Exercise 1. Let \((X,d)\) be a \(\delta\)-hyperbolic metric space, \(l \in \mathbb{N}\), and \(\gamma : [0,l] \to X\) be a \((K,L)\)-quasi-geodesics. Consider a curve \(\gamma' : [0,l] \to X\) having the following properties: for every \(i \in [0,l-1] \cap \mathbb{Z}\), the image of \(\gamma'|_{[i,i+1]}\) is a geodesics with starting point \(\gamma(i)\) and end-point \(\gamma(i+1)\), and for all \(x,y \in [i,i+1]\), \(d(\gamma'(x),\gamma'(y)) = d(\gamma(i),\gamma(i+1))\). Show that there exists \(K',L'\) depending only on \(\delta,K,L\) such that

1. \(\gamma'\) is a \((K',L')\)-quasi-geodesics.
2. For all \(s,t \in [0,l]\), \(\ell(\gamma'|_{[s,t]}) \leq K'd(\gamma'(s),\gamma'(t)) + L'\) for all \(s,t \in [0,L]\)
3. The image of \(\gamma'\) is in the neighborhood with radius \(K'+L'\) of the image of \(\gamma\).

Exercise 2. Prove that the geometric realization of a tree is a 0-hyperbolic space.

Exercise 3. Let \(\hat{\mathbb{C}}\) be the Riemann sphere, as in exercise sheet 4. Consider the unit disc \(D = \{z \in \mathbb{C} \mid |z| < 1\}\) and the upper half plane \(\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}\). The Poincaré metric on \(D\) is defined by \(4dzd\bar{z}/(1-|z|^2)^2\). The Poincaré metric on \(\mathbb{H}\) is defined by \(dzd\bar{z}/y^2\), where \(z = x+iy\).

1. Find a Möbius transformation that maps \(D\) onto \(\mathbb{H}\).
2. Prove that every Möbius transformation that maps \(D\) onto \(\mathbb{H}\) transforms the Poincaré metric of \(D\) in the Poincaré metric on \(\mathbb{H}\).
3. Compute the distance between the points \(i\) and \(ib\) for the Poincaré metric for \(\mathbb{H}\) (here \(b \in \mathbb{R}_{>0}\)).
4. Given four distinct points \(x,y,z,w \in \hat{\mathbb{C}}\), let \([x,y,z,w] \in \hat{\mathbb{C}}\) be defined by \((z_1-z_2)(z_3-z_4)/(z_2-z_3)(z_1-z_4)\), the cross-ratio between the four points. Prove that this quantity is invariant by Möbius transformations.
5. For every \(z,w \in D\), consider the unique circle containing \(z,w\) and perpendicular to \(\partial D\). Denote by \(z^*,w^*\) the two extremes of the circle, on the side of \(z,w\) respectively. Prove that the distance between \(z\) and \(w\) in \(D\) is equal to \(\log([z,w^*,w,z^*])\). Show that the same formula holds for \(\mathbb{H}\).

Exercise 4. Prove that the free product \(\mathbb{Z}/m\mathbb{Z}*\mathbb{Z}/n\mathbb{Z}\) is a hyperbolic group.