RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 27.12.2013

Exercise sheet 10
Christmas exercises — Bonus sheet
To hand in by Friday January 10, 13:00

Given a non-empty tree T, a decoration of T is a labelling of every edge and of some vertices of T with colors taken from a set of three colors $F = \{1, 2, 3\}$. A vertex that has a color is called a ball.

A path of a single color is a sequence of edges $e_1, \dots, e_n$ of the same color, such that for $1 < i < n$, e_i shares one extreme with e_{i-1} and the other extreme with e_{i+1}. The extremes of a path are the vertices of e_1 and e_n that are not in common with e_2 and e_{n-1}. A light chain is a maximal path of a single color, i.e. a path of a single color that is not included in a longer path.

A perfect decoration is a decoration satisfying the following properties:

A Light chains have lenght exactly 3.

B A vertex is a ball of a certain color if and only if it is an extreme of a light chain of that color.

C At every vertex there are edges of all colors.

D For every vertex v, in the closed ball with center v and radius 1 there is exactly one vertex of every color.

Exercise 1. Prove that no non-empty tree admits a perfect decoration.

An admissible decoration is a decoration satisfying conditions A and B above, and the following conditions

C' At every vertex that is not a ball, there are edges of all colors.

D' For every vertex v that is not a ball, in the closed ball with center v and radius 1 there is exactly one vertex of every color.

E In every light chain, the two inner vertices are not balls.

Exercise 2. There exists a unique non-empty tree (up to isomorphisms) admitting an admissible decoration, and also the admissible decoration is unique (up to decoration-preserving isomorphisms). Draw a picture of a finite portion of this tree.

We will call the decorated tree constructed in the previous exercise a Christmas tree, and we will denote it by T. The group of automorphisms of this tree will be denoted by $\mathrm{Aut}(T)$, and the group of decoration-preserving auromorphisms will be denoted by $\mathrm{Dec}(T)$.

Exercise 3. The group $\mathrm{Aut}(T)$ is uncountable and is not finitely generated. The group $\mathrm{Dec}(T)$ is countable. Is it finitely generated?

Exercise 4. Is it true that all finitely generated subgroups of $\mathrm{Aut}(T)$ are of polynomial growth? Is it true that they are all of polynomial or exponential growth?

Exercise 5. Can you find infinitely many pairwise non-quasi-isometric trees?