RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 12

Symmetric spaces

Exercise 1. Let M be a Hadamard manifold, $C \subset M$ be a closed convex subset. Prove that

(a) For every $x \in M$ there exists a unique point $\pi_C(x)$ such that for every $q \in M$ it holds

$$d(\pi_C(x), x) \leq d(q, x).$$

(b) Show that the map $\pi_C : M \to C$ is 1-Lipschitz.

(c) Show that the function $x \mapsto d(x, \pi_C(x))$ is convex.

Exercise 2. Let M a Riemannian symmetric space of non-compact type.

(a) Show that for every geodesic $\gamma : \mathbb{R} \to M$ the parallel set $P(\gamma)$ is a totally geodesic submanifold of M. In particular it is a Riemannian symmetric space.

(b) Denote by $G_0 := \mathrm{Isom}^0(P(\gamma))$ and let $g = \sigma_{\gamma(1)}\sigma_{\gamma(0)} \in G_0$ denote a transvection along γ. Show that for every $h \in G_0$, it holds $hg = gh$.
(*Hint*: use that if h is a transvection, then $h\gamma$ is a geodesic parallel to γ, and that G_0 is generated by transvections).

Exercise 3. Let $M = S_n$ be the symmetric space associated to $SL(n, \mathbb{R})$ (see Sheet 8, Exercise 1). Recall that there is a Cartan decomposition of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ such that $\mathfrak{p}$ corresponds to symmetric matrices and $\mathfrak{k}$ corresponds to antisymmetric matrices.

(a) What is the maximal abelian subalgebra of $\mathfrak{p}$ containing $X_1 = \mathrm{diag}(n, n-2, \ldots, -n)$?

(b) What is the parallel set of the geodesic with tangent vector X_1?

(c) What is the centralizer of the element $X_2 = \mathrm{diag}(n-1, -1, \ldots, -1)$?

(d) Describe the parallel set of the geodesic with tangent vector X_2.

(e) Deduce that there is no $g \in O(n)$ with $gX_1 = X_2$.