**Exercise 1.** Let $B$ be a set, and consider the free group $F(B)$. Prove that $F(B)$ is finitely generated if and only if $B$ is finite.

**Exercise 2.** Let $G$ be a group, $N \triangleleft G$ a normal subgroup and $Q = G/N$ the quotient group. Denote by $\pi : G \to Q$ the projection to the quotient, and assume that there exists a homomorphism $s : Q \to G$ such that $\pi \circ s$ is the identity on $Q$ (such an $s$ is called a split of $\pi$). Prove that $G$ is the semi-direct product of $N$ and $Q$.

**Exercise 3 (The Petersen graph).** Prove that the pictured graph, called the Petersen graph, is not the Cayley graph of any group, with reference to any set of generators. (Hint: you can use, without the need of proving it, the classification of the groups with ten elements, up to isomorphism: there are only the cyclic group and the dihedral group. Without this classification, the exercise requires more work.)

![Petersen graph](image)

**Exercise 4 (Cayley graphs).** Prove that the set $S = \{(5, 2), (2, 1)\}$ generates $\mathbb{Z}^2$, and draw a finite portion of the corresponding Cayley graph.

**Exercise 5.** Consider an equilateral triangle $T$ in the plane $\mathbb{R}^2$, with vertices $A, B, C$. Let $\Gamma$ be the group generated by the reflections around the three sides of $T$, this is a subgroup of the group of isometries of the plane.

1. Consider the set of triangles $\{g(T) \mid g \in \Gamma\}$, and show that they form a tiling of the plane (i.e. their union is the plane, and their interior parts are pairwise disjoint).

2. Find the orbits of the points $A, B, C$ for the action of $\Gamma$ on the plane. Do they belong to the same orbit?

3. Prove that, if $g(T) = T$, then $g = e$.

4. Find the stabiliser of every point of the plane.

5. Describe the Cayley graph of $\Gamma$ with reference to some set of generators. (Hint: you can use the tiling.)

6. (Bonus) As a bonus exercise, you can also prove that $\Gamma$ admits the following presentation:

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 \rangle$$