Exercise 1. Let $G$ be a finitely generated group and let $S \subset G$ be a finite generating set.

1. Show that the growth function $\beta_{G,S}$ is sub-multiplicative:
   \[ \forall r, r' \in \mathbb{N}, \quad \beta_{G,S}(r + r') \leq \beta_{G,S}(r) \beta_{G,S}(r') \]

2. Prove that, if $G$ is infinite, $\beta_{G,S}$ is strictly increasing.

Exercise 2 (Generalised growth functions).

1. Show that for all $a, b \in \mathbb{R}_{>1}$, we have
   \[ (x \to a^x) \sim (x \to b^x). \]

2. Show that for all $a \in \mathbb{R}_{>1}$ and $p \in \mathbb{R}_{>0}$, we have
   \[ (x \to a^x) \succ (x \to x^p) \quad \text{and} \quad (x \to a^x) \not\succ (x \to x^p) \]

3. Find a generalised growth function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with
   \[ f \prec (x \to e^x), \quad f \not\prec (x \to e^x) \quad \text{and} \quad \forall a \in \mathbb{R}_{>0}, \quad (x \to x^a) \prec f. \]

Exercise 3 (Heisenberg group). Consider the subset
\[ H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \mid x, y, z \in \mathbb{Z} \right\} \]

1. Prove that $H$ is a subgroup of $SL(3, \mathbb{Z})$.

2. Consider the maps $i : \mathbb{Z} \to H$ and $\pi : H \to \mathbb{Z}^2$ defined by
   \[ i(z) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) = (x, y). \]

Prove that they are group homomorphisms, that $\ker \pi = i(\mathbb{Z})$, and that there is no homomorphism $s : \mathbb{Z}^2 \to H$ such that $\pi s$ is the identity of $\mathbb{Z}^2$. 
3. Prove that $H$ admits the following presentation

$$(X, Y, Z \mid [X, Y] = Z, [X, Z] = e, [Y, Z] = e)$$

with

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

4. Prove that $[Y^{-n}, X^n] = Z^{n^2}$.

5. Given $S = \{X, Y\}$, prove that for all $m, n, k \in \mathbb{Z}$, $d_S(x^m y^n z^k, e) \leq |m| + |n| + 6 \sqrt{|k|}$.

6. Prove that $|m| + |n| \leq d_S(x^m y^n z^k, e)$ and $\sqrt{|k|} \leq d_S(x^m y^n z^k, e)$.

7. Prove that the growth function $\beta_{H,S}$ is quasi-equivalent to a polynomial of degree 4.