RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometrische Gruppentheorie
Heidelberg, 08.11.2013

Exercise sheet 4
Möbius transformations
To hand in by Friday November 15, 13:00

Exercise 1 (The Riemann sphere)**.** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. For every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$ and every element $z \in \hat{\mathbb{C}}$, let Az be the element $\frac{az+b}{cz+d} \in \hat{\mathbb{C}}$, with the convention that if $cz + d = 0$, then $Az = \infty$, and that $A\infty = \frac{a}{c}$ if $c \neq 0$ and ∞ if $c = 0$. Consider the map $\pi : \mathbb{C}^2 \setminus \{0\} \to \hat{\mathbb{C}}$ that sends $\begin{pmatrix} x \\ y \end{pmatrix}$ to x/y if $y \neq 0$, and to ∞ if $y = 0$.

1. Prove that for every $A \in GL(2, \mathbb{C})$ and $v \in \mathbb{C}^2 \setminus \{0\}$, $\pi(Av) = A\pi(v)$.
2. Prove that $GL(2, \mathbb{C}) \times \hat{\mathbb{C}} \ni (A, z) \to Az \in \hat{\mathbb{C}}$ is a group action.
3. Find the set of all elements of $GL(2, \mathbb{C})$ that fix every point of $\hat{\mathbb{C}}$, and prove that it is a normal subgroup. The quotient by that subgroup is denoted $PGL(2, \mathbb{C})$, and called the group of Möbius transformations.
4. Prove that the action is transitive.
5. Prove that v is an eigenvector for the matrix A if and only if $\pi(v)$ is in the fixed set of A.
6. Describe the stabilisers of all the points of $\hat{\mathbb{C}}$. Are they all isomorphic?
7. Give a classification of the elements of $GL(2,)$ up to conjugation.
8. Describe the fixed set of every element of $GL(2, \mathbb{C})$.

Exercise 2 (Circles)**.** A generalised circle in $\hat{\mathbb{C}}$ is either a circle in $\mathbb{C}$ (the set of points equidistant from a given point), or a straight line union with $\{\infty\}$. Every generalised circle disconnects $\hat{\mathbb{C}}$ in two parts, that are called discs.

1. Prove the following identity: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (bc - ad)/c^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$.
2. Prove that the image of a generalised circle for a matrix A is again a generalised circle.
3. Prove that for every two discs D_1, D_2, there exists a matrix A such that $A(D_1) = D_2$.

Exercise 3 (Classical Schottky group)**.** Let $A_1, B_1, A_2, B_2 \subset \hat{\mathbb{C}}$ be four discs whose closures are pair-wise disjoint. Choose two matrices $T_1, T_2 \in GL(2, \mathbb{C})$ such that $T_1\left(\hat{\mathbb{C}} \setminus \overline{A_1}\right) = B_1$ and $T_2\left(\hat{\mathbb{C}} \setminus \overline{A_2}\right) = B_2$. The subgroup of $GL(2, \mathbb{C})$ generated by T_1, T_2 is called a Schottky group of rank 2.

1. Prove that every Schottky group is free.
2. Find explicitly generators for 2 distinct Schottky groups of rank 2.