RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometrische Gruppentheorie
Heidelberg, 29.11.2013

Exercise sheet 7
Growth types
To hand in by Friday December 6, 13:00

Exercise 1. Let G be a finitely generated group and let $S \subset G$ be a finite generating set.

1. Show that the growth function $\beta_{G,S}$ is sub-multiplicative:

$$\forall r, r' \in \mathbb{N}, \quad \beta_{G,S}(r + r') \leq \beta_{G,S}(r)\beta_{G,S}(r')$$

2. Prove that, if G is infinite, $\beta_{G,S}$ is strictly increasing.

Exercise 2 (Generalised growth functions)**.**

1. Show that for all $a, b \in \mathbb{R}_{>1}$, we have

$$(x \to a^x) \sim (x \to b^x).$$

2. Show that for all $a \in \mathbb{R}_{>1}$ and $p \in \mathbb{R}_{>0}$, we have

$$(x \to a^x) \succ (x \to x^p) \text{ and } (x \to a^x) \not\prec (x \to x^p)$$

3. Find a generalised growth function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with

$$f \prec (x \to e^x), \quad f \not\sim (x \to e^x) \text{ and } \forall a \in \mathbb{R}_{>0}, \ (x \to x^a) \prec f.$$

Exercise 3 (Heisenberg group)**.** Consider the subset

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \mid x, y, z \in \mathbb{Z} \right\}$$

1. Prove that H is a subgroup of $SL(3, \mathbb{Z})$.

2. Consider the maps $i : \mathbb{Z} \to H$ and $\pi : H \to \mathbb{Z}^2$ defined by

$$i(z) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = (x, y).$$

Prove that they are group homomorphisms, that $\ker \pi = i(\mathbb{Z})$, and that there is no homomorphism $s : \mathbb{Z}^2 \to H$ suct that πs is the identity of $\mathbb{Z}^2$.

3. Prove that H admits the following presentation

$$\langle X, Y, Z \mid [X, Y] = Z, [X, Z] = e, [Y, Z] = e \rangle$$

with

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Prove that $[Y^{-n}, X^n] = Z^{n^2}$.

5. Given $S = \{X, Y\}$, prove that for all $m, n, k \in \mathbb{Z}$, $d_S(x^m y^n z^k, e) \leq |m| + |n| + 6\sqrt{|k|}$.

6. Prove that $|m| + |n| \leq d_S(x^m y^n z^k, e)$ and $\sqrt{|k|} \leq d_S(x^m y^n z^k, e)$.

7. Prove that the growth function $\beta_{H,S}$ is quasi-equivalent to a polynomial of degree 4.