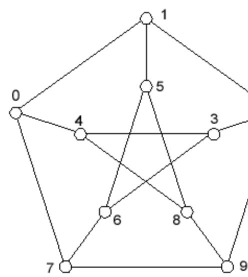
Vorlesung Geometrische Gruppentheorie
Heidelberg, 01.11.2013

Exercise sheet 3
Graphs and actions
To hand in by Friday November 8, 13:00

Exercise 1. Let B be a set, and consider the free group $F(B)$. Prove that $F(B)$ is finitely generated if and only if B is finite.

Exercise 2. Let G be a group, $N \lhd G$ a normal subgroup and $Q = G/N$ the quotient group. Denote by $\pi : G \to Q$ the projection to the quotient, and assume that there exists a homomorphism $s : Q \to G$ such that $\pi \circ s$ is the identity on Q (such an s is called a split of π). Prove that G is the semi-direct product of N and Q.

Exercise 3 (The Petersen graph)**.** Prove that the pictured graph, called the Petersen graph, is not the Cayley graph of any group, with reference to any set of generators. (Hint: you can use, without the need of proving it, the classification of the groups with ten elements, up to isomorphism: there are only the cyclic group and the dihedral group. Without this classification, the exercise requires more work.)



Exercise 4 (Cayley graphs)**.** Prove that the set $S = \{(5,2),(2,1)\}$ generates $\mathbb{Z}^2$, and draw a finite portion of the corresponding Cayley graph.

Exercise 5. Consider an equilateral triangle T in the plane $\mathbb{R}^2$, with vertices A, B, C. Let Γ be the group generated by the reflections around the three sides of T, this is a subgroup of the group of isometries of the plane.

1. Consider the set of triangles $\{g(T) \mid g \in \Gamma\}$, and show that they form a tiling of the plane (i.e. their union is the plane, and their interior parts are pairwise disjoint).
2. Find the orbits of the points A, B, C for the action of Γ on the plane. Do they belong to the same orbit?
3. Prove that, if $g(T) = T$, then $g = e$.
4. Find the stabiliser of every point of the plane.
5. Describe the Cayley graph of Γ with reference to some set of generators. (Hint: you can use the tiling.)
6. (Bonus) As a bonus exercise, you can also prove that Γ admits the following presentation:

$$\left\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 \right\rangle$$