RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometrische Gruppentheorie
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Exercise sheet 11
Hyperbolicity
To hand in by Friday January 17, 13:00

Exercise 1. Let (X, d) be a δ-hyperbolic metric space, $l \in \mathbb{N}$, and $\gamma : [0, l] \to X$ be a (K, L)-quasi-geodesics. Consider a curve $\gamma' : [0, l] \to X$ having the following properties: for every $i \in [0, l-1] \cap \mathbb{Z}$, the image of $\gamma'|_{[i,i+1]}$ is a geodesics with starting point $\gamma(i)$ and end-point $\gamma(i+1)$, and for all $x, y \in [i, i+1]$, $\frac{d(\gamma'(x),\gamma'(y))}{d(x,y)} = d(\gamma(i), \gamma(i+1))$. Show that there exists K', L' depending only on δ, K, L such that

1. γ' is a (K', L')-quasi-geodesics.
2. For all $s, t \in [0, l]$, $\ell(\gamma'|_{[s,t]}) \leq K' d(\gamma'(s), \gamma'(t)) + L'$ for all $s, t \in [0, L]$
3. The image of γ' is in the neighborhood with radius $K' + L'$ of the image of γ.

Exercise 2. Prove that the geometric realization of a tree is a 0-hyperbolic space.

Exercise 3. Let $\hat{\mathbb{C}}$ be the Riemann sphere, as in exercise sheet 4. Consider the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and the upper half plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The Poincaré metric on D is defined by $\frac{4dzd\bar{z}}{(1-|z|^2)^2}$. The Poincaré metric on H is defined by $\frac{dzd\bar{z}}{y^2}$, where $z = x + iy$.

1. Find a Möbius transformation that maps D onto H.
2. Prove that every Möbius transformation that maps D onto H transforms the Poincaré metric of D in the Poincaré metric on H.
3. Compute the distance between the points i and ib for the Poincaré metric for H (here $b \in \mathbb{R}_{>0}$).
4. Given four distinct points $x, y, z, w \in \hat{\mathbb{C}}$, let $[x, y, z, w] \in \hat{\mathbb{C}}$ be defined by $\frac{(z_1-z_2)(z_3-z_4)}{(z_3-z_2)(z_1-z_4)}$, the cross-ratio between the four points. Prove that this quantity is invariant by Möbius transformations.
5. For every $z, w \in D$, consider the unique circle containing z, w and perpendicular to ∂D. Denote by z^*, w^* the two extremes of the circle, on the side of z, w respectively. Prove that the distance between z and w in D is equal to $\log([z, w*, w, z*])$. Show that the same formula holds for H.

Exercise 4. Prove that the free product $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ is a hyperbolic group.