part of the course. You can just skip this if you don't feel confortable with the material.

[2] The compact-open topology on the space of continuous functions from X to Y, is the topology generated by the following open sets: given $K \subset X$ compact and $U \subset Y$ open, let $\mathcal{U}_{K,U} = \{f \mid f(K) \subset U\}$