RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 13

Symmetric spaces

Exercise 1. For the following groups and rays, compute the stabilizers G_ξ and the homogeneous space G/G_ξ:

(a) $G = SL_n(\mathbb{R})$, $\xi = \Phi_x(\mathrm{diag}(n-1, -1, \ldots, -1))$.

(b) $G = SL_n(\mathbb{R})$, $\xi = \Phi_x(\mathrm{diag}(n-k, \ldots, n-k, -k, \ldots, -k))$ (here $n-k$ has multiplicity k).

(c) $G = SL_n(\mathbb{R})$, $\xi = \Phi_x(\mathrm{diag}(n-1, n-3, \ldots, 1-n))$.

(d) $G = SO_0(p,q)$, $\xi = \Phi_x(X)$ where $X_{i,j} = 1$ if and only if $(i,j) \in \{(1, p+1), (p+1, 1)\}$ and is zero otherwise.

(e) $G = SO_0(p,q)$, $\xi = \Phi_x(X)$ where $X_{i,j} = 1$ if and only if $(i,j) \in \{(k, p+k), (k+1, k) | k \in \{1, \ldots, p\}\}$ and is zero otherwise.

Here $x \in G/K$ is the basepoint whose induced Cartan decomposition is standard.

Exercise 2. Let M be a Riemannian symmetric space of non compact type, $p \in M$ a basepoint, ξ be a point in $\partial_\infty(M)$ and let $\gamma(t) : \mathbb{R}^+ \to M$ be a geodesic ray in ξ with $\gamma(0) = p$. Recall that the Buseman function β_ξ is defined by

$$\beta_\xi(q) = \lim_{t\to\infty} d(q, \gamma(t)) - t.$$

(a) Show that

$$\beta_\xi^{-1}((-\infty, 0])) = \bigcup B(\gamma(t), t).$$

These sets are called *horoballs*.

(b) What is $\partial_\infty \mathbb{R}^n$? What are horoballs in $\mathbb{R}^n$?

(c) Show that the horoballs in M are contained in the image, under the exponential map, of the corresponding horoballs in T_pM.

Exercise 3. (Iwasawa decomposition)

(a) Determine the factors K, A, N of the Iwasawa decomposition relative to a regular point ξ for the following Lie groups: $SL_n(\mathbb{R})$, $Sp(2n, \mathbb{R})$.

(b) Given a unimodular basis of $\mathbb{R}^n$, let M be the matrix of change of basis, and let M' be the matrix of change of basis of the orthogonal basis obtained from the Gram-Schmidt process. Verify that M' is the K-factor of the Iwasawa decomposition of M.

(c) If we start with a symplectic basis of $\mathbb{R}^{2n}$, does the orthogonal basis obtained from the Gram-Schmidt process assume a special form?